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On volumes of quasi-arithmetic hyperbolic lattices

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Abstract We prove that the covolume of any quasi-arithmetic hyperbolic lattice (a notion that generalizes the definition of arithmetic subgroups) is a rational multiple of the covolume of an arithmetic subgroup. As a corollary, we obtain a good description for the shape of the volumes of most of the known hyperbolic n-manifolds with $n > 3$.

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1 Introduction

1.1. Let $\mathbf{H}^n$ be the hyperbolic n-space, with group of isometries $G = \mathrm{PO}(n, 1)$. Let k be a number field with ring of integers $\mathcal{O}_k$. An absolutely simple adjoint algebraic k-group $\mathbf{G}$ will be called *admissible* (for G) if $\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong G \times K$, where K is compact (possibly trivial). Assuming $n \geq 4$ this forces k to be totally real, and we can fix an inclusion $k \subset \mathbb{R}$ such that $G = \mathbf{G}(\mathbb{R})$. By the theorem of Borel and Harish-Chandra, any subgroup $\Gamma_0 \subset \mathbf{G}(\mathbb{R})$ commensurable with $\mathbf{G}(\mathcal{O}_k)$ is a lattice in G. Such a subgroup is called *arithmetic*. Since we assume that $\mathbf{G}$ is adjoint, we have necessarily $\Gamma_0 \subset \mathbf{G}(k)$; see [9, Proposition 1.2].

Following Vinberg [25] we call *quasi-arithmetic* a lattice of G that is obtained as a subgroup of $\mathbf{G}(k)$ for $\mathbf{G}$ admissible. We call "properly quasi-arithmetic" such a lattice if it is not arithmetic. First examples were obtained by Vinberg [25], who considered reflection groups. The construction of Belolipetsky and Thomson [2] proves the

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existence of infinitely many commensurability classes of properly quasi-arithmetic hyperbolic lattices in any dimension $n > 2$; see Thomson [23].

Remark 1.1 Suppose that $\Gamma' \subset \mathbf{G}(\mathbb{R})$ is commensurable with the quasi-arithmetic lattice $\Gamma \subset \mathbf{G}(k)$. Then it follows from the work of Vinberg [26] that $\Gamma' \subset \mathbf{G}(k)$ (using the fact that $\mathbf{G}$ is adjoint). In particular, this shows that our definition of quasi-arithmeticity coincides with the one from [25] and [23].

Remark 1.2 It follows from Weil's local rigidity theorem that any lattice $\Gamma \subset G$ can be embedded in $\mathbf{G}(k)$ for k some number field and $\mathbf{G}$ some k-group such that $\mathbf{G}(\mathbb{R}) = G$ (see [28, Ch. 1, Sect. 6.2]). However, in general $\mathbf{G}$ does not have to be admissible.

1.2. It is clear from the definition that the covolume of an arithmetic subgroup $\Gamma_0 \subset \mathbf{G}(k)$ is commensurable with the covolume of $\mathbf{G}(\mathcal{O}_k)$ (as lattices in $\mathbf{G}(\mathbb{R})$). In this paper we prove that this holds for any quasi-arithmetic lattice $\Gamma \subset \mathbf{G}(k)$.

Theorem 1.3 *Let $\Gamma \subset \mathbf{G}(k)$ be a quasi-arithmetic lattice. Then the covolume of Γ is a rational multiple of the covolume of $\mathbf{G}(\mathcal{O}_k)$.*

For n even this is an obvious consequence of the generalized Gauss-Bonnet formula. However, we obtain the result as a consequence of Theorem 1.8 below, which is of interest for even dimensions as well. In this stronger form our theorem also has application to the study of arithmetic lattices (see Corollary 1.9). Moreover—and despite the seemingly particular nature of properly quasi-arithmetic lattices—we will show how this notion permits to better understand both arithmetic lattices (Corollary 1.5) and non-quasi-arithmetic lattices (Sect. 1.4).

Remark 1.4 For $n = 3$ the result stated in Theorem 1.3 appears as a particular consequence of the known theory about the Bloch invariant; see [18] and [17, Section 12.7]. Very briefly, for a hyperbolic 3-manifold M its *Bloch invariant* $\beta(M)$ takes value in the *Bloch group* $\mathcal{B}(k) \otimes \mathbb{Q}$, which has dimension 1 when $\mathbf{G}$ is admissible (in this case the *invariant trace field* k must have exactly one complex place). The result then follows by applying the Borel regulator on $\beta(M)$ (which gives the volume). The theory has been generalized for higher dimensions [18, Section 8], however with $\beta(M)$ taking values in higher (pre-)Bloch groups of $\mathbb{C}$ (or $\overline{\mathbb{Q}}$)—instead of k. This does not permit direct volume comparisons, the vector spaces involved having infinite dimensions. See also Goncharov [13], who defines a similar invariant in the K-theory groups $K_n(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$.

The covolume of $\mathbf{G}(\mathcal{O}_k)$ can be computed up to a rational in terms of invariants of $\mathbf{G}$ and k (see Ono [19]; Prasad [20]). We discuss in Sect. 2 the particular case of nonuniform lattices. We also include there numerical comparisons for two properly quasi-arithmetic lattices—two reflection groups in dimensions 5 and 7—that illustrates Theorem 1.3.

1.3. As a corollary of Theorem 1.3 we obtain the following result. We do not know if it can be obtained by a more direct proof—even in the arithmetic case.

Corollary 1.5 *Let $M = \Gamma\backslash\mathbf{H}^n$ be an orientable hyperbolic manifold with $\Gamma \subset \mathbf{G}(k)$ quasi-arithmetic. Suppose that M contains a totally geodesic (connected) separating* *hypersurface S of finite volume, and let $M^+ \subset M$ be one of the two parts delimited by S. We denote by $H \subset \mathbf{H}^n$ the hyperplane that covers S, and we suppose that the reflection through H belongs to $\mathbf{G}(k)$. Then* $\mathrm{vol}(M^+)$ *is a rational multiple of* $\mathrm{vol}(M)$.

Proof Let us write $\Gamma^+ \subset \Gamma$ for the subgroup corresponding to the fundamental group of M^+. We consider the manifold V that is obtained by gluing two copies of M^+ along the boundary S. We have that V is a complete hyperbolic manifold that can be written as $V = \Lambda\backslash\mathbf{H}^n$, where $\Lambda \subset \mathbf{G}(\mathbb{R})$ is the subgroup generated by $\Gamma^+ \cup g\Gamma^+ g^{-1}$ and g is the reflection through H. Obviously V has finite volume, so that $\Lambda \subset \mathbf{G}(k)$ is a quasi-arithmetic lattice. By Theorem 1.3 we conclude that $\mathrm{vol}(V) = 2\,\mathrm{vol}(M^+)$ is a rational multiple of $\mathrm{vol}(M)$. □

1.4. The following list covers all the *currently known* hyperbolic lattices for $n > 3$ (up to commensurability). We indicate the relevant information about their volumes (for n odd):

1. Arithmetic lattices, for which the volumes are precisely computed up to rationals (see [19,20] and Sect. 2).
2. Quasi-arithmetic lattices; Theorem 1.3 shows that up to commensurability their covolumes are the same as in (1).
3. Lattices that come from the *interbreeding* constructions (see Gromov and Piateski-Shapiro [14], and generalizations [12,21]); note that those are not quasi-arithmetic [23, Theorem 1.6]. From their construction and Corollary 1.5 we obtain that their volumes are rational linear combinations of volumes from (1), in any case (the result being already clear when the construction only involves nonseparating hypersurfaces).
4. Non-quasi-arithmetic reflection groups; we do not have any information about what shape their volumes can take in general. Note however that some of these groups are obtained by the interbreeding construction; see for instance Vinberg [27].

Remark 1.6 For $n = 3$ hyperbolic manifolds can be obtained by performing Dehn filling on link complements, and this possibly provides lattices that are not of any of the types listed above.

1.5. The idea of the proof of Theorem 1.3 is easily summarized in the case when Γ is uniform. Up to passing to a subgroup of finite index, we may assume that Γ is torsion-free and contained in the connected component G°. Then $M = \Gamma\backslash\mathbf{H}^n$ is a compact orientable manifold, and this provides a "fundamental class" $[\Gamma] \in H_n(\Gamma) \cong H_n(M) \cong \mathbb{Z}$ (which corresponds to the generator with the same orientation as $\mathbf{H}^n$). Denoting by $j : \Gamma \to G$ the inclusion map, the result follows immediately from the two following observations:

1. the induced map j_* on group homology factors as $H_n(\Gamma) \to H_n(\mathbf{G}(k)) \to H_n(G)$, and the middle term is known to have rank one; see Proposition 4.2.
2. there is a linear map $v : H_n(G) \to \mathbb{R}$ (independent of Γ) such that $v(j_*([\Gamma])) = \mathrm{vol}(M)$; see Sect. 3.

1.6. The main effort of this article is to present a proof that includes the more difficult case of nonuniform lattices. We first need to recall a convenient way to consider the fundamental class in the general case. We refer to Sect. 3 for details. Let $\Omega = \partial \mathbf{H}^n$ be the geometric boundary of $\mathbf{H}^n$, endowed with the G-action. For a subgroup $S \subset G$, we define its homology *relative to* Ω by

$$H_n(S, \Omega) = H_{n-1}(S, J\Omega), \tag{1.1}$$

where $J\Omega$ is the kernel of the augmentation map $\mathbb{Z}\Omega \to \mathbb{Z}$ (in particular we can see $J\Omega$ as an S-module). For any torsion-free lattice $\Gamma \subset G^\circ$ we have $H_n(\Gamma, \Omega) \cong \mathbb{Z}$, and we denote by $[\Gamma] \in H_n(\Gamma, \Omega)$ the generator corresponding to the positive orientation (see Definition 3.3 and Remark 3.4). For Γ uniform there is a canonical isomorphism $H_n(\Gamma, \Omega) \cong H_n(\Gamma)$ and we recover the usual notion of the fundamental class.

For any lattice $\Gamma \subset G$ we consider the map $j_* : H_n(\Gamma, \Omega) \to H_n(G, \Omega)$ on the relative homology, induced by the inclusion $j : \Gamma \to G$. The following result implicitly appears in Neumann and Yang [18, Sect. 3–4]. We discuss the proof in Sect. 3.

Proposition 1.7 *There exists a linear map* $v : H_n(G, \Omega) \to \mathbb{R}$ *such that for any torsion-free lattice* $\Gamma \subset G^\circ$ *one has* $v(j_*([\Gamma])) = \mathrm{vol}(\Gamma \backslash \mathbf{H}^n)$.

1.7. We will write $\mathbf{G}(k)^+$ for the intersection $\mathbf{G}(k) \cap G^\circ$. After passing to a finite index torsion-free subgroup, Theorem 1.3 is a direct consequence of the following result, together with Proposition 1.7.

Theorem 1.8 *Let* $\mathbf{G}$ *be an admissible k-group. There exists a rank one* $\mathbb{Z}$*-submodule* $L \subset H_n(G, \Omega)$ *such that* $j_*([\Gamma]) \in L$ *for any torsion-free quasi-arithmetic lattice* $\Gamma \subset \mathbf{G}(k)^+$.

The proof of Theorem 1.8 is established in Sects. 5, 6. In view of Remark 1.4, we will assume $n > 3$ in the proofs. This permits to use a uniform notation (the main difference for $n = 3$ is that the field of definition k is not totally real).

1.8. One feature of our approach is that the fundamental classes are compared in a $\mathbb{Z}$-module, namely $L \subset H_n(G, \Omega)$. This contrasts with the Bloch invariant approach, which considers $\mathbb{Q}$-vector spaces (cf. Remark 1.4). To illustrate the advantage in doing so, we note the following simple corollary concerning the distribution of covolumes within a commensurability class of arithmetic subgroups. Recall that such a class contains infinitely many maximal subgroups (see [9, Prop. 1.4 (iv)]).

Corollary 1.9 *Let* $\mathbf{G}$ *be an admissible k-group. There exists a number* $c > 0$ *such that for any arithmetic subgroup* $\Gamma \subset \mathbf{G}(k)$ *we have that* $\mathrm{vol}(\Gamma \backslash \mathbf{H}^n)$ *is an integral multiple of c.*

Proof Since $\mathbf{G}$ has trivial center, we have that $\mathbf{G} \cong \mathrm{Ad}\,\mathbf{G}$ can be identified as a subgroup of $\mathrm{GL}(\mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra (defined over k) of $\mathbf{G}$. Using Weil's restriction of scalars, we can then identify $\mathbf{G}(k)$ with the rational points of an algebraic $\mathbb{Q}$-subgroup $H \subset \mathrm{GL}(\mathfrak{g}_0)$, where $\mathfrak{g}_0$ is a $\mathbb{Q}$-form of $\mathfrak{g}$. Under this identification, any arithmetic subgroup $\Gamma \subset \mathbf{G}(k)$ corresponds to an arithmetic subgroup (defined over $\mathbb{Q}$) of $H(\mathbb{Q})$. In particular (see [28, Ch. 1 Prop. 7.2]), Γ stabilizes a $\mathbb{Z}$-lattice $L \subset \mathfrak{g}_0$, and choosing a basis of L we can embed $\Gamma \subset \mathrm{GL}_m(\mathbb{Z})$. For a fixed field k, this integer m is independent of the arithmetic subgroup $\Gamma \subset \mathbf{G}(k)$.

From Minkowski's lemma we thus have an upper bound $A > 0$ such that any arithmetic lattice $\Gamma \subset \mathbf{G}(k)$ contains a torsion-free subgroup $\Gamma_0 \subset \Gamma$ of index $[\Gamma : \Gamma_0] \leq A$. Moreover, by doubling A we ensure that $\Gamma_0 \subset \mathbf{G}(k)^+$. By Theorem 1.8 and Proposition 1.7 there exists $c' > 0$ such that for all these subgroups Γ_0 we have $\mathrm{vol}(\Gamma_0 \backslash \mathbf{H}^n) \in c'\mathbb{Z}$. The result follows by choosing for c the number c' divided by the lowest multiple common to all integers $\leq A$. □

In dimension $n = 3$ this result was obtained by Borel [7], as a consequence of his volume formula; it answered positively a question of Thurston [24, 6.7.6] (the case of nonarithmetic lattices being solved by the commensurator theorem of Margulis). The work of Borel and Prasad [9], which relies on the volume formula [20], provides a lot of information about the volume distribution for arithmetic lattices in very generic situations—including the case of $\mathrm{PO}(n, 1)$. However it does not seem that Corollary 1.9 could be easily obtained from their results.

Remark 1.10 Our result also shows that all covolumes of the quasi-arithmetic *torsion-free* lattices $\Gamma \subset \mathbf{G}(k)$ are integral multiples of a single number. However it is not clear if this holds true for lattices containing torsion.

2 Volume computations for the nonuniform case

2.1. Let $\mathbf{G}$ be an admissible k-group for $G = \mathrm{PO}(n, 1)$, and suppose that $\Gamma \subset \mathbf{G}(k)$ is a nonuniform lattice. Then Γ must contain some nontrivial unipotent elements, which means that $\mathbf{G}$ is isotropic. For $n > 3$ this is only possible if $k = \mathbb{Q}$, and $\mathbf{G}$ corresponds to the adjoint group of $\mathrm{SO}(f)$ for f a quadratic form over $\mathbb{Q}$ (cf. [16, Lemma 2.2]).

From now on suppose that n is odd, with $n = 2m - 1$. Let us define

$$\delta = (-1)^m \mathrm{disc}(f), \tag{2.1}$$

where $\mathrm{disc}(f) \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is the discriminant of f, and consider the field $\ell = \mathbb{Q}(\sqrt{\delta})$. Let D_ℓ be its discriminant, and ζ_ℓ its Dedekind zeta function.

2.2. For a certain natural normalization of the Haar measure on a semisimple Lie group G, the covolume of any arithmetic subgroup of G can be obtained up to a rational by using Prasad's volume formula [20] (the formula also permits precise computations in many cases). Its application in the case $G = \mathrm{PO}(n, 1)$ (n odd) is worked out for instance in [1, Sect. 2.6–7]. The difference between Prasad's normalization of the measure and the hyperbolic volume on $\mathbf{H}^n$ is explained in [1, Sect. 2.1]. Together with Theorem 1.3 one can deduce the value $\mathrm{vol}(\Gamma \backslash \mathbf{H}^n)$ up to a rational for any quasi-arithmetic lattice $\Gamma \subset G$. We give in the following proposition the values for the case $k = \mathbb{Q}$.

Proposition 2.1 *Let* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *be a nonuniform quasi-arithmetic lattice of* $\mathrm{PO}(n,1)$ *with* $n \geq 5$ *odd, and let* $\ell/\mathbb{Q}$ *be the field extension defined above in Sect. 2.1.*

(1) If $\ell = \mathbb{Q}$, *then the covolume of* Γ *is a rational multiple of the Riemann zeta function evaluated at* $m = \frac{n+1}{2}$*:*

$$\mathrm{vol}(\Gamma\backslash\mathbf{H}^n) \ \in \ \zeta(m)\cdot\mathbb{Q}^\times;$$

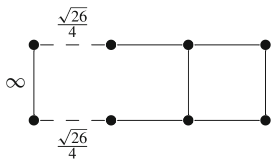
(2) otherwise we have

$$\mathrm{vol}(\Gamma\backslash\mathbf{H}^n) \ \in \ |D_\ell|^{n/2}\cdot\frac{\zeta_\ell(m)}{\zeta(m)}\cdot\mathbb{Q}^\times.$$

Remark 2.2 The quotient ζ_ℓ/ζ might alternatively be described as a Dirichlet L-function.

Remark 2.3 Prasad's formula also provides similar volume formulas in the compact case, i.e., for $k \neq \mathbb{Q}$. In this case ζ is to be replaced by ζ_k, and ℓ is a quadratic extension of k (except for the special case of "triality forms"). Both discriminants D_ℓ and D_k then appear in the formula.

2.3. We consider the 5-dimensional hyperbolic polytope $P_5 \subset \mathbf{H}^5$ that corresponds to the Coxeter diagram given in (2.2) (see [25, Sect. 4] for the notation). Let us denote by $\Delta_5 \subset \mathrm{PO}(5,1)$ the discrete subgroup generated by the reflections through the hyperplanes delimiting P_5.



(2.2)

This polytope appears in the list obtained by Mike Roberts in [22], which contains many new examples of hyperbolic Coxeter polytopes of finite volume. The finiteness of $\mathrm{vol}(P_5)$ implies that Δ_5 is a lattice – nonuniform since P_5 is noncompact. Using a geometric integration Steve Tschantz has computed the following numerical approximation for the volume of the polytope P_5:

$$\mathrm{vol}(P_5) \approx 0.024133068794582269990. \tag{2.3}$$

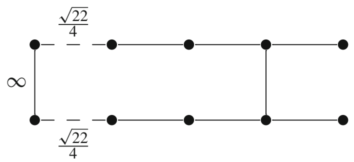
2.4. The Gram matrix of P_5 can be immediately deduced from the diagram (2.2). From this matrix one can obtain (with the procedure used in the proof of [25, Theorem 2]) a basis $\{u_i\}$ of the Minkowski space $\mathbb{R}^{5,1}$ for which $(u_i, u_j) \in \mathbb{Q}$ and such that each u_i is orthogonal to a hyperplane delimiting P_5. With this one easily checks that Δ_5 embeds as a subgroup of $\mathrm{O}(f,\mathbb{Q})$ for f some quadratic form of discriminant $\mathrm{disc}(f) = -13(\mathbb{Q}^\times)^2$. In particular Δ_5 is quasi-arithmetic; indeed, *properly* quasi-arithmetic by applying [25, Theorem 2]. From Proposition 2.1 we conclude that

$\mathrm{vol}(\Delta_5\backslash\mathbf{H}^5) = \mathrm{vol}(P_5)$ is a rational multiple of $13^{5/2}\zeta_\ell(3)/\zeta(3)$, where $\ell = \mathbb{Q}(\sqrt{13})$. Numerical comparison with (2.3) (we use Pari/GP to evaluate the zeta functions) then suggests the equality

$$\mathrm{vol}(\Delta_5\backslash\mathbf{H}^5) = \frac{1}{23040} \cdot 13^{5/2} \cdot \frac{\zeta_\ell(3)}{\zeta(3)}. \tag{2.4}$$

Remark 2.4 The numerical match between (2.4) and (2.3), together with the simplicity of the rational factor (note that $23040 = 2^9 \cdot 3^2 \cdot 5$), leave little doubt for the correctness of (2.4). However, at this point we do not see any way to give a rigorous proof of this equality.

2.5. Let $P_7 \subset \mathbf{H}^7$ be the hyperbolic Coxeter polytope with the diagram:



(2.5)

As well as P_5 this polytope was found in [22]. We proceed as in Sect. 2.4: the corresponding reflection group $\Delta_7 \subset \mathrm{PO}(7, 1)$ is properly quasi-arithmetic, defined as a subgroup of an algebraic $\mathbb{Q}$-group determined by a quadratic form with discriminant -11. Steve Tschantz has computed the following approximation:

$$\begin{aligned} \mathrm{vol}(P_7) &\approx 0.000181338 \\ &\approx \frac{11^{7/2}}{2^{13} \cdot 3^4 \cdot 5 \cdot 7} \cdot \frac{\zeta_\ell(4)}{\zeta(4)}, \end{aligned} \tag{2.6}$$

with $\ell = \mathbb{Q}(\sqrt{-11})$. Again, this agrees with Proposition 2.1.

3 Fundamental class and volume

The purpose of this section is to prove Proposition 1.7, mostly by following the approach of Neumann and Yang [18, Sect. 3–4] (see also Kuessner [15]). In the following text $M = \Gamma\backslash\mathbf{H}^n$ denotes a finite-volume orientable hyperbolic manifold. As in Sect. 1, we denote by Ω the geometric boundary of $\mathbf{H}^n$.

3.1. A point $x \in \Omega$ is a *cusp* of Γ if it is the fixed point of a parabolic element of Γ. Let $\mathcal{C} \subset \Omega$ be the set of cusps of Γ. When M is compact then $\mathcal{C}$ is empty. Let Z be the end compactification of M, i.e., Z is obtained by adjoining a point c_i to each of the (finitely many) cusps of M. We consider a triangulation of Z, and we suppose (as we may) that each c_i is a vertex of this triangulation. This triangulation lifts to the covering space $X = \mathbf{H}^n \cup \mathcal{C}$. We denote by $C_\bullet(Z)$ the chain complex defined by the triangulation of Z, and by $C_\bullet(X)$ the chain complex of its lift. Then $C_\bullet(X)$ is a $\mathbb{Z}\Gamma$-complex, and $C_\bullet(X)_\Gamma = C_\bullet(Z)$. In particular,

$$H_n(C_\bullet(X)_\Gamma) = H_n(Z) \cong \mathbb{Z}. \tag{3.1}$$

3.2. For a free $\mathbb{Z}$-module with basis S, we denote by JS the kernel of the augmentation map $\mathbb{Z}S \to \mathbb{Z}$, which by definition sends any $x \in S$ to 1. Let $X_0 \subset X$ denotes the set of vertices of the lifted triangulation. Then X_0 is a $\mathbb{Z}$-basis of $C_0(X)$, and we have that JX_0 is a Γ-submodule of $C_0(X)$.

Lemma 3.1 *The Γ-module JX_0 is isomorphic to $J\mathcal{C} \oplus F$, where F is some free Γ-module.*

Proof The group Γ acts freely on $X_0 \cap \mathbf{H}^n$. In particular, if $\mathcal{C}$ is empty then JX_0 is a free $\mathbb{Z}\Gamma$-module. When $\mathcal{C}$ is not empty it suffices to take as F the submodule generated by elements of the form $x - a$, with $x \in X_0 \cap \mathbf{H}^n$ and $a \in \mathcal{C}$. □

By definition, the relative homology $H_n(\Gamma, \mathcal{C})$ is $H_{n-1}(\Gamma, J\mathcal{C})$. We then have the following.

Proposition 3.2 $H_n(\Gamma, \mathcal{C}) = H_n(C_\bullet(X)_\Gamma)$.

Proof Since X is contractible and Γ acts freely on $\mathbf{H}^n$, we have that $C_{\bullet \geq 1}(X)$ is a free $\mathbb{Z}\Gamma$-resolution of JX_0. Thus $H_{n-1}(\Gamma, JX_0) = H_n(C_\bullet(X)_\Gamma)$, and the former equals $H_{n-1}(\Gamma, J\mathcal{C})$ by Lemma 3.1. □

The proposition, together with (3.1), justifies the following.

Definition 3.3 For Γ as above, we define its *fundamental class* $[\Gamma]$ to be the generator of $H_n(\Gamma, \mathcal{C})$ that can by represented by a sum of positively oriented simplices from $C_n(X)$.

Remark 3.4 The natural map $H_n(\Gamma, \mathcal{C}) \to H_n(\Gamma, \Omega)$ is an isomorphism (see [15, Lemma 2.2.5]). This justifies an option to take as an equivalent definition for $[\Gamma]$ the (positively oriented) generator of $H_n(\Gamma, \Omega)$, as we did in Sect. 1.6.

3.3. We consider the free $\mathbb{Z}$-module $S_j(\Omega)$ that is generated by the $(j+1)$-tuples $(x_0, \dots, x_j)$ of distinct elements in Ω modulo the relations

$$(x_0, \dots, x_j) = \operatorname{sgn}(\sigma)(x_{\sigma(0)}, \dots, x_{\sigma(j)}),$$

for any permutation σ. Geometrically, these generators correspond to the ideal geodesic simplices in $\mathbf{H}^n$ (with orientation). With the standard boundary map, the chain complex $S_\bullet(\Omega)$ gives a resolution of $\mathbb{Z}$. Moreover, the isometry group G acts on this complex, so that $S_{\bullet \geq 1}(\Omega)$ is a (non-free) $\mathbb{Z}G$-resolution of $J\Omega$.

Let τ denote the inclusion $J\mathcal{C} \to J\Omega$. Since $S_{\bullet \geq 1}(\Omega)$ is acyclic, for any free $\mathbb{Z}\Gamma$-resolution $D_\bullet \to J\mathcal{C}$ the map τ extends uniquely up to homotopy to a $\mathbb{Z}\Gamma$-chain complex map (see [11, Lemma I.74]):

$$\begin{array}{ccc} D_\bullet & \longrightarrow & J\mathcal{C} \\ \downarrow & & \downarrow \tau \\ S_{\bullet \geq 1}(\Omega) & \longrightarrow & J\Omega \end{array} \tag{3.2}$$

In particular, this induces a canonical map $\tau_* : H_n(\Gamma, \mathcal{C}) \to H_n(S_\bullet(\Omega)_\Gamma)$. For $D_\bullet = C_{\bullet \geq 1}(X)$, the chain complex map may be explicitly given as follows. Take a set of Γ-representatives of points of $X_0 \cap \mathbf{H}^n$ and send them to arbitrarily chosen distinct points in $\Omega \setminus \mathcal{C}$. Obviously such a choice determines uniquely a $\mathbb{Z}\Gamma$-map $C_j(X) \to S_j(\Omega)$ for any $j \geq 1$.

3.4. Let $\nu : S_n(\Omega) \to \mathbb{R}$ be the linear map that assigns to any n-simplex its signed hyperbolic volume. Then ν is zero on boundary elements: if $b = \partial c$ for some $c \in S_{n+1}(\Omega)$ then $\nu(b) = 0$. Moreover, ν is obviously G-invariant, so that for any subgroup $A \subset G$ we obtain an induced map $\nu_* : H_n(S_\bullet(\Omega)_A) \to \mathbb{R}$. For the case $A = \Gamma$, we can state the following (see [18, end of the proof of Lemma 4.2]).

Proposition 3.5 *We have* $\nu_*(\tau_*([\Gamma])) = \mathrm{vol}(M)$.

3.5. Consider $J\Omega$ as a $\mathbb{Z}G$-module, and let ι denote the identity $J\Omega \to J\Omega$. Similarly as for τ, it induces a canonical map $\iota_* : H_n(G, \Omega) \to H_n(S_\bullet(\Omega)_G)$. Since τ agrees with ι on its domain of definition, we have that the left square in the following diagram is commutative. That the right square is also commutative is obvious. Recall that $j : \Gamma \to G$ denotes the inclusion.

$$\begin{array}{ccccc} H_n(\Gamma, \mathcal{C}) & \xrightarrow{\tau_*} & H_n(S_\bullet(\Omega)_\Gamma) & \xrightarrow{\nu_*} & \mathbb{R} \\ \downarrow{\scriptstyle j_*} & & \downarrow & & \downarrow{\scriptstyle id} \\ H_n(G, \Omega) & \xrightarrow{\iota_*} & H_n(S_\bullet(\Omega)_G) & \xrightarrow{\nu_*} & \mathbb{R}. \end{array} \tag{3.3}$$

Proof of Proposition 1.7 Set $v = \nu_* \circ \iota_*$. Then the result follows by combining Proposition 3.5 with the fact that the diagram 3.3 is commutative. □

4 Homology of algebraic groups

4.1. We need to recall the following theorem, which appears in [5, Prop. XIII.3.9] for the anisotropic case, and in [10, Theorem 2.1] for $\widetilde{\mathbf{G}}$ isotropic. Its proof combines the work of Borel, Garland, Yang, and uses the work of Blasius et al. [4] in the isotropic case. Here $H^\bullet_{\mathrm{ct}}$ denotes the continuous cohomology.

Theorem 4.1 *Let $\widetilde{\mathbf{G}}$ be a simply connected absolutely simple k-group. The natural map*

$$H^\bullet_{\mathrm{ct}}(\widetilde{\mathbf{G}}(k \otimes_\mathbb{Q} \mathbb{R}), \mathbb{R}) \to H^\bullet(\widetilde{\mathbf{G}}(k), \mathbb{R}) \tag{4.1}$$

is an isomorphism.

4.2. Let $\widetilde{G} = \widetilde{\mathbf{G}}(k \otimes_\mathbb{Q} \mathbb{R})$, for $\widetilde{\mathbf{G}}$ as above. We have that $\widetilde{G}$ is connected. It is known that

$$H^\bullet_{\mathrm{ct}}(\widetilde{G}, \mathbb{R}) = H^\bullet(X_u, \mathbb{R}), \tag{4.2}$$

where X_u denotes the *compact dual* symmetric space associated with $\widetilde{G}$; see Borel [6, Sect. 10.2]. We can now prove the following.

Proposition 4.2 *Let* $\mathbf{G}$ *be an admissible* k*-group for* $\mathrm{PO}(n, 1)$*. Then* $H_n(\mathbf{G}(k))$ *has rank one.*

Proof Let $\widetilde{\mathbf{G}}$ be the simply connected (algebraic) cover of $\mathbf{G}$, and denote by π the covering map $\widetilde{\mathbf{G}} \to \mathbf{G}$ and by C the center of $\widetilde{\mathbf{G}}$. From Galois cohomology we have an exact sequence

$$1 \to \widetilde{\mathbf{G}}(k)/C(k) \xrightarrow{\pi} \mathbf{G}(k) \to A \to 1, \tag{4.3}$$

where A is defined as the kernel of the map $H^1(k, C) \to H^1(k, \widetilde{\mathbf{G}})$. The Lyndon–Hochschild–Serre spectral sequence (see [11, Sect. VII.6]) applied to (4.3) takes the form

$$E^2_{pq} = H_p(A, H_q(\widetilde{\mathbf{G}}(k)/C(k), \mathbb{R})) \Rightarrow H_{p+q}(\mathbf{G}(k), \mathbb{R}). \tag{4.4}$$

In all cases we have that $H^1(k, C)$ is torsion abelian of finite exponent. Thus A may be written as a direct limit of finite groups, and it follows by exchanging homology and direct limit that $H_p(A, -)$ is zero in (4.4) unless $p = 0$. This shows that $H_n(\mathbf{G}(k), \mathbb{R})$ has the same dimension as $H_n(\widetilde{\mathbf{G}}(k)/C(k), \mathbb{R})$. Moreover, since C is finite, the spectral sequence

$$E^2_{pq} = H_p(\widetilde{\mathbf{G}}(k)/C(k), H_q(C(k), \mathbb{R})) \Rightarrow H_{p+q}(\widetilde{\mathbf{G}}(k), \mathbb{R}) \tag{4.5}$$

further shows that this dimension equals $\dim(H_n(\widetilde{\mathbf{G}}(k), \mathbb{R}))$. For $\mathbf{G}$ admissible we have that the compact dual symmetric space X_u of $\widetilde{\mathbf{G}}(k \otimes_{\mathbb{Q}} \mathbb{R})$ has the same dimension as $\mathbf{H}^n$ (more precisely, X_u is the n-sphere). It then follows from (4.1) and (4.2) that $H_n(\mathbf{G}(k))$ has rank one. □

5 Algebraic structure at cusps

5.1. Let $G = \mathrm{PO}(n, 1)$, and take a point x on the boundary Ω of $\mathbf{H}^n$. Using the upper half space model with $x = \infty$ and the description of its isometry group as conformal maps (cf. for instance [3, Ch. A]) one sees that the stabilizer G_x decomposes as a product

$$G_x = U \cdot A \cdot S, \tag{5.1}$$

where

- $U \cong \mathbb{R}^{n-1}$ corresponds to the horospherical translations fixing x;
- $A \cong \mathbb{R}_{>0}$ corresponds to the homotheties centered at 0;
- and $S \cong \mathrm{O}(n)$ is the rotation group around the axis $(0, x)$.

5.2. Let $\mathbf{G}$ be an admissible k-group, so that $\mathbf{G}(\mathbb{R}) = G = \mathrm{PO}(n, 1)$. We suppose that $n > 3$. Let $\Gamma \subset \mathbf{G}(k)$ be a torsion-free lattice, and suppose that $x \in \Omega$ is a cusp of Γ. In particular, $\mathbf{G}$ must be isotropic with $k = \mathbb{Q}$ (cf. Sect. 2.1). The "cusp subgroup" Γ_x acts discretely and cocompactly on $\mathbb{R}^{n-1}$. By Bieberbach theorem there exists a normal finite index subgroup $\Gamma'_x \subset \Gamma_x$ that consists only of horospherical translations. In the notation of Sect. 5.1: $\Gamma'_x \subset U$, with $\Gamma'_x \cong \mathbb{Z}^{n-1}$. We may assume that Γ'_x is maximal abelian. Then there are only finitely many possibilities for the finite group Γ_x / Γ'_x (see [28, Ch. 4 Sect. 1.1]). It follows that there exists an integer N such that $[\Gamma_x : \Gamma'_x]$ divides N for every cusp x of Γ.

5.3. Let $\mathbf{U}$ be the Zariski closure of Γ'_x in $\mathbf{G}$. By the construction, $\mathbf{U}$ is a $\mathbb{Q}$-subgroup with $\mathbf{U}(\mathbb{R}) = U$. Let $\mathbf{P}$ be the normalizer of $\mathbf{U}$ in $\mathbf{G}$. This is a parabolic subgroup defined over $\mathbb{Q}$, with $\mathbf{P}(\mathbb{R}) = G_x$. We have a Levi decomposition $\mathbf{P} = \mathbf{U} \cdot \mathbf{L}$, where $\mathbf{L}$ is a reductive $\mathbb{Q}$-group. We denote by $\mathbf{Z}$ the connected component of the center of $\mathbf{L}$. By the construction it is a torus defined over $\mathbb{Q}$.

Proposition 5.1 *The $\mathbb{Q}$-torus $\mathbf{Z}$ is one-dimensional and split. It acts by conjugation on $\mathbf{U}$ as follows (for $g \in \mathbf{Z}$, $b \in \mathbf{U}$):*

$$gbg^{-1} = \lambda(g)b, \tag{5.2}$$

where λ is a nontrivial $\mathbb{Q}$-character of $\mathbf{Z}$.

Proof The decomposition (5.1) for $\mathbf{P}(\mathbb{R})$ shows that $\mathbf{Z}(\mathbb{R}) \cong \mathbb{R}^\times$, explicitly given by the group A extended by the rotation $-I \in S$. This shows the existence of a nontrivial $\mathbb{R}$-isomorphism λ of $\mathbf{Z}$ such that (5.2) holds for any $g \in \mathbf{Z}$ and $b \in \mathbf{U}$. But by construction $\mathbf{Z}$ is a $\mathbb{Q}$-group that normalizes $\mathbf{U}$, so that for $g \in \mathbf{Z}(\mathbb{Q})$ and $b \in \mathbf{U}(\mathbb{Q})$ we must have $\lambda(g)b \in \mathbf{U}(\mathbb{Q})$. This forces $\lambda(g) \in \mathbb{Q}^\times$. Since $\mathbf{Z}(\mathbb{Q})$ is Zariski-dense in $\mathbf{Z}$ (see [8, Cor. 18.3]) we conclude that λ is defined over $\mathbb{Q}$, and so $\mathbf{Z}$ is $\mathbb{Q}$-split. □

Lemma 5.2 *Under the map induced by the inclusion, $H_{n-1}(\mathbf{U}(\mathbb{Q}))$ has trivial image in $H_{n-1}(\mathbf{P}(\mathbb{Q}))$.*

Proof The conjugation induces a trivial action of $\mathbf{P}(\mathbb{Q})$ on its homology $H_{n-1}(\mathbf{P}(\mathbb{Q}))$ (see [11, Prop. II.6.2]), so that the map $H_{n-1}(\mathbf{U}(\mathbb{Q})) \to H_{n-1}(\mathbf{P}(\mathbb{Q}))$ factors as follows:

$$\begin{array}{ccc} H_{n-1}(\mathbf{U}(\mathbb{Q})) & \longrightarrow & H_{n-1}(\mathbf{U}(\mathbb{Q}))_{\mathbf{P}(\mathbb{Q})} \\ & \searrow & \downarrow \\ & & H_{n-1}(\mathbf{P}(\mathbb{Q})), \end{array} \tag{5.3}$$

where $H_{n-1}(\mathbf{U}(\mathbb{Q}))_{\mathbf{P}(\mathbb{Q})}$ denotes the module of co-invariants. Since $\mathbf{U}(\mathbb{Q}) \cong \mathbb{Q}^{n-1}$ we have $H_{n-1}(\mathbf{U}(\mathbb{Q})) = \mathbb{Q}$ (see [11, Th. V.6.4]), and the action of $\mathbf{Z}$ on $H_{n-1}(\mathbf{U}(\mathbb{Q}))$ induced by conjugation is explicitly given by

$$(g, u) \mapsto (\lambda(g))^{n-1} \cdot u.$$

It easily follows that the module $H_{n-1}(\mathbf{U}(\mathbb{Q}))_{\mathbf{P}(\mathbb{Q})}$ is trivial. □

5.4. For a cusp $x \in \Omega$ of a torsion-free Γ, we have that $H_{n-1}(\Gamma_x) \cong \mathbb{Z}$ (since $\Gamma_x \backslash \mathbb{R}^{n-1}$ is a compact manifold). Let us denote by $[\Gamma_x] \in H_{n-1}(\Gamma_x)$ a generator.

Proposition 5.3 *Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a nonuniform quasi-arithmetic lattice. There exists an integer N such that for any cusp x of Γ the image of $[\Gamma_x]$ in $H_{n-1}(\mathbf{P}(\mathbb{Q}))$ is annihilated by N.*

Proof Let $\Gamma'_x \cong \mathbb{Z}^{n-1}$ be the maximal abelian subgroup of Γ_x. The natural map $H_{n-1}(\Gamma'_x) \to H_{n-1}(\Gamma_x)$ corresponds to $\mathbb{Z} \to \mathbb{Z}$ with $1 \mapsto s$, where $s = [\Gamma_x : \Gamma'_x]$ is the index. Thus a generator of $H_{n-1}(\Gamma'_x)$ is mapped to $s \cdot [\Gamma_x] \in H_{n-1}(\Gamma_x)$. But by construction $\Gamma'_x \subset \mathbf{U}(\mathbb{Q})$, and applying Lemma 5.2 we see that $s \cdot [\Gamma_x]$ has zero image in $H_{n-1}(\mathbf{P}(\mathbb{Q}))$. Thus choosing N as in Sect. 5.2 the result follows. □

6 Conclusion of the proof

6.1. Let $\Gamma \subset \mathbf{G}(k)^+$ be a torsion-free quasi-arithmetic lattice. The $\mathbf{G}(k)$-module $J\Omega$ fits into the following exact sequence:

$$0 \to J\Omega \to \mathbb{Z}\Omega \to \mathbb{Z} \to 0. \tag{6.1}$$

From this we obtain the commutative diagram with exact rows:

$$\begin{array}{ccccc} H_n(\Gamma) & \longrightarrow & H_{n-1}(\Gamma, J\Omega) & \longrightarrow & H_{n-1}(\Gamma, \mathbb{Z}\Omega) \\ \downarrow & & \downarrow \alpha_* & \searrow^{\varphi} & \downarrow \\ H_n(\mathbf{G}(k)) & \xrightarrow{\delta} & H_{n-1}(\mathbf{G}(k), J\Omega) & \longrightarrow & H_{n-1}(\mathbf{G}(k), \mathbb{Z}\Omega). \end{array} \tag{6.2}$$

The vertical maps are induced by the inclusion $\alpha : \Gamma \to \mathbf{G}(k)$. Recall that the fundamental class $[\Gamma]$ corresponds to a generator of $H_n(\Gamma, \Omega) = H_{n-1}(\Gamma, J\Omega) \cong \mathbb{Z}$. Consider the map φ defined in (6.2).

Proposition 6.1 *Let the integer N be as in Proposition 5.3. Then $\varphi(N \cdot [\Gamma]) = 0$.*

Proof By Shapiro's lemma the module $H_{n-1}(\Gamma, \mathbb{Z}\Omega)$ decomposes as a direct sum of modules $H_{n-1}(\Gamma_x)$, indexed by the set of Γ-orbits of points $x \in \Omega$. But $H_{n-1}(\Gamma_x) = 0$ unless x is a cusp, so that the sum is actually indexed by the quotient set $\Gamma \backslash \mathcal{C}$.

Let us assume that $\mathbf{G}$ is isotropic (so that $k = \mathbb{Q}$). Since all cusps are conjugate by the action of $\mathbf{G}(\mathbb{Q})$, we have by Shapiro's lemma:

$$H_{n-1}(\mathbf{G}(\mathbb{Q}), \mathbb{Z}\Omega) = H_{n-1}(\mathbf{P}(\mathbb{Q})),$$

where $\mathbf{P}$ is constructed as in Sect. 5.3 for some cusp $x \in \Omega$. From Proposition 5.3 we have that the image of $H_{n-1}(\Gamma, \mathbb{Z}\Omega) = \bigoplus_{x \in \Gamma \backslash \mathcal{C}} H_{n-1}(\Gamma_x)$ in $H_{n-1}(\mathbf{G}(\mathbb{Q}), \mathbb{Z}\Omega)$ is annihilated by N. In particular, $N \cdot \varphi([\Gamma]) = 0$. □

Proof of Theorem 1.8 Let L_0 be the image of $H_n(\mathbf{G}(k))$ in $H_{n-1}(\mathbf{G}(k), \Omega)$ under the connecting map δ [cf. (6.2)]. By Proposition 4.2, L_0 has rank one. It follows from the exactness of the second row in (6.2) and Proposition 6.1 that $N \cdot \alpha_*([\Gamma]) \in L_0$ for any torsion-free lattice $\Gamma \subset \mathbf{G}(k)^+$. Let L_1 be the image of L_0 in $H_n(G, \Omega)$, and denote by L the submodule of $H_n(G, \Omega)$ generated by the elements $j_*([\Gamma])$ for torsion-free lattices $\Gamma \subset \mathbf{G}(k)^+$. Then $N \cdot L \subset L_1$, so that $\mathrm{rank}(L) = \mathrm{rank}(N \cdot L) = 1$ (note that this rank cannot be zero by Proposition 1.7). □

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