



Intrinsic curvature of curves and surfaces and a Gauss–Bonnet theorem in the Heisenberg group

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Abstract We use a Riemannian approximation scheme to define a notion of *intrinsic Gaussian curvature* for a Euclidean C^2 -smooth surface in the Heisenberg group \mathbb{H} away from characteristic points, and a notion of *intrinsic signed geodesic curvature* for Euclidean C^2 -smooth curves on surfaces. These results are then used to prove a Heisenberg version of the Gauss–Bonnet theorem. An application to Steiner’s formula for the Carnot–Carathéodory distance in \mathbb{H} is provided.

Keywords Heisenberg group · Sub-Riemannian geometry · Riemannian approximation · Gauss–Bonnet theorem · Steiner formula

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1 Introduction

A full understanding of the notion of curvature has been at the core of studies in differential geometry since the foundational works of Gauss and Riemann. The aim of this paper is to propose a suitable candidate for the notion of *intrinsic Gaussian curvature* for Euclidean C^2 -smooth surfaces in the first Heisenberg group \mathbb{H} , adopting the so called *Riemannian approximation scheme*, which has proved to be a very powerful tool to address sub-Riemannian issues.

Referencing the seminal work of Gauss, we recall that to a compact and oriented Euclidean C^2 -smooth regular surface $\Sigma \subset \mathbb{R}^3$ we can attach the notions of *mean curvature* and *Gaussian curvature* as symmetric polynomials of the second fundamental form. To be more precise, for every $p \in \Sigma$ we have a well-defined *outward unit normal* vector field, $N(p) : \Sigma \rightarrow \mathbb{S}^2$, usually called the *Gauss normal map*. For every $p \in \Sigma$, the differential of the Gauss normal map $dN(p) : T_p\Sigma \rightarrow T_{N(p)}\mathbb{S}^2$, defines a positive definite and symmetric quadratic form on $T_p\Sigma$ whose two real eigenvalues are usually called *principal curvatures* of Σ at p . The arithmetic mean of these principal curvatures is the *mean curvature* and their product is the *Gaussian curvature*. The importance of the latter became particularly clear after Gauss’ famous *Theorema Egregium*, which asserts that Gaussian curvature is intrinsic and is also an isometric invariant of the surface Σ .

The notions of curvature, as briefly recalled above, can be extended to far more general situations, for instance to submanifolds of higher codimension in \mathbb{R}^n , and also to the broader geometrical context provided by Riemannian geometry, as was done by Riemann. In particular, we will consider 2-dimensional Riemannian manifolds isometrically embedded into 3-dimensional Riemannian manifolds. We refer to Sect. 5 for details.

Our interest in the study of curvatures of surfaces in \mathbb{H} is motivated by the still ongoing studies in the context of sub-Riemannian manifolds or more specific structures like Carnot groups, whose easiest example is provided by the first Heisenberg group \mathbb{H} . Restricting our attention to \mathbb{H} , there is a currently accepted notion of *horizontal mean curvature* \mathcal{H}_0 at non-characteristic points of Euclidean regular surfaces. This notion has been considered by Pauls [28] via the method of Riemannian approximants, but has also been proved to be equivalent to other notions of mean curvature appearing in the literature (e.g. [12] or [19]).

The method of Riemannian approximants relies on a famous result due to Gromov, which states that the metric space (\mathbb{H}, d_{cc}) can be obtained as the pointed Gromov–Hausdorff limit of a family of metric spaces $(\mathbb{R}^3, g_\epsilon)$, where g_ϵ is a suitable family of Riemannian metrics. The Riemannian approximation scheme has also proved to be a very efficient tool in more analytical settings, for instance, in the study of estimates for fundamental solutions of the

sub-Laplacian $\Delta_{\mathbb{H}}$ (e.g. [7, 17]) as well as regularity theory for sub-Riemannian curvature flows (e.g. [8]). The preceding represents only a small sample of the many applications of the Riemannian approximation method in sub-Riemannian geometric analysis, and we refer the reader to the previously cited papers for more information and references to other work in the literature. The monograph [9] provides a detailed description of the Riemannian approximation scheme in the setting of the Heisenberg group.

Let us denote by X_1, X_2 and X_3 the left-invariant vector fields which span the Lie algebra \mathfrak{h} of \mathbb{H} . In particular, $[X_1, X_2] = X_3$. In order to exploit the contact nature of \mathbb{H} it is customary to define an inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ which makes $\{X_1, X_2\}$ an orthonormal basis. A possible way to define a Riemannian scalar product is to set $X_3^\epsilon := \sqrt{\epsilon} X_3$ for every $\epsilon > 0$, and then to extend $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ to a scalar product $\langle \cdot, \cdot \rangle_\epsilon$ which makes $\{X_1, X_2, X_3^\epsilon\}$ an orthonormal basis. The family of metric spaces $(\mathbb{R}^3, g_\epsilon)$ converges to (\mathbb{H}, d_{cc}) in the pointed Gromov–Hausdorff sense.

Within this family of Riemannian manifolds, we can now perform computations adopting the unique Levi-Civita connection associated to the family of Riemannian metrics g_ϵ . Obviously, all the results are expected to depend on the positive constant ϵ . The plan is to extract horizontal notions out of the computed objects and to study their asymptotics in ϵ as $\epsilon \rightarrow 0^+$. This is the technique adopted in [28] to define a notion of horizontal mean curvature.

It is natural to ask whether such a method can be employed to study the curvature of curves, and especially to articulate an appropriate notion of *intrinsic Gaussian curvature*. One attempt in this direction has been carried out in [10], where the authors proposed a notion of horizontal second fundamental form in relation with H -convexity. A different notion of intrinsic Gaussian curvature for graphs has been suggested in [18].

Our approach follows closely the classical theory of Riemannian geometry and leads us to the following notion of *intrinsic curvature* for a Euclidean C^2 -smooth and regular curve $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ defined on an interval $[a, b]$ and taking values in \mathbb{H} :

$$k_\gamma^0 = \begin{cases} \frac{|\dot{\gamma}_1 \ddot{\gamma}_2 - \dot{\gamma}_2 \ddot{\gamma}_1|}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^{3/2}}, & \text{if } \gamma(t) \text{ is a horizontal point of } \gamma, \\ \frac{\sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}}{|\omega(\dot{\gamma})|}, & \text{if } \gamma(t) \text{ is not a horizontal point of } \gamma. \end{cases} \tag{1.1}$$

Here $\omega = dx_3 - \frac{1}{2}(x_1 dx_2 - x_2 dx_1)$ is the standard contact form on \mathbb{R}^3 . We stress that, when dealing with *purely* horizontal curves, the above notion of curvature is already known and appears frequently in the literature.

An analogous procedure allow us to define also a notion of *intrinsic signed geodesic curvature* for Euclidean C^2 -smooth and regular curves $\gamma = (\gamma_1, \gamma_2, \gamma_3) : [a, b] \rightarrow \Sigma \subset \mathbb{H}$ living on a surface $\Sigma = \{x \in \mathbb{H} : u(x) = 0\}$, with $u \in C^2(\mathbb{R}^3)$. This notion takes the form

$$k_{\gamma, \Sigma}^{0,s} = \begin{cases} \frac{\bar{p}\dot{\gamma}_1 + \bar{q}\dot{\gamma}_2}{|\omega(\dot{\gamma})|}, & \text{if } \gamma(t) \text{ is a non-horizontal point,} \\ 0, & \text{if } \gamma(t) \text{ is a horizontal point,} \end{cases} \tag{1.2}$$

where $\nabla_{\mathbb{H}}u = (X_1u, X_2u)$, $\bar{p} = \frac{X_1u}{\|\nabla_{\mathbb{H}}u\|_{\mathbb{H}}}$ and $\bar{q} = \frac{X_2u}{\|\nabla_{\mathbb{H}}u\|_{\mathbb{H}}}$. We refer to Sects. 3 and 4 for precise statements and definitions.

In the same spirit we introduce a notion of *intrinsic Gaussian curvature* \mathcal{K}_0 away from characteristic points. We will work with Euclidean C^2 -smooth surfaces $\Sigma = \{x \in \mathbb{H} : u(x) = 0\}$, whose characteristic set $C(\Sigma)$ is defined as the set of points $x \in \Sigma$ where $\nabla_{\mathbb{H}}u(x) = (0, 0)$. The explicit expression of \mathcal{K}_0 reads as follows:

$$\mathcal{K}_0 = - \left(\frac{X_3 u}{\|\nabla_{\mathbb{H}u}\|_{\mathbb{H}}} \right)^2 - \left(\frac{X_2 u}{\|\nabla_{\mathbb{H}u}\|_{\mathbb{H}}} \right) X_1 \left(\frac{X_3 u}{\|\nabla_{\mathbb{H}u}\|_{\mathbb{H}}} \right) + \left(\frac{X_1 u}{\|\nabla_{\mathbb{H}u}\|_{\mathbb{H}}} \right) X_2 \left(\frac{X_3 u}{\|\nabla_{\mathbb{H}u}\|_{\mathbb{H}}} \right). \quad (1.3)$$

A few remarks are now in order. The Riemannian approximation scheme which we use, can in general depend upon the choice of the complement to the horizontal distribution. Nevertheless, in the context of \mathbb{H} the choice which we have adopted is rather natural. We want also to stress that the existence of the limit defining the intrinsic curvature of a surface depends crucially on the cancellation of certain divergent quantities in the limit as $\epsilon \rightarrow 0^+$. As presented here, such cancellation stems from the specific choice of the adapted frame bundle on the surface, and on symmetries of the underlying left-invariant group structure on the Heisenberg group. It is an interesting question to understand to what extent similar phenomena hold in other sub-Riemannian geometric structures. We plan to return to this problem in future work.

We want also to remark that the quantity in (1.3) cannot easily be viewed as a symmetric polynomial of any kind of horizontal Hessian. Moreover, the expression of \mathcal{K}_0 written above resembles one of the integrands, the one which would be expected to replace the classical Gaussian curvature, appearing in the Heisenberg Steiner's formula proved in [5]. The discrepancy between these two quantities will be the object of further investigation.

The definition of an appropriate notion of intrinsic Gaussian curvature leads to the question of proving a suitable Heisenberg version of the celebrated Gauss–Bonnet Theorem, which is the first main result of this paper. For a surface $\Sigma = \{x \in \mathbb{H} : u(x) = 0\}$, with $u \in C^2(\mathbb{R}^3)$, our main theorem is as follows.

Theorem 1.1 *Let $\Sigma \subset \mathbb{H}$ be a regular surface with finitely many boundary components $(\partial\Sigma)_i$, $i \in \{1, \dots, n\}$, given by Euclidean C^2 -smooth regular and closed curves $\gamma_i : [0, 2\pi] \rightarrow (\partial\Sigma)_i$. Let \mathcal{K}_0 be the intrinsic Gaussian curvature of Σ , and $k_{\gamma_i, \Sigma}^{0,s}$ the intrinsic signed geodesic curvature of γ_i relative to Σ . Suppose that the characteristic set $C(\Sigma)$ satisfies $\mathcal{H}_E^1(C(\Sigma)) = 0$, and that $\|\nabla_{\mathbb{H}u}\|_{\mathbb{H}}^{-1}$ is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure near the characteristic set $C(\Sigma)$. Then*

$$\int_{\Sigma} \mathcal{K}_0 d\mathcal{H}_{cc}^3 + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i, \Sigma}^{0,s} d\dot{\gamma}_i = 0.$$

The sharpness of the assumption made on the 1-dimensional Euclidean Hausdorff measure $\mathcal{H}_E^1(C(\Sigma))$ of the characteristic set $C(\Sigma)$ is discussed in Sect. 6, while comments on the local summability asked for $\|\nabla_{\mathbb{H}u}\|_{\mathbb{H}}^{-1}$ are postponed to Sect. 8. The measure $d\dot{\gamma}_i$ on the i th boundary curve $(\partial\Sigma)_i$ in the statement of Theorem 1.1 is the limit of scaled length measures in the Riemannian approximants. We remark that this measure vanishes along purely horizontal boundary curves.

Gauss–Bonnet type theorems have previously been obtained by Diniz and Veloso [22] for non-characteristic surfaces in \mathbb{H} , and by Agrachev [2] and Agrachev et al. [1] for almost-Riemannian and contact sub-Riemannian structures. We would also like to mention the results obtained by Bao and Chern [6] in Finsler spaces.

The notion of *horizontal mean curvature* has featured in a long and ongoing research program concerning the study of *constant mean curvature* surfaces in \mathbb{H} , especially in relation to *Pansu's isoperimetric problem* (e.g. [20, 25, 27, 29] or [9]). A simplified version of the aforementioned Gauss–Bonnet Theorem 1.1, i.e., when we consider a compact, oriented, Euclidean C^2 -surface with no boundary, or with boundary consisting of fully horizontal

curves, ensures that the only compact surfaces with constant *intrinsic Gaussian curvature* have $\mathcal{K}_0 = 0$.

Our main application concerns a Steiner’s formula for non-characteristic surfaces. This result (see Theorem 7.3) is a simplification of the Steiner’s formula recently proved in [5].

The structure of the paper is as follows. In Sect. 2 we provide a short introduction to the first Heisenberg group \mathbb{H} and the notation which we will use throughout the paper, with a special focus to the Riemannian approximation scheme. In Sects. 3 and 4 we adopt the Riemannian approximation scheme to derive the expression (1.1) for the intrinsic curvature of Euclidean C^2 -smooth curves in \mathbb{H} , and the expression (1.2) for the intrinsic geodesic curvature of curves on surfaces. In Sect. 5, we will derive the expression (1.3) for the *intrinsic Gaussian curvature*. In Sect. 6 we prove Theorem 1.1 and its corollaries. Section 7 contains the proof of Steiner’s formula for non-characteristic surfaces. In Sect. 8 we present a Fenchel-type theorem for horizontal closed curves (see Theorem 8.5) and we pose some questions. One of the more interesting and challenging questions concerns the summability of the intrinsic Gaussian curvature \mathcal{K}_0 with respect to the Heisenberg perimeter measure near isolated characteristic points. This summability issue is closely related to the open problem posed in [21] concerning the summability of the horizontal mean curvature \mathcal{H}_0 with respect to the Riemannian surface measure near the characteristic set. To end the paper, we add an “Appendix” where we collect several examples of surfaces for which we compute explicitly the intrinsic Gaussian curvature \mathcal{K}_0 .

2 Notation and background

Let \mathbb{H} be the first Heisenberg group where the non-commutative group law is given by

$$(y_1, y_2, y_3) * (x_1, x_2, x_3) = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 - \frac{1}{2}(x_1y_2 - x_2y_1) \right).$$

The corresponding Lie algebra of left-invariant vector fields admits a 2-step stratification, $\mathfrak{h} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$, where $\mathfrak{v}_1 = \text{span}\{X_1, X_2\}$ and $\mathfrak{v}_2 = \text{span}\{X_3\}$ for $X_1 = \partial_{x_1} - \frac{1}{2}x_2\partial_{x_3}$, $X_2 = \partial_{x_2} + \frac{1}{2}x_1\partial_{x_3}$ and $X_3 = [X, Y] = \partial_{x_3}$. On \mathbb{H} we consider also the standard contact form of \mathbb{R}^3 :

$$\omega = dx_3 - \frac{1}{2}(x_1dx_2 - x_2dx_1).$$

The left-invariant vector fields X_1 and X_2 play a major role in the theory of the Heisenberg group because they span a two-dimensional plane distribution $H\mathbb{H}$, known as the *horizontal distribution*, which is also the kernel of the contact form ω :

$$H_x\mathbb{H} := \text{span}\{X_1(x), X_2(x)\} = (\text{Ker } \omega)(x), \quad x \in \mathbb{H}.$$

This smooth distribution of planes is a subbundle of the tangent bundle of \mathbb{H} , and it is a non integrable distribution because $[X_1, X_2] = X_3 \notin H\mathbb{H}$. We can define an inner product $\langle \cdot, \cdot \rangle_{x,\mathbb{H}}$ on $H\mathbb{H}$, so that for every $x \in \mathbb{H}$, $\{X_1(x), X_2(x)\}$ forms an orthonormal basis of $H_x\mathbb{H}$. We will then denote by $\| \cdot \|_{x,\mathbb{H}}$ the horizontal norm induced by the scalar product $\langle \cdot, \cdot \rangle_{x,\mathbb{H}}$. In both cases, we will omit the dependence on the base point $x \in \mathbb{H}$ when it is clear.

Definition 2.1 An absolutely continuous curve $\gamma : [a, b] \subset \mathbb{R} \rightarrow \mathbb{H}$ is said to be horizontal if $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{H}$ for a.e. $t \in [a, b]$.

Definition 2.2 Let $\gamma : [a, b] \rightarrow \mathbb{H}$ a horizontal curve. The horizontal length $l_{\mathbb{H}}(\gamma)$ of γ is defined as

$$l_{\mathbb{H}}(\gamma) := \int_a^b \|\dot{\gamma}\|_{\mathbb{H}} dt.$$

It is standard to equip the Heisenberg group \mathbb{H} with a path-metric known as the Carnot–Carathéodory, or cc , distance:

Definition 2.3 Let $x, y \in \mathbb{H}$, with $x \neq y$. The cc distance between x, y is defined as

$$d_{cc}(x, y) := \inf\{l_{\mathbb{H}}(\gamma) \mid \gamma : [a, b] \rightarrow \mathbb{H}, \gamma(a) = x, \gamma(b) = y\}$$

Dilations of the Heisenberg group are defined as follows:

$$\delta_r(x_1, x_2, x_3) = (rx_1, rx_2, r^2x_3), \quad r > 0 \quad (2.1)$$

It is easy to verify that dilations are compatible with the group operation:

$$\delta_r(y * x) = \delta_r(y) * \delta_r(x), \quad x, y \in \mathbb{H}, \quad r > 0,$$

and that the cc distance is homogeneous of order one with respect to dilations:

$$d_{cc}(\delta_r(x), \delta_r(y)) = r d_{cc}(x, y), \quad x, y \in \mathbb{H}, \quad r > 0.$$

The scaling behavior of the left-invariant vector fields X_1, X_2, X_3 with respect to dilations is as follows:

$$X_1(f \circ \delta_r) = r X_1 f \circ \delta_r, \quad X_2(f \circ \delta_r) = r X_2 f \circ \delta_r, \quad X_3(f \circ \delta_r) = r^2 X_3 f \circ \delta_r.$$

We are now ready to implement the Riemannian approximation scheme. First, let us define $X_3^\epsilon := \sqrt{\epsilon} X_3$ for $\epsilon > 0$. We define a family of Riemannian metrics $(g_\epsilon)_{\epsilon > 0}$ on \mathbb{R}^3 such that $\{X_1, X_2, X_3^\epsilon\}$ becomes an orthonormal basis. The choice of this specific family of Riemannian metrics on \mathbb{R}^3 is indicated by the following theorem.

Theorem 2.4 (Gromov) *The family of metric spaces $(\mathbb{R}^3, g_\epsilon)$ converges to (\mathbb{H}, d_{cc}) in the pointed Gromov–Hausdorff sense as $\epsilon \rightarrow 0^+$.*

This deep result continues to hold even for more general Carnot groups, but there is one additional feature which is valid for \mathbb{H} :

Proposition 2.5 *Any length minimizing horizontal curve γ joining $x \in \mathbb{H}$ to the origin $0 \in \mathbb{H}$ is the uniform limit as $\epsilon \rightarrow 0^+$ of geodesic arcs joining x to 0 in the Riemannian manifold $(\mathbb{R}^3, g_\epsilon)$.*

For both results, we refer to [9, Chapter 2].

Continuing with notation, the scalar product that makes $\{X_1, X_2, X_3^\epsilon\}$ an orthonormal basis will be denoted by $\langle \cdot, \cdot \rangle_\epsilon$. Explicitly, this means that, given $V = v_1 X_1 + v_2 X_2 + v_3 X_3$ and $W = w_1 X_1 + w_2 X_2 + w_3 X_3$,

$$\langle V, W \rangle_\epsilon = v_1 w_1 + v_2 w_2 + \frac{1}{\epsilon} v_3 w_3.$$

Obviously, if we write V and W in the $\{X_1, X_2, X_3^\epsilon\}$ basis, i.e., $V = v_1 X_1 + v_2 X_2 + v_3^\epsilon X_3^\epsilon$ where $v_3^\epsilon = \frac{v_3}{\sqrt{\epsilon}}$ (and similarly for W), we have

$$\langle V, W \rangle_\epsilon = v_1 w_1 + v_2 w_2 + v_3^\epsilon w_3^\epsilon = v_1 w_1 + v_2 w_2 + \frac{1}{\epsilon} v_3 w_3.$$

The following relations allow us to switch from the standard basis $\{e_1, e_2, e_3\}$ to $\{X_1, X_2, X_3^\epsilon\}$ and vice versa:

$$\begin{cases} e_1 = X_1 + \frac{1}{2\sqrt{\epsilon}}x_2X_3^\epsilon, \\ e_2 = X_2 - \frac{1}{2\sqrt{\epsilon}}x_1X_3^\epsilon, \\ e_3 = \frac{1}{\sqrt{\epsilon}}X_3^\epsilon, \end{cases} \quad \text{and} \quad \begin{cases} X_1 = e_1 - \frac{1}{2}x_2e_3, \\ X_2 = e_2 + \frac{1}{2}x_1e_3, \\ X_3^\epsilon = \sqrt{\epsilon}e_3. \end{cases}$$

In exponential coordinates, the metric g_ϵ is represented by the 3×3 symmetric matrix $(g_\epsilon)_{ij} := \langle e_i, e_j \rangle_\epsilon$, for $i, j = 1, 2, 3$. In particular,

$$g_\epsilon(x_1, x_2, x_3) = \begin{pmatrix} 1 + \frac{1}{4\epsilon}x_2^2 & -\frac{1}{4\epsilon}x_1x_2 & \frac{1}{2\epsilon}x_2 \\ -\frac{1}{4\epsilon}x_1x_2 & 1 + \frac{1}{4\epsilon}x_1^2 & -\frac{1}{2\epsilon}x_1 \\ \frac{1}{2\epsilon}x_2 & -\frac{1}{2\epsilon}x_1 & \frac{1}{\epsilon} \end{pmatrix}.$$

Then $\det(g_\epsilon(x)) = \frac{1}{\epsilon}$ and

$$g_\epsilon^{-1}(x_1, x_2, x_3) = \begin{pmatrix} 1 & 0 & -\frac{x_2}{2} \\ 0 & 1 & \frac{x_1}{2} \\ -\frac{1}{2}x_2 & \frac{1}{2}x_1 & \frac{4\epsilon+(x_1^2+x_2^2)}{4} \end{pmatrix}$$

Following the classical notation of Riemannian geometry, we will denote by g_{ij} the elements of the matrix g_ϵ , and by g^{ij} the elements of its inverse g_ϵ^{-1} .

A standard computational tool in Riemannian geometry is the notion of *affine connection*.

Definition 2.6 Let $\mathcal{X}(M)$ be the set of C^∞ -smooth vector fields on a Riemannian manifold M . Let $\mathcal{D}(M)$ be the ring of real-valued C^∞ -smooth functions on M . An affine connection ∇ on M is a mapping

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M),$$

usually denoted by $(X, Y) \mapsto \nabla_X Y$, such that:

- (i) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$.
- (ii) $\nabla_X(Y + Z) = \nabla_X(Y) + \nabla_X(Z)$.
- (iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

for every $X, Y, Z \in \mathcal{X}(M)$ and for every $f, g \in \mathcal{D}(M)$.

It is well known that every Riemannian manifold is equipped with a privileged affine connection: the Levi-Civita connection ∇ . This is the unique affine connection which is compatible with the given Riemannian metric and symmetric, i.e.,

$$X\langle Y, Z \rangle_\epsilon = \langle \nabla_X Y, Z \rangle_\epsilon + \langle Y, \nabla_X Z \rangle_\epsilon$$

and

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for every $X, Y, Z \in \mathcal{X}(M)$. A direct proof of this fact yields the famous *Koszul identity*:

$$\begin{aligned} \langle Z, \nabla_X Y \rangle_\epsilon &= \frac{1}{2} (X \langle Y, Z \rangle_\epsilon + Y \langle Z, X \rangle_\epsilon - Z \langle X, Y \rangle_\epsilon \\ &\quad - \langle [X, Z], Y \rangle_\epsilon - \langle [Y, Z], X \rangle_\epsilon - \langle [X, Y], Z \rangle_\epsilon) \end{aligned} \quad (2.2)$$

for $X, Y, Z \in \mathcal{X}(M)$.

It is possible to write the Levi-Civita connection ∇ in a local frame by making use of the Christoffel symbols Γ_{ij}^m . In our case, due to the specific nature of the Riemannian manifold $(\mathbb{R}^3, g_\epsilon)$, we can use a global chart given by the identity map of \mathbb{R}^3 . The Christoffel symbols are uniquely determined by

$$\nabla_{e_i} e_j = \Gamma_{ij}^m e_m, \quad i, j, m = 1, 2, 3.$$

Lemma 2.7 *The Christoffel symbols Γ_{ij}^m of the Levi-Civita connection ∇ of $(\mathbb{R}^3, g_\epsilon)$ are given by*

$$\Gamma_{ij}^1 = \begin{cases} 0, & (i, j) \in \{(1, 1), (1, 3), (3, 1), (3, 3)\}, \\ \frac{1}{4\epsilon} x_2, & (i, j) \in \{(1, 2), (2, 1)\}, \\ -\frac{1}{2\epsilon} x_1, & (i, j) = (2, 2), \\ \frac{1}{2\epsilon}, & (i, j) \in \{(2, 3), (3, 2)\}, \end{cases} \quad (2.3)$$

$$\Gamma_{ij}^2 = \begin{cases} -\frac{1}{2\epsilon} x_2, & (i, j) = (1, 1), \\ \frac{1}{4\epsilon} x_1, & (i, j) \in \{(1, 2), (2, 1)\}, \\ -\frac{1}{2\epsilon}, & (i, j) \in \{(1, 3), (3, 1)\}, \\ 0, & (i, j) \in \{(2, 2), (2, 3), (3, 2), (3, 3)\}, \end{cases} \quad (2.4)$$

and

$$\Gamma_{ij}^3 = \begin{cases} -\frac{1}{4\epsilon} x_1 x_2, & (i, j) = (1, 1), \\ \frac{1}{8\epsilon} (x_1^2 - x_2^2), & (i, j) \in \{(1, 2), (2, 1)\}, \\ -\frac{1}{4\epsilon} x_1, & (i, j) \in \{(1, 3), (3, 1)\}, \\ \frac{1}{4\epsilon} x_1 x_2, & (i, j) = (2, 2), \\ -\frac{1}{4\epsilon} x_2, & (i, j) \in \{(2, 3), (3, 2)\}, \\ 0, & (i, j) = (3, 3). \end{cases} \quad (2.5)$$

Proof It is a direct computation using

$$\Gamma_{ij}^m = \frac{1}{2} \sum_{k=1}^3 \left\{ \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right\} g^{km},$$

for $i, j, m = 1, 2, 3$. □

We now compute the Levi-Civita connection ∇ associated to the Riemannian metric g_ϵ .

Lemma 2.8 *The action of the Levi-Civita connection ∇ of $(\mathbb{R}^3, g_\epsilon)$ on the vectors X_1, X_2 and X_3^ϵ is given by*

$$\begin{aligned} \nabla_{X_1} X_2 &= -\nabla_{X_2} X_1 = \frac{1}{2} X_3, \\ \nabla_{X_1} X_3^\epsilon &= \nabla_{X_3^\epsilon} X_1 = -\frac{1}{2\sqrt{\epsilon}} X_2, \\ \nabla_{X_2} X_3^\epsilon &= \nabla_{X_3^\epsilon} X_2 = \frac{1}{2\sqrt{\epsilon}} X_1. \end{aligned}$$

Proof It follows from a direct application of the Koszul identity (2.2), which here simplifies to

$$\langle Z, \nabla_X Y \rangle_\epsilon = -\frac{1}{2} \left(\langle [X, Z], Y \rangle_\epsilon + \langle [Y, Z], X \rangle_\epsilon + \langle [X, Y], Z \rangle_\epsilon \right).$$

□

To make the paper self-contained, we recall here the definitions of Riemann curvature tensor R and of sectional curvature.

Definition 2.9 The Riemann curvature tensor R of a Riemannian manifold M is a mapping $R(X, Y) : \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ defined as follows

$$R(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z, \quad Z \in \mathcal{X}(M).$$

Remark 2.10 Note that the Riemann curvature tensor R satisfies the functional property

$$R(fX, Y)Z = R(X, fY)Z = R(X, Y)(fZ) = fR(X, Y)Z, \tag{2.6}$$

for every $X, Y, Z \in \mathcal{X}(M)$ and every $f \in \mathcal{D}(M)$.

Definition 2.11 Let M be a Riemannian manifold and let $\Pi \subset T_p M$ be a two-dimensional subspace of the tangent space $T_p M$. Let $\{E_1, E_2\}$ be two linearly independent vectors in Π . The sectional curvature $K(E_1, E_2)$ of M is defined as

$$K(E_1, E_2) := \frac{\langle R(E_1, E_2)E_1, E_2 \rangle}{|E_1 \wedge E_2|^2},$$

where \wedge denotes the usual wedge product.

One of the main reasons to introduce the notion of affine connection, is to be able to differentiate smooth vector fields along curves: this operation is known as *covariant differentiation* (see [24], Chapter 2). Formally, let $Z = z_1 e_1 + z_2 e_2 + z_3 e_3$ be a smooth vector field (written in the standard basis of \mathbb{R}^3) along a curve $\gamma = \gamma(t)$. The covariant derivative of Z along the curve γ is given by

$$D_t Z = \sum_{m=1}^3 \left\{ \frac{dz_m}{dt} + \sum_{i,j=1}^3 \Gamma_{ij}^m z_j \frac{dx_i}{dt} \right\} e_m,$$

where the x_i 's are the coordinates of γ in a local chart and Γ_{ij}^m are the Christoffel symbols introduced before.

In particular, if $Z = \dot{\gamma}$, we have

$$D_t \dot{\gamma} = \sum_{m=1}^3 \left\{ \dot{\gamma}_m + \sum_{i,j=1}^3 \Gamma_{ij}^m \dot{\gamma}_i \dot{\gamma}_j \right\} e_m. \tag{2.7}$$

3 Riemannian approximation of curvature of curves

Let us define the objects we are going to study in this section.

Definition 3.1 Let $\gamma : [a, b] \rightarrow (\mathbb{R}^3, g_\epsilon)$ be a Euclidean C^1 -smooth curve. We say that γ is regular if

$$\dot{\gamma}(t) \neq 0, \quad \text{for every } t \in [a, b].$$

Moreover, we say that $\gamma(t)$ is a horizontal point of γ if

$$\omega(\dot{\gamma}(t)) = \dot{\gamma}_3(t) - \frac{1}{2} (\gamma_1(t)\dot{\gamma}_2(t) - \gamma_2(t)\dot{\gamma}_1(t)) = 0.$$

Definition 3.2 Let $\gamma : [a, b] \subset \mathbb{R} \rightarrow (\mathbb{R}^3, g_\epsilon)$ be a Euclidean C^2 -smooth regular curve in the Riemannian manifold $(\mathbb{R}^3, g_\epsilon)$. The curvature k_γ^ϵ of γ at $\gamma(t)$ is defined as

$$k_\gamma^\epsilon := \sqrt{\frac{\|D_t \dot{\gamma}\|_\epsilon^2}{\|\dot{\gamma}\|_\epsilon^4} - \frac{\langle D_t \dot{\gamma}, \dot{\gamma} \rangle_\epsilon^2}{\|\dot{\gamma}\|_\epsilon^6}}. \tag{3.1}$$

We stress that the above definition is well posed, indeed by Cauchy-Schwarz,

$$\frac{\|D_t \dot{\gamma}\|_\epsilon^2}{\|\dot{\gamma}\|_\epsilon^4} - \frac{\langle D_t \dot{\gamma}, \dot{\gamma} \rangle_\epsilon^2}{\|\dot{\gamma}\|_\epsilon^6} \geq \frac{\|\dot{\gamma}\|_\epsilon^2 \|D_t \dot{\gamma}\|_\epsilon^2 - \|\dot{\gamma}\|_\epsilon^2 \|D_t \dot{\gamma}\|_\epsilon^2}{\|\dot{\gamma}\|_\epsilon^6} = 0.$$

Remark 3.3 We recall that in Riemannian geometry the standard definition of curvature for a curve γ parametrized by arc length is $k_\gamma^\epsilon := \|D_t \dot{\gamma}\|_\epsilon$. The one we gave before is just more practical to perform computations for curves with an arbitrary parametrization.

Let us briefly recall the definition of g_ϵ -geodesics, cf. [9, Chapter 2]. For a Euclidean C^2 -smooth regular curve $\gamma : [a, b] \rightarrow (\mathbb{R}^3, g_\epsilon)$, define its *penalized energy functional* E_ϵ to be

$$E_\epsilon(\gamma) := \int_a^b \left(|\dot{\gamma}_1(t)|^2 + |\dot{\gamma}_2(t)|^2 + \frac{1}{\epsilon} |\omega(\dot{\gamma}(t))|^2 \right) dt.$$

Using a standard variational argument, we can derive the system of Euler-Lagrange equations for the functional E_ϵ : we will call g_ϵ -geodesics the critical points, which are actually curves, of the functional E_ϵ . In other words, we will say that γ is a g_ϵ -geodesic, if for every $t \in [a, b]$ it holds that

$$\begin{cases} \ddot{\gamma}_1(t) = -\frac{1}{\epsilon} \dot{\gamma}_2(t)\omega(\dot{\gamma}(t)), \\ \ddot{\gamma}_2(t) = \frac{1}{\epsilon} \dot{\gamma}_1(t)\omega(\dot{\gamma}(t)), \\ (\omega(\dot{\gamma}(t))|_{\gamma(t)})' = 0. \end{cases} \tag{3.2}$$

We are now ready to present a first result concerning the curvature k_γ^ϵ .

Lemma 3.4 Let $\gamma : [a, b] \rightarrow (\mathbb{R}^3, g_\epsilon)$ be a Euclidean C^2 -smooth regular curve in the Riemannian manifold $(\mathbb{R}^3, g_\epsilon)$. Then

$$k_\gamma^\epsilon = \sqrt{\frac{(\dot{\gamma}_1 + \frac{1}{\epsilon} \dot{\gamma}_2 \omega(\dot{\gamma}))^2 + (\dot{\gamma}_2 - \frac{1}{\epsilon} \dot{\gamma}_1 \omega(\dot{\gamma}))^2 + \frac{1}{\epsilon} \omega(\dot{\gamma})^2}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \frac{1}{\epsilon} \omega(\dot{\gamma})^2)^2} - \frac{(\dot{\gamma}_1 \dot{\gamma}_1 + \dot{\gamma}_2 \dot{\gamma}_2 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \omega(\ddot{\gamma}))^2}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \frac{1}{\epsilon} \omega(\dot{\gamma})^2)^3}}. \tag{3.3}$$

In particular, if $\gamma(t)$ is a horizontal point of γ ,

$$k_\gamma^\epsilon = \sqrt{\frac{\ddot{\gamma}_1^2 + \ddot{\gamma}_2^2}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^2} - \frac{(\ddot{\gamma}_1\dot{\gamma}_1 + \ddot{\gamma}_2\dot{\gamma}_2)^2}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^3}} = \frac{|\ddot{\gamma}_2\dot{\gamma}_1 - \dot{\gamma}_1\ddot{\gamma}_2|}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^{3/2}}. \tag{3.4}$$

Proof We first compute the covariant derivative of $\dot{\gamma}$ as in (2.7), using (2.3), (2.4) and (2.5). In components, with respect to the standard basis of \mathbb{R}^3 , we have

$$\begin{aligned} (D_t\dot{\gamma})_1 &= \ddot{\gamma}_1 + \frac{\gamma_2}{2\epsilon}\dot{\gamma}_1\dot{\gamma}_2 - \frac{\gamma_1}{2\epsilon}\dot{\gamma}_2^2 + \frac{1}{\epsilon}\dot{\gamma}_3\dot{\gamma}_2 = \ddot{\gamma}_1 + \frac{1}{\epsilon}\dot{\gamma}_2\omega(\dot{\gamma}), \\ (D_t\dot{\gamma})_2 &= \ddot{\gamma}_2 - \frac{\gamma_2}{2\epsilon}\dot{\gamma}_1^2 + \frac{\gamma_1}{2\epsilon}\dot{\gamma}_1\dot{\gamma}_2 - \frac{1}{\epsilon}\dot{\gamma}_3\dot{\gamma}_1 = \ddot{\gamma}_2 - \frac{1}{\epsilon}\dot{\gamma}_1\omega(\dot{\gamma}), \\ (D_t\dot{\gamma})_3 &= \ddot{\gamma}_3 - \frac{\gamma_1\gamma_2}{4\epsilon}\dot{\gamma}_1^2 + \frac{(\gamma_1^2 - \gamma_2^2)}{4\epsilon}\dot{\gamma}_1\dot{\gamma}_2 - \frac{\gamma_1}{2\epsilon}\dot{\gamma}_1\dot{\gamma}_3 + \frac{\gamma_1\gamma_2}{4\epsilon}\dot{\gamma}_2^2 - \frac{\gamma_2}{2\epsilon}\dot{\gamma}_2\dot{\gamma}_3. \end{aligned} \tag{3.5}$$

Now we express $D_t\dot{\gamma}$ in the basis $\{X_1, X_2, X_3^\epsilon\}$:

$$D_t\dot{\gamma} = \left(\ddot{\gamma}_1 + \frac{1}{\epsilon}\dot{\gamma}_2\omega(\dot{\gamma})\right) X_1 + \left(\ddot{\gamma}_2 - \frac{1}{\epsilon}\dot{\gamma}_1\omega(\dot{\gamma})\right) X_2 + \left(\frac{1}{\sqrt{\epsilon}}\omega(\ddot{\gamma})\right) X_3^\epsilon, \tag{3.6}$$

where

$$\omega(\ddot{\gamma}(t)) = \ddot{\gamma}_3(t) - \frac{1}{2}(\gamma_1(t)\ddot{\gamma}_2(t) - \gamma_2(t)\ddot{\gamma}_1(t)),$$

coincides with the expression $(\omega(\dot{\gamma}))'$ in (3.2). Recalling that

$$\dot{\gamma} = \dot{\gamma}_1 X_1 + \dot{\gamma}_2 X_2 + \left(\frac{1}{\sqrt{\epsilon}}\omega(\dot{\gamma})\right) X_3^\epsilon, \tag{3.7}$$

we compute

$$(D_t\dot{\gamma}, \dot{\gamma})_\epsilon = \ddot{\gamma}_1\dot{\gamma}_1 + \ddot{\gamma}_2\dot{\gamma}_2 + \frac{1}{\epsilon}\omega(\dot{\gamma})\omega(\ddot{\gamma}).$$

Therefore

$$k_\gamma^\epsilon = \sqrt{\frac{(\ddot{\gamma}_1 + \frac{1}{\epsilon}\dot{\gamma}_2\omega(\dot{\gamma}))^2 + (\ddot{\gamma}_2 - \frac{1}{\epsilon}\dot{\gamma}_1\omega(\dot{\gamma}))^2 + \frac{1}{\epsilon}\omega(\ddot{\gamma})^2}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \frac{1}{\epsilon}\omega(\dot{\gamma})^2)^2} - \frac{(\dot{\gamma}_1\dot{\gamma}_1 + \dot{\gamma}_2\dot{\gamma}_2 + \frac{1}{\epsilon}\omega(\dot{\gamma})\omega(\ddot{\gamma}))^2}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \frac{1}{\epsilon}\omega(\dot{\gamma})^2)^3}}. \tag{3.8}$$

On the other hand, if $\gamma(t)$ is a horizontal point for γ , then $\omega(\dot{\gamma}(t)) = 0$ and

$$D_t\dot{\gamma} = \ddot{\gamma}_1 X_1 + \ddot{\gamma}_2 X_2.$$

It follows that

$$k_\gamma^\epsilon = \sqrt{\frac{\ddot{\gamma}_1^2 + \ddot{\gamma}_2^2}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^2} - \frac{(\dot{\gamma}_1\dot{\gamma}_1 + \dot{\gamma}_2\dot{\gamma}_2)^2}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^3}},$$

as desired. □

Remark 3.5 The pointwise notion of curvature provided by (3.3) is continuous along the curve γ . Moreover, we want to stress that this notion is independent of the parametrization chosen.

We now use the previous results to study curvatures of curves in the Heisenberg group \mathbb{H} .

Definition 3.6 Let $\gamma : [a, b] \rightarrow \mathbb{H}$ be a Euclidean C^2 -smooth regular curve. We define the intrinsic curvature k_γ^0 of γ at $\gamma(t)$ to be

$$k_\gamma^0 := \lim_{\epsilon \rightarrow 0^+} k_\gamma^\epsilon,$$

if the limit exists.

It is clear that, a priori, the above limit may not exist. In order to deal with the asymptotics as $\epsilon \rightarrow 0^+$ of the quantities involved, let us introduce the following notation: for continuous functions $f, g : (0, +\infty) \rightarrow \mathbb{R}$,

$$f(\epsilon) \sim g(\epsilon), \quad \text{as } \epsilon \rightarrow 0^+ \iff \lim_{\epsilon \rightarrow 0^+} \frac{f(\epsilon)}{g(\epsilon)} = 1. \tag{3.9}$$

Lemma 3.7 Let $\gamma : [a, b] \rightarrow \mathbb{H}$ be a Euclidean C^2 -smooth regular curve. Then

$$k_\gamma^0 = \begin{cases} \frac{|\dot{\gamma}_1 \ddot{\gamma}_2 - \dot{\gamma}_2 \ddot{\gamma}_1|}{\sqrt{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^3}}, & \text{if } \gamma(t) \text{ is a horizontal point of } \gamma, \\ \frac{\sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}}{|\omega(\dot{\gamma})|}, & \text{if } \gamma(t) \text{ is not a horizontal point of } \gamma. \end{cases} \tag{3.10}$$

Proof The first result follows from the fact that the expression (3.4) for the curvature at horizontal points in $(\mathbb{R}^3, g_\epsilon)$, is independent of ϵ . For the other case, we need to study the asymptotics in ϵ . Using the notation introduced in (3.9), we have

$$\begin{aligned} \|D_t \dot{\gamma}\|_\epsilon &\sim \frac{1}{\epsilon} \sqrt{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)} |\omega(\dot{\gamma})|, \quad \text{as } \epsilon \rightarrow 0^+, \\ \|\dot{\gamma}\|_\epsilon &\sim \frac{1}{\sqrt{\epsilon}} |\omega(\dot{\gamma})|, \quad \text{as } \epsilon \rightarrow 0^+, \\ \langle D_t \dot{\gamma}, \dot{\gamma} \rangle_\epsilon &\sim \frac{1}{\epsilon} \omega(\dot{\gamma}) \omega(\ddot{\gamma}), \quad \text{as } \epsilon \rightarrow 0^+. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\|D_t \dot{\gamma}\|_\epsilon^2}{\|\dot{\gamma}\|_\epsilon^4} &\rightarrow \frac{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}{\omega(\dot{\gamma})^2}, \quad \text{as } \epsilon \rightarrow 0^+, \\ \frac{\langle D_t \dot{\gamma}, \dot{\gamma} \rangle_\epsilon^2}{\|\dot{\gamma}\|_\epsilon^6} &\sim \frac{\epsilon^3 \omega(\dot{\gamma})^2 \omega(\ddot{\gamma})^2}{\epsilon^2 \omega(\dot{\gamma})^6} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0^+. \end{aligned}$$

Altogether,

$$k_\gamma^0 = \lim_{\epsilon \rightarrow 0^+} k_\gamma^\epsilon = \frac{\sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}}{|\omega(\dot{\gamma})|},$$

as desired. □

Remark 3.8 We stress that the g_ϵ -length of a non horizontal curve always blows up once we take the limit as $\epsilon \rightarrow 0^+$. The remarkable fact provided by Lemma 3.7 is that this never occurs for the curvature k_γ^ϵ . We notice that, as in Remark 3.5, the intrinsic curvature k_γ^0 is independent of the parametrization. In the horizontal case, the quantity k_γ^0 is the absolute value of the *signed horizontal curvature* of a horizontal curve which arises in the study of horizontal mean curvature. See Remark 5.4 for more details.

There is another fact to notice. If we consider a Euclidean C^2 -smooth regular curve $\gamma : [a, b] \rightarrow \mathbb{H}$, which is partially horizontal and partially not, the quantity $k_\gamma^0(t)$ in (3.10), need not be a continuous function (in contrast to Remark 3.5). Moreover, in view of the independence of k_γ^0 of the parametrization, when we approach a horizontal point of γ from non-horizontal points of γ , we always find a singularity. Let us clarify the last sentences with an example.

Example 3.9 Consider the planar curve

$$\gamma(\theta) := (\cos(\theta) + 1, \sin(\theta), 0), \quad \theta \in [0, 2\pi).$$

The curve γ is horizontal only for $\theta = \pi$, indeed $\omega(\dot{\gamma}) = -\frac{1+\cos(\theta)}{2}$, which vanishes only for $\theta = \pi$. The horizontal curvature k_γ^0 is a pointwise notion, therefore we have

$$k_\gamma^0(\theta)|_{\theta=\pi} = \sqrt{\frac{\ddot{\gamma}_1^2 + \ddot{\gamma}_2^2}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^2} - \frac{(\dot{\gamma}_1\dot{\gamma}_2 + \ddot{\gamma}_1\ddot{\gamma}_2)^2}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^3}} \Big|_{\theta=\pi} = 1,$$

and for $\theta \in [0, 2\pi) \setminus \{\pi\}$,

$$k_\gamma^0(\theta) = \frac{\sqrt{\dot{\gamma}_1^2(\theta) + \dot{\gamma}_2^2(\theta)}}{|\omega(\dot{\gamma}(\theta))|} = \frac{2}{|1 + \cos(\theta)|} \rightarrow +\infty, \quad \text{as } \theta \rightarrow \pi.$$

In the classical theory of differential geometry of smooth space curves in \mathbb{R}^3 , there is a famous *rigidity theorem* (see for instance [23]) stating that every curve is characterized by its curvature and its torsion, up to rigid motions. Similar questions are addressed in [14] and [15] but with a different approach, viewing the first Heisenberg group \mathbb{H} as flat 3-dimensional manifold carrying a pseudo-hermitian structure.

We now have a notion of *horizontal curvature* for Euclidean C^2 -smooth regular curves in (\mathbb{H}, d_{cc}) . A first result in the direction of the classical rigidity theorem is the following.

Proposition 3.10 *Let $\gamma : [a, b] \rightarrow (\mathbb{H}, d_{cc})$ be a Euclidean C^2 -smooth regular curve. The intrinsic curvature k_γ^0 of γ is invariant under left translation and rotations around the x_3 -axis of γ .*

Proof Fix a point $g = (g_1, g_2, g_3) \in \mathbb{H}$. Define the curve $\tilde{\gamma}$ as the left translation by g of γ ,

$$\tilde{\gamma}(t) := L_g(\gamma(t)) = \left(\gamma_1(t) + g_1, \gamma_2(t) + g_2, \gamma_3(t) + g_3 - \frac{1}{2}(\gamma_1(t)g_2 - \gamma_2(t)g_1) \right),$$

$$t \in [a, b].$$

It is clear that

$$\dot{\tilde{\gamma}}_i = \dot{\gamma}_i \quad \text{and} \quad \ddot{\tilde{\gamma}}_i = \ddot{\gamma}_i, \quad i = 1, 2,$$

and therefore $k_\gamma^0 = k_{\tilde{\gamma}}^0$.

For the second assertion, fix an angle $\theta \in [0, 2\pi)$ and define the curve $\bar{\gamma}$ as the rotation by θ of γ around the x_3 -axis,

$$\bar{\gamma}(t) := (\gamma_1(t) \cos(\theta) + \gamma_2(t) \sin(\theta), -\gamma_1 \sin(\theta) + \gamma_2 \cos(\theta), \gamma_3), \quad t \in [a, b].$$

An easy computation shows that in this case we have

$$\dot{\bar{\gamma}}_1^2 + \dot{\bar{\gamma}}_2^2 = \dot{\gamma}_1^2 + \dot{\gamma}_2^2, \quad \ddot{\bar{\gamma}}_1^2 + \ddot{\bar{\gamma}}_2^2 = \ddot{\gamma}_1^2 + \ddot{\gamma}_2^2, \quad \dot{\bar{\gamma}}_1\ddot{\bar{\gamma}}_1 + \dot{\bar{\gamma}}_2\ddot{\bar{\gamma}}_2 = \dot{\gamma}_1\ddot{\gamma}_1 + \dot{\gamma}_2\ddot{\gamma}_2,$$

and therefore $k_\gamma^0 = k_{\bar{\gamma}}^0$. □

Remark 3.11 The behavior of the curvature k_γ^0 under dilations is as follows: if γ is a C^2 smooth regular curve and $r > 0$, then

$$k_{\delta_r \gamma}^0 = \frac{1}{r} k_\gamma^0.$$

Here $\delta_r \gamma$ denotes the curve $(r\gamma_1, r\gamma_2, r^2\gamma_3)$ when $\gamma = (\gamma_1, \gamma_2, \gamma_3)$.

4 Riemannian approximation of geodesic curvature of curves on surfaces

In order to prove a Heisenberg version of the Gauss–Bonnet Theorem, we need the concept of *intrinsic signed geodesic curvature*. This section will be devoted to the study of curvature of curves living on surfaces.

Let us fix once for all the assumptions we will make on the surface Σ in this and the coming section. We will say that a surface $\Sigma \subset (\mathbb{R}^3, g_\epsilon)$, or $\Sigma \subset \mathbb{H}$, is regular if

$$\Sigma \text{ is a Euclidean } C^2\text{-smooth compact and oriented surface.} \tag{4.1}$$

In particular we will assume that there exists a Euclidean C^2 -smooth function $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that

$$\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : u(x_1, x_2, x_3) = 0\}$$

and $\nabla_{\mathbb{R}^3} u \neq 0$. As in [9, Section 4.2], our study will be local and away from characteristic points of Σ . For completeness, we recall that a point $x \in \Sigma$ is called *characteristic* if

$$\nabla_{\mathbb{H}} u(x) = (0, 0). \tag{4.2}$$

The presence or absence of characteristic points will be stated explicitly. For estimates of the size of the characteristic set, see [4].

To fix notation (following the one adopted in [9, Chapter 4]), let us define first

$$p := X_1 u, \quad q := X_2 u \quad \text{and} \quad r := X_3^\epsilon u.$$

We then define

$$\begin{aligned} l &:= \|\nabla_{\mathbb{H}} u\|_{\mathbb{H}}, \quad \bar{p} := \frac{p}{l} \quad \text{and} \quad \bar{q} := \frac{q}{l}, \\ l_\epsilon &:= \sqrt{(X_1 u)^2 + (X_2 u)^2 + (X_3^\epsilon u)^2}, \quad \bar{r}_\epsilon := \frac{r}{l_\epsilon}, \\ \bar{p}_\epsilon &:= \frac{p}{l_\epsilon} \quad \text{and} \quad \bar{q}_\epsilon := \frac{q}{l_\epsilon}. \end{aligned} \tag{4.3}$$

In particular, $\bar{p}^2 + \bar{q}^2 = 1$. It is clear from (4.2) that these functions are well defined at every non-characteristic point of Σ .

Definition 4.1 Let $\Sigma \subset (\mathbb{R}^3, g_\epsilon)$ be a regular surface and let $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ be as before. The Riemannian unit normal v_ϵ to Σ is

$$v_\epsilon := \frac{\nabla_\epsilon u}{\|\nabla_\epsilon u\|_\epsilon} = \bar{p}_\epsilon X_1 + \bar{q}_\epsilon X_2 + \bar{r}_\epsilon X_3^\epsilon,$$

where $\nabla_\epsilon u$ is the Riemannian gradient of u .

Definition 4.2 Let $\Sigma \subset (\mathbb{R}^3, g_\epsilon)$ be a regular surface and let $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ be as before. For every point $g \in \Sigma$, we introduce the orthonormal basis $\{E_1(g), E_2(g)\}$ for $T_g \Sigma$, where

$$E_1(g) := \bar{q}(g) X_1(g) - \bar{p}(g) X_2(g)$$

and

$$E_2(g) := \bar{r}_\epsilon(g) \bar{p}(g) X_1(g) + \bar{r}_\epsilon(g) \bar{q}(g) X_2(g) - \frac{l}{l_\epsilon}(g) X_3^\epsilon(g).$$

On $T_g \Sigma$ we define a linear transformation $J_\epsilon : T_g \Sigma \rightarrow T_g \Sigma$ such that

$$J_\epsilon(E_1(g)) := E_2(g), \quad \text{and} \quad J_\epsilon(E_2(g)) := -E_1(g). \tag{4.4}$$

In the following we will omit the dependence on the point $g \in \Sigma$ when it is clear.

Let us spend a few words on the choice of the basis of the tangent plane $T_g \Sigma$. The vector E_1 is a horizontal vector given by $J(\nabla_{\mathbb{H}u} / \|\nabla_{\mathbb{H}u}\|_H)$, where J is the linear operator acting on horizontal vector fields by $J(aX_1 + bX_2) = bX_1 - aX_2$. This vector is called by some authors *characteristic direction* and plays an important role in the study of minimal surfaces in \mathbb{H} and in the study of the properties of the characteristic set $\text{char}(\Sigma)$, see for instance [12, 13] and [11].

In order to perform computations on the surface Σ , we need a Riemannian metric and a connection. Classically, see for instance [26] or [24], there is a standard way to define a Riemannian metric $g_{\epsilon, \Sigma}$ on Σ so that Σ is isometrically immersed in $(\mathbb{R}^3, g_\epsilon)$. Once we have a Riemannian metric on the surface Σ , we can define the unique Levi-Civita connection ∇^Σ on Σ related to the Riemannian metric $g_{\epsilon, \Sigma}$. This procedure is equivalent to the following definition.

Definition 4.3 Let $\Sigma \subset (\mathbb{R}^3, g_\epsilon)$ be a Euclidean C^2 -smooth surface. For every $U, V \in T_g \Sigma$ we define $\nabla_U^\Sigma V$ to be the tangential component of ∇ , namely

$$\nabla_U^\Sigma V = \Pi(\nabla_U V),$$

where $\Pi : \mathbb{R}^3 \rightarrow T\Sigma$.

Note that to compute $\nabla_U V$ we are considering an extension of both U and V to \mathbb{R}^3 .

As a notational remark, given a Euclidean C^2 -smooth and regular curve $\gamma \subset \Sigma$, we will denote the covariant derivative of $\dot{\gamma}$ with respect to the $g_{\epsilon, \Sigma}$ -metric, as $D_t^\Sigma \dot{\gamma}$.

We are now interested in detecting the curvature of a Euclidean C^2 -smooth regular curve γ on a given regular surface $\Sigma \subset \mathbb{H}$.

We start with a technical lemma that will simplify our treatment in the ensuing discussion.

Lemma 4.4 For $\bar{p}, \bar{q}, l_\epsilon$ and \bar{r}_ϵ as before, we have

$$l_\epsilon \rightarrow \|\nabla_{\mathbb{H}u}\|, \quad \text{as } \epsilon \rightarrow 0^+, \tag{4.5}$$

$$\bar{r}_\epsilon \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0^+, \tag{4.6}$$

$$\frac{\bar{r}_\epsilon}{l_\epsilon} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0^+, \tag{4.7}$$

$$\frac{\bar{r}_\epsilon}{\sqrt{\epsilon} l_\epsilon} \rightarrow \frac{X_3 u}{\|\nabla_{\mathbb{H}u}\|^2}, \quad \text{as } \epsilon \rightarrow 0^+, \tag{4.8}$$

$$\frac{1}{\epsilon} \bar{r}_\epsilon^2 \rightarrow \frac{(X_3 u)^2}{\|\nabla_{\mathbb{H}u}\|^2}, \quad \text{as } \epsilon \rightarrow 0^+, \tag{4.9}$$

$$\frac{\bar{r}_\epsilon}{\epsilon} \sim \frac{(X_3 u)}{\sqrt{\epsilon} \|\nabla_{\mathbb{H}u}\|}, \quad \text{as } \epsilon \rightarrow 0^+ \tag{4.10}$$

Proof All the limits and asymptotics follow directly from the definitions in (4.3). □

Lemma 4.5 *Let $\gamma : [a, b] \rightarrow \Sigma$ be a Euclidean C^2 -smooth, regular and non-horizontal curve. The covariant derivative $D_t^\Sigma \dot{\gamma}$ of $\dot{\gamma}$ with respect to the $g_{\epsilon, \Sigma}$ -metric is given by*

$$D_t^\Sigma \dot{\gamma} = (D_t^\Sigma \dot{\gamma})_1 E_1 + (D_t^\Sigma \dot{\gamma})_2 E_2,$$

where

$$\begin{aligned} (D_t^\Sigma \dot{\gamma})_1 &= \bar{q} \left(\ddot{\gamma}_1 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_2 \right) - \bar{p} \left(\ddot{\gamma}_2 - \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_1 \right), \\ (D_t^\Sigma \dot{\gamma})_2 &= \bar{r}_\epsilon \bar{p} \left(\ddot{\gamma}_1 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_2 \right) + \bar{r}_\epsilon \bar{q} \left(\ddot{\gamma}_2 - \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_1 \right) - \frac{l}{\sqrt{\epsilon} l_\epsilon} \omega(\ddot{\gamma}). \end{aligned} \tag{4.11}$$

Moreover, if $\gamma : [a, b] \rightarrow \Sigma$ is Euclidean C^2 -smooth, regular and horizontal curve, we have

$$D_t^\Sigma \dot{\gamma} = (\bar{q} \ddot{\gamma}_1 - \bar{p} \ddot{\gamma}_2) E_1 + \bar{r}_\epsilon (\bar{p} \ddot{\gamma}_1 + \bar{q} \ddot{\gamma}_2) E_2, \tag{4.12}$$

Proof The covariant derivative $D_t \dot{\gamma}$ in the $\{X_1, X_2, X_3\}$ basis is as in (3.6). Projecting $D_t \dot{\gamma}$ via Π into the tangent plane $T_g \Sigma$, we find

$$\begin{aligned} D_t^\Sigma \dot{\gamma} &:= \langle D_t \dot{\gamma}, E_1 \rangle_\epsilon E_1 + \langle D_t \dot{\gamma}, E_2 \rangle_\epsilon E_2 \\ &= \left[\bar{q} \left(\ddot{\gamma}_1 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_2 \right) - \bar{p} \left(\ddot{\gamma}_2 - \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_1 \right) \right] E_1 \\ &\quad + \left[\bar{r}_\epsilon \bar{p} \left(\ddot{\gamma}_1 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_2 \right) + \bar{r}_\epsilon \bar{q} \left(\ddot{\gamma}_2 - \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_1 \right) - \frac{l}{\sqrt{\epsilon} l_\epsilon} \omega(\ddot{\gamma}) \right] E_2. \end{aligned}$$

The situation is much simpler if γ is horizontal, indeed it suffices to set $\omega(\dot{\gamma}) = \omega(\ddot{\gamma}) = 0$ in the previous expression. □

Definition 4.6 Let $\Sigma \subset (\mathbb{R}^3, g_\epsilon)$ be a regular surface. Let $\gamma : [a, b] \rightarrow \Sigma$ be a Euclidean C^2 -smooth and regular curve, and let D_t^Σ be the covariant derivative of $\dot{\gamma}$ with respect to the Riemannian metric $g_{\epsilon, \Sigma}$. The geodesic curvature $k_{\gamma, \Sigma}^\epsilon$ of γ at the point $\gamma(t)$ is defined to be

$$k_{\gamma, \Sigma}^\epsilon := \sqrt{\frac{\|D_t^\Sigma \dot{\gamma}\|_{\epsilon, \Sigma}^2}{\|\dot{\gamma}\|_{\epsilon, \Sigma}^4} - \frac{\langle D_t^\Sigma \dot{\gamma}, \dot{\gamma} \rangle_{\epsilon, \Sigma}^2}{\|\dot{\gamma}\|_{\epsilon, \Sigma}^6}}.$$

Definition 4.7 Let $\Sigma \subset \mathbb{H}$ be a regular surface and let $\gamma : [a, b] \rightarrow \Sigma$ be a Euclidean C^2 -smooth and regular curve. The *intrinsic geodesic curvature* $k_{\gamma, \Sigma}^0$ of γ at the non-characteristic point $\gamma(t)$ is defined as

$$k_{\gamma, \Sigma}^0 := \lim_{\epsilon \rightarrow 0^+} k_{\gamma, \Sigma}^\epsilon,$$

if the limit exists.

As in Sect. 3, our first task is to show that the above limit does always exist.

Lemma 4.8 *Let $\Sigma \subset \mathbb{H}$ be a regular surface and let $\gamma : [a, b] \rightarrow \Sigma$ be a Euclidean C^2 -smooth and regular curve. Then*

$$k_{\gamma, \Sigma}^0 = \frac{|\bar{p} \dot{\gamma}_1 + \bar{q} \dot{\gamma}_2|}{|\omega(\dot{\gamma})|}$$

if $\gamma(t)$ is a non-horizontal point, while

$$k_{\gamma, \Sigma}^0 = 0$$

if $\gamma(t)$ is a horizontal point.

Proof First, let $\gamma(t)$ be a non-horizontal point of the curve γ . From Lemma 4.5, we know that

$$D_t^\Sigma \dot{\gamma} = \left(\bar{q} \left(\ddot{\gamma}_1 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_2 \right) - \bar{p} \left(\ddot{\gamma}_2 - \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_1 \right) \right) E_1 + \left(\bar{r}_\epsilon \bar{p} \left(\ddot{\gamma}_1 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_2 \right) + \bar{r}_\epsilon \bar{q} \left(\ddot{\gamma}_2 - \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_1 \right) - \frac{l}{\sqrt{\epsilon} l_\epsilon} \omega(\ddot{\gamma}) \right) E_2.$$

For $\dot{\gamma}$, recalling (3.7) we have

$$\dot{\gamma} = (\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2) E_1 + \left(\bar{r}_\epsilon \bar{p} \dot{\gamma}_1 + \bar{r}_\epsilon \bar{q} \dot{\gamma}_2 - \frac{l}{\sqrt{\epsilon} l_\epsilon} \omega(\dot{\gamma}) \right) E_2.$$

Recalling (4.10), we have

$$\begin{aligned} \|D_t^\Sigma \dot{\gamma}\|_{\epsilon, \Sigma}^2 &= \left(\bar{q} \left(\ddot{\gamma}_1 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_2 \right) - \bar{p} \left(\ddot{\gamma}_2 - \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_1 \right) \right)^2 \\ &\quad + \left(\bar{r}_\epsilon \bar{p} \left(\ddot{\gamma}_1 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_2 \right) + \bar{r}_\epsilon \bar{q} \left(\ddot{\gamma}_2 - \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_1 \right) - \frac{l}{\sqrt{\epsilon} l_\epsilon} \omega(\ddot{\gamma}) \right)^2 \\ &\sim \frac{1}{\epsilon^2} \omega(\dot{\gamma})^2 (\bar{p} \dot{\gamma}_1 + \bar{q} \dot{\gamma}_2)^2, \quad \text{as } \epsilon \rightarrow 0^+. \end{aligned}$$

In a similar way, we have that

$$\|\dot{\gamma}\|_{\epsilon, \Sigma} = \sqrt{(\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2)^2 + \left(\bar{r}_\epsilon \bar{p} \dot{\gamma}_1 + \bar{r}_\epsilon \bar{q} \dot{\gamma}_2 - \frac{l}{l_\epsilon} \frac{1}{\sqrt{\epsilon}} \omega(\dot{\gamma}) \right)^2} \sim \frac{1}{\sqrt{\epsilon}} |\omega(\dot{\gamma})|, \quad \text{as } \epsilon \rightarrow 0^+.$$

A bit more involved computation shows that

$$\langle D_t^\Sigma \dot{\gamma}, \dot{\gamma} \rangle_{\epsilon, \Sigma} \sim \frac{1}{\epsilon} M, \quad \text{as } \epsilon \rightarrow 0^+,$$

where M does not depend on ϵ . Therefore, at a non-horizontal point $\gamma(t)$, we have

$$k_{\gamma, \Sigma}^0 = \frac{|\bar{p} \dot{\gamma}_1 + \bar{q} \dot{\gamma}_2|}{|\omega(\dot{\gamma})|}.$$

Now let $\gamma(t)$ be a horizontal point of the curve γ . In this case,

$$\begin{aligned} D_t^\Sigma \dot{\gamma} &= (\bar{q} \ddot{\gamma}_1 - \bar{p} \ddot{\gamma}_2) E_1 + \bar{r}_\epsilon (\bar{p} \ddot{\gamma}_1 + \bar{q} \ddot{\gamma}_2) E_2, \\ \dot{\gamma} &= (\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2) E_1 + \bar{r}_\epsilon (\bar{p} \dot{\gamma}_1 + \bar{q} \dot{\gamma}_2) E_2. \end{aligned}$$

Recalling (4.5), we have

$$\begin{aligned} \|D_t^\Sigma \dot{\gamma}\|_{\epsilon, \Sigma}^2 &\rightarrow (\bar{q} \ddot{\gamma}_1 - \bar{p} \ddot{\gamma}_2)^2, \quad \text{as } \epsilon \rightarrow 0^+, \\ \|\dot{\gamma}\|_{\epsilon, \Sigma}^2 &\rightarrow (\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2)^2, \quad \text{as } \epsilon \rightarrow 0^+, \\ \langle D_t^\Sigma \dot{\gamma}, \dot{\gamma} \rangle_{\epsilon, \Sigma} &\rightarrow (\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2)(\bar{q} \ddot{\gamma}_1 - \bar{p} \ddot{\gamma}_2), \quad \text{as } \epsilon \rightarrow 0^+. \end{aligned}$$

Therefore

$$k_{\gamma, \Sigma}^0 = \sqrt{\frac{(\bar{q} \ddot{\gamma}_1 - \bar{p} \ddot{\gamma}_2)^2}{(\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2)^4} - \frac{(\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2)^2 (\bar{q} \ddot{\gamma}_1 - \bar{p} \ddot{\gamma}_2)^2}{(\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2)^6}} = 0,$$

as desired. □

Remark 4.9 In the last computation, it is possible to divide by the term $\bar{q}\dot{\gamma}_1 - \bar{p}\dot{\gamma}_2$, because it cannot be 0 when $\gamma(t)$ is a horizontal point. Indeed, let $g \in \Sigma$ be a non-characteristic point, then we know that

$$\dim(H_g \cap T_g \Sigma) = 1.$$

At a horizontal point $\gamma(t)$, we have that $\dot{\gamma} \in H_g \cup T_g \Sigma$. On the other hand, $E_1 \in H_g \cap T_g \Sigma$ as well. Therefore, if $\bar{q}\dot{\gamma}_1 - \bar{p}\dot{\gamma}_2 = 0$, this would imply that we would have

$$\langle \dot{\gamma}, E_1 \rangle_\epsilon = 0,$$

and therefore that either $\dim(H_g \cap T_g \Sigma) = 2$ or $\dot{\gamma} = 0$, which would lead to a contradiction.

For planar curves it is possible to define a notion of *signed curvature*. Actually, it is possible to define an analogous concept for curves living on arbitrary two-dimensional Riemannian manifolds (see for instance [26, Chapter 9]).

Definition 4.10 Let $\Sigma \subset (\mathbb{R}^3, g_\epsilon)$ be a regular surface and let $\gamma : [a, b] \rightarrow \Sigma$ be a Euclidean C^2 -smooth and regular curve. Let ∇^Σ be the Levi-Civita connection on Σ related to the Riemannian metric $g_{\epsilon, \Sigma}$. For every $g \in \Sigma$, let $\{E_1(g), E_2(g)\}$ be an orthonormal basis of $T_g \Sigma$. The signed geodesic curvature $k_{\gamma, \Sigma}^{\epsilon, s}$ of γ at the point $\gamma(t)$ is defined as

$$k_{\gamma, \Sigma}^{\epsilon, s} := \frac{\langle D_t^\Sigma \dot{\gamma}, J_\epsilon(\dot{\gamma}) \rangle_{\epsilon, \Sigma}}{\|\dot{\gamma}\|_{\epsilon, \Sigma}^3},$$

where J_ϵ is the linear transformation on $T_g \Sigma$ defined in (4.4).

Remark 4.11 If we take the absolute value of $k_{\gamma, \Sigma}^{\epsilon, s}$, we recover precisely $k_{\gamma, \Sigma}^\epsilon$. Indeed, by definition of J_ϵ , $\{\dot{\gamma}/\|\dot{\gamma}\|_\epsilon, J_\epsilon(\dot{\gamma})/\|\dot{\gamma}\|_\epsilon\}$ is another orthonormal basis of $T_g \Sigma$ oriented as $\{E_1(g), E_2(g)\}$. Since $D_t^\Sigma \dot{\gamma}$ is defined as the projection of $D_t \dot{\gamma}$ onto $T_g \Sigma$, we have

$$D_t^\Sigma \dot{\gamma} = \langle D_t^\Sigma \dot{\gamma}, \frac{\dot{\gamma}}{\|\dot{\gamma}\|_\epsilon} \rangle_\epsilon \frac{\dot{\gamma}}{\|\dot{\gamma}\|_\epsilon} + \langle D_t^\Sigma \dot{\gamma}, \frac{J_\epsilon(\dot{\gamma})}{\|\dot{\gamma}\|_\epsilon} \rangle_\epsilon \frac{J_\epsilon(\dot{\gamma})}{\|\dot{\gamma}\|_\epsilon}.$$

In particular,

$$\frac{\|D_t^\Sigma \dot{\gamma}\|_\epsilon^2}{\|\dot{\gamma}\|_\epsilon^4} = \frac{\langle D_t^\Sigma \dot{\gamma}, \dot{\gamma} \rangle_{\epsilon, \Sigma}^2}{\|\dot{\gamma}\|_\epsilon^6} + \frac{\langle D_t^\Sigma \dot{\gamma}, J_\epsilon(\dot{\gamma}) \rangle_\epsilon^2}{\|\dot{\gamma}\|_\epsilon^6}$$

and so

$$|k_{\gamma, \Sigma}^{\epsilon, s}| = \frac{|\langle D_t^\Sigma \dot{\gamma}, J_\epsilon(\dot{\gamma}) \rangle_\epsilon|}{\|\dot{\gamma}\|_\epsilon^3} = \sqrt{\frac{\|D_t^\Sigma \dot{\gamma}\|_\epsilon^2}{\|\dot{\gamma}\|_\epsilon^4} - \frac{\langle D_t^\Sigma \dot{\gamma}, \dot{\gamma} \rangle_\epsilon^2}{\|\dot{\gamma}\|_\epsilon^6}} = k_{\gamma, \Sigma}^\epsilon.$$

Definition 4.12 Let $\Sigma \subset \mathbb{H}$ be a regular surface and let $\gamma : [a, b] \rightarrow \Sigma$ be a Euclidean C^2 -smooth and regular curve. The *intrinsic signed geodesic curvature* $k_{\gamma, \Sigma}^{0, s}(t)$ of γ at the non-characteristic point $\gamma(t)$ is defined as

$$k_{\gamma, \Sigma}^{0, s} := \lim_{\epsilon \rightarrow 0^+} k_{\gamma, \Sigma}^{\epsilon, s}.$$

Again, we need to show that the above limit actually exists.

Proposition 4.13 Let $\Sigma \subset \mathbb{H}$ be a regular surface and let $\gamma : [a, b] \rightarrow \Sigma$ be a Euclidean C^2 -smooth and regular curve. Then

$$k_{\gamma, \Sigma}^{0, s} = \frac{\bar{p}\dot{\gamma}_1 + \bar{q}\dot{\gamma}_2}{|\omega(\dot{\gamma})|}$$

if $\gamma(t)$ is a non-horizontal point, and

$$k_{\gamma, \Sigma}^{0,s} = 0$$

if $\gamma(t)$ is a horizontal point.

Proof We already know that

$$D_t^\Sigma \dot{\gamma} = \left(\bar{q} \left(\dot{\gamma}_1 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_2 \right) - \bar{p} \left(\dot{\gamma}_2 - \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_1 \right) \right) E_1 + \left(\bar{r}_\epsilon \bar{p} \left(\dot{\gamma}_1 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_2 \right) + \bar{r}_\epsilon \bar{q} \left(\dot{\gamma}_2 - \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_1 \right) - \frac{l}{l_\epsilon \sqrt{\epsilon}} \omega(\dot{\gamma}) \right) E_2.$$

and

$$\dot{\gamma} = (\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2) E_1 + \left(\bar{r}_\epsilon \bar{p} \dot{\gamma}_1 + \bar{r}_\epsilon \bar{q} \dot{\gamma}_2 - \frac{l}{l_\epsilon \sqrt{\epsilon}} \omega(\dot{\gamma}) \right) E_2,$$

and therefore, by definition of J_ϵ ,

$$J_\epsilon(\dot{\gamma}) = - \left(\bar{r}_\epsilon \bar{p} \dot{\gamma}_1 + \bar{r}_\epsilon \bar{q} \dot{\gamma}_2 - \frac{l}{l_\epsilon \sqrt{\epsilon}} \omega(\dot{\gamma}) \right) E_1 + (\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2) E_2.$$

After some simplifications, we get

$$\langle D_t^\Sigma \dot{\gamma}, J_\epsilon(\dot{\gamma}) \rangle_{\epsilon, \Sigma} = \bar{r}_\epsilon \dot{\gamma}_1 \left(\dot{\gamma}_2 - \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_2 \right) (\bar{p}^2 + \bar{q}^2) - \bar{r}_\epsilon \dot{\gamma}_2 \left(\dot{\gamma}_1 + \frac{1}{\epsilon} \omega(\dot{\gamma}) \dot{\gamma}_1 \right) (\bar{p}^2 + \bar{q}^2) + \frac{l}{l_\epsilon \sqrt{\epsilon}} \left[\omega(\dot{\gamma}) \left(\bar{q} \dot{\gamma}_1 + \frac{1}{\epsilon} \bar{q} \omega(\dot{\gamma}) \dot{\gamma}_2 - \bar{p} \dot{\gamma}_2 + \frac{1}{\epsilon} \bar{p} \omega(\dot{\gamma}) \dot{\gamma}_1 \right) - \bar{q} \omega(\dot{\gamma}) \dot{\gamma}_1 + \bar{p} \omega(\dot{\gamma}) \dot{\gamma}_2 \right]$$

Therefore, exploiting Lemma 4.4, for a non-horizontal curve γ , we have

$$\langle D_t^\Sigma \dot{\gamma}, J_\epsilon(\dot{\gamma}) \rangle_{\epsilon, \Sigma} \sim \left(\frac{1}{\epsilon} \right)^{3/2} \omega(\dot{\gamma})^2 (\bar{q} \dot{\gamma}_2 + \bar{p} \dot{\gamma}_1), \quad \text{as } \epsilon \rightarrow 0^+.$$

Recalling that $\|\dot{\gamma}\|_{\epsilon, \Sigma} \sim \frac{1}{\sqrt{\epsilon}} |\omega(\dot{\gamma})|$, as $\epsilon \rightarrow 0^+$, we find

$$\frac{\langle D_t^\Sigma \dot{\gamma}, J_\epsilon(\dot{\gamma}) \rangle_{\epsilon, \Sigma}}{\|\dot{\gamma}\|_{\epsilon, \Sigma}^3} \rightarrow \frac{(\bar{p} \dot{\gamma}_1 + \bar{q} \dot{\gamma}_2)}{|\omega(\dot{\gamma})|}, \quad \text{as } \epsilon \rightarrow 0^+.$$

As for the previous results, the case of γ horizontal is slightly easier, because $\omega(\dot{\gamma}) = \omega(\ddot{\gamma}) = 0$. Therefore,

$$D_t^\Sigma \dot{\gamma} = (\bar{q}(\dot{\gamma}_1) - \bar{p}(\dot{\gamma}_2)) E_1 + \bar{r}_\epsilon (\bar{p}(\dot{\gamma}_1) + \bar{q}(\dot{\gamma}_2)) E_2, \\ \dot{\gamma} = (\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2) E_1 + \bar{r}_\epsilon (\bar{p} \dot{\gamma}_1 + \bar{q} \dot{\gamma}_2) E_2, \\ J_\epsilon(\dot{\gamma}) = -\bar{r}_\epsilon (\bar{p} \dot{\gamma}_1 + \bar{q} \dot{\gamma}_2) E_1 + (\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2) E_2$$

Hence, from Lemma 4.4,

$$\langle D_t^\Sigma \dot{\gamma}, J_\epsilon(\dot{\gamma}) \rangle_\epsilon = \bar{r}_\epsilon \left[-(\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2) (\bar{p} \dot{\gamma}_1 + \bar{q} \dot{\gamma}_2) + (\bar{p} \dot{\gamma}_1 + \bar{q} \dot{\gamma}_2) (\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2) \right] \rightarrow 0, \\ \text{as } \epsilon \rightarrow 0^+,$$

as desired. □

Remark 4.14 The intrinsic geodesic curvature (both signed and unsigned) is invariant under isometries of \mathbb{H} (left translations and rotations about the x_3 -axis), and scales by the factor $\frac{1}{r}$ under the dilation δ_r . Compare Proposition 3.10 and Remark 3.11. We omit the elementary proofs of these facts.

5 Riemannian approximation of curvatures of surfaces

In this section we want to study curvatures of regular surfaces $\Sigma \subset \mathbb{H}$. As in the previous sections, the idea will be to compute first the already known curvatures of a regular 2-dimensional Riemannian manifold $\Sigma \subset (\mathbb{R}^3, g_\epsilon)$, and then try to derive appropriate Heisenberg notions taking the limit as $\epsilon \rightarrow 0^+$.

First, we need to define the *second fundamental form* II^ϵ of the embedding of Σ into $(\mathbb{R}^3, g_\epsilon)$:

$$II^\epsilon = \begin{pmatrix} \langle \nabla_{E_1} v_\epsilon, E_1 \rangle_\epsilon & \langle \nabla_{E_1} v_\epsilon, E_2 \rangle_\epsilon \\ \langle \nabla_{E_2} v_\epsilon, E_1 \rangle_\epsilon & \langle \nabla_{E_2} v_\epsilon, E_2 \rangle_\epsilon \end{pmatrix}. \tag{5.1}$$

The explicit computation of the second fundamental form in our case can be found in [9]. For sake of completeness, we recall here the complete statement.

Theorem 5.1 ([9], Theorem 4.3) *The second fundamental form II^ϵ of the embedding of Σ into $(\mathbb{R}^3, g_\epsilon)$ is given by*

$$II^\epsilon = \begin{pmatrix} \frac{l}{l_\epsilon}(X_1(\bar{p}) + X_2(\bar{q})) & -\frac{1}{2\sqrt{\epsilon}} - \frac{l_\epsilon}{l} \langle E_1, \nabla_H(\bar{r}_\epsilon) \rangle_\epsilon \\ -\frac{1}{2\sqrt{\epsilon}} - \frac{l_\epsilon}{l} \langle E_1, \nabla_H(\bar{r}_\epsilon) \rangle_\epsilon & -\frac{l^2}{l_\epsilon^2} \langle E_2, \nabla_H(\bar{l}) \rangle_\epsilon + X_3^\epsilon(\bar{r}_\epsilon) \end{pmatrix}. \tag{5.2}$$

The Riemannian mean curvature \mathcal{H}_ϵ of Σ is

$$\mathcal{H}_\epsilon := \text{tr}(II^\epsilon),$$

while the Riemannian Gaussian curvature \mathcal{K}_ϵ is

$$\mathcal{K}_\epsilon := \bar{\mathcal{K}}_\epsilon(E_1, E_2) + \det(II^\epsilon).$$

Here $\bar{\mathcal{K}}_\epsilon(E_1, E_2)$ denotes the sectional curvature of $(\mathbb{R}^3, g_\epsilon)$ in the plane generated by E_1 and E_2 . We need to spend a few words about the last definition, which is actually a result known as *Gauss equation*. The geometric meaning is that the second fundamental form II^ϵ is the right object to measure the discrepancy between the two sectional curvatures, of the ambient manifold, and of the isometrically immersed one.

Proposition 5.2 *Away from characteristic points, the horizontal mean curvature \mathcal{H}_0 of $\Sigma \subset \mathbb{H}$ is given by*

$$\mathcal{H}_0 = \lim_{\epsilon \rightarrow 0^+} \mathcal{H}_\epsilon = \text{div}_H \left(\frac{\nabla_H u}{\|\nabla_H u\|_H} \right), \tag{5.3}$$

and the intrinsic Gaussian curvature \mathcal{K}_0 is given by

$$\begin{aligned} \mathcal{K}_0 &= \lim_{\epsilon \rightarrow 0^+} \mathcal{K}_\epsilon \\ &= - \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right)^2 - \left(\frac{X_2 u}{\|\nabla_H u\|_H} \right) X_1 \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right) + \left(\frac{X_1 u}{\|\nabla_H u\|_H} \right) X_2 \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right). \end{aligned} \tag{5.4}$$

In (5.3) the expression div_H denotes the *horizontal divergence* of a horizontal vector field, which is defined as follows: for a horizontal vector field $V = a X_1 + b X_2$,

$$\text{div}_H(V) = X_1(a) + X_2(b).$$

Proof By definition,

$$\mathcal{H}_\epsilon = \text{trace}(II^\epsilon) = \frac{l}{l_\epsilon}(X_1(\bar{p}) + X_2(\bar{q})) - \frac{l^2}{l_\epsilon^2} \langle E_2, \nabla_H(\frac{l}{l}) \rangle_\epsilon + X_3^\epsilon(\bar{r}_\epsilon).$$

We recall that $r = \sqrt{\epsilon} X_3 u$ and therefore in the limit as $\epsilon \rightarrow 0^+$, we obtain

$$\mathcal{H}_0 = \lim_{\epsilon \rightarrow 0^+} \mathcal{H}_\epsilon = X_1 \left(\frac{X_1 u}{\|\nabla_H u\|_H} \right) + X_2 \left(\frac{X_2 u}{\|\nabla_H u\|_H} \right).$$

As we have already observed in the computation of \mathcal{H}_0 , the term

$$-\frac{l^2}{l_\epsilon^2} \langle E_2, \nabla_H(\frac{l}{l}) \rangle_\epsilon + X_3^\epsilon(\bar{r}_\epsilon)$$

tends to zero as $\epsilon \rightarrow 0^+$, and therefore in analyzing $\det(II^\epsilon)$ we can focus only on the term

$$-\left(-\frac{1}{2\sqrt{\epsilon}} - \frac{l_\epsilon}{l} \langle E_1, \nabla_H(\bar{r}_\epsilon) \rangle_\epsilon \right)^2.$$

Clearly,

$$-\left(-\frac{1}{2\sqrt{\epsilon}} - \frac{l_\epsilon}{l} \langle E_1, \nabla_H(\bar{r}_\epsilon) \rangle_\epsilon \right)^2 \sim -\frac{1}{4\epsilon} - \langle E_1, \nabla_H \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right) \rangle_\epsilon, \quad \text{as } \epsilon \rightarrow 0^+.$$

It remains to compute the sectional curvature $\bar{\mathcal{K}}_\epsilon(E_1, E_2)$. By Definition 2.11, the functional property in Remark 2.10, and orthonormality of the basis $\{E_1, E_2\}$, we have

$$\bar{\mathcal{K}}_\epsilon(E_1, E_2) = -\frac{1}{4\epsilon} - \frac{1}{\epsilon} \bar{r}_\epsilon^2.$$

Hence

$$\begin{aligned} \mathcal{K}_0 &= \lim_{\epsilon \rightarrow 0^+} \mathcal{K}_\epsilon \\ &= -\left(\frac{X_3 u}{\|\nabla_H u\|_H} \right)^2 - \left(\frac{X_2 u}{\|\nabla_H u\|_H} \right) X_1 \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right) + \left(\frac{X_1 u}{\|\nabla_H u\|_H} \right) X_2 \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right), \end{aligned}$$

as desired. □

Remark 5.3 As already mentioned in the introduction, it is clear from the previous computations that the existence of the limit defining the *intrinsic Gaussian curvature* \mathcal{K}_0 is based on the cancellation of two divergent terms (as $\epsilon \rightarrow 0^+$), one coming from the sectional curvature $\bar{\mathcal{K}}_\epsilon(E_1, E_2)$, and the other coming from the determinant of the second fundamental form II^ϵ . We remark once more that such cancellation depends on both the choice of the frame $\{E_1, E_2\}$ on the surface Σ , and on intrinsic symmetries of the Heisenberg group \mathbb{H} .

Remark 5.4 It is well known (see, for instance [9]) that the horizontal mean curvature of Σ at a non-characteristic point x coincides, up to a choice of sign, with the signed horizontal curvature of the Legendrian curve γ in Σ through x . This is the unique horizontal curve γ defined locally near x , $\gamma : (-\delta, \delta) \rightarrow \Sigma$, such that $\gamma(0) = x$ and $\dot{\gamma}(0) = J(\nabla_H u / \|\nabla_H u\|_H) \in H_x \mathbb{H} \cap T_x \Sigma$. The signed horizontal curvature of $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is given by

$$\frac{\dot{\gamma}_1 \ddot{\gamma}_2 - \dot{\gamma}_2 \ddot{\gamma}_1}{\sqrt{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^3}}.$$

Observe that the absolute value of this expression coincides with the intrinsic curvature k_γ^0 introduced in Definition 3.6, when considered on horizontal curves.

Remark 5.5 The horizontal mean curvature \mathcal{H}_0 and intrinsic Gauss curvature \mathcal{K}_0 have been given in terms of a defining function u for the surface Σ . The preceding remark shows that \mathcal{H}_0 is independent of the choice of the defining function. Concerning \mathcal{K}_0 , we note that K_ϵ is independent of the choice of the defining function because it is the Riemannian Gaussian curvature of $(\Sigma, g_\epsilon|_\Sigma)$, therefore the same holds true for its limit \mathcal{K}_0 as $\epsilon \rightarrow 0^+$.

We conclude this section with a brief discussion of the local summability of the horizontal Gaussian curvature \mathcal{K}_0 with respect to the Heisenberg perimeter measure near isolated characteristic points. This observation will be used in our subsequent study of the validity of an intrinsic Gauss–Bonnet theorem in \mathbb{H} . Without loss of generality, suppose that the origin is an isolated characteristic point of Σ . Consider a neighborhood U of the origin on Σ . Due to the expression (5.4), and since the Heisenberg perimeter measure $d\sigma_H$ is given by

$$d\sigma_H = \frac{\|\nabla_H u\|_H}{\|\nabla_{\mathbb{R}^3} u\|_{\mathbb{R}^3}} d\mathcal{H}_{\mathbb{R}^3}^2,$$

there exists a positive constant $C = C(U) > 0$ such that

$$|\mathcal{K}_0| d\sigma_H \leq \frac{C}{\|\nabla_H u\|_H} d\mathcal{H}_{\mathbb{R}^3}^2. \tag{5.5}$$

Finer results turn out to be particularly difficult to achieve. We postpone further discussion of this topic to Sect. 8.

6 A Gauss–Bonnet theorem

The goal of this section is to prove Theorem 1.1. Before doing that, we need to recall a couple of technical results concerning, respectively the Riemannian length measure, and the Riemannian surface measure. Let us first consider the case of a curve $\gamma : [a, b] \rightarrow (\mathbb{R}^3, g_\epsilon)$ in the Riemannian manifold $(\mathbb{R}^3, g_\epsilon)$. We define the Riemannian length measure,

$$d\dot{\gamma}_\epsilon := \gamma_\# (\|\dot{\gamma}\|_\epsilon d\mathcal{L}_{[a,b]}^1) = \|\dot{\gamma}\|_\epsilon dt.$$

Lemma 6.1 *Let $\gamma : [a, b] \rightarrow (\mathbb{R}^3, g_\epsilon)$ be a Euclidean C^2 -smooth and regular curve. Then*

$$\lim_{\epsilon \rightarrow 0^+} \sqrt{\epsilon} \int_\gamma d\dot{\gamma}_\epsilon = \int_a^b |\omega(\dot{\gamma})| dt =: \int_\gamma d\dot{\gamma}, \tag{6.1}$$

Proof As we already saw, $\|\dot{\gamma}\|_\epsilon = \sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + L\omega(\dot{\gamma})^2}$, hence by the definition of $d\dot{\gamma}_\epsilon$ and the dominated convergence theorem,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \sqrt{\epsilon} \int_\gamma d\dot{\gamma}_\epsilon &= \int_a^b \lim_{\epsilon \rightarrow 0^+} \sqrt{\epsilon} \|\dot{\gamma}\|_\epsilon dt = \int_a^b \lim_{\epsilon \rightarrow 0^+} \sqrt{\epsilon} \sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \frac{1}{\epsilon} \omega(\dot{\gamma})^2} dt \\ &= \int_a^b |\omega(\dot{\gamma})| dt \end{aligned}$$

as desired. □

Remark 6.2 It is clear that this scaled measure vanishes on fully horizontal curves.

Let us also recall a technical result concerning the scaled limit of the Riemannian surface measure.

Proposition 6.3 ([9], Chapter 5.1) *Let $\Sigma \subset (\mathbb{R}^3, g_\epsilon)$ be a Euclidean C^2 -smooth surface. Let $d\sigma_\epsilon$ denote the surface measure on Σ with respect to the Riemannian metric g_ϵ . Let M be the 2×3 matrix*

$$M := \begin{pmatrix} 1 & 0 & -\frac{x_2}{2} \\ 0 & 1 & \frac{x_1}{2} \end{pmatrix}. \tag{6.2}$$

If $\Sigma = \{u = 0\}$ with $u \in C^2(\mathbb{R}^3)$, then

$$\lim_{\epsilon \rightarrow 0^+} \sqrt{\epsilon} \int_{\Sigma} d\sigma_\epsilon = \int_{\Sigma} \frac{\|\nabla_{\mathbb{H}u}\|_{\mathbb{H}}}{\|\nabla_{\mathbb{R}^3}u\|_{\mathbb{R}^3}} d\mathcal{H}^2_{\mathbb{R}^3} = \int_{\Sigma} d\mathcal{H}^3_{cc}, \tag{6.3}$$

where $d\mathcal{H}^2_{\mathbb{R}^3}$ denotes the Euclidean 2-Hausdorff measure and $d\mathcal{H}^3_{cc}$ the 3-dimensional Hausdorff measure with respect to the cc metric d_{cc} . If $\Sigma = f(D)$ with

$$f = f(\mathbf{u}, \mathbf{v}) : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3,$$

and Euclidean normal vector to Σ given by $\vec{n}(\mathbf{u}, \mathbf{v}) = (f_{\mathbf{u}} \times f_{\mathbf{v}})(\mathbf{u}, \mathbf{v})$, then

$$\lim_{\epsilon \rightarrow 0^+} \sqrt{\epsilon} \int_{\Sigma} d\sigma_\epsilon = \int_D \|M\vec{n}\|_{\mathbb{R}^2} d\mathbf{u}d\mathbf{v}. \tag{6.4}$$

The classical Gauss–Bonnet theorem for a regular surface $\Sigma \subset (\mathbb{R}^3, g_\epsilon)$ with boundary components given by Euclidean C^2 -smooth and regular curves γ_i (see for instance [26, Chapter 9] or [24]) states that

$$\int_{\Sigma} \mathcal{K}_\epsilon d\sigma_\epsilon + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i, \Sigma}^{\epsilon, s} d\dot{\gamma}_i \epsilon = 2\pi \chi(\Sigma), \tag{6.5}$$

where \mathcal{K}_ϵ is the Gaussian curvature of Σ , $k_{\gamma_i, \Sigma}^{\epsilon, s}$ is the signed geodesic curvature of the i^{th} boundary component γ_i , $d\dot{\gamma}_i \epsilon = \|\dot{\gamma}_i\|_\epsilon d\theta$ and $\chi(\Sigma)$ is the Euler characteristic of Σ . It is clear that for a regular surface $\Sigma \subset (\mathbb{R}^3, g_\epsilon)$ without boundary, (6.5) simplifies to

$$\int_{\Sigma} \mathcal{K}_\epsilon d\sigma_\epsilon = 2\pi \chi(\Sigma).$$

Recalling the considerations made on the Riemannian surface measure $d\sigma_\epsilon$, it is natural to multiply (6.5) by a factor $\sqrt{\epsilon}$,

$$\int_{\Sigma} \mathcal{K}_\epsilon \sqrt{\epsilon} d\sigma_\epsilon + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i, \Sigma}^{\epsilon, s} \sqrt{\epsilon} d\dot{\gamma}_i \epsilon = 2\pi \sqrt{\epsilon} \chi(\Sigma), \tag{6.6}$$

and then hope to derive a Gauss–Bonnet Theorem as a limit as $\epsilon \rightarrow 0^+$. The most difficult task is to take care of the possible presence of characteristic points on Σ . In order to deal with them, we provide the following general definition:

Definition 6.4 ((R)-property) Let $S \subset \mathbb{R}^2$ be any set in \mathbb{R}^2 . We say that the set S satisfies the removability (R)-property, if for every $\delta > 0$, there exist a number $n = n(\delta) \in \mathbb{N}$ and smooth simple closed curves $\gamma_1, \dots, \gamma_n : I \subset \mathbb{R} \rightarrow \mathbb{R}^2$, such that

$$S \subset \bigcup_{i=1}^n \text{int}(\gamma_i), \quad \text{and} \quad \sum_{i=1}^n \text{length}(\gamma_i) \leq \delta,$$

where $\text{length}_E(\gamma)$ denotes the usual Euclidean length of a curve in \mathbb{R}^2 .

It is clear that if the set S consists of finitely many isolated points, then it satisfies the (R)-property. A more complicated example is provided by the self-similar Cantor set $C^{(2)}(\lambda)$ in \mathbb{R}^2 with scaling ratio $\lambda < 1/4$. Here $C^{(2)}(\lambda) = C^{(1)}(\lambda) \times C^{(1)}(\lambda)$, where $C^{(1)}(\lambda)$ denotes the unique nonempty compact subset of \mathbb{R} which is fully invariant under the action of the two contractive similarities $f_1(x) = \lambda x$ and $f_2(x) = \lambda x + 1 - \lambda$.

Proposition 6.5 *When $\lambda < \frac{1}{4}$, the self-similar Cantor set $C^{(2)}(\lambda)$ satisfies the (R)-property.*

Proof At the n^{th} stage of the iterative construction of $C^{(2)}(\lambda)$ we have 4^n pieces. We can surround every such piece with a smooth closed curve γ_n^k . For sake of simplicity let us take a Euclidean circle whose radius r_n^k is comparable to λ^n . Therefore

$$\sum_{k=1}^{4^n} \text{length}_E(\gamma_n^k) \lesssim 4^n \lambda^n \rightarrow 0, \quad \text{as } n \rightarrow +\infty,$$

because $\lambda < 1/4$. □

Lemma 6.6 *If $S \subset \mathbb{R}^2$ is compact and such that $\mathcal{H}_E^1(S) = 0$ then S satisfies the (R)-property.*

Proof First, since S is compact, we can take a finite covering of S made of Euclidean balls $\{B(g_i, r_i)\}_{i=1, \dots, m}$, with $g_i \in S$ for every $i = 1, \dots, m$, and such that $\sum_{i=1}^m r_i \leq \delta$. Enlarging these balls by a factor which will not depend on δ , we can assume that S lies entirely in the interior of the union of these balls. Now we define the surrounding curves, as the boundaries of the unions of these balls. By construction, the Euclidean length of such curves is comparable to δ . □

Remark 6.7 The converse of Lemma 6.6 also holds true. In other words, the validity of the (R)-property for a compact set $S \subset \mathbb{R}^2$ is equivalent to $\mathcal{H}_E^1(S) = 0$.

Before proving Theorem 1.1, let us make another useful remark. Let us denote by Π the projection $\Pi : \mathbb{H} \rightarrow \mathbb{R}^2$ onto the first two components. Consider a surface $\Sigma \subset \mathbb{H}$ as in the hypothesis of Theorem 1.1. Then

$$\mathcal{H}_E^1(C(\Sigma)) = \mathcal{H}_E^1(\Pi(C(\Sigma))).$$

In particular, if $\mathcal{H}_E^1(C(\Sigma)) = 0$, then its projection $\Pi(C(\Sigma))$ satisfies the (R)-property.

Proof of Theorem 1.1 The proof of Theorem 1.1 will be a combination of different steps.

Step 1 First we consider the case of a regular surface without characteristic points. Precisely, let $\Sigma \subset \mathbb{H}$ be a regular surface without characteristic points, and with finitely many boundary components $(\partial\Sigma)_i, i \in \{1, \dots, n\}$, given by Euclidean C^2 -smooth closed curves $\gamma_i : [0, 2\pi] \rightarrow (\partial\Sigma)_i$. We may assume that none of the boundary components are fully horizontal. Let \mathcal{K}_0 be the intrinsic Gaussian curvature of Σ , and let $k_{\gamma_i, \Sigma}^{0,s}$ be the intrinsic signed geodesic curvature of γ_i , for every $i \in \{1, \dots, n\}$. Then

$$\int_{\Sigma} \mathcal{K}_0 d\mathcal{H}_{cc}^3 + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i, \Sigma}^{0,s} d\dot{\gamma}_i = 0.$$

□

Proof of Step 1 The results will follow passing to the limit as $\epsilon \rightarrow 0^+$ in

$$\int_{\Sigma} \mathcal{K}_{\epsilon} \sqrt{\epsilon} d\sigma_{\epsilon} + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i, \Sigma}^{\epsilon, s} \sqrt{\epsilon} d\gamma_i = 2\pi \sqrt{\epsilon} \chi(\Sigma).$$

To do this, we need to apply Lebesgue’s dominated convergence theorem. Let us start with the integral of \mathcal{K}_{ϵ} . We take a partition of unity $\{\varphi_i\}$, $i = 1, \dots, m$. Calling $\Sigma_i := \text{supp}(\varphi_i) \cap \Sigma$, for every $i = 1, \dots, m$, we have

$$\int_{\Sigma} \mathcal{K}_{\epsilon} \sqrt{\epsilon} d\sigma_{\epsilon} = \sum_{i=1}^m \int_{\Sigma_i} \mathcal{K}_{\epsilon} \varphi_i \sqrt{\epsilon} d\sigma_{\epsilon}.$$

Let us choose a parametrization of every Σ_i , $\psi_i : D_i \rightarrow \Sigma_i$, then, for every $i = 1, \dots, m$, it holds that

$$\int_{\Sigma_i} \mathcal{K}_{\epsilon} \varphi_i \sqrt{\epsilon} d\sigma_{\epsilon} = \int_{D_i} \mathcal{K}_{\epsilon} \varphi_i |M_{\epsilon} \vec{n}| dv dw,$$

where

$$M_{\epsilon} := \begin{pmatrix} 1 & 0 & -\frac{x_2}{2} \\ 0 & 1 & \frac{x_1}{2} \\ 0 & 0 & \sqrt{\epsilon} \end{pmatrix},$$

and \vec{n} denotes the Euclidean normal vector to Σ .

It is now sufficient to check whether there exist two positive constants M_1 and M_2 , independent of ϵ , such that

$$|\mathcal{K}_{\epsilon}| \leq M_1, \quad \text{and} \quad |M_{\epsilon} \vec{n}| \leq M_2.$$

The second estimate is proved in Proposition 6.3, see [9]. For the first one, we recall that the explicit expression of \mathcal{K}_{ϵ} is given by

$$\begin{aligned} \mathcal{K}_{\epsilon} = & -\frac{1}{\epsilon} \bar{r}_{\epsilon}^2 - \left(\frac{l}{\epsilon}\right)^3 (X_1 \bar{p} + X_2 \bar{q}) \langle E_2, \nabla_{\mathbb{H}} \left(\frac{r}{l}\right) \rangle_{\epsilon} + \frac{l}{\epsilon} (X_1 \bar{p} + X_2 \bar{q}) X_3^{\epsilon} \bar{r}_{\epsilon} \\ & - \left(\frac{l}{\epsilon}\right)^4 \left(\langle E_1, \nabla_{\mathbb{H}} \left(\frac{r}{l}\right) \rangle_{\epsilon}\right)^2 - \frac{1}{\sqrt{\epsilon}} \left(\frac{l}{\epsilon}\right)^2 \langle E_1, \nabla_{\mathbb{H}} \left(\frac{r}{l}\right) \rangle_{\epsilon}. \end{aligned} \tag{6.7}$$

Since $l_{\epsilon} = \|\nabla_{\epsilon} u\|_{\epsilon} = \sqrt{(X_1 u)^2 + (X_2 u)^2 + (\sqrt{\epsilon} X_3 u)^2} \geq \|\nabla_{\mathbb{H}} u\|_{\mathbb{H}}$, we have that

$$\frac{l}{l_{\epsilon}} = \frac{\|\nabla_{\mathbb{H}} u\|_{\mathbb{H}}}{\|\nabla_{\epsilon} u\|_{\epsilon}} \leq 1.$$

Since $\epsilon < 1$, it also holds that

$$l_{\epsilon} \leq \sqrt{(X_1 u)^2 + (X_2 u)^2 + (X_3 u)^2}.$$

Moreover, since $u \in C^2(\mathbb{R}^3)$ and Σ is a compact surface without characteristic points, there exists a positive constant $C_1 > 0$ such that

$$|X_1 \bar{p} + X_2 \bar{q}| = \left| X_1 \left(\frac{X_1 u}{\|\nabla_{\mathbb{H}} u\|_{\mathbb{H}}} \right) + X_2 \left(\frac{X_2 u}{\|\nabla_{\mathbb{H}} u\|_{\mathbb{H}}} \right) \right| \leq C_1.$$

Therefore we have the following list of estimates:

$$\frac{1}{\epsilon} \bar{r}_{\epsilon}^2 = \frac{1}{\epsilon} \frac{(X_3^{\epsilon} u)^2}{l_{\epsilon}^2} \leq \left(\frac{X_3 u}{\|\nabla_{\mathbb{H}} u\|_{\mathbb{H}}} \right)^2 \leq C_2.$$

$$\begin{aligned} \left| \langle E_2, \nabla_H \left(\frac{r}{l} \right) \rangle_\epsilon \right| &= \left| \bar{r}_\epsilon \bar{p} X_1 \left(\frac{r}{l} \right) + \bar{r}_\epsilon \bar{q} X_2 \left(\frac{r}{l} \right) \right| \\ &= \epsilon |X_3 u| \left| \frac{X_1 u}{\|\nabla_H u\|_H} X_1 \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right) + \frac{X_2 u}{\|\nabla_H u\|_H} X_2 \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right) \right| \\ &\leq |X_3 u| \left| \frac{X_1 u}{\|\nabla_H u\|_H} X_1 \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right) + \frac{X_2 u}{\|\nabla_H u\|_H} X_2 \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right) \right| \leq C_3. \end{aligned}$$

$$\begin{aligned} \frac{1}{\sqrt{\epsilon}} \left| \langle E_1, \nabla_H \left(\frac{r}{l} \right) \rangle_\epsilon \right| &= \frac{1}{\sqrt{\epsilon}} \left| \bar{q} X_1 \left(\frac{\sqrt{\epsilon} X_3 u}{\|\nabla_H u\|_H} \right) - \bar{p} X_2 \left(\frac{\sqrt{\epsilon} X_3 u}{\|\nabla_H u\|_H} \right) \right| \\ &\leq \left| \frac{X_2 u}{\|\nabla_H u\|_H} X_1 \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right) \right| + \left| \frac{X_1 u}{\|\nabla_H u\|_H} X_2 \left(\frac{X_3 u}{\|\nabla_H u\|_H} \right) \right| \leq C_4. \end{aligned}$$

Similarly,

$$|X_3^\epsilon \bar{r}_\epsilon| \leq C_5.$$

Altogether,

$$|\mathcal{K}_\epsilon| \leq C_2 + C_1 \cdot C_3 + C_1 \cdot C_5 + C_4^2 + C_4 =: M_1,$$

as desired.

It remains to see what happens for the boundary integrals. Without loss of generality, let us assume we are given a surface Σ with only one boundary component, given by a smooth curve γ . We need to estimate

$$\left| \frac{\sqrt{\epsilon} \langle D_t^\Sigma \dot{\gamma}, J_\epsilon(\dot{\gamma}) \rangle_\epsilon}{\|\dot{\gamma}\|_\epsilon^2} \right|,$$

where

$$\begin{aligned} \langle D_t^\Sigma \dot{\gamma}, J_\epsilon(\dot{\gamma}) \rangle_\epsilon &= \bar{r}_\epsilon (\dot{\gamma}_1 \dot{\gamma}_2 - \dot{\gamma}_2 \dot{\gamma}_1) \\ &\quad + \frac{1}{\sqrt{\epsilon}} \frac{l}{l_\epsilon} \left[\omega(\dot{\gamma}) (\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2) + \omega(\dot{\gamma}) (\bar{q} \dot{\gamma}_1 - \bar{p} \dot{\gamma}_2) \right] \\ &\quad - \frac{1}{\epsilon} \omega(\dot{\gamma}) \bar{r}_\epsilon (2\dot{\gamma}_1 \dot{\gamma}_2) + \left(\frac{1}{\epsilon} \right)^{3/2} \frac{l}{l_\epsilon} \omega(\dot{\gamma})^2 (\bar{p} \dot{\gamma}_1 + \bar{q} \dot{\gamma}_2), \end{aligned}$$

and

$$\|\dot{\gamma}\|_\epsilon^2 = \dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \frac{1}{\epsilon} \omega(\dot{\gamma})^2.$$

Note that there exists a positive constant $C_0 > 0$, such that

$$\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \frac{1}{\epsilon} \omega(\dot{\gamma})^2 \geq \dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \omega(\dot{\gamma})^2 \geq C_0 > 0.$$

Now, we have

$$\begin{aligned} \frac{\sqrt{\epsilon} |\bar{r}_\epsilon (\dot{\gamma}_1 \dot{\gamma}_2 - \dot{\gamma}_2 \dot{\gamma}_1)|}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \frac{1}{\epsilon} \omega(\dot{\gamma})^2)} &\leq \frac{\epsilon |\dot{\gamma}_1 \dot{\gamma}_2 - \dot{\gamma}_2 \dot{\gamma}_1| |X_3 u|}{\|\nabla_H u\|_H (\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + L\omega(\dot{\gamma})^2)} \\ &\leq \frac{|X_3 u|}{\|\nabla_H u\|_H} \frac{|\dot{\gamma}_1 \dot{\gamma}_2 - \dot{\gamma}_2 \dot{\gamma}_1|}{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \omega(\dot{\gamma})^2} \leq C_6. \end{aligned}$$

$$\begin{aligned} & \frac{1}{\sqrt{\epsilon}} \frac{l}{l_\epsilon} \frac{\sqrt{\epsilon} | [\omega(\dot{\gamma}) (\bar{q}\dot{\gamma}_1 - \bar{p}\dot{\gamma}_2) + \omega(\dot{\gamma}) (\bar{q}\dot{\gamma}_1 - \bar{p}\dot{\gamma}_2)] |}{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \frac{1}{\epsilon} \omega(\dot{\gamma})^2)} \\ & \leq \frac{|\omega(\dot{\gamma})| |\bar{q}\dot{\gamma}_2 - \bar{p}\dot{\gamma}_2| + |\omega(\dot{\gamma})| |\bar{q}\dot{\gamma}_1 - \bar{p}\dot{\gamma}_2|}{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \omega(\dot{\gamma})^2} \leq C_7. \\ & \frac{2\sqrt{\epsilon} |\omega(\dot{\gamma})| |\bar{r}_\epsilon| |\dot{\gamma}_1 \dot{\gamma}_2|}{\epsilon (\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \frac{1}{\epsilon} \omega(\dot{\gamma})^2)} \leq \frac{2|\omega(\dot{\gamma})| \|X_{3u}\| |\dot{\gamma}_1 \dot{\gamma}_2|}{\|\nabla_{\mathbb{H}u}\|_{\mathbb{H}} (\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \omega(\dot{\gamma})^2)} \leq C_8. \\ & \frac{l}{l_\epsilon} \frac{\sqrt{\epsilon} \omega(\dot{\gamma})^2 |\bar{p}\dot{\gamma}_1 + \bar{q}\dot{\gamma}_2|}{\epsilon^{3/2} (\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \frac{1}{\epsilon} \omega(\dot{\gamma})^2)} \leq \frac{\epsilon^{3/2} \omega(\dot{\gamma})^2}{\epsilon^{3/2} \omega(\dot{\gamma})^2} |\bar{p}\dot{\gamma}_1 + \bar{q}\dot{\gamma}_2| = |\bar{p}\dot{\gamma}_1 + \bar{q}\dot{\gamma}_2| \leq C_9. \end{aligned}$$

The behavior of the measure has been already treated in Lemma 6.1. □

Step 2 Due to Lemma 6.6, we can surround the projection of the characteristic set $\Pi(C(\Sigma))$ with smooth simple closed curves $\{\beta_j\}_{j=1, \dots, n(\delta)}$ such that

$$\sum_{j=1}^{n(\delta)} \text{length}_E(\beta_j) \leq \delta. \tag{6.8}$$

We can now work with a new surface Σ_δ which has no characteristic points, and boundary components which are given by the curves γ_i 's and the curves β_j 's. Step 1 tells us that for every $\delta > 0$,

$$\int_{\Sigma_\delta} \mathcal{K}_0 d\mathcal{H}_{cc}^3 + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i, \Sigma_\delta}^{0,s} d\dot{\gamma}_i = - \sum_{j=1}^{n(\delta)} k_{\beta_j, \Sigma_\delta}^{0,s} d\dot{\beta}_j,$$

which, combined with (6.8), implies that for every $\delta > 0$,

$$\left| \int_{\Sigma_\delta} \mathcal{K}_0 d\mathcal{H}_{cc}^3 + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i, \Sigma_\delta}^{0,s} d\dot{\gamma}_i \right| = \left| \sum_{j=1}^{n(\delta)} k_{\beta_j, \Sigma_\delta}^{0,s} d\dot{\beta}_j \right| \leq \delta,$$

and this completes the proof. □

Corollary 6.8 *Let $\Sigma \subset \mathbb{H}$ be a regular surface without boundary, or with boundary components given by Euclidean C^2 -smooth horizontal curves. Assume that the characteristic set $C(\Sigma)$ satisfies $\mathcal{H}^1(C(\Sigma)) = 0$ and that $\|\nabla_{\mathbb{H}u}\|_{\mathbb{H}}^{-1}$ is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure, near the characteristic set $C(\Sigma)$. Then the intrinsic Gaussian curvature \mathcal{K}_0 cannot be an always positive or negative function. In particular, Σ cannot have constant non-zero intrinsic Gaussian curvature \mathcal{K}_0 .*

Proof If Σ has no boundary, then by Theorem 1.1 we have

$$\int_{\Sigma} \mathcal{K}_0 d\mathcal{H}_{cc}^3 = 0,$$

and therefore \mathcal{K}_0 cannot have a sign.

The same holds for boundary components given by horizontal curves because the intrinsic signed curvature $k_{\gamma, \Sigma}^{0,s}$ of a horizontal curve γ is 0. □

Remark 6.9 At the moment we do not know any example of a regular surface with intrinsic Gaussian curvature \mathcal{K}_0 constantly equal to zero. On the other hand, if we remove the requirement of Σ being compact, all vertically ruled (smooth) surfaces have $\mathcal{K}_0 = 0$ at every point.

The following examples show that the assumption made in Theorem 1.1 about the 1-dimensional Euclidean Hausdorff measure of the characteristic set $C(\Sigma)$ is sharp.

Example 6.10 Let $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H} : u(x_1, x_2, x_3) = 0\}$, with

$$u(x_1, x_2, x_3) = \begin{cases} x_3 - \frac{x_1 x_2}{2} + x_2 \exp((x_1 + 1)^{-2}), & x_1 < -1, x_2, x_3 \in \mathbb{R}, \\ x_3 - \frac{x_1 x_2}{2}, & x_1 \in [-1, 1], x_2, x_3 \in \mathbb{R}, \\ x_3 - \frac{x_1 x_2}{2} + x_2 \exp((x_1 - 1)^{-2}), & x_1 > 1, x_2, x_3 \in \mathbb{R}. \end{cases} \quad (6.9)$$

We have that

$$C(\Sigma) = \{(x_1, 0, 0) : x_1 \in [-1, 1]\}.$$

The idea now is to consider the projection of $C(\Sigma)$ and to surround it with a curve in \mathbb{R}^2 . Then, we will lift it to the surface Σ , exploiting that Σ is given by a graph. For $\delta > 0$, define

$$\gamma(\theta) := \bigcup_{i=1}^5 \gamma_i(\theta), \quad \theta \in [0, 2\pi],$$

where

$$\begin{aligned} \gamma_1(\theta) &:= \left(1 + \delta \cos\left(\frac{\pi}{2 \tan(\delta)} \theta\right), \delta \sin\left(\frac{\pi}{2 \tan(\delta)} \theta\right) \right), \quad \theta \in [0, \tan(\delta)), \\ \gamma_2(\theta) &:= \left(-\frac{2}{\pi - 2 \tan(\delta)} \theta + \frac{\pi}{\pi - 2 \tan(\delta)}, \delta \right), \quad \theta \in [\tan(\delta), \pi - \tan(\delta)), \\ \gamma_3(\theta) &:= (-1 + \delta \cos(\alpha(\theta)), \delta \sin(\alpha(\theta))), \quad \theta \in [\pi - \tan(\delta), \pi + \tan(\delta)), \\ \gamma_4(\theta) &:= (t(\theta), -\delta), \quad \theta \in [\pi + \tan(\delta), 2\pi - \tan(\delta)), \\ \gamma_5(\theta) &:= (1 + \delta \cos(\beta(\theta)), \delta \sin(\beta(\theta))), \quad \theta \in [2\pi - \tan(\delta), 2\pi), \end{aligned}$$

with

$$\begin{aligned} \alpha(\theta) &:= \frac{\pi}{2 \tan(\delta)} \theta + \pi - \frac{\pi^2}{2 \tan(\delta)}, \\ t(\theta) &:= \frac{2}{\pi - 2 \tan(\delta)} \theta - \frac{3\pi}{\pi - 2 \tan(\delta)}, \\ \beta(\theta) &:= \frac{\pi}{\pi - 2 \tan(\delta)} \theta + 2\pi \left(1 - \frac{\pi}{2 \tan(\delta)} \right). \end{aligned}$$

Now we need to control the integral of the signed curvature. It will be made of five pieces. Three of them (namely for $i = 1, 3, 5$), will behave exactly like when we deal with isolated characteristic points, because the velocity of those parts goes to 0 as $\delta \rightarrow 0$. It remains to check the other two integrals. We can re-parametrize those two parts as follows:

$$\gamma_2(s) = (-s, \delta), \quad \text{and} \quad \gamma_4(s) = (s, -\delta), \quad s \in [-1, 1].$$

Because of the results concerning the signed curvature, it does not matter what happens to the third component. In this situation, we get that

$$\dot{\gamma}_2(s) = (-1, 0), \quad \text{and} \quad \dot{\gamma}_4(s) = (1, 0), \quad s \in [-1, 1],$$

and

$$\bar{p}(x_1, x_2, x_3) = -\frac{x_2}{|x_2|}, \quad \bar{q}(x_1, x_2, x_3) = 0.$$

Therefore $\bar{p}|_{\gamma_2} = 1$ and $\bar{p}|_{\gamma_4} = -1$. Then

$$\int_{-1}^1 \bar{p}|_{\gamma_2}(\dot{\gamma}_2)_1 ds + \int_{-1}^1 \bar{p}|_{\gamma_4}(\dot{\gamma}_4)_1 ds = -2.$$

Example 6.11 Let $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H} : x_3 = \frac{x_1 x_2}{2}\}$. The projection of the characteristic set $\Pi(C(\Sigma))$ is the 1-dimensional line $\{x_2 = 0\}$. Consider the curve

$$\gamma(t) = \left(\cos(t), \sin(t), \frac{\sin(2t)}{4} \right), \quad t \in [0, 2\pi],$$

which lives on the surface Σ as boundary component of a new surface $\tilde{\Sigma}$ which is now bounded. Simple computations show that in this case

$$\mathcal{K}_0 = 0, \quad k_{\gamma, \tilde{\Sigma}}^{0,s} = \frac{1}{|\sin(t)|}, \quad \omega(\dot{\gamma}) = \frac{\cos(2t)}{2} - \frac{1}{2}.$$

Therefore,

$$\int_{\tilde{\Sigma}} \mathcal{K}_0 d\mathcal{H}_{cc}^3 + \int_{\gamma} k_{\gamma, \tilde{\Sigma}}^{0,s} d\dot{\gamma} = 4,$$

in contrast with the statement of Theorem 1.1.

We end this section with an explicit example.

Example 6.12 (Korányi sphere) Consider the Korányi sphere $\mathbb{S}_{\mathbb{H}}$,

$$\mathbb{S}_{\mathbb{H}} := \{(x_1, x_2, x_3) \in \mathbb{H} : (x_1^2 + x_2^2)^2 + 16x_3^2 - 1 = 0\}.$$

A parametrization of $\mathbb{S}_{\mathbb{H}}$ is

$$f(\varphi, \theta) := \left(\sqrt{\cos(\varphi)} \cos(\theta), \sqrt{\cos(\varphi)} \sin(\theta), \frac{\sin(\varphi)}{4} \right), \quad \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right), \theta \in [0, 2\pi).$$

In particular, recalling the definition (6.2) of M and after some computations, we get that

$$\|M(f_\varphi \times f_\theta)\|_{\mathbb{R}^2} = \frac{\sqrt{\cos(\varphi)}}{4}.$$

A direct computation shows that the intrinsic Gaussian curvature of $\mathbb{S}_{\mathbb{H}}$ is given by

$$\mathcal{K}_0 = -\frac{2}{x_1^2 + x_2^2} + 6(x_1^2 + x_2^2) = -\frac{2}{\cos(\varphi)} + 6 \cos(\varphi), \tag{6.10}$$

which is locally summable around the isolated characteristic points with respect to the Heisenberg perimeter measure.

Thus, by a special instance of Theorem 1.1

$$\int_{\mathbb{S}_{\mathbb{H}}} \kappa_0 d\mathcal{H}_{cc}^3 = 0. \tag{6.11}$$

Equation 6.11 can also be verified directly using (6.10).

Further examples can be found in the ‘‘Appendix’’.

7 Application: a simplified Steiner’s formula

The main application of the Gauss–Bonnet Theorem concerns a simplification to the Steiner’s formula recently proved in [5] concerning cc neighborhoods of surfaces without characteristic points. Let us first recall some notation and background from [5].

Let $\Omega \subset \mathbb{H}$ be an open, bounded and regular domain with smooth boundary $\partial\Omega$. Let δ_{cc} be the signed cc -distance function from $\partial\Omega$. The ϵ -neighborhood Ω_ϵ of Ω with respect to the cc -metric is given by

$$\Omega_\epsilon := \Omega \cup \{g \in \mathbb{H} : 0 \leq \delta_{cc}(g) < \epsilon\}. \tag{7.1}$$

Remark 7.1 In this section, we make no use of the Riemannian approximation scheme. We therefore proceed to reuse the variable ϵ in its traditional role as a small distance parameter, as in (7.1).

We define the *iterated divergences* of δ_{cc} as follows:

$$\operatorname{div}_{\mathbb{H}}^{(i)}(\delta_{cc}) = \begin{cases} 1, & i = 0 \\ \operatorname{div}_{\mathbb{H}}(\operatorname{div}_{\mathbb{H}}^{(i-1)} \delta_{cc} \cdot \nabla_{\mathbb{H}} \delta_{cc}), & i \geq 1 \end{cases}$$

Finally we put

$$\begin{aligned} A &:= \Delta_{\mathbb{H}} \delta_{cc}, & B &:= -(X_3 \delta_{cc})^2, & C &:= (X_1 \delta_{cc})(X_{32} \delta_{cc}) - (X_2 \delta_{cc})(X_{31} \delta_{cc}), \\ D &:= X_{33} \delta_{cc}, & E &:= (X_{31} \delta_{cc})^2 + (X_{32} \delta_{cc})^2. \end{aligned} \tag{7.2}$$

In order to make the paper self-contained, we recall a technical result from [5]. Define the operator g acting on smooth real valued functions as

$$g(\alpha) := \langle \nabla_{\mathbb{H}} \alpha, \nabla_{\mathbb{H}} \delta \rangle.$$

For a real-valued function h we have

$$\operatorname{div}_{\mathbb{H}}(h \nabla_{\mathbb{H}} \delta_{cc}) = h \Delta_{\mathbb{H}} \delta_{cc} + \langle \nabla_{\mathbb{H}} h, \nabla_{\mathbb{H}} \delta_{cc} \rangle = h A + g(h). \tag{7.3}$$

Note that g is linear and satisfies the Leibniz rule, i.e.

$$\begin{aligned} g(\alpha + \beta) &= g(\alpha) + g(\beta), \\ g(\alpha \beta) &= g(\alpha)\beta + \alpha g(\beta). \end{aligned}$$

The following lemma holds (see [5], Lemma 4.2). The proof involves a number of lengthy calculations using higher derivatives of the cc distance function.

Lemma 7.2 *The following relations hold:*

$$g(1) = 0, \tag{7.4}$$

$$g(A) = B + 2C - A^2, \tag{7.5}$$

$$g(B) = 0, \tag{7.6}$$

$$g(C) = D - AC, \tag{7.7}$$

$$g(D) = -E, \tag{7.8}$$

$$g(E) = -2AE + 2CD. \tag{7.9}$$

We now state the main result of this section, which gives a simplification to the main theorem of [5] (a Steiner’s formula for the cc distance function).

Theorem 7.3 *Let $\Omega \subset \mathbb{H}$ be an open, bounded and regular domain whose boundary $\partial\Omega$ is a Euclidean C^2 -smooth compact and oriented surface with no characteristic points. For sufficiently small $\epsilon > 0$ define the ϵ -neighborhood Ω_ϵ of Ω with respect to the cc -metric as in (7.1). Then*

$$\begin{aligned} \mathcal{L}^3(\Omega_\epsilon) &= \mathcal{L}^3(\Omega) + \epsilon \mathcal{H}_{cc}^3(\partial\Omega) + \frac{\epsilon^2}{2} \int_{\partial\Omega} A d\mathcal{H}_{cc}^3 + \frac{\epsilon^3}{3!} \int_{\partial\Omega} C d\mathcal{H}_{cc}^3 \\ &+ \sum_{j=1}^{+\infty} \left(\int_{\partial\Omega} (B^{j-1} D) d\mathcal{H}_{cc}^3 \right) \frac{\epsilon^{2j+2}}{(2j+2)!} \\ &+ \sum_{j=1}^{+\infty} \left(\int_{\partial\Omega} (B^{j-1} (AD - E)) d\mathcal{H}_{cc}^3 \right) \frac{\epsilon^{2j+3}}{(2j+3)!}, \end{aligned}$$

where A, B, C, D and E are given as in (7.2).

Proof Let us write $\partial_f \Omega_\epsilon := \{g \in \mathbb{H} : \delta_{cc}(g) = \epsilon\}$. The evolution of the non-characteristic set $\partial\Omega$ can be explicitly described if we know the defining function of $\partial\Omega$, see [3]. In our situation, we can assume that $\partial\Omega = \{g \in \mathbb{H} : \delta_{cc}(g) = 0\}$. In particular, the results from [3] tell us that there exists a map $\mathcal{N} : [0, \epsilon] \times \partial\Omega \rightarrow \mathbb{R}^3$, such that $\mathcal{N}(\cdot, g) : [0, \epsilon] \rightarrow \mathbb{R}^3$ is continuous, $\mathcal{N}(\epsilon, \cdot) \rightarrow \partial_f \Omega_\epsilon$ is smooth and $\mathcal{N}(0, g) = \text{id}|_{\partial\Omega}(g)$. We claim that there exists a continuous map $a(\epsilon, g) := \text{angle}(T_{\mathcal{N}(\epsilon, g)} \partial_f \Omega_\epsilon; H_{\mathcal{N}(\epsilon, g)})$. Since $\partial\Omega$ has no characteristic points, $a(0, g) > 0$ for every $g \in \partial\Omega$. The continuity of $a(\cdot, \cdot)$ implies that there exists $\epsilon_0, 0 < \epsilon_0 < \epsilon$, so that $a(s, g) > 0$ for every $g \in \partial\Omega$ and for every $s \in (0, \epsilon_0)$. In particular this shows that we can choose a sufficiently small $\epsilon > 0$ so that $\partial_f \Omega_\epsilon$ is still a non-characteristic set.

We will use the following proposition from [5] (see Proposition 3.3) which can be proved with the help of the sub-Riemannian divergence formula.

Proposition 7.4 *Let $h : \Omega_\epsilon \rightarrow \mathbb{R}$ be a C^1 function. Then the vector field $h \nabla_H \delta_{cc} : \Omega_\epsilon \rightarrow \mathbb{R}^3$ satisfies*

$$\int_{\{s < \delta_{cc} < t\}} \text{div}_H(h \nabla_H \delta_{cc}) d\mathcal{L}^3 = \int_{\delta_{cc}^{-1}(t)} h d\mathcal{H}_{cc}^3 - \int_{\delta_{cc}^{-1}(s)} h d\mathcal{H}_{cc}^3.$$

We are interested in a Taylor series expansion of the function

$$\epsilon \mapsto \mathcal{L}^3(\Omega_\epsilon),$$

about $\epsilon = 0$. The analyticity of this function has been already proved in [5], therefore let us denote by $f^{(i)}(\epsilon)$ the i^{th} derivative of $\epsilon \mapsto \mathcal{L}^3(\Omega_\epsilon)$. The first three elements of the expansion

are obtained as in [5]. For the other terms, we need to recall that by Theorem 3.4 of [5], for every $i \geq 0$,

$$f^{(i)}(s) = \int_{\delta_{cc}^{-1}(s)} (\operatorname{div}_H^{(i-1)} \delta_{cc}) d\mathcal{H}_{cc}^3, \quad \text{for every } s \in [0, \epsilon_0). \tag{7.10}$$

In particular, for $i = 3$, $\operatorname{div}_H^{(2)} \delta_{cc} = B + 2C$ and the expression $B + C$ coincides with the horizontal Gaussian curvature \mathcal{K}_0 . Therefore we can apply the Gauss–Bonnet Theorem 1.1 to obtain

$$f^{(3)}(s) = \int_{\delta_{cc}^{-1}(s)} C d\mathcal{H}_{cc}^3$$

and hence

$$f^{(3)}(0) = \int_{\partial\Omega} C d\mathcal{H}_{cc}^3.$$

We now claim that

$$f^{(2j+2)}(s) = \int_{\delta_{cc}^{-1}(s)} (B^{j-1}D) d\mathcal{H}_{cc}^3 \quad \text{for } j \geq 1 \tag{7.11}$$

and

$$f^{(2j+3)}(s) = \int_{\delta_{cc}^{-1}(s)} (B^{j-1}(AD - E)) d\mathcal{H}_{cc}^3 \quad \text{for } j \geq 1, \tag{7.12}$$

from which the indicated values of the coefficients in the series expansion of $\epsilon \mapsto \mathcal{L}^3(\Omega_\epsilon)$ are obtained by setting $s = 0$.

The formulas in (7.11) and (7.12) can be obtained inductively by evaluating difference quotient approximations to the indicated derivatives using the inductive hypothesis and the divergence formula in Proposition 7.4. First,

$$\begin{aligned} f^{(4)}(s) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(f^{(3)}(s + \epsilon) - f^{(3)}(s) \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\int_{\delta_{cc}^{-1}(s+\epsilon)} C d\mathcal{H}_{cc}^3 - \int_{\delta_{cc}^{-1}(s)} C d\mathcal{H}_{cc}^3 \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{\{s < \delta_{cc} < s+\epsilon\}} \operatorname{div}_H(C \nabla_H \delta_{cc}) d\mathcal{L}^3 = \int_{\delta_{cc}^{-1}(s)} \operatorname{div}_H(C \nabla_H \delta_{cc}) d\mathcal{H}_{cc}^3. \end{aligned}$$

By (7.3) and Lemma 7.2, $\operatorname{div}_H(C \nabla_H \delta_{cc}) = CA + g(C) = AC + (D - AC) = D$ and so (7.11) holds in the case $j = 1$.

Similarly, for $j \geq 2$,

$$\begin{aligned} f^{(2j+2)}(s) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(f^{(2j+1)}(s + \epsilon) - f^{(2j+1)}(s) \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\int_{\delta_{cc}^{-1}(s+\epsilon)} (B^{j-2}(AD - E)) d\mathcal{H}_{cc}^3 - \int_{\delta_{cc}^{-1}(s)} (B^{j-2}(AD - E)) d\mathcal{H}_{cc}^3 \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{\{s < \delta_{cc} < s+\epsilon\}} \operatorname{div}_H(B^{j-2}(AD - E) \nabla_H \delta_{cc}) d\mathcal{L}^3 \\ &= \int_{\delta_{cc}^{-1}(s)} \operatorname{div}_H(B^{j-2}(AD - E) \nabla_H \delta_{cc}) d\mathcal{H}_{cc}^3. \end{aligned}$$

By (7.3) and Lemma 7.2,

$$\begin{aligned}
 \operatorname{div}_H(B^{j-2}(AD - E) \nabla_H \delta_{cc}) &= B^{j-2}(AD - E)A + g(B^{j-2}(AD - E)) \\
 &= B^{j-2}(A^2D - AE) + B^{j-2}(Ag(D) + Dg(A) - g(E)) \\
 &= B^{j-2}(A^2D - AE - AE + BD + 2CD - A^2D \\
 &\quad + 2AE - 2CD) \\
 &= B^{j-1}D
 \end{aligned}$$

and so (7.11) holds in the case $j \geq 2$.

A similar computation establishes (7.12) for all $j \geq 1$. This completes the proof. □

8 Questions and remarks

As is clear from the proof of the Gauss–Bonnet theorem, it is of crucial interest to prove local summability of the intrinsic Gaussian curvature \mathcal{K}_0 around isolated characteristic points, with respect to the Heisenberg perimeter measure. For the horizontal mean curvature \mathcal{H}_0 of Σ this is an established result, see [21]. In the same work [21], it is showed that the situation could change dramatically if we address the problem of local integrability of \mathcal{H}_0 with respect to the Riemannian surface measure, near the characteristic set. In this case, it is conjectured that we should have locally integrability if we are close to an *isolated characteristic point* of Σ , and a counterexample in the case in which the characteristic set $\operatorname{char}(\Sigma)$ is 1-dimensional is presented.

Question 8.1 Is the intrinsic Gaussian curvature \mathcal{K}_0 locally summable with respect to the Heisenberg perimeter measure, near the characteristic set?

Recalling (5.5), it is clear that the local summability of \mathcal{K}_0 is closely related to the integrability of \mathcal{H}_0 near isolated characteristic points with respect to the Riemannian surface measure. As far as we know, the best results in this direction are those of [21], which provide a class of examples where we have local integrability. In the same spirit of [21], we have the following result.

Proposition 8.2 *Let $\Sigma \subset \mathbb{H}$ be a Euclidean C^2 -smooth surface. Suppose Σ has cylindrical symmetry near an isolated characteristic point g , then $\mathcal{K}_0 \in L^1(\Sigma, d\mathcal{H}_{cc}^3)$.*

Proof Without loss of generality we can assume that the isolated characteristic point is the origin $0 = (0, 0, 0)$, and that locally around 0 the surface Σ is given by the 0-level set of the function

$$u(x_1, x_2, x_3) := x_3 - f\left(\frac{x_1^2 + x_2^2}{4}\right),$$

where $f \in C^2$. For simplicity, let us denote by $f(r) := f\left(\frac{x_1^2 + x_2^2}{4}\right)$. Then,

$$X_3u = 1, \quad X_1u = -\frac{1}{2}(x_2 + x_1 f'(r)) \quad \text{and} \quad X_2u = \frac{1}{2}(x_1 - x_2 f'(r)),$$

therefore $\|\nabla_{\mathbb{H}^n} u\|_{\mathbb{H}} = \frac{1}{2}\sqrt{x_1^2 + x_2^2}\sqrt{1 + (f'(r))^2}$. In order to compute the intrinsic Gaussian curvature \mathcal{K}_0 we need

$$\begin{aligned} X_1 \left(\frac{1}{\|\nabla_{\mathbb{H}^n} u\|_{\mathbb{H}}} \right) &= 2\partial_{x_1} \left((x_1^2 + x_2^2)^{-\frac{1}{2}}(1 + (f')^2)^{-\frac{1}{2}} \right) \\ &= -\frac{2x_1(1 + (f')^2) + f' f'' x_1(x_1^2 + x_2^2)}{(x_1^2 + x_2^2)^{3/2}(1 + (f')^2)^{3/2}}, \end{aligned}$$

and

$$\begin{aligned} X_2 \left(\frac{1}{\|\nabla_{\mathbb{H}^n} u\|_{\mathbb{H}}} \right) &= 2\partial_{x_2} \left((x_1^2 + x_2^2)^{-\frac{1}{2}}(1 + (f')^2)^{-\frac{1}{2}} \right) \\ &= -\frac{2x_2(1 + (f')^2) + f' f'' x_2(x_1^2 + x_2^2)}{(x_1^2 + x_2^2)^{3/2}(1 + (f')^2)^{3/2}}. \end{aligned}$$

After some simplifications we get

$$\mathcal{K}_0 = -\frac{2}{(x_1^2 + x_2^2)(1 + (f')^2)} + \frac{f' f''}{(1 + (f')^2)^2}, \tag{8.1}$$

which is summable. □

A generalization of (8.1) appears in Example 8.10.

It is obvious that the local summability of $\|\nabla_{\mathbb{H}^n} u\|_{\mathbb{H}}^{-1}$ implies the local summability of the intrinsic Gaussian curvature \mathcal{K}_0 , but it is not necessary, as showed by the following example.

Example 8.3 Let $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H} : u(x_1, x_2, x_3) = 0\}$, for

$$u(x_1, x_2, x_3) = x_3 - \frac{x_1 x_2}{2} - \exp(-(x_1^2 + x_2^2)^{-2}).$$

The origin $0 = (0, 0, 0)$ is an isolated characteristic point of Σ , indeed

$$X_1 u = -x_2 - \frac{4x_1 \exp(-(x_1^2 + x_2^2)^{-2})}{(x_1^2 + x_2^2)^3}, \quad \text{and} \quad X_2 u = -\frac{4x_2 \exp(-(x_1^2 + x_2^2)^{-2})}{(x_1^2 + x_2^2)^3}.$$

Switching to polar coordinates $x = r \cos(\theta)$, $y = r \sin(\theta)$, we have

$$\begin{aligned} \|\nabla_{\mathbb{H}^n} u\|_{\mathbb{H}}^2 &= r^2 \sin^2(\theta) + \frac{16 \exp(-2r^{-4})}{r^{10}} + \frac{4r^2 \sin(2\theta) \exp(-r^{-4})}{r^6} \\ &\leq r^2 \sin^2(\theta) + \exp(-r^{-4}). \end{aligned}$$

Therefore we are interested in the summability of the following integral

$$\int_0^\epsilon \int_0^{2\pi} \frac{r}{\sqrt{r^2 \sin^2(\theta) + \exp(-r^{-4})}} d\theta dr. \tag{8.2}$$

Now, setting $g(r) := r^{-1} \exp(-r^{-2})$, we have

$$\begin{aligned} \int_0^\epsilon \int_0^{2\pi} \frac{r}{\sqrt{r^2 \sin^2(\theta) + \exp(-r^{-4})}} d\theta dr &\gtrsim \int_0^\epsilon \int_0^{2\pi} \frac{1}{|\sin(\theta)| + g(r)} d\theta dr \\ &\geq \int_0^\epsilon \left(\int_0^\delta \frac{1}{|\sin(\theta)| + g(r)} d\theta \right) dr \\ &\approx \int_0^\epsilon \left(\int_0^\delta \frac{1}{\theta + g(r)} d\theta \right) dr \\ &= \int_0^\epsilon \log \left(1 + \frac{\delta}{g(r)} \right) dr, \end{aligned}$$

which is divergent.

Unfortunately this does not provide an example of a surface with isolated characteristic points whose intrinsic Gaussian curvature \mathcal{K}_0 is *not* locally integrable with respect to the Heisenberg perimeter measure. Indeed, after quite long computations, one can prove that in this case the intrinsic Gaussian curvature \mathcal{K}_0 is actually locally integrable.

Our second question relates to the possible connection existing between the intrinsic Gaussian curvature \mathcal{K}_0 and one of the integrands appearing in the localized Steiner’s formula proved in [5]. In particular, if we consider a Euclidean C^2 -smooth and regular surface $\Sigma \subset \mathbb{H}$ a defining function u such that $\|\nabla_{\mathbb{H}} u\|_{\mathbb{H}} = 1$, then we have that

$$\mathcal{K}_0 = B + C,$$

where B and C are defined as in Sect. 7.

Question 8.4 Is there any explanation for the discrepancy between the horizontal Gaussian curvature \mathcal{K}_0 and the fourth integrand appearing in the Steiner’s formula proved in [5]?

We want to mention that the expression we got for the intrinsic Gaussian curvature \mathcal{K}_0 appears also in the paper [19] in the study of stability properties of minimal surfaces in \mathbb{H} , and also in the upcoming manuscript [16].

We end this paper with a Fenchel-type theorem for fully horizontal curves.

Theorem 8.5 *Let $\gamma : [a, b] \rightarrow \mathbb{H}$ be a Euclidean C^2 -smooth, regular, closed and fully horizontal curve. Then*

$$\int_\gamma k_\gamma^0 d\dot{\gamma}_{\mathbb{H}} > 2\pi, \tag{8.3}$$

where $d\dot{\gamma}_{\mathbb{H}}$ is the standard Heisenberg length measure of Definition 2.2.

Proof Define the projected curve

$$\tilde{\gamma}(t) := \Pi(\gamma)(t) = (\gamma_1(t), \gamma_2(t), 0)_{\{e_1, e_2, e_3\}}.$$

Then $\tilde{\gamma}$ is a planar Euclidean C^2 -smooth, regular and closed curve whose curvature k coincides precisely with the intrinsic curvature k_γ^0 of γ . Due to the fact that the curve is horizontal,

$$\int_\gamma k_\gamma^0 d\dot{\gamma}_{\mathbb{H}} = \int_{\tilde{\gamma}} k d\dot{\gamma}_{\mathbb{R}^2},$$

where $d\dot{\gamma}_{\mathbb{R}^2}$ denotes the standard Euclidean length measure in \mathbb{R}^2 .

The classical Fenchel Theorem (see [23], Chapter 5.7) assures that

$$\int_{\tilde{\gamma}} k d\tilde{\gamma}_{\mathbb{R}^2} \geq 2\pi,$$

and states that equality is achieved if and only if the curve $\tilde{\gamma}$ is convex. It is a well known fact concerning horizontal curves that the projection of a closed fully horizontal curve γ has enclosed oriented area equal to 0. Therefore its projection $\tilde{\gamma}$ cannot be a convex curve, and the inequality has to be strict. This completes the proof. \square

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Appendix: Examples

We want to collect here a list of explicit examples where we compute the intrinsic Gaussian curvature explicitly.

Example 8.6 Any vertically ruled surface Euclidean C^2 -smooth surface Σ has vanishing intrinsic Gaussian curvature, i.e. if

$$\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H} : f(x_1, x_2) = 0\},$$

for $f \in C^2(\mathbb{R}^2)$, then $\mathcal{K}_0 = 0$. In particular, every vertical plane has constant intrinsic Gaussian curvature $\mathcal{K}_0 = 0$.

Example 8.7 The horizontal plane through the origin, $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H} : x_3 = 0\}$, has

$$\mathcal{K}_0 = -\frac{2}{(x_1^2 + x_2^2)}.$$

Example 8.8 The Korányi sphere, $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H} : (x_1^2 + x_2^2)^2 + 16x_3^2 - 1 = 0\}$, has

$$\mathcal{K}_0 = -\frac{2}{(x_1^2 + x_2^2)} + 6(x_1^2 + x_2^2).$$

Example 8.9 Let $\alpha > 0$. The paraboloid, $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H} : x_3 = \alpha(x_1^2 + x_2^2)\}$, has

$$\mathcal{K}_0 = -\frac{1}{1 + 16\alpha^2} \frac{2}{(x_1^2 + x_2^2)}.$$

Example 8.10 Every surface Σ given as a x_3 -graph, $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H} : x_3 = f(x_1, x_2)\}$, with $f \in C^2(\mathbb{R}^2)$, has

$$\mathcal{K}_0 = -\frac{2}{\|\nabla_{\mathbb{H}u}\|_H^2} + \frac{1}{\|\nabla_{\mathbb{H}u}\|_H^4} (\text{Hess } f) (\nabla_{\mathbb{H}u}, J\nabla_{\mathbb{H}u}),$$

where $u(x_1, x_2, x_3) := x_3 - f(x_1, x_2)$.

For x_3 -graphs, we have another useful result that provides a sufficient condition for the intrinsic Gaussian curvature \mathcal{K}_0 to vanish.

Lemma 8.11 *Let $\Sigma \subset \mathbb{H}$ be as before. Let $g \in \Sigma$ and suppose that Σ is a Euclidean C^2 -smooth x_3 -graph and X_1u and X_2u are linearly dependent in a neighborhood of g . Then $\mathcal{K}_0(g) = 0$.*

Proof In the case of a x_3 -graph, $X_3u = 1$ and we have

$$\mathcal{K}_0 = -\frac{1}{|\nabla_0 u|^2} + \frac{X_2uX_1(\frac{1}{2}|\nabla_0 u|^2) - X_1uX_2(\frac{1}{2}|\nabla_0 u|^2)}{|\nabla_0 u|^4}.$$

Assume that $aX_1u + bX_2u =$ in a neighborhood of g . Let us suppose that $b \neq 0$; the case $a \neq 0$ is similar. Without loss of generality, assume that $b = 1$. We expand

$$\mathcal{K}_0 = \frac{-(X_1u)^2 - (X_2u)^2 + X_1uX_2uX_1X_1u + (X_2u)^2X_1X_2u - (X_1u)^2X_2X_1u - X_1uX_2uX_2X_2u}{|\nabla_0 u|^4} \tag{8.4}$$

and use the identities $X_2u = aX_1u$,

$$\begin{aligned} X_1X_2u &= aX_1X_1u, \\ X_2X_1u &= X_1X_2u - X_3u = aX_1X_1u - 1 \end{aligned}$$

and

$$X_2X_2u = aX_2X_1u = a - a^2X_1X_1u$$

to rewrite the numerator of (8.4) entirely in terms of X_1u and X_1X_1u . A straightforward computation shows that the expression for \mathcal{K}_0 vanishes. The case when $a \neq 0$ is similar. \square

Example 8.12 Every surface Σ given as a x_1 -graph, $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{H} : x_1 = f(x_2, x_3)\}$ with $f \in C^2(\mathbb{R}^2)$, has

$$\begin{aligned} \mathcal{K}_0 &= -\frac{f_3^2}{\|\nabla_{\mathbb{H}u}\|_H^2} + \frac{(x_1^2 - x_2^2)f_{33}}{8\|\nabla_{\mathbb{H}u}\|_H^4} \left(x_1 f_3 + \frac{x_1x_2 f_3^2}{2} \right) - \frac{f_{23}(1 + \frac{x_2}{2} f_3)}{\|\nabla_{\mathbb{H}u}\|_H^2} \\ &\quad + \frac{1}{\|\nabla_{\mathbb{H}u}\|_H^4} \left(\frac{x_1^2 f_3^3}{8} + \frac{(1 + \frac{x_2}{2} f_3)}{2} f_3 \right), \end{aligned}$$

where $u(x_1, x_2, x_3) := x_1 - f(x_2, x_3)$. A similar result holds for x_2 -graphs.

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