Monotonicity of facet numbers of random convex hulls

Gilles Bonnet*, Julian Grote†, Daniel Temesvari‡,
Christoph Thäle§, Nicola Turchi¶ and Florian Wespi‖

Abstract

Let $X_1, \dots, X_n$ be independent random points that are distributed according to a probability measure on $\mathbb{R}^d$ and let P_n be the random convex hull generated by $X_1, \dots, X_n$ ($n \geq d+1$). For natural classes of probability distributions and by means of Blaschke-Petkantschin formulae from integral geometry it is shown that the mean facet number of P_n is strictly monotonically increasing in n.

Keywords. Blaschke-Petkantschin formula, mean facet number, random convex hull
MSC. 52A22, 52B05, 53C65, 60D05

1 Introduction and main result

Fix a space dimension $d \geq 2$. For an integer $n \geq d+1$, let $X_1, \dots, X_n$ be independent random points that are chosen according to an absolutely continuous probability distribution on $\mathbb{R}^d$. By P_{n-1} and P_n we denote the random convex hulls generated by $X_1, \dots, X_{n-1}$ and $X_1, \dots, X_n$, respectively. In our present text we are interested in the mean number of facets $\mathbb{E} f_{d-1}(P_{n-1})$ and $\mathbb{E} f_{d-1}(P_n)$ of P_{n-1} and P_n. More specifically, we ask the following monotonicity question:

Is it true that $\mathbb{E} f_{d-1}(P_{n-1}) \leq \mathbb{E} f_{d-1}(P_n)$*?*

This question has been put forward and answered positively by Devillers, Glisse, Goaoc, Moroz and Reitzner [7] for random points that are uniformly distributed in a convex body $K \subset \mathbb{R}^d$ if $d = 2$ and, if $d \geq 3$, under the additional assumptions that the boundary of K is twice differentiable with strictly positive Gaussian curvature and that n is sufficiently large, that is, $n \geq n(K)$, where $n(K)$ is a constant depending on K. Moreover, an affirmative answer was obtained by Beermann [4] if the random points are chosen with respect to the standard Gaussian distribution on $\mathbb{R}^d$ or according to the uniform distribution in the d-dimensional unit ball for all $d \geq 2$. Beermann's proof essentially relies on a Blaschke-Petkantschin formula, a well known change-of-variables formula

*Faculty of Mathematics, Ruhr University Bochum, Germany. E-mail: gilles.bonnet@rub.de
†Faculty of Mathematics, Ruhr University Bochum, Germany. E-mail: julian.grote@rub.de
‡Faculty of Mathematics, Ruhr University Bochum, Germany. E-mail: daniel.temesvari@rub.de
§Faculty of Mathematics, Ruhr University Bochum, Germany. E-mail: christoph.thaele@rub.de
¶Faculty of Mathematics, Ruhr University Bochum, Germany. E-mail: nicola.turchi@rub.de
‖Institute of Mathematical Statistics and Actuarial Science, University of Bern, Switzerland. E-mail: florian.wespi@stat.unibe.ch

in integral geometry. Our aim in this text is to generalize her approach to other and more general probability distributions on $\mathbb{R}^d$. In fact, we will be able to characterize all absolutely continuous rotationally symmetric distributions on $\mathbb{R}^d$ whose densities satisfy a natural scaling property (see (9) below), to which the methodology based on the Blaschke-Petkantschin formula can be applied and for which we can answer positively the monotonicity question posed above for any of these distributions. Moreover, we will apply our results to study similar monotonicity questions for a class of spherical convex hulls generated by random points on a half-sphere, which comprises as a special case the model recently studied by Bárány, Hug, Reitzner and Schneider [3].

To present our main result formally, we introduce four classes of probability measures:

- **G** is the class of centred Gaussian distributions on $\mathbb{R}^d$ with density proportional to

$$x \mapsto \exp\Big(-\frac{\|x\|^2}{2\sigma^2}\Big), \qquad \sigma > 0,$$

- **H** is the class of heavy-tailed distributions on $\mathbb{R}^d$ with density proportional to

$$x \mapsto \Big(1 + \frac{\|x\|^2}{\sigma^2}\Big)^{-\beta}, \qquad \beta > d/2, \sigma > 0,$$

- **B** is the class of beta-type distributions on the d-dimensional centred ball $\mathbb{B}^d_\sigma$ of radius σ with density proportional to

$$x \mapsto \Big(1 - \frac{\|x\|^2}{\sigma^2}\Big)^{\beta}, \qquad \beta > -1, \sigma > 0,$$

- **U** comprises the uniform distributions on the $(d-1)$-dimensional centred spheres $\mathbb{S}^{d-1}_\sigma$ with radius $\sigma > 0$.

It will turn out that the classes **G**, **H**, **B** and **U** contain precisely the absolutely continuous rotationally symmetric probability distributions on $\mathbb{R}^d$, whose densities satisfy the natural scaling property (9) below, for which monotonicity of the mean facet number of the associated random convex hulls can be shown by means of arguments based on a Blaschke-Petkantschin formula, see the discussion at the end of Section 4 for further details. In fact, our result shows that even the stronger strict monotonicity holds.

Theorem 1 *Let $X_1, \ldots, X_n \in \mathbb{R}^d$, $n \geq d+1$, be independent and identically distributed according to a probability measure belonging to one of the classes* **G**, **H**, **B** *or* **U**. *Then,*

$$\mathbb{E} f_{d-1}(P_n) > \mathbb{E} f_{d-1}(P_{n-1}).$$

It should be emphasized that strict monotonicity of $n \mapsto f_{d-1}(P_n)$ cannot hold pathwise (except for the trivial case $n = d+1$), since the addition of a further random point can reduce the facet number arbitrarily as the additional point might 'see' much more than d vertices of the already constructed random convex hull. For this reason, the expectation in Theorem 1 is essential.
We would also like to remark that monotonicity questions related to the volume of random convex hulls have recently attracted some interest in convex geometry because of their connection to the famous slicing problem. Namely, if K and L are two compact

convex sets in $\mathbb{R}^d$ with interior points, let V_K and V_L be the volume of the convex hull of $d+1$ independent random points uniformly distributed in K and L, respectively. One is interested in the question whether the set inclusion $K \subseteq L$ implies the inequality $\mathbb{E}V_K \leq \mathbb{E}V_L$. In particular, the work of Rademacher [11] shows that this is false in general whenever $d \geq 4$. Higher moments were treated by Reichenwallner and Reitzner [12], and we refer to the discussion therein for further details and background material.

Our text is structured as follows. In Section 2 we recall the necessary background material from integral geometry and introduce some more notation. Moreover, in Section 3 we develop several auxiliary results that prepare for the proof of Theorem 1, which is presented in Section 4. We also discuss in Section 4 the limitations and possible extensions of the method we use. The final Section 5 contains the application of Theorem 1 to random convex hulls on a half-sphere.

2 Background material from integral geometry

2.1 General notation

We denote by $A(d,q)$ the Grassmannian of all q-dimensional affine subspaces of $\mathbb{R}^d$, where $q \in \{0,1,\ldots,d\}$. It is a locally compact, homogeneous space with respect to the group of Euclidean motions in $\mathbb{R}^d$. The corresponding locally finite, motion invariant measure is denoted by μ_q, which is normalized in such a way that

$$\mu_q(\{H \in A(d,q) : H \cap \mathbb{B}^d \neq \emptyset\}) = \kappa_{d-q},$$

see [14]. Here, $\mathbb{B}^d$ is the centred d-dimensional unit ball, $\kappa_{d-q} = \frac{\pi^{(d-q)/2}}{\Gamma(1+\frac{d-q}{2})}$ is the $(d-q)$-dimensional volume of $\mathbb{B}^{d-q}$ and $\Gamma(\,\cdot\,)$ indicates the Gamma function. Moreover, the Beta function is given by

$$B(a,b) := \int_0^1 s^{a-1}(1-s)^{b-1}\,\mathrm{d}s = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad a,b > 0.$$

We shall denote by $\mathbb{S}^{d-1}$ the $(d-1)$-dimensional unit sphere and abbreviate by $\omega_d = d\kappa_d$ its total spherical Lebesgue measure. For a subspace $H \in A(d,q)$, we let λ_H be the Lebesgue measure on H.
For a set $K \subset \mathbb{R}^d$, we shall write $\mathcal{H}^q_K$ for the q-dimensional Hausdorff measure on K. The operators $\mathrm{conv}(\,\cdot\,)$ and $\mathrm{span}(\,\cdot\,)$ denote the convex and linear hull of the argument. We will use the notation $\Delta_{d-1}(x_1,\ldots,x_d)$ to indicate the $(d-1)$-dimensional volume of the convex hull of d points $x_1,\ldots,x_d$.

2.2 Blaschke-Petkantschin formulae

Our proof of Theorem 1 heavily relies on Blaschke-Petkantschin formulae from integral geometry. First, we rephrase a special case of the affine Blaschke-Petkantschin formula in $\mathbb{R}^d$, which appears as Theorem 7.2.7 in [14].

Proposition 2 *Let $f : (\mathbb{R}^d)^d \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\int\limits_{(\mathbb{R}^d)^d} f(x_1,\dots,x_d)\,\mathrm{d}(x_1,\dots,x_d)$$
$$= \frac{\omega_d}{2}(d-1)! \int\limits_{A(d,d-1)} \int\limits_{H^d} f(x_1,\dots,x_d)\Delta_{d-1}(x_1,\dots,x_d)\,\lambda_H^d(\mathrm{d}(x_1,\dots,x_d))\,\mu_{d-1}(\mathrm{d}H).$$

Besides the affine Blaschke-Petkantschin formula in $\mathbb{R}^d$ we need its spherical counterpart, which is a special case of Theorem 1 in [15] and can also be found as Theorem 4 in [10].

Proposition 3 *Let $f : (\mathbb{S}^{d-1})^d \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\int\limits_{(\mathbb{S}^{d-1})^d} f(x_1,\dots,x_d)\,\mathcal{H}^{d(d-1)}_{(\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1,\dots,x_d)) = (d-1)! \int\limits_{A(d,d-1)} \int\limits_{(H\cap\mathbb{S}^{d-1})^d} f(x_1,\dots,x_d)$$
$$\times\,\Delta_{d-1}(x_1,\dots,x_d)(1-h^2)^{-\frac{d}{2}}\,\mathcal{H}^{d(d-2)}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1,\dots,x_d))\,\mu_{d-1}(\mathrm{d}H),$$

where h denotes the distance from H to the origin.

2.3 A slice integration formula

Finally, we will make use of the following special case of the spherical slice integration formula taken from Theorem A.4 in [2].

Proposition 4 *Let $f : \mathbb{S}^{d-1} \to \mathbb{R}$ be a non-negative measurable function. Then,*

$$\int\limits_{\mathbb{S}^{d-1}} f(x)\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}x) = \int\limits_{-1}^{1} (1-t^2)^{\frac{d-3}{2}} \int\limits_{\mathbb{S}^{d-2}} f(t,\sqrt{1-t^2}\,y)\,\mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}y)\,\mathrm{d}t.$$

3 Auxiliary results

3.1 An estimate for integrals of concave functions

A version of the next lemma was already stated in [4], but without proof. For the sake of completeness we include here the short argument.

Lemma 5 *Let $h : \mathbb{R} \to \mathbb{R}$ be a non-negative measurable function which is strictly positive on a set of positive Lebesgue measure. Further, let $g : \mathbb{R} \to \mathbb{R}$ be an affine function with negative slope and root $s^* \in [0,1]$. Moreover, let $L : \mathbb{R} \to \mathbb{R}$ be positive and strictly concave on $[0,1]$. Then,*

$$\int\limits_0^1 h(s)g(s)L(s)^{d-1}\,\mathrm{d}s > \int\limits_0^1 h(s)g(s)\ell(s)^{d-1}\,\mathrm{d}s, \tag{1}$$

where $\ell(s) = \frac{L(s^*)}{s^*}s$.

Proof. We start by exploiting the positivity and strict concavity of L. For $s \in [0, s^*)$, it implies that

$$L(s) = L\left(\frac{s}{s^*}s^*\right) > \frac{s}{s^*}L(s^*), \tag{2}$$

while for $s \in (s^*, 1]$, it gives

$$L(s) < \frac{s}{s^*}L(s^*). \tag{3}$$

Since g has a negative slope, it is positive on $[0, s^*)$ and negative on $(s^*, 1]$. Splitting the integral on the left hand side of (1) at the point s^* and using (2) and (3), respectively, yields

$$\begin{aligned}
&\int_0^1 h(s)g(s)L(s)^{d-1}\,\mathrm{d}s \\
&= \int_0^{s^*} h(s)g(s)L(s)^{d-1}\,\mathrm{d}s + \int_{s^*}^1 h(s)g(s)L(s)^{d-1}\,\mathrm{d}s \\
&> \int_0^{s^*} h(s)g(s)\left(\frac{s}{s^*}L(s^*)\right)^{d-1}\,\mathrm{d}s + \int_{s^*}^1 h(s)g(s)\left(\frac{s}{s^*}L(s^*)\right)^{d-1}\,\mathrm{d}s \\
&= \int_0^1 h(s)g(s)\ell(s)^{d-1}\,\mathrm{d}s.
\end{aligned}$$

This completes the argument. □

3.2 Computation of marginal densities

Recall the definitions of the distribution classes **H**, **B** and **U** (since random convex hulls of Gaussian points have already been treated in [4], we concentrate on the classes **H**, **B** and **U**). It will turn out that it suffices to consider the cases where the scale parameters σ are equal to 1. From now on we restrict to these cases and denote the density of a distribution in **H** by

$$p_{\mathbf{H},\beta}(x) = \pi^{-d/2}\frac{\Gamma(\beta)}{\Gamma(\beta - \frac{d}{2})}(1 + \|x\|^2)^{-\beta}, \qquad x \in \mathbb{R}^d, \beta > d/2, \tag{4}$$

that of a distribution in **B** by

$$p_{\mathbf{B},\beta}(x) = \pi^{-d/2}\frac{\Gamma(\frac{d}{2} + \beta + 1)}{\Gamma(\beta + 1)}(1 - \|x\|^2)^{\beta}, \qquad x \in \mathbb{B}^d, \beta > -1,$$

and note that the uniform distribution on $\mathbb{S}^{d-1}$ has density

$$p_{\mathbf{U}}(x) = \frac{1}{\omega_d}, \qquad x \in \mathbb{S}^{d-1},$$

with respect to the spherical Lebesgue measure. The next lemma provides formulas for the densities of the one-dimensional marginals of these distributions and shows that the classes **B** and **H** are in some sense closed under one-dimensional projections. Since all distributions we consider are rotationally symmetric, it is sufficient to consider projections onto the first coordinate. We would like to emphasize that the proof of Lemma 6 uses in an essential way the scaling property (9) below of the involved densities.

Lemma 6 *Let $\Pi : \mathbb{R}^d \to \mathbb{R}, (x_1, \dots, x_d) \mapsto x_1$ be the projection onto the first coordinate.*

(i) Let $\mathbb{P} \in \mathbf{H}$ be a distribution with density $p_{\mathbf{H},\beta}$ for some $\beta > d/2$. Then, the image measure of $\mathbb{P}$ under Π has density

$$f_{\mathbf{H},\beta}(x) = \pi^{-1/2} \frac{\Gamma\left(\beta - \frac{d-1}{2}\right)}{\Gamma\left(\beta - \frac{d}{2}\right)} (1 + x^2)^{\frac{d-1}{2} - \beta}, \qquad x \in \mathbb{R}.$$

(ii) Let $\mathbb{P} \in \mathbf{B}$ be a distribution with density $p_{\mathbf{B},\beta}$ for some $\beta > -1$. Then, the image measure of $\mathbb{P}$ under Π has density

$$f_{\mathbf{B},\beta}(x) = \pi^{-1/2} \frac{\Gamma\left(\beta + 1 + \frac{d}{2}\right)}{\Gamma\left(\beta + \frac{d+1}{2}\right)} (1 - x^2)^{\frac{d-1}{2} + \beta}, \qquad x \in [-1, 1].$$

(iii) Let $\mathbb{P} \in \mathbf{U}$ be the uniform distribution on $\mathbb{S}^{d-1}$. Then, the image measure of $\mathbb{P}$ under Π has density

$$f_{\mathbf{U}}(x) = \pi^{-1/2} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} (1 - x^2)^{\frac{d-3}{2}}, \qquad x \in [-1, 1].$$

Proof. To prove (i) we put $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $y := (x_2, \dots, x_d)$ and also define $c_{\mathbf{H},d,\beta} := \pi^{-d/2} \frac{\Gamma(\beta)}{\Gamma(\beta - d/2)}$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} c_{\mathbf{H},d,\beta} \left(1 + \|x\|^2\right)^{-\beta} \mathrm{d}(x_2, \dots, x_d) \\
&= \int_{\mathbb{R}^{d-1}} c_{\mathbf{H},d,\beta} (1 + x_1^2)^{-\beta} \left(1 + \frac{\|y\|^2}{1 + x_1^2}\right)^{-\beta} \mathrm{d}y \\
&= (1 + x_1^2)^{-\beta} \int_{\mathbb{R}^{d-1}} c_{\mathbf{H},d,\beta} \left(1 + \|z\|^2\right)^{-\beta} (1 + x_1^2)^{\frac{d-1}{2}} \mathrm{d}z \\
&= (1 + x_1^2)^{\frac{d-1}{2} - \beta} \frac{c_{\mathbf{H},d,\beta}}{c_{\mathbf{H},d-1,\beta}} \int_{\mathbb{R}^{d-1}} c_{\mathbf{H},d-1,\beta} \left(1 + \|z\|^2\right)^{-\beta} \mathrm{d}z \\
&= \frac{c_{\mathbf{H},d,\beta}}{c_{\mathbf{H},d-1,\beta}} (1 + x_1^2)^{\frac{d-1}{2} - \beta},
\end{aligned}$$

where we used the substitution $z = y / \sqrt{1 + x_1^2}$. Plugging in the constants yields the desired result.

Next, we consider the distribution with density $p_{\mathbf{B},\beta}$. For $x = (x_1, \ldots, x_d) \in \mathbb{B}^d$, we put again $y := (x_2, \ldots, x_d)$ and abbreviate $c_{\mathbf{B},d,\beta} := \pi^{-d/2} \frac{\Gamma(\frac{d}{2}+\beta+1)}{\Gamma(\beta+1)}$. Then, similarly as above, we compute

$$
\begin{aligned}
&\int\limits_{\mathbb{B}^{d-1}} c_{\mathbf{B},d,\beta} \left(1 - \|x\|^2\right)^{\beta} \mathrm{d}(x_2, \ldots, x_d) \\
&= \int\limits_{\mathbb{B}^{d-1}} c_{\mathbf{B},d,\beta} (1 - x_1^2)^{\beta} \left(1 - \frac{\|y\|^2}{1 - x_1^2}\right)^{\beta} \mathrm{d}y \\
&= (1 - x_1^2)^{\beta} \int\limits_{\mathbb{B}^{d-1}} c_{\mathbf{B},d,\beta} \left(1 - \|z\|^2\right)^{\beta} (1 - x_1^2)^{\frac{d-1}{2}} \mathrm{d}z \\
&= (1 - x_1^2)^{\frac{d-1}{2}+\beta} \frac{c_{\mathbf{B},d,\beta}}{c_{\mathbf{B},d-1,\beta}} \int\limits_{\mathbb{B}^{d-1}} c_{\mathbf{B},d-1,\beta} \left(1 - \|z\|^2\right)^{\beta} \mathrm{d}z \\
&= \frac{c_{\mathbf{B},d,\beta}}{c_{\mathbf{B},d-1,\beta}} (1 - x_1^2)^{\frac{d-1}{2}+\beta},
\end{aligned}
$$

where we used the substitution $z = y/\sqrt{1 - x_1^2}$. Again, simplification of the constants yields the desired result.
Finally, we consider the case of the uniform distribution on $\mathbb{S}^{d-1}$. We denote by F the distribution function of the image measure of $\mathbb{P} = \omega_d^{-1} \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}$ under the orthogonal projection map Π and let $x_1 \in [-1, 1]$. Using the slice integration formula from Proposition 4, we obtain

$$
\begin{aligned}
\mathrm{F}(x_1) &= \frac{1}{\omega_d} \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}} \left(\left\{ u \in \mathbb{S}^{d-1} : \Pi(u) \in [-1, x_1] \right\} \right) \\
&= \frac{1}{\omega_d} \int\limits_{\mathbb{S}^{d-1}} \mathbb{1}\{\Pi(u) \in [-1, x_1]\} \, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\
&= \frac{1}{\omega_d} \int\limits_{-1}^{x_1} (1 - t^2)^{\frac{d-3}{2}} \int\limits_{\mathbb{S}^{d-2}} \mathcal{H}^{d-2}_{\mathbb{S}^{d-2}}(\mathrm{d}y) \, \mathrm{d}t \\
&= \frac{\omega_{d-1}}{\omega_d} \int\limits_{-1}^{x_1} (1 - t^2)^{\frac{d-3}{2}} \, \mathrm{d}t.
\end{aligned}
$$

Differentiation with respect to x_1, together with the definitions of ω_d and ω_{d-1}, complete the proof. □

In what follows, we shall denote by $F_{\mathbf{H},\beta}$, $F_{\mathbf{B},\beta}$ and $F_{\mathbf{U}}$ the distribution functions corresponding to the densities $f_{\mathbf{H},\beta}$, $f_{\mathbf{B},\beta}$ and $f_{\mathbf{U}}$ computed in Lemma 6, respectively.

4 Proof of Theorem 1 and discussion

4.1 Proof of Theorem 1

Based on the results from the two previous sections we are now able to present the proof of our main result. In what follows, we denote all constants by c. Unless otherwise stated, they only depend on the space dimension d and the parameter β of the underlying probability distribution. Their value might change from instance to instance.

Proof of Theorem 1. For the classes $\mathbf{G}$, $\mathbf{H}$, $\mathbf{B}$ and $\mathbf{U}$ it is sufficient to consider the case that the scale parameter σ is equal to 1, since the mean facet number is invariant under rescalings.

For the distributions from class $\mathbf{G}$ the result is known from [4, Theorem 5.3.1].

Next, we consider the heavy-tailed distributions on $\mathbb{R}^d$ with density $p_{\mathbf{H},\beta}(x) = c_{\mathbf{H},d,\beta}(1+\|x\|^2)^{-\beta}$, where $\beta > d/2$ and $c_{\mathbf{H},d,\beta} = \pi^{-d/2}\frac{\Gamma(\beta)}{\Gamma(\beta-d/2)}$. We follow the ideas from [4] and start with the equality

$$\begin{aligned}\mathbb{E} f_{d-1}(P_n) &= \mathbb{E} \sum_{1\le i_1<\ldots<i_d\le n} \mathbb{1}\{\mathrm{conv}(X_{i_1},\ldots,X_{i_d}) \text{ is a facet of } P_n\} \\ &= \binom{n}{d} \mathbb{P}(\mathrm{conv}(X_1,\ldots,X_d) \text{ is a facet of } P_n),\end{aligned} \tag{5}$$

which holds due to the fact that the random points $X_1,\ldots,X_n$ are independent and identically distributed. Let us denote by $H \in A(d,d-1)$ the affine hull of the $(d-1)$-dimensional simplex P_d spanned by $X_1,\ldots,X_d$. In the case that P_d is a facet of P_n, all the remaining points $X_{d+1},\ldots,X_n$ have to lie in one of the (open) half-spaces determined by H. If we denote by Π_H the orthogonal projection onto $H^\perp$, the orthogonal complement of H, we observe that P_d is a facet of P_n if and only if the point $\Pi_H(P_d)$ is not contained in the interior of the interval $\Pi_H(P_n)$ on $H^\perp$. Therefore, using Lemma 6, the affine Blaschke-Petkantschin formula from Proposition 2 and the abbreviation $F^* = F_{\mathbf{H},\beta}(\Pi_H(P_d))$, we get for the probability that P_d is a facet of P_n,

$$\begin{aligned}&\mathbb{P}(\mathrm{conv}(X_1,\ldots,X_d) \text{ is a facet of } P_n) \\ &= \int_{(\mathbb{R}^d)^d} \mathbb{P}(E(x_1,\ldots,x_d)) \prod_{i=1}^d p_{\mathbf{H},\beta}(x_i)\,\mathrm{d}(x_1,\ldots,x_d) \\ &= \int\limits_{(\mathbb{R}^d)^d} \Big((1-F^*)^{n-d} + (F^*)^{n-d}\Big) \prod_{i=1}^d c_{\mathbf{H},d,\beta}(1+\|x_i\|^2)^{-\beta}\,\mathrm{d}(x_1,\ldots,x_d) \\ &= c \int\limits_{A(d,d-1)} \int\limits_{H^d} \Big((1-F^*)^{n-d} + (F^*)^{n-d}\Big)\, \Delta_{d-1}(x_1,\ldots,x_d) \prod_{i=1}^d c_{\mathbf{H},d,\beta}(1+\|x_i\|^2)^{-\beta} \\ &\qquad \times \lambda_H^d(\mathrm{d}(x_1,\ldots,x_d))\, \mu_{d-1}(\mathrm{d}H),\end{aligned}$$

where $E(x_1,\ldots,x_d)$ stands for the event that all remaining random points $X_{d+1},\ldots,X_n$ are located in the same open half-space determined by H. Next, we use the theorem of Pythagoras to decompose, for each $i \in \{1,\ldots,d\}$, the norm $\|x_i\|$. Namely, writing $\|\cdot\|_H$ for the Euclidean norm in $H \in A(d,d-1)$ and h for the distance from H to the origin in $\mathbb{R}^d$, we have that

$$\|x_i\|^2 = \|x_i\|_H^2 + h^2.$$

Therefore and as already used in the proof of Lemma 6, the last term of the integrand can be rewritten as

$$(1+\|x_i\|^2)^{-\beta} = (1+h^2+\|x_i\|_H^2)^{-\beta} = (1+h^2)^{-\beta}\left(1+\frac{\|x_i\|_H^2}{1+h^2}\right)^{-\beta}. \tag{6}$$

Moreover, since each hyperplane $H = H(u,h)$ is uniquely determined by its unit normal vector $u \in \mathbb{S}^{d-1}$ and its distance $h \in [0,\infty)$ to the origin, the integration over $A(d,d-1)$ can be replaced by a twofold integral over $\mathbb{S}^{d-1}$ and $[0,\infty)$. Using the substitutions $y_i = x_i/\sqrt{1+h^2}$ with $\lambda_H(\mathrm{d}x_i) = (1+h^2)^{(d-1)/2}\lambda_H(\mathrm{d}y_i)$, the rotational invariance of the underlying probability measure, and writing $F(h)$ for $F_{\mathbf{H},\beta}(h)$ as well as $f(h)$ for $f_{\mathbf{H},\beta}(h)$, gives in view of Lemma 6 that

$$\begin{aligned}
&\mathbb{P}(\mathrm{conv}(X_1,\ldots,X_d) \text{ is a facet of } P_n)\\
&= c\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_{H^d}\Big((1-F(h))^{n-d}+F(h)^{n-d}\Big)\,\Delta_{d-1}(x_1,\ldots,x_d)\\
&\qquad\times(1+h^2)^{-d\beta}\prod_{i=1}^d c_{\mathbf{H},d,\beta}\left(1+\frac{\|x_i\|_H^2}{1+h^2}\right)^{-\beta}\lambda_H^d(\mathrm{d}(x_1,\ldots,x_d))\,\mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)\\
&= c\int_{\mathbb{S}^{d-1}}\int_0^\infty\Big((1-F(h))^{n-d}+F(h)^{n-d}\Big)\,(1+h^2)^{-d\left(\beta-\frac{d-1}{2}\right)+\frac{d-1}{2}}\,\mathrm{d}h\,\mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u)\\
&\qquad\times\int_{H^d}\Delta_{d-1}(y_1,\ldots,y_d)\prod_{i=1}^d c_{\mathbf{H},d-1,\beta}\left(1+\|y_i\|_H^2\right)^{-\beta}\lambda_H^d(\mathrm{d}(y_1,\ldots,y_d))\\
&= c\int_0^\infty\Big((1-F(h))^{n-d}+F(h)^{n-d}\Big)\,(1+h^2)^{-d\left(\beta-\frac{d-1}{2}\right)+\frac{d-1}{2}}\,\mathrm{d}h\\
&= c\int_{-\infty}^\infty(1-F(h))^{n-d}f(h)^d(1+h^2)^{\frac{d-1}{2}}\,\mathrm{d}h,
\end{aligned}$$

where we also used the fact that the integral over H^d is a finite constant given by Equation (72) in [10] and which only depends on the space dimension d and on β.
Write now $s = F(h)$ and $L(s) = f\left(F^{-1}(s)\right)\sqrt{1+(F^{-1}(s))^2}$ to obtain

$$\mathbb{P}(\mathrm{conv}(X_1,\ldots,X_d) \text{ is a facet of } P_n) = c\int_0^1(1-s)^{n-d}L(s)^{d-1}\,\mathrm{d}s.$$

Thus, combination of the above computation with (5) yields the representation

$$\begin{aligned}
&\mathbb{E}f_{d-1}(P_n)-\mathbb{E}f_{d-1}(P_{n-1})\\
&\qquad= c\int_0^1\left[\binom{n}{d}(1-s)-\binom{n-1}{d}\right](1-s)^{n-d-1}L(s)^{d-1}\,\mathrm{d}s.
\end{aligned}\tag{7}$$

In order to apply Lemma 5, we have to verify that $L(s)$ is strictly concave on $(0,1)$. We prove this by showing that the second derivative of $L(s)$ is negative. So, let $c_{\mathbf{H},\beta} := \pi^{-1/2}\frac{\Gamma(\beta-\frac{d-1}{2})}{\Gamma(\beta-\frac{d}{2})}$ and recall that $f(x) = c_{\mathbf{H},\beta}(1+x^2)^{\frac{d-1}{2}-\beta}$ from Lemma 6. Furthermore, from the definition of F it follows that

$$\left(F^{-1}(s)\right)' = \frac{1}{f\left(F^{-1}(s)\right)} = \frac{1}{c_{\mathbf{H},\beta}\left(1+(F^{-1}(s))^2\right)^{\frac{d-1}{2}-\beta}}.\tag{8}$$

We recall that

$$L(s) = f\left(F^{-1}(s)\right)\sqrt{1+(F^{-1}(s))^2} = c_{\mathbf{H},\beta}\left(1+(F^{-1}(s))^2\right)^{\frac{d}{2}-\beta}.$$

Hence, using (8), the first derivative of $L(s)$ is

$$\begin{aligned}L'(s) &= c_{\mathbf{H},\beta}\left(\frac{d}{2}-\beta\right)\left(1+(F^{-1}(s))^2\right)^{\frac{d-2}{2}-\beta} 2F^{-1}(s)\left(F^{-1}(s)\right)' \\ &= 2\left(\frac{d}{2}-\beta\right)\left(1+(F^{-1}(s))^2\right)^{-\frac{1}{2}} F^{-1}(s)\end{aligned}$$

and, thus, for the second derivative we find that

$$\begin{aligned}L''(s) = 2\left(\frac{d}{2}-\beta\right)&\Bigg[\left(1+(F^{-1}(s))^2\right)^{-\frac{1}{2}}\left(F^{-1}(s)\right)' \\ &\quad -\frac{1}{2}\left(1+(F^{-1}(s))^2\right)^{-\frac{3}{2}} 2\left(F^{-1}(s)\right)^2\left(F^{-1}(s)\right)'\Bigg] \\ &= 2\left(\frac{d}{2}-\beta\right)\left(F^{-1}(s)\right)'\left[\left(1+(F^{-1}(s))^2\right)^{-\frac{1}{2}} - \left(1+(F^{-1}(s))^2\right)^{-\frac{3}{2}}\left(F^{-1}(s)\right)^2\right] \\ &= \frac{2}{c_{\mathbf{H},\beta}}\left(\frac{d}{2}-\beta\right)\left(1+(F^{-1}(s))^2\right)^{\beta-1-\frac{d}{2}}\left[1+(F^{-1}(s))^2-(F^{-1}(s))^2\right] \\ &= -\frac{2}{c_{\mathbf{H},\beta}}\left(\beta-\frac{d}{2}\right)\left(1+(F^{-1}(s))^2\right)^{\beta-1-\frac{d}{2}} \\ &< 0,\end{aligned}$$

where the last inequality follows from the fact that $\beta > d/2$. As a consequence, we can apply Lemma 5 to deduce that

$$\begin{aligned}&\mathbb{E}f_{d-1}(P_n) - \mathbb{E}f_{d-1}(P_{n-1}) \\ &= c\int_0^1 \left[\binom{n}{d}(1-s) - \binom{n-1}{d}\right](1-s)^{n-d-1}L(s)^{d-1}\,\mathrm{d}s \\ &> c\left(\frac{L(d/n)}{d/n}\right)^{d-1}\binom{n}{d}\int_0^1 (1-s)^{n-d-1}s^{d-1}\left((1-s)-\frac{n-d}{n}\right)\mathrm{d}s \\ &= c\left(\frac{L(d/n)}{d/n}\right)^{d-1}\binom{n}{d}\left(\mathrm{B}(d,n-d+1)-\frac{n-d}{n}\mathrm{B}(d,n-d)\right) \\ &= 0,\end{aligned}$$

where we used the well-known relation $\mathrm{B}(d,n-d+1) = \frac{n-d}{n}\mathrm{B}(d,n-d)$ for the beta function.

As the next case we consider the class $\mathbf{B}$ of beta-type distribution on the unit ball $\mathbb{B}^d$ with density $f_{\mathbf{B},\beta}$ for some $\beta > -1$. In this case the proof follows almost line by line the proof for $\mathbf{H}$, up to some minor modifications. In particular, (7) stays the same except that now $L(s) = f\left(F^{-1}(s)\right)\sqrt{1-(F^{-1}(s))^2}$, where $F(h) = F_{\mathbf{B},\beta}(h)$ and $f(h) = f_{\mathbf{B},\beta}(h)$. Therefore, it follows that

$$L''(s) = -\frac{2}{c_{\mathbf{B},\beta}}\left(\beta + \frac{d}{2}\right)\left(1-(F^{-1}(s))^2\right)^{-\beta-1-\frac{d}{2}},$$

where the constant $c_{\mathbf{B},\beta}$ is $c_{\mathbf{B},\beta} := \pi^{-1/2}\Gamma(\beta+1+\frac{d}{2})\Gamma(\beta+\frac{d+1}{2})^{-1}$. Since $F^{-1}(s) \in (-1,1)$, we obtain $L''(s) < 0$ and can conclude as in the proof for the class **H** presented above.

Finally, we consider the case of the uniform distribution on $\mathbb{S}^{d-1}$. Here we get by applying the spherical Blaschke-Petkantschin formula from Proposition 3 and using the abbreviations $F(h) = F_{\mathbf{U}}(h)$ and $f(h) = f_{\mathbf{U}}(h)$,

$$\begin{aligned}
&\mathbb{P}(\mathrm{conv}(X_1,\dots,X_d) \text{ is a facet of } P_n) \\
&= c \int_{A(d,d-1)} \int_{(H\cap\mathbb{S}^{d-1})^d} \left((1-F(h))^{n-d} + F(h)^{n-d}\right) \Delta_{d-1}(x_1,\dots,x_d)(1-h^2)^{-\frac{d}{2}} \\
&\qquad \times \mathcal{H}^{d(d-2)}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1,\dots,x_d))\, \mu_{d-1}(\mathrm{d}H) \\
&= c \int_{\mathbb{S}^{d-1}} \int_0^1 \int_{(H\cap\mathbb{S}^{d-1})^d} \left((1-F(h))^{n-d} + F(h)^{n-d}\right) \Delta_{d-1}(x_1,\dots,x_d)(1-h^2)^{-\frac{d}{2}} \\
&\qquad \times \mathcal{H}^{d(d-2)}_{(H\cap\mathbb{S}^{d-1})^d}(\mathrm{d}(x_1,\dots,x_d))\, \mathrm{d}h\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\
&= c \int_{\mathbb{S}^{d-1}} \int_0^1 \left((1-F(h))^{n-d} + F(h)^{n-d}\right) (1-h^2)^{d\frac{d-2}{2}+\frac{d-1}{2}-\frac{d}{2}}\, \mathrm{d}h\, \mathcal{H}^{d-1}_{\mathbb{S}^{d-1}}(\mathrm{d}u) \\
&\qquad \times \int_{(\mathbb{S}^{d-2})^d} \Delta_{d-1}(y_1,\dots,y_d)\, \mathcal{H}^{d(d-2)}_{(\mathbb{S}^{d-2})^d}(\mathrm{d}(y_1,\dots,y_d)),
\end{aligned}$$

where the substitution $x_i = y_i\sqrt{1-h^2}$ with $\mathcal{H}^{d-2}_{H\cap\mathbb{S}^{d-1}}(\mathrm{d}x_i) = (1-h^2)^{\frac{d-2}{2}}\mathcal{H}^{d-2}_{H\cap\mathbb{S}^{d-1}}(\mathrm{d}y_i)$ was used. In particular, this transforms the integration over $(H\cap\mathbb{S}^{d-1})^d$ into a d-fold integral over the unit sphere in H, which in turn has been identified with $\mathbb{S}^{d-2}$ due to rotational invariance. Since the integral over $(\mathbb{S}^{d-2})^d$ is a known positive constant only depending on d (the precise value can be deduced from [14, Theorem 8.2.3], for example), we get by rotational invariance of the underlying distribution that

$$\begin{aligned}
&\mathbb{P}(\mathrm{conv}(X_1,\dots,X_d) \text{ is a facet of } P_n) \\
&= c \int_0^1 \left((1-F(h))^{n-d} + F(h)^{n-d}\right)(1-h^2)^{d\frac{d-3}{2}+\frac{d-1}{2}}\, \mathrm{d}h \\
&= c \int_{-1}^1 (1-F(h))^{n-d} f(h)^d (1-h^2)^{\frac{d-1}{2}}\, \mathrm{d}h.
\end{aligned}$$

As a consequence, also for the uniform distribution on $\mathbb{S}^{d-1}$ we arrive at an expression of the form (7), this time with $L(s) = f(F^{-1}(s))\sqrt{1-(F^{-1}(s))^2}$. From this point on, the proof can be completed as in the case of the distribution class **H** or **B**. This completes the argument. □

4.2 Discussion

Let $p : \mathbb{R}^d \to [0, \infty)$ denote a probability density. By a careful inspection of the proof of Theorem 1 we see that the following properties of the density p have been used there. First of all, we used that p is spherically symmetric, that is, $p(x)$ only depends on $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ via $\|x\|$. By abuse of notation, we shall write $p(r) : (0, \infty) \to [0, \infty)$ with $r^2 = x_1^2 + \dots + x_d^2$ for the radial part of the density p. This was essential to apply the Blaschke-Petkantschin formulae, which use the invariant hyperplane measure μ_{d-1}. Moreover, given $H \in A(d, d-1)$ with distance h to the origin, we have used that we can find $\varphi(h), \psi(h) > 0$ such that

$$p(\sqrt{r^2 + h^2}) = \varphi(h)\, p\Big(\frac{r}{\psi(h)}\Big) \tag{9}$$

for all $r > 0$. For example, for the density $p_{\mathbf{H},\beta}$, $\beta > d/2$, the scaling property (9) is satisfied with $\varphi(h) = (1 + h^2)^{-\beta}$ and $\psi(h) = \sqrt{1 + h^2}$, see (6). This scaling property was essential when we separated what happens within H from the contribution that arises from the distance of H to the origin. However, all rotationally symmetric densities with (almost everywhere differentiable) radial part satisfying the scaling property (9) with an (almost everywhere differentiable) function ψ have been classified by Miles [10] (see p. 376 there) and Ruben and Miles [13]. They precisely correspond to the distributions in the classes **G**, **H**, **B** as well as to the exceptional distributions in **U**, for which Theorem 1 is formulated.

On the other hand, this does not mean that **G**, **H**, **B** and **U** contain the only rotationally symmetric distributions on $\mathbb{R}^d$ for which such computations are possible. For example, the density with radial part $p_{\beta,j}(r) = c_{\beta,d,j}\, r^{2j}/(1 + r^2)^{\beta}$, $r > 0$, $j \in \{0, 1, 2, \dots\}$ and $\beta > j + d/2$, which does not belong to the class **H** whenever $j > 0$, satisfies the following generalization of the scaling property (9):

$$p_{\beta,j}(\sqrt{r^2 + h^2}) = \sum_{k=0}^{j} \varphi_k(h) p_{\beta,k}\Big(\frac{r}{\psi(h)}\Big)$$

with

$$\varphi_k(h) = \binom{j}{k} h^{2(k-j)} (1 + h^2)^{-\beta} \quad (k = 0, \dots, j) \qquad \text{and} \qquad \psi(h) = \sqrt{1 + h^2}.$$

One can check that the 1-dimensional marginal density of $p_{\beta,j}$ equals

$$f_{\beta,j}(x_1) = \sum_{k=0}^{j} \binom{j}{k} \frac{c_{\beta,d,j}}{c_{\beta,d-1,k}} x_1^{2(k-j)} (1 + x_1^2)^{k + \frac{d-1}{2} - \beta},$$

and that from here on the argument based on the affine Blaschke-Petkantschin formula can be applied term-by-term. Unfortunately, the computations in such and similar situations become quite involved. Moreover, to classify *all* rotationally symmetric densities on $\mathbb{R}^d$ for which these computations can be performed seems to be out of reach.

One might also ask whether the method based on Blaschke-Petkantschin formulae yields monotonicity of the mean facet number in such situations where the random points $X_1, \dots, X_n$ are independent with distributions belonging to one of the classes **G**, **H**, **B** and **U**, but not necessarily the same (a so-called mixed case). That is, some of the X_i's are Gaussian, some distributed according to a distribution in $\mathbf{H}$ etc. (but within each class we choose every time the same scale parameter σ). Unfortunately, the method breaks down. The reason is that each distribution class requires its individual substitution, which is adapted to its respective scaling property (9). The resulting different rescalings in the hyperplane H distort the relationship between the $(d-1)$-volume in H before and after the transformation, cf. [13].

5 Random convex hulls on a half-sphere

In this section we consider an application of Theorem 1 to convex hulls generated by random points on a half-sphere. We fix $d \geq 2$, denote by $\mathbb{S}^d$ the d-dimensional unit sphere in $\mathbb{R}^{d+1}$ and define the half-sphere

$$\mathbb{S}^d_+ = \{y = (y_1, \dots, y_{d+1}) \in \mathbb{S}^d : y_{d+1} > 0\}.$$

Furthermore, we let $\mathbf{S}$ be the class of probability distributions on $\mathbb{S}^d_+$ that have density

$$p_{\mathbf{S},\alpha}(y) = c_{\mathbf{S},\alpha}\, y_{d+1}^{\alpha}, \qquad y = (y_1, \dots, y_{d+1}) \in \mathbb{S}^d_+, \quad \alpha > -1,$$

with respect to the spherical Lebesgue measure on $\mathbb{S}^d_+$. Here, $c_{\mathbf{S},\alpha} > 0$ is a suitable normalization constant. In particular, choosing $\alpha = 0$ shows that the uniform distribution on $\mathbb{S}^d_+$ belongs to the class $\mathbf{S}$.

For fixed $\alpha > -1$ and $n \geq d+1$ we let $X_1, \dots, X_n$ be independent random points that are distributed on $\mathbb{S}^d_+$ according to the density $p_{\mathbf{S},\alpha}$. By S_n we denote the *spherical* convex hull of $X_1, \dots, X_n$, that is, the smallest spherically convex set in $\mathbb{S}^d_+$ containing the points $X_1, \dots, X_n$. For the special choice $\alpha = 0$, this model has recently been studied in [3]. In particular, it is shown in [3] that for this choice of α the mean number of facets $\mathbb{E} f_{d-1}(S_n)$ of the spherical random polytope S_n converges to a finite constant only depending on d, as $n \to \infty$ (a similar result is in fact valid for all distributions in $\mathbf{S}$, see [1, 6, 8]). As a special case, our next result shows the somewhat surprising fact that this limit is approached in a strictly monotone way.

Theorem 7 *Let $X_1, \dots, X_n$, $n \geq d+1$, be independent and identically distributed according to a probability measure belonging to the class $\mathbf{S}$. Then,*

$$\mathbb{E} f_{d-1}(S_n) > \mathbb{E} f_{d-1}(S_{n-1}).$$

Proof. Let $g \colon \mathbb{R}^d \to \mathbb{S}^d_+$ be the mapping defined as

$$g(x) = \left(\frac{x_1}{\sqrt{1+\|x\|^2}}, \dots, \frac{x_d}{\sqrt{1+\|x\|^2}}, \frac{1}{\sqrt{1+\|x\|^2}} \right),$$

with inverse given by

$$g^{-1}(y) = \left(\frac{y_1}{y_{d+1}}, \dots, \frac{y_d}{y_{d+1}} \right)$$

(this is known as the gnomonic projection). Let $\mathrm{D}g$ be the Jacobian matrix of g and put $J_g(x) := \sqrt{\det \mathrm{D}g(x)^{\mathsf{T}} \mathrm{D}g(x)}$. Then, it holds that

$$J_g(x) = (1+\|x\|^2)^{-\frac{d+1}{2}},$$

see [5, Proposition 4.2]. Moreover, for a measurable subset $A \subset \mathbb{R}^d$ and a measurable function $f\colon A \to \mathbb{R}$ the area formula [9, Theorem 3.2.3] says that

$$\int_A f(x)\,\mathrm{d}x = \int_{g(A)} f \circ g^{-1}(y)(J_g \circ g^{-1}(y))^{-1}\,\mathcal{H}^d_{\mathbb{S}^d_+}(\mathrm{d}y).$$

Next, we notice that $1 + \|g^{-1}(y)\|^2 = y_{d+1}^{-2}$ and apply the formula with $f(x) = p_{\mathbf{H},\beta}(x)$ for some $\beta > d/2$:

$$\int_A c_{\mathbf{H},d,\beta}\,(1+\|x\|^2)^{-\beta}\,\mathrm{d}x = \int_{g(A)} c_{\mathbf{H},d,\beta}\,y_{d+1}^{2\beta-d-1}\,\mathcal{H}^d_{\mathbb{S}^d_+}(\mathrm{d}y),$$

where $c_{\mathbf{H},d,\beta} = \pi^{-d/2}\Gamma(\beta)/\Gamma(\beta-\frac{d}{2})$ is the normalization constant of the density $p_{\mathbf{H},\beta}$ defined in (4). As a result, we see that the density $p_{\mathbf{S},2\beta-d-1}$ on $\mathbb{S}^d_+$ is the push-forward of the density $p_{\mathbf{H},\beta}$ on $\mathbb{R}^d$ under g. Note also that $2\beta - d - 1 > -1$ since $\beta > d/2$ and that the uniform measure on the half-sphere corresponds to the choice $\beta = (d+1)/2$. The above discussion shows the following. Let P_n be the random convex hull in $\mathbb{R}^d$ generated by n independent points with density $p_{\mathbf{H},\beta}$. Then, the push-forward of P_n has the same distribution as the spherical random polytope S_n with $\alpha = 2\beta - d - 1$. Moreover, the facets of P_n are in one-to-one correspondence with those of S_n. As a consequence, the mean facet number of the spherical random polytope S_n is the same as the mean facet number of the random convex hull P_n, i.e.,

$$\mathbb{E} f_{d-1}(S_n) = \mathbb{E} f_{d-1}(P_n).$$

Thus, the monotonicity follows from Theorem 1. □

Acknowledgement

We would like to thank Zakhar Kabluchko (Münster) for a number of useful conversations and for bringing references [8] and [13] to our attention. Thanks go also to Matthias Reitzner (Osnabrück) for his valuable remarks on an earlier version of this text. We thank the anonymous referee for helpful hints and suggestions.
JG, DT and NT have been supported by the Deutsche Forschungsgemeinschaft (DFG) via RTG 2131 *High-dimensional Phenomena in Probability – Fluctuations and Discontinuity*.

References

[1] Aldous, D., Fristedt, D., Griffin, P. and Pruitt, W.: *The number of extreme points in the convex hull of a random sample*. J. Appl. Probab. **28**, 287–304 (1991).

[2] Axler, S., Bourdon, P. and Ramey, W.: *Harmonic Function Theory*. Volume **137** of Graduate Texts in Mathematics, Springer, New York, second edition (2001).

[3] Bárány, I., Hug, D., Reitzner, M. and Schneider, R.: *Random points in halfspheres*. Random Structures Algorithms **50**, 3–22 (2017).

[4] Beermann, M.: *Random Polytopes*. Dissertation, University of Osnabrück (2014).

[5] Besau, F. and Werner, W.: *The spherical convex floating body*. Adv. Math. **301**, 867–901 (2016).

[6] Carnal, H.: *Die konvexe Hülle von n rotationssymmetrisch verteilten Punkten*. Z. Wahrscheinlichkeitstheorie verw. Geb. **15**, 168–176 (1970).

[7] Devillers, O., Glisse, M., Goaoc, X., Moroz, G. and Reitzner, M.: *The monotonicity of f-vectors of random polytopes*. Electron. Commun. Probab. **18**, article 23 (2013).

[8] Eddy, W.F. and Gale, J.D.: *The convex hull of a spherically symmetric sample*. Adv. in Appl. Probab. **13**, 751–763 (1981).

[9] Federer, H.: *Geometric Measure Theory*. Springer, Berlin (1969).

[10] Miles, R.E.: *Isotropic random simplices*. Adv. in Appl. Probab. **3**, 353-382 (1971).

[11] Rademacher, L.: *On the monotonicity of the expected volume of a random simplex*. Mathematika **58**, 77–91 (2012).

[12] Reichenwallner, B. and Reitzner, M.: *On the monotonicity of the moments of volumes of random simplices*. Mathematika **62**, 949–958 (2016).

[13] Ruben, H. and Miles, R.E.: *A canonical decomposition of the probability measure of sets of isotropic random points in* $\mathbb{R}^n$. J. Multivariate Anal. **10**, 1–18 (1980).

[14] Schneider, R. and Weil, W.: *Stochastic and Integral Geometry*. Springer, Berlin (2008).

[15] Zähle, M.: *A kinematic formula and moment measures of random sets*. Math. Nachr. **149**, 325–340 (1990).