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# ORTHOGONAL AND UNITARY TENSOR DECOMPOSITION FROM AN ALGEBRAIC PERSPECTIVE

BY

Ada Boralevi

*Dipartimento di Scienze Matematiche, Politecnico di Torino*
*Corso Duca degli Abruzzi 24, 10129 Torino, Italy*
*e-mail: ada.boralevi@polito.it*

AND

Jan Draisma

*Mathematisches Institut, Universität Bern*
*Sidlerstrasse 5, 3012 Bern, Switzerland*
*and*
*Eindhoven University of Technology, The Netherlands*
*e-mail: jan.draisma@math.unibe.ch*

AND

Emil Horobeţ

*Department of Mathematics and Computer Science*
*Sapientia Hungarian University of Transylvania, 540485 Târgu-Mureş, Romania*
*e-mail: horobetemil@gmail.com*

AND

Elina Robeva

*Department of Mathematics, Massachusetts Institute of Technology*
*Room 2-378, 77 Massachusetts Ave, Cambridge, MA 02139, USA*
*e-mail: erobeva@gmail.com*



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ABSTRACT

While every matrix admits a singular value decomposition, in which the terms are pairwise orthogonal in a strong sense, higher-order tensors typically do not admit such an orthogonal decomposition. Those that do have attracted attention from theoretical computer science and scientific computing. We complement this existing body of literature with an algebro-geometric analysis of the set of orthogonally decomposable tensors.

More specifically, we prove that they form a real-algebraic variety defined by polynomials of degree at most four. The exact degrees, and the corresponding polynomials, are different in each of three times two scenarios: ordinary, symmetric, or alternating tensors; and real-orthogonal versus complex-unitary. A key feature of our approach is a surprising connection between orthogonally decomposable tensors and semisimple algebras—associative in the ordinary and symmetric settings and of compact Lie type in the alternating setting.

Contents

## 1. Introduction and results

By the singular value decomposition, any complex $(m \times n)$-matrix $A$ can be written as

$$A = \sum_{i=1}^{k} u_i v_i^T,$$

where $u_1, \dots, u_k \in \mathbb{C}^m$ and $v_1, \dots, v_k \in \mathbb{C}^n$ are sets of nonzero, pairwise orthogonal vectors with respect to the standard Hermitian forms on these spaces. The singular values $||u_i|| \cdot ||v_i||$, including their multiplicities, are uniquely determined by $A$, and if these singular values are all distinct, then the terms $u_i v_i^T$ are also uniquely determined.

If $m = n$ and $A$ is symmetric, then the $u_i$ and $v_i$ can be chosen equal. And if, on the other hand, $A$ is skew-symmetric, then $k$ is necessarily even, say $k = 2\ell$, and one can choose $v_i = u_{\ell+i}$ for $i = 1, \dots, \ell$ and $v_i = -u_{i-\ell}$ for $i = \ell + 1, \dots, n$, so that the terms can be grouped into pairs of the form $u_i v_i^T - v_i u_i^T$ for $i = 1, \dots, \ell$. Note that the two-dimensional spaces $\langle u_i, v_i \rangle_{\mathbb{C}}$ for $i = 1, \dots, \ell$ are pairwise perpendicular.

In this paper we consider higher-order tensors in a tensor product $V_1 \otimes \cdots \otimes V_d$ of finite-dimensional vector spaces $V_i$ over $K \in \{\mathbb{R}, \mathbb{C}\}$, where the tensor product is also over $K$. We assume that each $V_i$ is equipped with a positive-definite inner product $(\cdot|\cdot)$, Hermitian if $K = \mathbb{C}$.

*Definition 1:* A tensor in $V_1 \otimes \cdots \otimes V_d$ is called **orthogonally decomposable** (**odeco**, if $K = \mathbb{R}$) or **unitarily decomposable** (**udeco**, if $K = \mathbb{C}$) if it can be written as

$$\sum_{i=1}^{k} v_{i1} \otimes \cdots \otimes v_{id},$$

where for each $j$ the vectors $v_{1j}, \dots, v_{kj}$ are nonzero and pairwise orthogonal in $V_j$.

We use the adverb **unitarily** for $K = \mathbb{C}$ to stress that we have fixed Hermitian inner products rather than symmetric bilinear forms. Note that orthogonality implies that the number $k$ of terms is at most the minimum of the dimensions of the $V_i$, so odeco tensors form a rather low-dimensional subset of the space of all tensors; see Proposition 8.

Next we consider tensor powers of a single, finite-dimensional $K$-space $V$. We write $\mathrm{Sym}_d(V)$ for the subspace of $V^{\otimes d}$ consisting of all **symmetric tensors**, i.e., those fixed by all permutations of the tensor factors.

*Definition 2:* A tensor in $\mathrm{Sym}_d(V)$ is called **symmetrically odeco** (if $K = \mathbb{R}$) or **symmetrically udeco** (if $K = \mathbb{C}$) if it can be written as

$$\sum_{i=1}^{k} \pm v_i^{\otimes d}$$

where the vectors $v_1, \ldots, v_k$ are nonzero, pairwise orthogonal vectors in $V$.

The signs are only required when $K = \mathbb{R}$ and $d$ is even, as they can otherwise be absorbed into the $v_i$ by taking a $d$-th root of $-1$. Clearly, a symmetrically odeco or udeco tensor is symmetric and odeco or udeco in the earlier sense. The converse also holds; see Proposition 32.

Our third scenario concerns the space $\mathrm{Alt}_d(V) \subseteq V^{\otimes d}$ consisting of all **alternating tensors**, i.e., those $T$ for which $\pi T = \mathrm{sgn}(\pi)T$ for each permutation $\pi$ of $[d] := \{1, \ldots, d\}$. The simplest alternating tensors are the alternating product tensors

$$v_1 \wedge \cdots \wedge v_d := \sum_{\pi \in S_d} \mathrm{sgn}(\pi) v_{\pi(1)} \otimes \cdots \otimes v_{\pi(d)}.$$

Such a tensor is nonzero if and only if the vectors $v_1, \ldots, v_d$ are linearly independent, and it changes only by a scalar factor upon replacing these vectors by another basis of the space $\langle v_1, \ldots, v_d \rangle$. We say that this subspace is **represented** by the alternating product tensor.

*Definition 3:* A tensor in $\mathrm{Alt}_d(V)$ is called **alternatingly odeco** or **alternatingly udeco** if it can be written as

$$\sum_{i=1}^{k} v_{i1} \wedge \cdots \wedge v_{id},$$

where the $k \cdot d$ vectors $v_{11}, \ldots, v_{kd}$ are nonzero and pairwise orthogonal.

Equivalently, this means that the tensor is a sum of $k$ alternating product tensors that represent pairwise orthogonal $d$-dimensional subspaces of $V$; by choosing orthogonal bases in each of these spaces one obtains a decomposition as above. In particular, $k$ is at most $\lfloor n/d \rfloor$. For $d \geq 3$, alternatingly odeco tensors are not odeco in the ordinary sense unless they are zero; see Remark 34.

By quantifier elimination, it follows that the set of odeco or udeco tensors is a semi-algebraic set in $V_1 \otimes \cdots \otimes V_d$, i.e., a finite union of subsets described by polynomial equations and (weak or strict) polynomial inequalities; here this space is considered as a real vector space even if $K = \mathbb{C}$. A simple compactness argument (see Proposition 6) also shows that they form a closed subset in the Euclidean topology, so that only weak inequalities are needed. However, our main result says that, in fact, only *equations* are needed, and that the same holds in the symmetrically or alternatingly odeco or udeco regimes.

THEOREM 4 (Main Theorem): *For each integer $d \geq 3$, for $K \in \{\mathbb{R}, \mathbb{C}\}$, and for all finite-dimensional inner product spaces $V_1, \ldots, V_d$ and $V$ over $K$, the odeco/udeco tensors in $V_1 \otimes \cdots \otimes V_d$, the symmetrically odeco/udeco tensors in $\mathrm{Sym}_d(V)$, and the alternatingly odeco/udeco tensors in $\mathrm{Alt}_d(V)$, form real algebraic varieties defined by polynomials of degrees given in the following table.*

| Degrees of equations | odeco (over $\mathbb{R}$) | udeco (over $\mathbb{C}$) |
|---|---|---|
| symmetric | 2 (associativity) | 3 (semi-associativity) |
| ordinary | 2 (partial associativity) | 3 (partial semi-associativity) |
| alternating | 2 (Jacobi) and 4 (cross) | 3 (Casimir) and 4 (cross) |

*Remark 5:* Several remarks are in order:

(1) Unlike for $d = 2$, for $d \geq 3$ the decomposition in Definitions 1, 2, and 3 is always unique in the sense that the terms are uniquely determined, regardless of whether some of their norms coincide; see Proposition 7.

(2) A direct consequence of the fact that we work with Hermitian forms is that even when $K = \mathbb{C}$ the varieties above are real algebraic only, except in the following three degenerate cases: $\dim V$ or some $\dim V_i$ equals zero; $\dim V_i = 1$ for some $i$, so that the set of odeco/udeco tensors equals the affine cone over the Segre product of $\mathbb{P}V_1, \ldots, \mathbb{P}V_d$; or $d > \dim V/2$ in the alternating case, so that the set of alternatingly odeco/udeco tensors equals the affine cone over the Grassmannian of $k$-subspaces of $V$. So apart from these cases, we need to allow polynomial equations involving coordinates with respect to a $\mathbb{C}$-basis together with their complex conjugates.

(3) We will describe the polynomials defining these varieties in detail later on, but here is a high-level perspective. In the odeco case, the equations of degree two guarantee that some algebra associated to a tensor is associative (in the ordinary and symmetric cases) or Lie (in the alternating case), and the equations of degree four come from a certain polynomial identity satisfied by the cross product on $\mathbb{R}^3$. These degree-four equations are not always required: e.g., for $\mathrm{Alt}_3(V)$ they can be discarded (leaving only our degree-two equations) if and only if the real vector space $V$ has dimension $\leq 7$ (see Remark 19).

(4) The udeco case is more involved: the equations of degree three express that some algebra with a bi-semilinear product is (partially) semi-associative in a sense to be defined below, or, in the case of alternatingly udeco tensors, that a variant of the Casimir operator commutes with the multiplication.

(5) The listed degrees are minimal in the sense that there are no linear equations in the odeco case and no quadratic equations in the udeco case—again, except in degenerate cases. Moreover, the equations of degree four for the alternating case cannot be simply discarded. But we do not know whether, instead of the degree-four equations, lower-degree equations might also suffice.

(6) More generally, we do not know whether the equations that we give generate the prime ideal of all polynomial equations vanishing on our real algebraic varieties.

The remainder of this paper is organised as follows. In Section 2 we discuss some background and earlier literature; in particular we show that our degree-two equations follow from those obtained by the fourth-named author in [Rob14], so that our Main Theorem implies [Rob14, Conjecture 3.2] at a set-theoretic level and over the real numbers.

In Section 3 we prove the Main Theorem for tensors of order three. We first treat odeco tensors, and then the more involved case of udeco tensors. The proofs for symmetrically odeco and udeco three-tensors are the simplest, and those for ordinary odeco and udeco three-tensors build upon them. The alternatingly odeco and udeco three-tensors require a completely separate treatment. Then, in Section 4 we derive the theorem for higher-order tensors for ordinary, symmetric, and alternating tensors consecutively. We conclude in Section 5 with some open questions.

## 2. Background

In this section we collect background results on orthogonally decomposable tensors, and connect our paper to earlier work on them.

PROPOSITION 6: *The set of (ordinary, symmetrically, or alternatingly) odeco or udeco tensors is closed in the Euclidean topology.*

*Proof.* We give the argument for symmetrically udeco tensors; the same works in the other cases. Thus consider the space $V = \mathbb{C}^n$ with the standard inner product, let $\mathrm{U}_n$ be the unitary group of that inner product, and consider the map

$$\varphi : \mathrm{U}_n \times \mathbb{P}V \to \mathbb{P}\,\mathrm{Sym}_d(V), \quad ((u_1|\ldots|u_n), [\lambda_1 : \ldots : \lambda_n]) \mapsto \left[\sum_{i=1}^{n} \lambda_i u_i^{\otimes d}\right].$$

Here $\mathbb{P}$ stands for projective space and $u_i$ is the $i$-th column of the unitary matrix $u$. The key point is that this map is well-defined and continuous, since the expression between the last square brackets is never zero by linear independence of the $u_i^{\otimes d}$. Now $\varphi$ is a continuous map whose source is a compact topological space, hence $\mathrm{im}\,\varphi$ is a closed subset of $\mathbb{P}\,\mathrm{Sym}_d(V)$. But then the pre-image of $\mathrm{im}\,\varphi$ in $\mathrm{Sym}_d(V) \setminus \{0\}$ is also closed, and so is the union of this pre-image with $\{0\}$. This is the set of symmetrically udeco tensors in $\mathrm{Sym}_d(V)$. ∎

PROPOSITION 7: *For $d \geq 3$, any (ordinary, symmetrically, or alternatingly) odeco or udeco tensor has a* unique *orthogonal decomposition.*

In the ordinary case this was proved in [ZG01, Theorem 3.2].

*Proof.* We give the argument for ordinary odeco tensors. Consider an orthogonal decomposition

$$T = \sum_{i=1}^{k} v_{i1} \otimes \cdots \otimes v_{id}$$

of an odeco tensor $T \in V_1 \otimes \cdots \otimes V_d$. Contracting $T$ with an arbitrary tensor $S \in V_3 \otimes \cdots \otimes V_d$ via the inner products on $V_3, \ldots, V_d$ leads to a tensor

$$T' = \sum_{i=1}^{k} \lambda_i v_{i1} \otimes v_{i2}$$

where $\lambda_i$ is the inner product of $S$ with $v_{i3} \otimes \cdots \otimes v_{id}$. Now the above is a singular value decomposition for the two-tensor $T'$, of which, for $S$ sufficiently general, the singular values $|\lambda_i| \cdot ||v_{i1}|| \cdot ||v_{i2}||$ are all distinct. Thus $v_{11}, \ldots, v_{k1}$ are, up to nonzero scalars, uniquely determined as the singular vectors (corresponding to the nonzero singular values) of the pairing of $T$ with a sufficiently general $S$. And these vectors determine the corresponding terms, since the $i$-th term equals $v_{i1}$ tensor the pairing of $T$ with $v_{i1}$, divided by $||v_{i1}||^2$.

The arguments in the symmetric or alternating case, as well as in the udeco case, are almost identical. We stress that, as permuting the first two factors commutes with contracting the last $d-2$ factors, the contraction of a symmetric or alternating tensor is a symmetric or alternating matrix. Also, in the alternating case, rather than contracting with a general $S$, we contract with a general alternating product tensor $S = u_1 \wedge \cdots \wedge u_{d-2}$. This has the effect of intersecting the space spanned by $v_{i1}, \ldots, v_{id}$ with the orthogonal complement of the space spanned by $u_1, \ldots, u_{d-2}$. ∎

The proof of this proposition yields a simple probabilistic algorithm for deciding whether a tensor is odeco or udeco, and for finding a decomposition when it exists: contract with a random $S$, compute the (unique) singular value decomposition of $T'$, and let $u_1, \ldots, u_k$ be the norm-one singular vectors of $T'$ in $V_1$. Then for $T$ to be odeco, the contraction $T_i$ of $T$ with each $u_i$ must be a rank-one tensor, and $\sum_i u_i \otimes T_i$ is the only candidate orthogonal decomposition for $T$. It then still needs to be checked that for each $j$ the factors of the $T_i$ in $V_j$ are pairwise perpendicular. For much more on algorithmic issues see [BLW15, Kol15, SRK09, ZG01].

The uniqueness of the orthogonal decomposition makes it easy to compute the dimensions of the real-algebraic varieties in our Main Theorem.

Proposition 8: *Let $n := \dim_K V$, $l := \lfloor \frac{n}{d} \rfloor$, and assume that the dimensions $n_i := \dim_K V_i$ are in increasing order $n_1 \leq \cdots \leq n_d$. Then, the dimensions of the real-algebraic varieties of odeco/udeco tensors are given in the following table.*

| *Dimension over* $\mathbb{R}$ | *odeco* | *udeco* |
|---|---|---|
| *symmetric* | $n + \binom{n}{2}$ | $2n + n(n-1)$ |
| *ordinary* | $n_1 + \sum_{j=1}^d \frac{n_1(2n_j - n_1 - 1)}{2}$ | $2n_1 + \sum_{j=1}^d n_1(2n_j - n_1 - 1)$ |
| *alternating* | $l + \frac{ld(2n-(l+1)d)}{2}$ | $2l + ld(2n - (l+1)d)$ |

*Proof.* In the symmetric case, a symmetrically odeco/udeco tensor encodes $n$ pairwise perpendicular points in $\mathbb{P}V$. For the first point we have $n-1$ degrees of freedom over $K$. The second point is chosen from the projective space orthogonal to the first point, so this yields $n-2$ degrees of freedom, etc. Summing up, we obtain $\binom{n}{2}$ degrees of freedom over $K$ for the points. In addition, we have $n$ scalars from $K$ for the individual terms. If $K = \mathbb{C}$ we multiply by two to obtain the real dimension. Since each odeco/udeco tensor has a unique decomposition, the dimension of the odeco/udeco variety is the same as the dimension of the space of $n$ pairwise orthogonal points and $n$ scalars.

The computation for the ordinary case is the same, except that only $n_1$ pairwise perpendicular projective points are chosen from each $V_j$.

In the alternating case, an alternatingly odeco/udeco tensor encodes $l$ pairwise perpendicular $d$-dimensional $K$-subspaces of $V$. The first space is an arbitrary point on the $d(n-d)$-dimensional Grassmannian of $d$-subspaces, the second an arbitrary point on the $d(n–2d)$-dimensional Grassmannian of $d$-subspaces in the orthogonal complement of the first, etc. Add $l$ degrees of freedom for the scalars, and if $K{=}\mathbb{C}$, multiply by 2. By uniqueness of the decomposition, the dimension of the odeco/udeco variety is preserved and as given in the table above. ∎

Over the last two decades, orthogonal tensor decomposition has been studied intensively from a scientific computing perspective (see, e.g., [Com94, Kol01, Kol03, CS09, Kol15]), though the alternating case has not received much attention so far. The paper [CS09] gives a characterisation of orthogonally decomposable tensors in terms of their higher-order SVD [DDV00], which is different from the real-algebraic characterisation in our Main Theorem. One of the interesting properties of an orthogonal tensor decomposition with $k$ terms is that discarding the $k-r$ terms with smallest norm yields the best rank-$r$ approximation to the tensor; see [VNVM14], where it is also proved that in general, tensors are not optimally truncatable in this manner.

In general, tensor decomposition is NP-hard [HL13]. The decomposition of odeco tensors, however, can be found efficiently. The vectors in the decomposition of an odeco tensor are exactly the attraction points of the tensor power method and are called **robust** eigenvectors. Because of their efficient decomposition, odeco tensors have been used in machine learning, in particular for learning latent variables in statistical models [AGH$^+$14]. More recent work in this direction concerns overcomplete latent variable models [AGJ14].

In [Rob14], the fourth-named author describes all eigenvectors of symmetrically odeco tensors in terms of the robust ones, and conjectures the equations defining the variety of symmetrically odeco tensors. Formulated for the case of ordinary tensors instead, this conjecture is as follows. Let $V_1, \ldots, V_d$ be real inner product spaces, and consider an odeco tensor $T \in V_1 \otimes \cdots \otimes V_d$ with orthogonal decomposition $T = \sum_{i=1}^k v_{i1} \otimes \cdots \otimes v_{id}$. Now take two copies of $T$, and contract them in their $l$-th components via the inner product $V_l \times V_l \to \mathbb{R}$. By orthogonality of the $v_{il},\ i = 1, \ldots, k$, after regrouping the tensor factors, the resulting tensor is

$$\sum_{i=1}^{k} \left( ||v_{il}||^2 \bigotimes_{j \neq l} (v_{ij} \otimes v_{ij}) \right) \in \bigotimes_{j \neq l} (V_j \otimes V_j);$$

we write $T *_l T$ for this tensor. It is clear from this expression that $T *_l T$ is multi-symmetric in the sense that it lies in the subspace $\bigotimes_{j \neq l} \mathrm{Sym}_2(V_j)$. The fourth-named author conjectured that this (or rather, its analogue in the symmetric setting) characterises (symmetrically) odeco tensors. This is now a theorem, which follows from the proof of our main theorem (see Remark 15).

Theorem 9: *$T \in V_1 \otimes \cdots \otimes V_d$ is odeco if and only if for all $l = 1, \ldots, d$ we have*

$$T *_l T \in \bigotimes_{j \neq l} \mathrm{Sym}_2(V_j).$$

This concludes the discussion of background to our results. We now proceed to prove the main theorem in the case of order-three tensors.

## 3. Tensors of order three

In all our proofs below, we will encounter a finite-dimensional vector space $A$ over $K = \mathbb{R}$ or $\mathbb{C}$ equipped with a positive-definite inner product $(\cdot|\cdot)$, as well as a bi-additive product $A \times A \to A$, $(x, y) \mapsto x \cdot y$ which is bilinear if $K = \mathbb{R}$ and bi-semilinear if $K = \mathbb{C}$. The product will be either commutative or anti-commutative. Moreover, the inner product will be compatible with the product in the sense that $(x \cdot y|z) = (z \cdot x|y)$. An **ideal** in $(A, \cdot)$ is a $K$-subspace $I$ such that $I \cdot A \subseteq I$—by (anti-)commutativity we then also have $A \cdot I \subseteq I$—and $A$ is called **simple** if $A \neq \{0\}$ and $A$ contains no nonzero proper ideals. We have the following well-known result.

Lemma 10: *The orthogonal complement $I^\perp$ of any ideal $I$ in $A$ is an ideal, as well. Consequently, $A$ splits as a direct sum of pairwise orthogonal simple ideals.*

*Proof.* We have $(A \cdot I^\perp | I) = (I \cdot A | I^\perp) = \{0\}$. The second statement follows by induction on $\dim A$. ∎

3.1. Symmetrically odeco three-tensors. In this subsection, we fix a finite-dimensional real inner product space $V$ and characterise symmetrically odeco tensors in $\mathrm{Sym}_3(V)$. We have $\mathrm{Sym}_3(V) \subseteq V^{\otimes 3} \cong (V^*)^{\otimes 2} \otimes V$, where the isomorphism comes from the linear isomorphism $V \to V^*$, $v \mapsto (v|\cdot)$. Thus a tensor $T \in \mathrm{Sym}_3(V)$ gives rise to a bilinear map $V \times V \to V$, $(u,v) \mapsto u \cdot v$, which has the following properties:

(1) $u \cdot v = v \cdot u$ for all $u, v \in V$ (commutativity, which follows from the fact that $T$ is invariant under permuting the first two factors); and
(2) $(u \cdot v | w) = (u \cdot w | v)$ (compatibility with the inner product, which follows from the fact that $T$ is invariant under permuting the last two factors).

Thus $T$ gives $V$ the structure of an $\mathbb{R}$-algebra equipped with a compatible inner product. The following lemma describes the quadratic equations from the Main Theorem.

Lemma 11: *If $T$ is symmetrically odeco, then $(V, \cdot)$ is associative.*

*Proof.* Write $T = \sum_{i=1}^k v_i^{\otimes 3}$ where $v_1, \ldots, v_k$ are pairwise orthogonal nonzero vectors. Then we find, for $x, y, z \in V$, that

$$x \cdot (y \cdot z) = x \cdot \left( \sum_i (v_i|y)(v_i|z) v_i \right) = \sum_i (v_i|x)(v_i|y)(v_i|z)(v_i|v_i) v_i = (x \cdot y) \cdot z,$$

where we have used that $(v_i|v_j) = 0$ for $i \neq j$ in the second equality. ∎

Proposition 12: *Conversely, if $(V, \cdot)$ is associative, then $T$ is symmetrically odeco.*

*Proof.* By Lemma 10, $V$ has an orthogonal decomposition $V = \bigoplus_i U_i$ where the subspaces $U_i$ are (nonzero) simple ideals. Correspondingly, $T$ decomposes as an element of $\bigoplus_i \mathrm{Sym}_3(U_i)$. Thus it suffices to prove that each $U_i$ is one-dimensional. This is certainly the case when the multiplication $U_i \times U_i \to U_i$ is zero, because then any one-dimensional subspace of $U_i$ is an ideal in $V$, hence equal to $U_i$ by simplicity. If the multiplication map is nonzero, then pick an element $x \in U_i$ such that the multiplication $M_x : U_i \to U_i,\ y \mapsto x \cdot y$ is nonzero. Then $\ker M_x$ is an ideal in $V$, because for $z \in V$ we have

$$x \cdot (\ker M_x \cdot z) = (x \cdot \ker M_x) \cdot z = \{0\},$$

where we use associativity. By simplicity of $U_i$, $\ker M_x = \{0\}$. Now define a new bilinear multiplication $*$ on $U_i$ via $y * z := M_x^{-1}(y \cdot z)$. This multiplication is commutative, has $x$ as a unit element, and we claim that it is also associative. Indeed,

$$\begin{aligned}((x \cdot y) * z) * (x \cdot v) =& M_x^{-1}(M_x^{-1}((x \cdot y) \cdot z) \cdot (x \cdot v)) \\ =& y \cdot z \cdot v = (x \cdot y) * (z * (x \cdot v)),\end{aligned}$$

where we used associativity and commutativity of $\cdot$ in the second equality. Since any element is a multiple of $x$, this proves associativity. Moreover, $(U_i, *)$ is simple; indeed, if $I$ is an ideal, then $M_x^{-1}(U_i \cdot I) \subseteq I$ and hence

$$U_i \cdot (x \cdot I) = (U_i \cdot x) \cdot I = U_i \cdot I \subseteq x \cdot I,$$

so that $x \cdot I$ is an ideal in $(U_i, \cdot)$; and therefore $I = \{0\}$ or $I = U_i$.

Now $(U_i, *)$ is a simple, associative $\mathbb{R}$-algebra with 1, hence isomorphic to a matrix algebra over a division ring. As it is also commutative, it is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. If it were isomorphic to $\mathbb{C}$, then it would contain a square root of $-1$, i.e., an element $y$ with $y * y = -x$, so that $y \cdot y = -x \cdot x$. But then

$$0 < (x \cdot y | x \cdot y) = (y \cdot y | x \cdot x) = -(x \cdot x | x \cdot x) < 0,$$

a contradiction. We conclude that $U_i$ is one-dimensional, as desired. ∎

Lemma 11 and Proposition 12 imply the Main Theorem for symmetrically odeco three-tensors, because the identity $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ expressing associativity translates into quadratic equations for the tensor $T$.

3.2. Ordinary odeco three-tensors. In this subsection, we consider a tensor $T$ in a tensor product $U \otimes V \otimes W$ of real, finite-dimensional inner product spaces. Via the inner products, $T$ gives rise to a bilinear map $U \times V \to W$, and similarly with the three spaces permuted. Consider the external direct sum $A := U \oplus V \oplus W$ of $U, V, W$, and equip $A$ with the inner product $(\cdot|\cdot)$ that restricts to the given inner products on $U, V, W$ and that makes these spaces pairwise perpendicular.

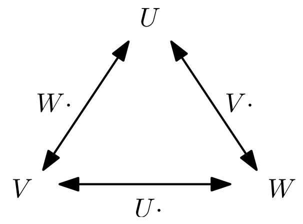


Figure 1. $U \cdot (V + W) = W + V$, and similarly with $U, V, W$ permuted.

Taking cue from the symmetric case, we construct a bilinear product

$$\cdot : A \times A \to A$$

as follows: the product in $A$ of two elements in $U$, or two elements in $V$, or in $W$, is defined as zero; $\cdot$ restricted to $U \times V$ is the map into $W$ given by $T$; etc.—see Figure 1. The tensor in $\mathrm{Sym}_3(A)$ describing the multiplication is the symmetric embedding of $T$ from [RV13].

As in the symmetrically odeco case, the algebra has two fundamental properties:

(1) it is commutative: $x \cdot y = y \cdot x$ by definition; and
(2) the inner product is compatible: $(x \cdot y|z) = (x \cdot z|y)$. For instance, if $x \in U, y \in V, z \in W$, then both sides equal the inner product of the tensor $x \otimes y \otimes z$ with $T$; and if $y, z \in W$, then both sides are zero both for $x \in U$ (so that $x \cdot y, x \cdot z \in V$, which is perpendicular to $W$), for $x \in W$ (so that $x \cdot y = x \cdot z = 0$), and for $x \in V$ (so that $x \cdot y, x \cdot z \in U \perp W$).

We are now interested only in **homogeneous** ideals $I \subseteq A$, i.e., ideals such that

$$I = (I \cap U) \oplus (I \cap V) \oplus (I \cap W).$$

We call $A$ **simple** if it is nonzero and does not contain proper, nonzero homogeneous ideals. We call an element of $A$ homogeneous if it belongs to one of $U, V, W$. Next, we derive a polynomial identity for odeco tensors.

Lemma 13: *If $T$ is odeco, then for all homogeneous $x, y, z$ where $x$ and $z$ belong to the same space ($U, V$, or $W$), we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.*

We refer to this property as **partial associativity**.

*Proof.* If $x, y, z$ all belong to the same space, then both products are zero. Otherwise, by symmetry, it suffices to check the case where $x, z \in U$ and $y \in V$. Let $T = \sum_i u_i \otimes v_i \otimes w_i$ be an orthogonal decomposition of $T$. Then we have

$$(x \cdot y) \cdot z = \left( \sum_i (u_i|x)(v_i|y) w_i \right) \cdot z = \sum_i (u_i|x)(v_i|y)(w_i|w_i)(z|u_i) v_i = x \cdot (y \cdot z),$$

where we have used that $(w_i|w_j) = 0$ for $i \neq j$ in the second equality. ∎

Proposition 14: *Conversely, if $(A, \cdot)$ is partially associative, then $T$ is odeco.*

*Proof.* By a version of Lemma 10 restricted to homogeneous ideals, $A$ is the direct sum of pairwise orthogonal, simple homogeneous ideals $I_i$. Accordingly, $T$ lies in $\bigoplus_i (I_i \cap U) \otimes (I_i \cap V) \otimes (I_i \cap W)$. Thus it suffices to prove that $T$ is odeco under the additional assumption that $A$ itself is simple and that $\cdot$ is not identically zero.

By symmetry, we may assume that $V \cdot (U + W) \neq \{0\}$. For $u \in U$, let $M_u : V + W \to W + V$ be multiplication with $u$. By commutativity and partial associativity, the $M_u$, for $u \in U$, all commute. By compatibility of $(\cdot|\cdot)$, each $M_u$ is symmetric with respect to the inner product on $V + W$, and hence orthogonally diagonalisable.

Consequently, $V + W$ splits as a direct sum of pairwise orthogonal simultaneous eigenspaces

$$(V + W)_\lambda := \{v + w \in V + W \mid u \cdot (v + w) = \lambda(u)(w + v) \text{ for all } u \in U\},$$

where $\lambda$ runs over $U^*$. Suppose we are given $v+w \in (V+W)_\lambda$ and $v'+w' \in (V+W)_\mu$ with $\lambda \neq \mu$. Then $v + w$ and $v' + w'$ are perpendicular and for each $u \in V$ we have

$$(u|(v + w) \cdot (v' + w')) = (u \cdot (v + w)|v' + w') = \lambda(u)(v + w|v' + w') = 0,$$

hence $(v + w) \cdot (v' + w') = 0$. We conclude that for each $\lambda$ the space

$$(V + W)_\lambda \oplus [(V + W)_\lambda \cdot (V + W)_\lambda]$$

is a homogeneous ideal in $A$. By simplicity and the fact that $M_u \neq 0$ for at least some $u$, $A$ is equal to this ideal for some nonzero $\lambda \in U^*$. Pick an $x \in U$ such that $\lambda(x) = 1$, so that $x \cdot (v + w) = w + v$ for all $v \in V$, $w \in W$. In particular, for $v, v' \in V$ we have $(M_x v|M_x v') = (M_x^2 v|v') = (v|v')$, so that the restrictions $M_x : V \to W$ and $M_x : W \to V$ are mutually inverse isometries.

By the same construction, we find an element $z \in W$ such that $z\cdot(u+v) = v+u$ for all $u \in U, v \in V$. Let $T'$ be the image of $T$ under the linear map

$$M_z \otimes 1_V \otimes M_x : U \otimes V \otimes W \to V \otimes V \otimes V,$$

via the isomorphism $V\otimes V\otimes V \simeq V^*\otimes V\otimes V^*$. We claim that $T'$ is symmetrically odeco. Indeed, let $* : V \times V \to V$ denote the bilinear map associated to $T'$. We verify the conditions from Section 3.1. First,

$$v * v' = (z \cdot v) \cdot (x \cdot v') = x \cdot ((z \cdot v) \cdot v') = x \cdot ((v' \cdot z) \cdot v) = (x \cdot v) \cdot (v' \cdot z) = v' * v,$$

where we have repeatedly used commutativity and partial associativity (e.g., in the second equality, to the elements $z \cdot v, x$ belonging to the same space $W$). Second, we have

$$(v*v'|v'') = ((z\cdot v)\cdot(x\cdot v')|v'') = ((z\cdot v)|v'\cdot(x\cdot v'')) = (v|(z\cdot v')\cdot(x\cdot v'')) = (v|v'*v'').$$

Hence $T'$ is, indeed, an element of $\mathrm{Sym}_3(V)$. Finally, we have

$$\begin{aligned}(v * v') * v'' =&(z \cdot ((z \cdot v) \cdot (x \cdot v'))) \cdot (x \cdot v'')\\ =&z \cdot ((x \cdot v'') \cdot ((z \cdot v) \cdot (x \cdot v')))\\ =&z \cdot (((x \cdot v'') \cdot (z \cdot v)) \cdot (x \cdot v'))\\ =&z \cdot ((v * v'') \cdot (x \cdot v'))\\ =&(v * v'') * v',\end{aligned}$$

which, together with commutativity, implies associativity of $*$. Hence $T'$ is (symmetrically) odeco by Proposition 14, and hence so is its image $T$ under the tensor product $M_z \otimes 1_V \otimes M_x$ of linear isometries. ∎

*Remark 15:* The condition that $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ for, say, $x, z \in W$ and $y \in V$ translates into the condition that the contraction $T *_1 T \in (V \otimes V) \otimes (W \otimes W)$ lies in $\mathrm{Sym}_2(V) \otimes \mathrm{Sym}_2(W)$. Thus Proposition 14 implies Theorem 9 in the case of three factors. The case of more factors follows from the case of three factors and flattenings as in Proposition 31.

3.3. Alternatingly odeco three-tensors. In this subsection we consider a tensor $T \in \mathrm{Alt}_3(V)$, where $V$ is a finite-dimensional real vector space with inner product $(\cdot|\cdot)$. Via

$$\mathrm{Alt}_3(V) \subseteq V^{\otimes 3} \cong (V^*)^{\otimes 2} \otimes V$$

such a tensor gives rise to a bilinear map $V \times V \to V$, $(u,v) \mapsto [u,v]$, which gives $V$ the structure of an algebra. Now,

(1) as the permutation $(1,2)$ maps $T$ to $-T$, we have $[u,v] = -[v,u]$; and
(2) as $(2,3)$ does the same, we have $([u,v]|w) = -([u,w]|v) = ([w,u]|v)$, so that the inner product is compatible with the product.

The following lemma gives the degree-two equations from the Main Theorem.

Lemma 16: *If $T$ is alternatingly odeco, then $[\cdot,\cdot]$ satisfies the Jacobi identity.*

*Proof.* Let $T = \sum_{i=1}^k u_i \wedge v_i \wedge w_i$ be an orthogonal decomposition of $T$, and set $V_i := \langle u_i, v_i, w_i \rangle$. Then $V$ splits as the direct sum of $k$ ideals $V_i$ and one further ideal

$$V_0 := \left( \bigoplus_{i=1}^k V_i \right)^{\perp}.$$

The restriction of the bracket to $V_0$ is zero, so it suffices to verify the Jacobi identity on each $V_i$. By scaling the bracket, which preserves both the Jacobi identity and the set of alternatingly odeco tensors, we achieve that $u_i, v_i, w_i$ can be taken of norm one. Then we have

$$[u_i, v_i] = w_i, \quad [v_i, w_i] = u_i, \quad \text{and} \quad [w_i, u_i] = v_i,$$

which we recognise as the multiplication table of $\mathbb{R}^3$ with the cross product $\times$, isomorphic to the Lie algebra $\mathfrak{so}_3(\mathbb{R})$. ∎

The following lemma gives the degree-four equations from the Main Theorem.

Lemma 17: *If $T$ is alternatingly odeco, then*

$$[x, [[x,y],[x,z]]] = 0$$

*for all $x, y, z \in V$.*

We refer to this identity as the **first cross product identity**.

*Proof.* By the proof of Lemma 16, if $T$ is odeco, then $V$ splits as an orthogonal direct sum of ideals $V_1, \ldots, V_k$ that are isomorphic, as Lie algebras with compatible inner products, to scaled copies of $\mathbb{R}^3$ with the cross product, and possibly an additional ideal $V_0$ on which the multiplication is trivial. Thus it suffices to prove that the lemma holds for $\mathbb{R}^3$ with the cross product. But there it is immediate: if $[[x,y],[x,z]]$ is nonzero, then the two arguments span the plane orthogonal to $x$, hence their cross product is a scalar multiple of $x$. ∎

We now prove the Main Theorem for alternatingly odeco three-tensors.

PROPOSITION 18: *Conversely, if the bracket* $[\cdot,\cdot]$ *on* $V$ *satisfies the Jacobi identity and the first cross product identity, then* $T$ *is alternatingly odeco.*

*Proof.* By Lemma 10 the space $V$ splits into pairwise orthogonal, simple ideals $V_i$. Correspondingly, $T$ lies in $\bigoplus_i \mathrm{Alt}_3 V_i$, where the sum is over those $V_i$ where the bracket is nonzero. These are simple real Lie algebras equipped with a compatible inner product, hence compact Lie algebras. Let $\mathfrak{g}$ be one of these, so $\mathfrak{g}$ satisfies the first cross product identity. Then so does the complex Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathbb{C} \otimes \mathfrak{g}$, which is semisimple. For $\mathfrak{g} \cong \mathfrak{so}_3(\mathbb{R})$, we have $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$, whose Dynkin diagram consists of a single node. The classification of simple compact Lie algebras (see, e.g., [Kna02]) implies that, if $\mathfrak{g}$ is not isomorphic to $\mathfrak{so}_3(\mathbb{R})$, then the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ contains at least a single, double, or triple edge. In the first and last case, $\mathfrak{g}_{\mathbb{C}}$ contains a copy of $\mathfrak{sl}_3(\mathbb{C})$; in the second case, $\mathfrak{g}_{\mathbb{C}}$ contains a copy of $\mathfrak{so}_5(\mathbb{C})$. But neither of these two Lie algebras satisfy the first cross product identity. For instance, in $\mathfrak{sl}_3(\mathbb{C})$ we have

$$[E_{11} - E_{33}, [[E_{11} - E_{33}, E_{12}], [E_{11} - E_{33}, E_{23}]]] = 2E_{13} \neq 0,$$

where $E_{ij}$ is the matrix with zeroes everywhere except for a 1 on position $(i, j)$; and a similar counterexample can be written down for $\mathfrak{so}_5(\mathbb{C})$. Hence $\mathfrak{g} \cong \mathfrak{so}_3(\mathbb{R})$ is three-dimensional, and $T$ is alternatingly odeco. ∎

*Remark 19:* The classification of simple compact Lie algebras shows that, after $\mathfrak{so}_3(\mathbb{R})$, the next smallest one is $\mathfrak{su}_3(\mathbb{C})$, of dimension 8—recall that $\mathfrak{so}_4(\mathbb{R})$, of dimension 6, is a direct sum of two copies of $\mathfrak{so}_3$, arising from left and right multiplication of quaternions by norm-one quaternions. Thus our degree-four equations from the cross product identity are not necessary for $\dim V < 8$.

3.4. SYMMETRICALLY UDECO THREE-TENSORS. In the complex udeco setting, all proofs towards the Main Theorem are more complicated than in the real case. The reason for this is that, as we will see below, the bi-additive map $(u, v) \mapsto u \cdot v$ associated to a tensor is no longer bi-linear. Rather, it is bi-semilinear, i.e., it satisfies

$$(cu) \cdot (dv) = \overline{cd}(u \cdot v)$$

for complex coefficients $c, d$. To appreciate why this causes trouble, consider the usual associativity identity:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

If the product is bi-semilinear, then the left-hand side depends linearly on both $x$ and $y$ and semilinearly on $z$, while the right-hand side depends linearly on $y$ and $z$ and semilinearly on $x$. Hence, except in trivial cases, one should not expect associativity to hold for bi-semilinear products. This explains the need for more complex polynomial identities (pun intended).

In this subsection, $V$ is a complex, finite-dimensional vector space equipped with a positive-definite Hermitian inner product $(\cdot|\cdot)$, and $T$ is an element of $\mathrm{Sym}_3(V)$. There is a canonical linear isomorphism $V \to V^s,\ v \mapsto (v|\cdot)$, where $V^s$ is the space of semilinear functions $V \to \mathbb{C}$. Through

$$\mathrm{Sym}_3(V) \subseteq V^{\otimes 3} \cong (V^s)^{\otimes 2} \otimes V,$$

the tensor $T$ gives rise to a bi-semilinear product $V \times V \to V,\ (u, v) \mapsto u \cdot v$. Moreover:

(1) since $T$ is invariant under permuting the first two factors, $\cdot$ is commutative; and
(2) since $T$ is invariant under permuting the last two factors, we find that $(u \cdot v|w) = (u \cdot w|v)$. Note that, in this identity, both sides are semilinear in all three vectors $u, v, w$.

The following lemma gives the degree-three equations of the Main Theorem.

Lemma 20: *If $T$ is symmetrically udeco, then for all $x, y, z, u \in V$ we have*

$$x \cdot (y \cdot (z \cdot u)) = z \cdot (y \cdot (x \cdot u))$$

*and*

$$(x \cdot y) \cdot (z \cdot u) = (x \cdot u) \cdot (z \cdot y).$$

We call a commutative operation $\cdot$ satisfying the identities in the lemma **semi-associative**. It is clear that any commutative and associative operation is also semi-associative, but the converse does not hold. Note that, since the product is bi-semilinear, both sides of the first identity depend semilinearly on $x, z, u$ but linearly on $y$, while both parts of the second identity depend linearly on all of $x, y, z, u$.

*Proof.* Let $T = \sum_i v_i^{\otimes 3}$ be an orthogonal decomposition of $T$. Then we have

$$z \cdot u = \sum_i (v_i|z)(v_i|u)v_i$$

and

$$y \cdot (z \cdot u) = \sum_i (v_i|y)(z|v_i)(u|v_i)(v_i|v_i)v_i$$

by the orthogonality of the $v_i$. We stress that the coefficient $(v_i|z)(v_i|u)$ has been transformed into its complex conjugate $(z|v_i)(u|v_i)$. Next, we find

$$x \cdot (y \cdot (z \cdot u)) = \sum_i (v_i|x)(y|v_i)(v_i|z)(v_i|u)(v_i|v_i)(v_i|v_i)v_i$$

and this expression is invariant under permuting $x, z, u$ in any manner. This proves the first identity.

For the second identity, we compute

$$(x \cdot y) \cdot (z \cdot u) = \sum_i (v_i|v_i)^2 (x|v_i)(y|v_i)(z|v_i)(u|v_i)v_i,$$

which is clearly invariant under permuting $x, y, z, u$ in any manner. ∎

Proposition 21: *Conversely, if $\cdot$ is semi-associative, then $T$ is symmetrically udeco.*

In fact, in the proof we only use the first identity. The second identity is used later on, for the case of ordinary udeco three-tensors.

*Example 22:* To see how the identities for semi-associativity, in Lemma 20, transform into equations for symmetrically udeco tensors we consider $(2\times 2\times 2)$-tensors. Let $\{e_1, e_2\}$ be an orthonormal basis of $\mathbb{C}^2$ and we represent an element of $\mathrm{Sym}_3(\mathbb{C}^2)$ by

$$\begin{aligned} T =& t_{3,0} e_1 \otimes e_1 \otimes e_1 \\ &+ t_{2,1}(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1) \\ &+ t_{1,2}(e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1) \\ &+ t_{0,3} e_2 \otimes e_2 \otimes e_2. \end{aligned}$$

Then the identities for semi-associativity are translated into two complex equations. If we separate the real and imaginary parts of these two complex equations then we get that the real algebraic variety of symmetrically udeco

$(2 \times 2 \times 2)$-tensors is given by the following four real equations (note that they are invariant under conjugation):

$$\begin{aligned}
f_1 = &- t_{1,2}^2\bar{t}_{1,2} + t_{0,3}t_{2,1}\bar{t}_{1,2} - t_{1,2}\bar{t}_{1,2}^2 - t_{1,2}t_{2,1}\bar{t}_{2,1} + t_{0,3}t_{3,0}\bar{t}_{2,1} + t_{1,2}\bar{t}_{0,3}\bar{t}_{2,1}\\
&- t_{2,1}\bar{t}_{1,2}\bar{t}_{2,1} - t_{3,0}\bar{t}_{2,1}^2 - t_{2,1}^2\bar{t}_{3,0} + t_{1,2}t_{3,0}\bar{t}_{3,0} + t_{2,1}\bar{t}_{0,3}\bar{t}_{3,0} + t_{3,0}\bar{t}_{1,2}\bar{t}_{3,0};\\
f_2 = &- t_{1,2}^2\bar{t}_{1,2} + t_{0,3}t_{2,1}\bar{t}_{1,2} + t_{1,2}\bar{t}_{1,2}^2 - t_{1,2}t_{2,1}\bar{t}_{2,1} + t_{0,3}t_{3,0}\bar{t}_{2,1} - t_{1,2}\bar{t}_{0,3}\bar{t}_{2,1}\\
&+ t_{2,1}\bar{t}_{1,2}\bar{t}_{2,1} + t_{3,0}\bar{t}_{2,1}^2 - t_{2,1}^2\bar{t}_{3,0} + t_{1,2}t_{3,0}\bar{t}_{3,0} - t_{2,1}\bar{t}_{0,3}\bar{t}_{3,0} - t_{3,0}\bar{t}_{1,2}\bar{t}_{3,0};\\
f_3 = &- t_{1,2}^2\bar{t}_{0,3} + t_{0,3}t_{2,1}\bar{t}_{0,3} - t_{1,2}t_{2,1}\bar{t}_{1,2} + t_{0,3}t_{3,0}\bar{t}_{1,2} - t_{0,3}\bar{t}_{1,2}^2 - t_{2,1}^2\bar{t}_{2,1}\\
&+ t_{1,2}t_{3,0}\bar{t}_{2,1} + t_{0,3}\bar{t}_{0,3}\bar{t}_{2,1} - t_{1,2}\bar{t}_{1,2}\bar{t}_{2,1} - t_{2,1}\bar{t}_{2,1}^2 + t_{1,2}\bar{t}_{0,3}\bar{t}_{3,0} + t_{2,1}\bar{t}_{1,2}\bar{t}_{3,0};\\
f_4 = &- t_{1,2}^2\bar{t}_{0,3} + t_{0,3}t_{2,1}\bar{t}_{0,3} - t_{1,2}t_{2,1}\bar{t}_{1,2} + t_{0,3}t_{3,0}\bar{t}_{1,2} + t_{0,3}\bar{t}_{1,2}^2 - t_{2,1}^2\bar{t}_{2,1}\\
&+ t_{1,2}t_{3,0}\bar{t}_{2,1} - t_{0,3}\bar{t}_{0,3}\bar{t}_{2,1} + t_{1,2}\bar{t}_{1,2}\bar{t}_{2,1} + t_{2,1}\bar{t}_{2,1}^2 - t_{1,2}\bar{t}_{0,3}\bar{t}_{3,0} - t_{2,1}\bar{t}_{1,2}\bar{t}_{3,0}.
\end{aligned}$$

The polynomials $f_1, f_2, f_3$ and $f_4$ generate a real codimension 2 variety, as expected by Proposition 8. We do not know whether the ideal generated by these polynomials is prime or not.

*Proof of Proposition 21.* By Lemma 10, $V$ is the direct sum of pairwise orthogonal, simple ideals $V_i$. Correspondingly, $T$ lies in $\bigoplus_i \mathrm{Sym}_3(V_i)$. We want to show that those ideals on which the multiplication is nonzero are one-dimensional. Thus we may assume that $V$ itself is simple with nonzero product.

Then the elements $x \in V$ for which the semilinear map $M_x : V \to V$, $y \mapsto x \cdot y$ is identically zero form a proper ideal in $V$, which is zero by simplicity. Hence for any nonzero $x \in V$ the map $M_x$ is nonzero.

Now consider, for nonzero $x \in V$, the space $W := \ker M_x$. We claim that $W$ is a proper ideal. First, $W$ also equals $\ker M_x^2$, because if $M_x^2 v = 0$, then

$$(M_x^2 v|v) = (x(xv)|v) = (xv|xv) = 0,$$

so $xv = 0$. We have

$$M_x^2(V \cdot W) = x \cdot (x \cdot (V \cdot W)) = V \cdot (x \cdot (x \cdot W)) = \{0\}.$$

Here we used semi-associativity in the second equality. So $V \cdot W \subseteq \ker M_x^2 = W$, as claimed. Hence $W$ is zero.

Fixing any nonzero $x \in V$, we define a new operation on $V$ by

$$y * z := M_x^{-1}(y \cdot z).$$

Since $M_x^{-1}$ is semilinear, $*$ is bilinear, commutative, and has $x$ as a unit element. We claim that it is also associative. For this we need to prove that

$$v \cdot M_x^{-1}(z \cdot y) = z \cdot M_x^{-1}(v \cdot y)$$

holds for all $y, z, v \in V$. Write $y = M_x^2 y'$, so that $x \cdot (x \cdot (z \cdot y')) = z \cdot y$ and $x \cdot (x \cdot (v \cdot y')) = v \cdot y$ by semi-associativity. Then the equation to be proved reads

$$v \cdot (x \cdot (z \cdot y')) = z \cdot (x \cdot (v \cdot y')),$$

which is another instance of semi-associativity.

Furthermore, any nonzero element $y \in V$ is invertible in $(V, *)$ with inverse $M_y^{-1}(x \cdot x)$. We conclude that $(V, *, +)$ is a finite-dimensional field extension of $\mathbb{C}$, hence equal to $\mathbb{C}$. ∎

3.5. Ordinary udeco three-tensors. In this subsection, $U, V, W$ are three finite-dimensional complex vector spaces equipped with Hermitian inner products $(\cdot|\cdot)$, and $T$ is a tensor in $U \otimes V \otimes W$. Then $T$ gives rise to bi-semilinear maps $U \times V \to W$, $V \times U \to W$, etc. Like for ordinary three-tensors in the real case, we equip $A := U \oplus V \oplus W$ with the bi-semilinear product $\cdot$ arising from these maps and with the inner product which restricts to the given inner products on $U, V$, and $W$, and is zero on all other pairs. By construction:

(1) $(A, \cdot)$ is commutative, and
(2) the inner product is compatible.

The following lemma gives the degree-three equations from the Main Theorem.

Lemma 23: *If $T$ is udeco, then*

(1) *for all $u, u', u'' \in U$ and $v \in V$ we have*

$$u \cdot (u' \cdot (u'' \cdot v)) = u'' \cdot (u' \cdot (u \cdot v));$$

(2) *for all $u \in U$, $v, v' \in V$, and $w \in W$ we have*

$$u \cdot (v \cdot (w \cdot v')) = w \cdot (v \cdot (u \cdot v')) \quad \textit{and} \quad (u \cdot v) \cdot (w \cdot v') = (u \cdot v') \cdot (w \cdot v);$$

*and the same relations hold with $U, V, W$ permuted in any manner.*

We call the product $\cdot$ **partially semi-associative** if it satisfies these conditions.

*Proof.* Let $T = \sum_i u_i \otimes v_i \otimes w_i$ be an orthogonal decomposition of $T$. Then we have

$$\begin{aligned} u'' \cdot v &= \sum_i (u_i|u'')(v_i|v)w_i, \\ u' \cdot (u'' \cdot v) &= \sum_i (u_i|u')(u''|u_i)(v|v_i)(w_i|w_i)v_i, \quad \text{and} \\ u \cdot (u' \cdot (u'' \cdot v)) &= \sum_i (u_i|u)(u'|u_i)(u_i|u'')(v_i|v)(w_i|w_i)(v_i|v_i)w_i, \end{aligned}$$

which is invariant under swapping $u$ and $u''$. The second identity is similar. For the last identity, we have

$$(u \cdot v) \cdot (w \cdot v') = \sum_i (u|u_i)(v|v_i)(w_i|w_i)(w|w_i)(v'|v_i)(u_i|u_i)v_i,$$

which is invariant under swapping $v$ and $v'$. ∎

The following proposition implies the Main Theorem for three-tensors over $\mathbb{C}$.

Proposition 24: *Conversely, if $\cdot$ is partially semi-associative, then $T$ is udeco.*

*Proof.* By a version of Lemma 10 for homogeneous ideals $I \subseteq A$, i.e., those for which

$$I = (I \cap U) \oplus (I \cap V) \oplus (I \cap W),$$

$A$ splits as a direct sum of nonzero, pairwise orthogonal, homogeneous ideals $I_i$ that each do not contain proper, nonzero homogeneous ideals, and $T$ lies in $\bigoplus_i (I_i \cap U) \otimes (I_i \cap V) \otimes (I_i \cap W)$, where the sum is over those $i$ on which the multiplication $\cdot$ is nontrivial. Thus we may assume that $A$ itself is nonzero, contains no proper nonzero ideals, and has nontrivial multiplication. We then need to prove that each of $U, V, W$ is one-dimensional.

Without loss of generality, $U \cdot V$ is a nonzero subset of $W$. The $u \in U$ for which the multiplication $M_u : V + W \to W + V$, $(v + w) \mapsto u \cdot v + u \cdot w$ is zero form a homogeneous, proper ideal in $A$, which is zero by simplicity.

Pick an $x \in U$, and let

$$Q := \ker M_x \subseteq V + W,$$

so that $Q \cdot Q \subseteq U$. We want to prove that $Q \oplus (Q \cdot Q)$ is a proper homogeneous ideal in $A$. First, $\ker M_x$ equals $\ker M_x^2$ because $0 = (x(xv)|v) = (xv|xv)$

implies $xv = 0$. Now $U \cdot Q \subseteq Q$ because

$$M_x^2(U \cdot Q) = x \cdot (x \cdot (U \cdot Q)) = U \cdot (x \cdot (x \cdot Q)) = \{0\}$$

by partial semi-associativity.

Next, let $R$ be the orthogonal complement of $Q$ in $V + W$. We have

$$(Q \cdot R|U) = (Q \cdot U|R) = \{0\},$$

so that $Q \cdot R = \{0\}$, and therefore

$$(V + W) \cdot Q = (Q + R) \cdot Q = Q \cdot Q.$$

It remains to check whether $V \cdot (Q \cdot Q) \subseteq Q$, and similarly for $W$. This is true since, for $v \in Q \cap V$ and $w \in Q \cap W$, we have

$$x \cdot (V \cdot (w \cdot v)) = w \cdot (V \cdot (x \cdot v)) = \{0\}$$

by partial semi-associativity. We have now proved that $Q \oplus (Q \cdot Q)$ is a proper homogeneous ideal in $A$. Hence $Q = 0$ by simplicity.

We conclude that $M_x$ is a bijection $V + W \to W + V$ for each nonzero $x \in U$. Similarly, $M_z$ is a bijection $U + V \to V + U$ for each nonzero $z \in W$. Fixing nonzero $x \in U$ and nonzero $z \in W$, define a new multiplication $*$ on $V$ by

$$v * v' := (x \cdot v) \cdot (z \cdot v') \in W \cdot U \subseteq V.$$

This operation is commutative by the third identity in partial associativity, and it is $\mathbb{C}$-linear. Moreover, for each nonzero $v' \in V$ and each $v'' \in V$ there is an element $v \in V$ such that $v * v' = v''$, namely, $M_x^{-1} M_{z \cdot v'}^{-1} v''$, which is well-defined since also the element $z \cdot v' \in U$ is nonzero. Thus $(V, *)$ is a commutative division algebra over $\mathbb{C}$, and by Hopf's theorem [Hop41], $\dim_{\mathbb{C}} V = 1$. ∎

3.6. Alternatingly udeco three-tensors. In this section, $V$ is a finite-dimensional complex inner product space. An alternating tensor

$$T \in \mathrm{Alt}_3(V) \subseteq V \otimes V \otimes V \cong V^s \otimes V^s \otimes V$$

gives rise to a bi-semilinear multiplication $V \times V \to V$, $(a, b) \mapsto [a, b]$ that satisfies

$$[a, b] = -[b, a] \quad \text{and} \quad ([a, b]|c) = -([a, c]|b).$$

Just like the multiplication did not become associative in the symmetrically udeco case, the bracket does not satisfy the Jacobi identity in the alternatingly udeco case. However, it does satisfy the following cross product identities.

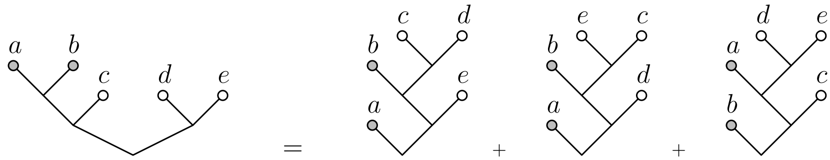


Figure 2. Quartic equations for alternatingly udeco tensors; see Lemma 25.

Lemma 25: *If $T$ is alternatingly udeco, then for all $a, b, c, d, e \in V$ we have*

$$[a, [[a, b], [a, c]]] = 0$$

*and*

$$[[[a, b], c], [d, e]] = [a, [[b, [c, d]], e]] + [a, [[b, [e, c]], d]] + [b, [[a, [d, e]], c]].$$

For a pictorial representation of the second identity see Figure 2.

*Proof.* In the alternatingly udeco case, the simple, nontrivial ideals of the algebra $(V, [.,.])$ are isomorphic, via an inner product preserving isomorphism, to $(\mathbb{C}^3, \gamma\times)$, where $\times$ is the semilinear extension to $\mathbb{C}^3$ of the cross product on $\mathbb{R}^3$ and where $\gamma$ is a scalar. Thus it suffices to prove the two identities for this three-dimensional algebra. Moreover, both identities are homogeneous in the sense that their validity for some $(a, b, c, d, e)$ implies their validity when any one of the variables is scaled by a complex number. Indeed, for the first identity this is clear, and for the second identity this follows since all four terms are semilinear in $a, b$ and linear in $c, d, e$. Hence both identities follow from their validity for the crossproduct and arbitrary $a, b, c, d, e \in \mathbb{R}^3$. ∎

The cross product identities yield real degree-four equations that vanish on the set of alternatingly odeco three-tensors. There are also degree-three equations, which arise as follows. Let $\mu : V \otimes V \to V$, $(a \otimes b) \to [a, b]$ be the semilinear multiplication, and let, conversely, $\psi : V \to V \otimes V$ be the semilinear map determined by $(c|[a, b]) = (a \otimes b|\psi(c))$—note that both sides are linear in $a, b, c$. Then let $H := \mu \circ \psi : V \to V$. Being the composition of two semilinear maps, this is a linear map, and it satisfies $(Ha|b) = (\psi(a)|\psi(b)) = (a|Hb)$. Hence $H$ is a positive semidefinite Hermitian map.

Lemma 26: *If $T$ is alternatingly udeco, then $[Hx, y] = [x, Hy]$ for all $x, y \in V$.*

*Proof.* Let $T = \sum_i u_i \wedge v_i \wedge w_i$ be an orthogonal decomposition of $T$. Then we have

$$\begin{aligned}[Hx, y] =& \left[\mu\left(\sum_i (w_i|x)u_i \wedge v_i - (v_i|x)u_i \wedge w_i + (u_i|x)v_i \wedge w_i\right), y\right]\\ =& \sum_i [2(w_i|x)(u_i|u_i)(v_i|v_i)w_i + 2(v_i|x)(u_i|u_i)(w_i|w_i)v_i\\ &\quad + 2(u_i|x)(v_i|v_i)(w_i|w_i)u_i, y]\\ =&2\sum_i ((w_i|x)(u_i|u_i)(v_i|v_i)(w_i|w_i)((u_i|y)v_i - (v_i|y)u_i)\\ &\quad + (v_i|x)(u_i|u_i)(w_i|w_i)(v_i|v_i)((w_i|y)u_i - (u_i|y)w_i)\\ &\quad + (u_i|x)(v_i|v_i)(w_i|w_i)(u_i|u_i)((v_i|y)w_i - (w_i|y)v_i)).\end{aligned}$$

Now we observe that the latter expression is skew-symmetric in $x$ and $y$, so that it equals $-[Hy, x] = [x, Hy]$. ∎

*Remark 27:* For a real, compact Lie algebra $\mathfrak{g}$, the positive semidefinite matrix $H$ constructed above is a (negative) scalar multiple of the Casimir element in its adjoint action [Kna02]; this is why we call the identity in the lemma the **Casimir identity**. Complexifying $\mathfrak{g}$ and its invariant inner product to a semilinear algebra with an invariant Hermitian inner product, we obtain an algebra satisfying the degree-three equations of the lemma. Hence, since in dimension at least 8 there exist other compact Lie algebras than direct sums of copies of $\mathbb{R}^3$ and of $\mathbb{R}^1$, these degree-three equations do not suffice to characterise alternatingly udeco three-tensors in general, though perhaps they do so for $\dim V \leq 7$.

*Example 28:* The lemma yields cubic equations satisfied by alternatingly udeco tensors. Here is one of these, with $V = \mathbb{C}^6$ and $t_{ijk}$ the coefficient of $e_i \otimes e_j \otimes e_k$:

$$\begin{aligned} t_{1,4,5}t_{2,3,4}\bar{t}_{1,3,5} &- t_{1,3,4}t_{2,4,5}\bar{t}_{1,3,5} + t_{1,2,4}t_{3,4,5}\bar{t}_{1,3,5} + t_{1,4,6}t_{2,3,4}\bar{t}_{1,3,6}\\ &- t_{1,3,4}t_{2,4,6}\bar{t}_{1,3,6} + t_{1,2,4}t_{3,4,6}\bar{t}_{1,3,6} - t_{1,4,6}t_{2,4,5}\bar{t}_{1,5,6}\\ &+ t_{1,4,5}t_{2,4,6}\bar{t}_{1,5,6} - t_{1,2,4}t_{4,5,6}\bar{t}_{1,5,6} + t_{2,4,6}t_{3,4,5}\bar{t}_{3,5,6}\\ &- t_{2,4,5}t_{3,4,6}\bar{t}_{3,5,6} + t_{2,3,4}t_{4,5,6}\bar{t}_{3,5,6} = 0.\end{aligned}$$

This equation was first discovered as follows: working in $V = (\mathbb{Z}/19)[i]^6$ instead of $\mathbb{C}^6$ (where it is important that 19 is 3 modulo 4 so that $-1$ has no square root in $\mathbb{Z}/19$), we implemented the Cayley transform to sample general unitary matrices from which we constructed alternatingly udeco tensors. We sampled as many as there are degree-three monomials in the 20 variables $t_{ijk}$ plus the 20 variables $\bar{t}_{ijk}$ (namely, $\binom{40+2}{3} = 11480$), and evaluated these monomials on the tensors. The 280-dimensional kernel of this matrix over $\mathbb{Z}/19$ turned out to have a basis consisting of vectors with entries $0, 1, 2, 17, 18$. The natural guess for lifting these equations to characteristic zero, respectively, yielded equations that vanish on general alternatingly udeco tensors in characteristic zero. A similar, but smaller computation shows that there are no degree-two equations; here the fact that these do not exist modulo 19 proves that they do not exist in characteristic zero either.

In a Lie algebra, if $[a, b] = 0$, then the left multiplications $L_a : V \to V$ and $L_b : V \to V$ commute. This is not true in our setting, since the Jacobi identity does not hold, but the following statement does hold.

Lemma 29: *Suppose that the bracket satisfies the second cross product identity in Lemma 25, and let $a, b, c \in V$ be such that $[a, c] = [b, c] = 0$. Then $[[a, b], c] = 0$.*

*Proof.* Compute the inner product

$$([[a, b], c]|[[a, b], c]) = -([[a, b], [[a, b], c]]|c) = ([[[a, b], c], [a, b]]|c)$$

and use the identity to expand the first factor in the last inner product as

$$[[[a, b], c], [a, b]] = [a, [[b, [c, a]], b]] + [a, [[b, [b, c]], a]] + [b, [[a, [a, b]], c]].$$

Now each of the terms on the right-hand side is of the form $[a, x]$ or $[b, y]$, and we have $([a, x]|c) = -([a, c], x) = 0$ and similarly $([b, y]|c) = 0$. Since the inner product is positive definite, this shows that $[[a, b], c] = 0$, as claimed. ∎

We now prove that the equations that we have found so far suffice.

Proposition 30: *Suppose that, conversely, $T \in \mathrm{Alt}_3(V)$ has the properties in Lemmas 25 and 26. Then $T$ is alternatingly udeco.*

*Proof.* If $a, b \in V$ belong to distinct eigenspaces of the Hermitian linear map $H$, then the property that $[Ha, b] = [a, Hb]$ implies that $[a, b] = 0$. Moreover, a fixed eigenspace of $H$ is closed under multiplication, as for $a, b$ in the eigenspace with eigenvalue $\lambda$ and $c$ in the eigenspace with eigenvalue $\mu \neq \lambda$, we have

$$\overline{\lambda}([a, b]|c) = ([Ha, b]|c) = -([Ha, c]|b) = -\overline{\mu}([a, c]|b) = \overline{\mu}([a, b]|c),$$

and hence $([a,b]|c) = 0$. Thus the eigenspaces of $H$ are ideals. We may replace $V$ by one of these, so that $H$ becomes a scalar. If the scalar is zero, then $T$ is zero and we are done, so we assume that it is nonzero, in which case we can scale $T$ (even by a positive real number) to achieve that $H = 1$.

Furthermore, by compatibility of the inner product and Lemma 10, $V$ splits further as a direct sum of simple ideals. So to prove the proposition, in addition to $H = 1$, we may assume that $V$ is a simple algebra and that the multiplication is not identically zero; in this case it suffices to prove that $V$ is three-dimensional. Let $x \in V$ be a nonzero element such that the semilinear left multiplication $L_x : V \to V$ has minimal possible rank. If its rank is zero, then $\langle x \rangle$ is an ideal, contrary to the assumptions. Hence $V_1 := L_x V$ is a nonzero space, and we set $V_0 := [V_1, V_1]$, the linear span of all products of two elements from $V_1$. We claim that $x \in V_0$. For this, we note that $V_1^\perp = \ker L_x$ and compute

$$(\psi(x)|V_1^\perp \otimes V) = ([V_1^\perp, V]|x) = ([\ker L_x, x]|V) = \{0\}.$$

Similarly, we find that $(\psi(x)|V \otimes V_1^\perp) = \{0\}$, so $\psi(x) \in V_1 \otimes V_1$ and therefore

$$x = Hx = \mu(\psi(x)) \in [V_1, V_1] = V_0,$$

as claimed.

Next, by the first cross product identity in Lemma 25, we find that $[x, V_0] = \{0\}$. This implies that $(V_0|V_1) = (V_0|[x,V]) = ([x,V_0]|V) = \{0\}$, so

$$V_0 \perp V_1.$$

Furthermore, by substituting $x + s$ for $x$ in that same identity and taking the part quadratic in $x$, we find the identity

$$[s,[[x,a],[x,b]]] + [x,[[s,a],[x,b]]] + [x,[[x,a],[s,b]]] = 0.$$

An arbitrary element of $[V, V_0]$ is a linear combination of terms of the left-most shape in this identity, hence the identity shows that $[V, V_0] \subseteq V_1$. Moreover, substituting for $s$ an element $[[x,c],[x,d]] \in V_0$ we find that the last two terms are zero, since $[s,a] \in V_1$ and $[x,[V_1,V_1]] = \{0\}$. Hence the first term is also zero, which shows that $[V_0, V_0] = \{0\}$.

Now let $V_2$ be the orthogonal complement $(V_0 \oplus V_1)^\perp$, so that $V$ decomposes orthogonally as $V_0 \oplus V_1 \oplus V_2$. We claim that $V_2$ is an ideal. First, we have $([V_0, V_2]|V) = (V_2|[V_0, V]) \subseteq (V_2|V_1) = \{0\}$, so

$$[V_0, V_2] = \{0\}.$$

By the first paragraph of the proof, $x$ is contained in $V_0$, hence in particular $[x, V_2] = 0$, so $\ker L_x$ contains $V_0 \oplus V_2$. For dimension reasons, equality holds: $\ker L_x = V_0 \oplus V_2$. Now Lemma 29 applied with $c = x$ yields that $\ker L_x$ is closed under multiplication, so in particular $[V_2, V_2] \subseteq V_0 \oplus V_2$. Since $([V_2, V_2]|V_0) = \{0\}$, we have $[V_2, V_2] \subseteq V_2$. Furthermore, we have

$$([V_1, V_2]|V_0 \oplus V_1) = (V_2|V_1 \oplus V_0) = \{0\},$$

so that $[V_1, V_2] \subseteq V_2$. This concludes the proof of the claim that $V_2$ is an ideal. By simplicity of $V$, $V_2 = \{0\}$ and hence $V = V_0 \oplus V_1$.

Now consider any $y \in V_0 \setminus \{0\}$. Then $\ker L_y \supseteq V_0 \oplus V_2$, and hence equality holds by maximality of $\dim \ker L_x$. But we can show more: let $v \in V_1$ be an eigenvector of the map $(L_x|_{V_1})^{-1}(L_y|_{V_1})$ (which is linear since it is the composition of two semilinear maps), say with eigenvalue $\lambda$. Then we have $[y, v] = [x, \lambda v] = [\lambda x, v]$. This means that the element $z := y - \lambda x \in V_0$ has $\ker L_z \supseteq V_0 \oplus V_2$, but also $v \in \ker L_z$. Hence the kernel of $L_z$ is strictly larger than that of $L_x$, and therefore $z = 0$. We conclude that $y = \lambda x$, and hence $V_0$ is one-dimensional.

Finally, consider a nonzero element $z \in V_1$. From $[z, V_1] \subseteq V_0 = \langle x \rangle$ we find that $L_z V$ is contained in $\langle x, [z, x] \rangle_{\mathbb{C}}$, i.e., $L_z$ has rank at most two. Hence, by minimality, the same holds for $L_x$. This means that $\dim V_1 \leq 2$, and hence

$$\dim V = \dim(V_0 \oplus V_1) \leq 3.$$

Since $T$ is nonzero, we find $\dim V = 3$, as desired. ∎

## 4. Higher-order tensors

In this section, building on the case of order three, we prove the Main Theorem for tensors of arbitrary order.

4.1. Ordinary tensors. Let $V_1, \ldots, V_d$ be finite-dimensional inner product spaces over $K \in \{\mathbb{R}, \mathbb{C}\}$. The key observation is the following. Let

$$J_1 \cup \cdots \cup J_e = \{1, \ldots, d\}$$

be a partition of $\{1, \ldots, d\}$. Then the natural **flattening** map

$$V_1 \otimes \cdots \otimes V_d \to \left( \bigotimes_{j \in J_1} V_j \right) \otimes \cdots \otimes \left( \bigotimes_{j \in J_e} V_j \right)$$

sends the set of order-$d$ odeco/udeco tensors into the set of order-$e$ odeco/udeco tensors, where the inner product on each factor $\bigotimes_{j \in J_\ell} V_j$ is the one induced from the inner products on the factors. The following proposition gives a strong converse to this observation.

PROPOSITION 31: *Let $T \in V_1 \otimes \cdots \otimes V_d$ be a tensor, where $d \geq 4$. Suppose that the flattenings of $T$ with respect to the three partitions*

(i) $\{1\}, \ldots, \{d-3\}, \{d-2\}, \{d-1, d\}$,
(ii) $\{1\}, \ldots, \{d-3\}, \{d-2, d-1\}, \{d\}$, *and*
(iii) $\{1\}, \ldots, \{d-3\}, \{d-2, d\}, \{d-1\}$

*are all odeco/udeco. Then so is $T$.*

The lower bound of 4 in this proposition is essential, because any flattening of a three-tensor is a matrix and hence odeco, but as we have seen in Section 3 not every three-tensor is odeco.

*Proof.* As the first two flattenings are odeco, we have orthogonal decompositions

$$T = \sum_{i=1}^{k} T_i \otimes u_i \otimes A_i = \sum_{\ell=1}^{r} T'_\ell \otimes B_\ell \otimes w_\ell$$

where $A_1, \ldots, A_k \in V_{d-1} \otimes V_d$ are pairwise orthogonal and nonzero, and so are $u_1, \ldots, u_k \in V_{d-2}$, and the $T_i$ are of the form $z_{i1} \otimes \cdots \otimes z_{i(d-3)}$ where for each $j$ the $z_{ij}$, $i = 1, \ldots, k$ are pairwise orthogonal and nonzero. Similarly for the factors in the second expression. Contracting $T$ with $T_i$ in the first $d-3$ factors yields a single term on the left (here we use that $d > 3$):

$$(T_i|T_i)u_i \otimes A_i = \sum_{\ell=1}^{r} (T'_\ell|T_i)B_\ell \otimes w_\ell.$$

For an index $\ell$ such that $(T'_\ell|T_i)$ is nonzero, by contracting with $w_\ell$ we find that $B_\ell$ is of rank one and, more specifically, of the form $u_i \otimes v_\ell$ with $v_\ell \in V_{d-1}$. There is at least one such index, since the left-hand side is nonzero. Moreover, since the $u_i$ are linearly independent for distinct $i$, we find that the set of $\ell$ with $(T'_\ell|T_i) \neq 0$ is disjoint from the set defined similarly for another value of $i$. Hence, $r \geq k$. By swapping the roles of the two decompositions we also find the opposite inequality, so that $r = k$, and after relabelling we find $B_i = u_i \otimes v_i$ for $i = 1, \ldots, k$ and certain nonzero vectors $v_i$. Hence we find

$$T = \sum_{i=1}^{k} T'_i \otimes u_i \otimes v_i \otimes w_i,$$

where we do not know yet whether the $v_i$ are pairwise perpendicular. However, applying the same reasoning to the second and third decompositions in the lemma, we obtain another decomposition

$$T = \sum_{i=1}^{k} T_i' \otimes u_i' \otimes v_i' \otimes w_i,$$

where we do know that the $v_i'$ are pairwise perpendicular (but not that the $u_i'$ are). Contracting with $T_i'$ we find that, in fact, both decompositions are equal and the $v_i$ are pairwise perpendicular, as required. ∎

*Proof of the Main Theorem for ordinary tensors.* It follows from Lemma 13 and Proposition 14 that ordinary odeco tensors of order three are characterised by degree-two equations. Similarly, by Lemma 23 and Proposition 24, ordinary udeco tensors of order three are characterised by degree-three equations. By Proposition 31 and the remarks preceding it, a higher-order tensor is odeco (udeco) if and only if certain of its flattenings are odeco (udeco). Thus the equations characterising lower-order odeco (udeco) tensors pull back, along linear maps, to equations characterising higher-order odeco (udeco) tensors. ∎

4.2. Symmetric tensors. In this section, $V$ is a finite-dimensional vector space over $K = \mathbb{R}$ or $\mathbb{C}$.

Proposition 32: *For $d \geq 3$, a tensor $T \in \mathrm{Sym}_d(V)$ is symmetrically odeco (udeco) if and only if it is odeco (udeco) when considered as an ordinary tensor in $V^{\otimes d}$.*

*Proof.* The “only if” direction is immediate, since a symmetric orthogonal decomposition is a fortiori an ordinary orthogonal decomposition. For the converse, consider an orthogonal decomposition

$$T = \sum_{i=1}^{k} v_{i1} \otimes \cdots \otimes v_{id},$$

where the $v_{ij}$ are nonzero vectors, pairwise perpendicular for fixed $j$. Since $T$ is symmetric, we have

$$T = \sum_{i} v_{i\pi(1)} \otimes \cdots \otimes v_{i\pi(d)}$$

for each $\pi \in S_d$. By uniqueness of the decomposition (Proposition 7), the terms in this latter decomposition are the same, up to a permutation, as the terms in the original decomposition. In particular, the unordered cardinality-$k$ sets of projective points $Q_j := \{[v_{1j}], \ldots, [v_{kj}]\} \subseteq \mathbb{P}V$ are identical for all $j = 1, \ldots, d$. Consider the integer $(k \times d)$-matrix $A$ with entries in $[k] := \{1, \ldots, k\}$ determined by $a_{ij} = m$ if $[v_{ij}] = [v_{m1}]$. This matrix has all integers $1, \ldots, k$ in each column, in increasing order in the first column, and furthermore has the property that for each $(d \times d)$-permutation matrix $\pi$ there exists a $(k \times k)$-permutation matrix $\sigma$ such that $\sigma A = A\pi$. To conclude the proof we only need to prove the following claim, namely that, for $d \geq 3$, the only such $(k \times d)$-matrix is the matrix whose $i$-th row consists entirely of copies of $i$.

Claim: *Let $k \geq 1$ and $d \geq 3$ be natural numbers. Let $S_k$ act on $S_k^d$ diagonally from the left by left multiplication and let $S_d$ act on $S_k^d$ from the right by permuting the terms. Consider an element*

$$A := (\mathrm{id}, \tau_2, \ldots, \tau_d) \in S_k^d,$$

*where* id *is the identity permutation. Suppose that for each $\pi \in S_d$ there exists a $\sigma \in S_k$ such that $\sigma A = A\pi$. Then $A = (\mathrm{id}, \ldots, \mathrm{id})$.*

*Proof of claim.* For $j \in \{2, \ldots, d\}$ pick $\pi_j = (1, j)$ to be the transposition switching 1 and $j$. By the property imposed on $A$ there exists a $\sigma_j$ such that $\sigma_j A = A\pi_j$. In particular, $(A\pi_j)_1 = \tau_j$ equals $(\sigma_j A)_1 = \sigma_j$. So $\tau_j = \sigma_j$ for all $j \in \{2, \ldots, d\}$. Since $d \geq 3$, one can pick an index $l$ which is fixed by $\pi_j$, so that $\tau_l = (\sigma_j A)_l = \sigma_j \tau_l$. So then $\sigma_j = \mathrm{id} = \tau_j$. This concludes the proof of the claim, and thus that of Proposition 32. ∎

*Proof of the Main Theorem for symmetric tensors.* By the preceding proposition, the equations for odeco tensors in $V \otimes \cdots \otimes V$ pull back to equations characterising symmetrically odeco tensors in $\mathrm{Sym}_d V$ via the inclusion of the latter space into the former. Thus the Main Theorem for symmetric tensors follows from the Main Theorem for ordinary tensors, proved in the previous subsection. ∎

*Remark 33:* The proof of the Main Theorem in Section 3 for ordinary odeco three-tensors relies on the proof for symmetrically odeco three-tensors, so the proof above does not render that proof superfluous. On the other hand, the proof for ordinary udeco three-tensors does not rely on that for symmetrically udeco three-tensors, so in view of the proof above the latter could have been left out. We have decided to retain it for completeness.

*Remark 34:* The argument in the proposition also implies that an odeco/udeco tensor in $V^{\otimes d} \setminus \{0\}$ with $d \geq 3$ cannot be alternating: permuting tensor factors with a transposition must leave the decomposition intact up to a sign and a permutation of terms, but then the claim shows that in each term all vectors are equal, hence their alternating product is zero.

4.3. Alternating tensors. In this section we prove that an alternating tensor of order at least four is alternatingly odeco/udeco if and only if all its contractions with a vector are. Thus, let $V$ be a vector space over $K \in \{\mathbb{R}, \mathbb{C}\}$ and consider an orthogonal decomposition

$$T = \sum_{i=1}^{k} \lambda_i v_{i1} \wedge \cdots \wedge v_{id} \tag{1}$$

of an alternatingly odeco tensor $T \in \mathrm{Alt}_d V$, where $v_{11}, \ldots, v_{kd}$ form an orthonormal set of vectors in $V$ and where $\lambda_i \in K$. The following lemmas are straightforward exercises in differential geometry, and we omit their proofs.

Lemma 35: *Suppose that $K = \mathbb{R}$. Let $d \geq 3$ and $dk \leq n := \dim V$. The set $X$ of alternatingly odeco tensors in $\mathrm{Alt}_d V$ with exactly $k$ terms in their orthogonal decomposition is a smooth manifold of dimension $k + \frac{1}{2}dk(2n - (k+1)d)$ whose tangent space at a point $T$ is the direct sum of the following spaces:*

1. *$\bigoplus_{i=1}^{k} (\mathrm{Alt}_{d-1} V_i) \wedge V_0$ where $V_i = \langle v_{i1}, \ldots, v_{id} \rangle$ and $V_0 = (V_1 \oplus \cdots \oplus V_k)^{\perp}$;*
2. *$\bigoplus_{i=1}^{k} \mathrm{Alt}_d V_i$; and*
3. *$\langle \lambda_i (v_{i1} \wedge \cdots \wedge v_{ml} \wedge \cdots \wedge v_{id})$*
 *$- \lambda_m (v_{m1} \wedge \cdots \wedge v_{ij} \wedge \cdots \wedge v_{md}) \mid 1 \leq j, l \leq d$ and $i \neq m \rangle$,*
 *where $v_{ml}$ replaces $v_{ij}$ in the first term and vice versa in the second term.*

The three summands are obtained as follows: $X$ is the image of the Cartesian product of the manifold of $k \cdot d$-tuples of orthonormal vectors with $(\mathbb{R} \setminus \{0\})^k$ via

$$\phi : ((v_{ij})_{(i,j) \in [k] \times [d]}, \lambda) \mapsto \sum_i \lambda_i v_{i1} \wedge \cdots \wedge v_{id}.$$

Replacing a $v_{ij}$ by a $v_{ij} + \epsilon v_0$ with $v_0 \in V_0$ yields the first summand. Replacing $\lambda_i$ by $\lambda_i + \epsilon$ yields the second summand, and infinitesimally rotating $(v_{ij}, v_{ml})$ into $(v_{ij} + \epsilon v_{ml}, v_{ml} - \epsilon v_{ij})$ yields the last summand. The complex analogue of Lemma 35 is the following.

LEMMA 36: *Suppose that $K = \mathbb{C}$. Let $d \geq 3$ and $2k \leq n := \dim_{\mathbb{C}} V$. The set $X$ of alternatingly udeco tensors in $\mathrm{Alt}_d V$ with exactly $k$ terms in their orthogonal decomposition is a smooth manifold of dimension $2k + dk(2n - (k+1)d)$ whose tangent space at $T$ is the direct sum of the following spaces:*

(1) *the complex space $\bigoplus_{i=1}^{k} (\mathrm{Alt}_{d-1} V_i) \wedge V_0$ where $V_i = \langle v_{i1}, \ldots, v_{id} \rangle$ and $V_0 = (V_1 \oplus \cdots \oplus V_k)^{\perp}$;*

(2) *the complex space $\bigoplus_{i=1}^{k} \mathrm{Alt}_d V_i$;*

(3) *the real space*

$$\langle \lambda_i (v_{i1} \wedge \cdots \wedge v_{ml} \wedge \cdots \wedge v_{id}) \\ - \lambda_m (v_{m1} \wedge \cdots \wedge v_{ij} \wedge \cdots \wedge v_{md}) \mid 1 \leq j, l \leq d \text{ and } i \neq m \rangle_{\mathbb{R}},$$

*where $v_{ml}$ replaces $v_{ij}$ in the first term and vice versa in the second term; and*

(4) *the real space*

$$\langle \lambda_i (v_{i1} \wedge \cdots \wedge (\mathbf{i} v_{ml}) \wedge \cdots \wedge v_{id}) \\ + \lambda_m (v_{m1} \wedge \cdots \wedge (\mathbf{i} v_{ij}) \wedge \cdots \wedge v_{md}) \mid 1 \leq j, l \leq d \text{ and } i \neq m \rangle_{\mathbb{R}},$$

*where $\mathbf{i} v_{ml}$ replaces $v_{ij}$ in the first term and vice versa in the second term and where $\mathbf{i} \in \mathbb{C}$ is a square root of $-1$.*

The last summand arises from the infinitesimal unitary transformations sending $(u_{ij}, u_{ml})$ to $(u_{ij} + \mathbf{i} u_{ml}, u_{ml} + \mathbf{i} u_{ij})$.

PROPOSITION 37: *Let $V$ be a vector space over $K \in \{\mathbb{R}, \mathbb{C}\}$. Let $d \geq 3$ and let $S \in \mathrm{Alt}_{d+1} V$. Then $S$ is alternatingly odeco (or udeco) if and only if for each $v_0 \in V$ the contraction $(S|v_0)$ of $S$ with $v_0$ in the last factor is an alternatingly odeco (or udeco) tensor in $\mathrm{Alt}_d V$.*

*Proof.* The "only if" direction is immediate: contracting the terms in an orthogonal decomposition of $S$ with $v_0$ yields an orthogonal decomposition for $(S|v_0)$. Note that in this process the pairwise orthogonal $(d+1)$-spaces encoded by $S$ are replaced by their $d$-dimensional intersections with the hyperplane $v_0^{\perp}$, and discarded if they happen to be contained in that hyperplane.

Conversely, assume that all contractions of $S$ with a vector are alternatingly odeco. Among all $v_0 \in V$ choose one, say of norm 1, such that $T := (S|v_0)$ is odeco with the maximal number of terms, say $k$, and let $\lambda_i$ and the $v_{ij}$ be as in (1). Then $\Psi : v \mapsto (S|v)$ is a real-linear map from an open neighbourhood of

$v_0$ in $V$ into the set $X$ in the lemma, and hence its derivative at $v_0$, which is $\Psi$ itself, maps $V$ into the tangent space described in the lemma. Since contracting with $v_0$ maps $\mathrm{Alt}_{d+1}V$ into $\mathrm{Alt}_d(v_0^\perp)$, we may choose a basis $v_{00}, \dots, v_{0(n-kd)}$ of $V_0$ from the lemma that starts with $v_{00} := v_0$. Now we have

$$S = \left( \sum_{i=1}^{k} \lambda_i v_{i1} \wedge \cdots \wedge v_{id} \wedge v_{00} \right) + S'' =: S' + S''$$

where $(S''|v_{00}) = 0$. We have an orthonormal basis $(v_{ij})_{ij}$ of $V$ where $(i, j)$ runs through $A := ([k] \times [d]) \cup (\{0\} \times [n - kd])$, where $[k] := \{1, \dots, k\}$.

For a subset $I \subseteq A$ we write $v_I$ for the vector in $\mathrm{Alt}_{|I|}V$ obtained as the wedge product of the vectors labelled by $I$ (in some fixed linear order on $A$). The vectors $v_I$ with $|I| = d + 1$ form a $K$-basis of $\mathrm{Alt}_{d+1}V$, and similarly for those with $|I| = d$. Now $(S'|v)$ lies in the tangent space to $X$ at $T$ for all $v$ (indeed, in the sum of the first two summands in the lemma). Hence also $(S''|v)$ must lie in that tangent space. Expand $S''$ on the chosen basis:

$$S'' = \sum_{I \subseteq A, |I| = d+1} c_I v_I.$$

We claim that $c_I = 0$ unless $I$ contains one of the $k$ sets $\{i\} \times [d]$. Indeed, suppose that $c_I \neq 0$ and that $I$ does not contain any of these $k$ sets. Contracting $v_I$ with any $v_\alpha$ with $\alpha \in I$ yields $\pm v_J$ where $J := I \setminus \{\alpha\}$, hence $v_J$ appears with a nonzero coefficient in $(S''|v_\alpha)$. By the lemma we find that $J$ must contain a $(d - 1)$-subset of at least one of the sets $\{i\} \times [d]$. So in particular, there exists an $i$ such that $I$ itself contains a $(d - 1)$-subset of $\{i\} \times [d]$. Suppose first that this $i$ is unique, say equal to $i_0$. Then contracting $v_I$ with $v_{i_0,j}$ with $(i_0, j) \in I$ yields $\pm v_J$ where $J$ contains only at most $d - 2$ of the elements of each of the sets $\{i\} \times [d]$, a contradiction with the lemma. So this $i$ is not unique. Then $I$ contains $d - 1$ elements from each of at least two disjoint sets, so $2(d-1) \le d+1$, so $d \le 3$, and hence $d = 3$—here we use that $d \ge 3$. Without loss of generality, then, $I = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Now contracting $v_I$ with $v_{11}$ yields a scalar times $\pm v_{12} \wedge v_{21} \wedge v_{22}$, hence this term appears in $(S|v_{11})$. But (see the last one/two summand/summands in the tangent space for the odeco/udeco case, respectively) this term can only appear in a tangent vector if also the term $\pm v_{11} \wedge v_{23} \wedge v_{13}$ appears—which is impossible after contracting with $v_{11}$. This proves the claim.

We conclude that $S$ can be written as

$$S = \sum_{i=1}^{k} v_{i1} \wedge \cdots \wedge v_{id} \wedge w_i$$

for suitable vectors $w_i$ satisfying $(w_i|v_0) = \lambda_i$. Set $W_i := V_i + \langle w_i \rangle$. We need to show that the spaces $W_1, \ldots, W_k$ are pairwise perpendicular. For this, it suffices to show that, for $z$ in an open dense subset of $V$, the spaces $W_i' := W_i \cap z^\perp$ are pairwise perpendicular. We choose this open subset such that

(1) the contraction $(S|z)$ has an orthogonal decomposition with $k$ terms;
(2) the $k$ spaces $W_i'$ are $d$-dimensional and linearly independent;
(3) the tensor $((S|z)|v_0) = \pm((S|v_0)|z) \in \mathrm{Alt}_{d-1}V$, which by assumption is alternatingly odeco, has a unique orthogonal decomposition.

By Proposition 7, the last condition is void if $d > 3$. Now, each $W_i'' := W_i' \cap v_0^\perp$ is contained in $V_i$, so that $W_i'' \perp W_m''$ for all $i \neq m$. Now, by assumption, the tensor

$$(S|z) \in \bigoplus_{i=1}^{k} \mathrm{Alt}_d W_i'$$

is alternatingly odeco with $k$ terms. Let $U_1, \ldots, U_k$ be the $d$-dimensional, pairwise orthogonal spaces encoded by it. Then $((S|z)|v_0)$ has an orthogonal decomposition with terms in $\mathrm{Alt}_{d-1}(U_i \cap v_0^\perp)$. But we also have

$$((S|z)|v_0) \in \bigoplus_{i=1}^{k} \mathrm{Alt}_{d-1} W_i'',$$

where the $W_i''$ are pairwise perpendicular. So, since we assumed that this orthogonal decomposition is unique, after a permutation of the $U_i$ we have $U_i \cap v_0^\perp = W_i''$. Now let $u_{i1}, \ldots, u_{id}$ be an orthonormal basis of $U_i$, where the first $(d-1)$ form a basis of $W_i''$. Extend with $u_{01}, \ldots, u_{0(n-kd)}$ to an orthonormal basis of $V$. Arguing with respect to the basis $(u_I)_{|I|=d}$, we find that the map $V^k \to \mathrm{Alt}_d V$ that sends $(y_1, \ldots, y_k)$ to $\sum_{i=1}^{k} u_{i1} \wedge \cdots \wedge u_{i(d-1)} \wedge y_i$ is injective. Since

$$(S|z) = \sum_{i=1}^{k} \mu_i u_{i1} \wedge \cdots \wedge u_{id} = \sum_{i=1}^{k} \mu_i' u_{i1} \wedge \cdots \wedge u_{i(d-1)} \wedge w_i'$$

for suitable $w_i' \in W_i'$ and nonzero scalars $\mu_i, \mu_i'$, we find that $W_i' = U_i$, and hence the $W_i'$ are pairwise perpendicular, as desired. ∎

*Proof of the Main Theorem for alternating tensors.* In Lemmas 16, 17 and Proposition 18 we found that an alternating three-tensor is alternatingly odeco if and only if it satisfies certain polynomial equations of degrees 2 and 4. Correspondingly, Proposition 30 settles the Main Theorem for alternatingly udeco three-tensors. Proposition 37 yields that the pullbacks of the real polynomial equations characterising alternatingly odeco/udeco $d$-tensors along real-linear maps yield equations characterising alternatingly odeco/udeco $(d+1)$-tensors. These pullbacks have the same degrees as the original equations. ∎

## 5. Concluding remarks

We have established low-degree real-algebraic characterisations of orthogonally decomposable tensors in six different scenarios. While this is quite a satisfactory result, at least three questions remain.

First, do the equations that we have found generate the ideals of the real-algebraic varieties at hand? We are somewhat optimistic in the ordinary and symmetric odeco case, because of evidence in [Rob14] for the case of symmetrically odeco $(2\times 2\times\cdots\times 2)$-tensors. But in general we believe that representation theory of the orthogonal and unitary groups should be used to approach this question.

Second, and related to this, our main result can be read as a finiteness result for an infinite class of varieties in the spirit of Snowden's Delta-modules [Sno13]. Can the methods of [SS15], tailored to the orthogonal and unitary groups that preserve orthogonally decomposable tensors, lead to more refined finiteness results on equations and higher-order syzygies?

Third, a potentially interesting line of research, which we have not yet pursued further, concerns a form of (non-associative, non-commutative) elimination. To make this somewhat precise, suppose that we are given a number of polynomial identities defining a class of algebras over $\mathbb{R}$. Now consider the functor that assigns to such an algebra $A$ the space $\mathbb{C}\otimes_{\mathbb{R}} A$ equipped with the semilinear extension of the product, and that assigns to an algebra homomorphism its linear extension. What polynomial identities are satisfied by the image of our class under this functor? Above we have implicitly seen that commutative, associative $\mathbb{R}$-algebras are mapped to commutative, semi-associative $\mathbb{C}$-algebras. But in the case of real Lie algebras we do not know a characterisation of the outcome—this is why we needed more ad hoc methods for alternatingly udeco three-tensors.

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## References

[AGH+14] A. Anandkumar, R. Ge, D. Hsu, S. M. Kakade and M. Telegarski, *Tensor decompositions for learning latent variable models*, Journal of Machine Learning Research **15** (2014), 2773–2832.

[AGJ14] A. Anandkumar, R. Ge and M. Janzamin, *Sample complexity analysis for learning overcomplete latent variable models through tensor methods*, Preprint, arXiv:1408.0553.

[BLW15] K. Batselier, H. Liu and N. Wong, *A constructive algorithm for decomposing a tensor into a finite sum of orthonormal rank-1 terms*, SIAM Journal on Matrix Analysis and Applications **36** (2015), 1315–1337.

[Com94] P. Comon, *Independent component analysis, a new concept?*, Signal Processing **36** (1994), 287–314.

[CS09] J. Chen and Y. Saad, *On the tensor SVD and the optimal low rank orthogonal approximation of tensors*, SIAM Journal on Matrix Analysis and Applications **30** (2009), 1709–1734.

[DDV00] L. De Lathauwer, B. De Moor and J. Vandewalle, *A multilinear singular value decomposition*, SIAM Journal on Matrix Analysis and Applications **21** (2000), 1253–1278.

[HL13] C. Hillar and L.-H. Lim, *Most tensor problems are NP hard*, Journal of the ACM **60** (2013), no. 6, Art. 45.

[Hop41] H. Hopf, *Ein topologischer Beitrag zur reellen Algebra*, Commentarii Mathematici Helvetici **13** (1941), 219–239.

[Kna02] A. W. Knapp, *Lie Groups Bbeyond an Introduction*, Progress in Mathematics, Vol. 140, Birkhäuser Boston, Boston, MA, 2002.

[Kol01] T. G. Kolda, *Orthogonal tensor decompositions*, SIAM Journal on Matrix Analysis and Applications **23** (2001), 243–255.

[Kol03] T. G. Kolda, *A counterexample to the possibility of an extension of the eckart-young low-rank approximation theorem for the orthogonal rank tensor decomposition*, SIAM Journal on Matrix Analysis and Applications **24** (2003), 762–767.

[Kol15] T. G. Kolda, *Symmetric orthogonal tensor decomposition is trivial*, Preprint, `arXiv:1503.01375`.

[Rob14] E. Robeva, *Orthogonal decomposition of symmetric tensors*, SIAM Journal on Matrix Analysis and Applications **37** (2016), 86–102.

[RV13] S. Ragnarsson and C. F. Van Loan, *Block tensors and symmetric embeddings*, Linear Algebra and its Applications **438** (2013), 853–874.

[Sno13] A. Snowden, *Syzygies of Segre embeddings and $\Delta$-modules*, Duke Mathematical Journal **162** (2013), 225–277.

[SRK09] J. Salmi, A. Richter and V. Koivunen, *Sequential unfolding svd for tensors with applications in array signal processing*, IEEE Transactions on Signal Processing **57** (2009), 4719–4733.

[SS15] S. V. Sam and A. Snowden, *Stability patterns in representation theory*, Forum of Mathematics. Sigma **3** (2015), e11.

[VNVM14] N. Vannieuwenhoven, J. Nicaise, R. Vandebril and K. Meerbergen, *On generic nonexistence of the Schmidt–Eckart–Young decomposition for complex tensors*, SIAM Journal on Matrix Analysis and Applications **35** (2014), 886–903.

[ZG01] T. Zhang and G. H. Golub, *Rank-one approximation to high order tensors*, SIAM Journal on Matrix Analysis and Applications **23** (2001), 534–550.