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# Hadronic light-by-light contribution to $(\mathbf{g}-\mathbf{2})_\mu$: a dispersive approach

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**Abstract.** After a brief introduction on ongoing experimental and theoretical activities on $(g-2)_\mu$, we report on recent progress in approaching the calculation of the hadronic light-by-light contribution with dispersive methods. General properties of the four-point function of the electromagnetic current in QCD, its Lorentz decomposition and dispersive representation are discussed. On this basis a numerical estimate for the pion box contribution and its rescattering corrections is obtained. We conclude with an outlook for this approach to the calculation of hadronic light-by-light.



## 1 Introduction

The measured value of the anomalous magnetic moment of the muon $a_\mu$, obtained by the BNL E821 experiment [1], represents a puzzle for the standard model (SM): it differs by about three standard deviations from the calculated value (see e.g. [2, 3]). Taken at face value this is a serious discrepancy, but before claiming a real crisis for the SM or plain discovery of new physics, it is important to make sure that systematic effects, either on the theory or on the experimental side, have not been underestimated. Experimentally this requires redoing the measurement, ideally in a completely new setting. This is the aim of the Muon $g-2$ experiment [4] which has started to run at Fermilab and aims to reduce the final uncertainty reached by the BNL E821 experiment by about a factor four. This experiment is reusing the same ring and conceptual design of the Brookhaven experiment. A completely different setting has instead been adopted by the J-PARC E34 experiment [5], which however still needs to be approved and will start running only in a few years from now.

On the theory side there has been impressive progress in recent years in the calculation of pure QED contributions [6–9]. Also electroweak contributions, which are known to two loops, are under good control and have survived further more recent tests and double checks [10, 11]. None of these improvements or checks has had any significant impact neither on the central value nor on the error, essentially because their contributions to the theoretical uncertainty is negligible with respect to that coming from hadronic contributions. The latter are indeed the most crucial (for the error estimate) and critical (for the central value) contributions to the SM value and a lot of theoretical activity has

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been and still is devoted to improving their estimate. Quite a substantial part of it concerns lattice calculations.

Hadronic contributions can be classified based on the order in $\alpha$ and the topology. At order $\alpha^2$ there is the hadronic vacuum polarization contribution to the vertex-correction diagram, usually called *tout court* hadronic vacuum polarization (HVP). At order $\alpha^3$, there are next-to-leading (NLO) order HVP diagrams as well as hadronic light-by-light (HLbL) contributions. At order $\alpha^4$ there are NNLO HVP diagrams and NLO HLbL diagrams. Going beyond LO for each topology is important for the central value estimate and for checking that there are no surprises, but for the uncertainty estimate it is the LO contributions which matter. As it turns out, the two topologies contribute about the same to the total uncertainty, even though HLbL is suppressed by one order of $\alpha$. The reason is simple: the calculation of the HVP is based on an exact relation, derived from analyticity and unitarity, which allows one to express this contribution as an integral over the measurable cross section $\sigma(e^+e^- \to \text{hadrons})$. The latter cross section has actually been measured with high accuracy, especially at low energy, which is the region contributing with the highest weight, so that the calculation of the relevant integral can be performed with subpercent accuracy. For HLbL such a relation did not exist until recently and has been derived in a series of recent papers [12–16]. This is not as simple and effective as the one for HVP, where all intermediate states contribute in the same form: for HLbL different intermediate states appear in different integrals and the explicit form of these integrals has been derived only for up to two-particle intermediate states. Moreover, the measurable quantities that enter these integrals (*e.g.* the $\gamma^*\gamma^* \to \pi\pi$ helicity amplitudes) have not yet been measured other than in corners of the phase space (for two or one real photons). Complete estimates of the HLbL which have appeared so far are therefore based on models [17–22] (an alternative dispersive approach, where the whole muon form factor is treated dispersively has been proposed in Ref. [23]).

In this contribution I will briefly report on recent progress in the calculation of the HLbL contribution on the lattice and discuss in some more detail the dispersive approach we have developed, in particular illustrating our first numerical estimate of the pion box and the corresponding rescattering contribution. The status of the calculations of the HVP contribution is covered by the talk given by Christoph Lehner [24].

## 2 Hadronic light-by-light on the lattice

Two lattice collaborations have started different approaches to calculate the HLbL contribution. Until recent years it was not known how this contribution could be calculated on the lattice and at the present stage it is not clear which of the approaches tried so far will be the most effective and will reach the highest precision in the long run.

### 2.1 The RBC/UKQCD approach

The first attempt to perform a lattice calculation of the HLbL contribution to $(g-2)_\mu$ is due to Blum et al. [25]. The method proposed relied on putting on the lattice not only quarks and gluons, but also photons and muons and integrating over the two loops involving the muon and photon propagators with Monte Carlo methods on the lattice. The calculation is nontrivial, not only because the hadronic object to be considered is complicated, but also because, by choosing to include also the photons in the calculation, all the difficulties related to having massless photons on the lattice affect this calculation too. In particular one expects quite significant finite-volume effects due to the photons (see [26]). The first calculation only considered connected diagrams and was performed with unphysical quark masses, whereas a more recent one [27, 28], performed along the same lines, included the leading disconnected diagram and was done at the physical point. No attempt at taking the continuum limit was made yet, since the calculation was performed at a single lattice spacing.

The result obtained reads:

$$a_\mu^{\mathrm{HLbL}} = (5.35 \pm 1.35) \cdot 10^{-10} \quad , \tag{1}$$

which is about half of the most recent estimates based on models [29, 30]. Given the exploratory nature of the calculation and the fact that it has been performed at a single lattice spacing, it is premature to discuss about numerical differences. The main message to be taken from the impressive RBC/UKQCD effort is that the calculation is possible with current computers and that the chosen approach seems to work well. Further efforts in this direction promise to provide a number with a controlled uncertainty estimate, however the problems related to the finite-volume effects can be more efficiently solved with a different approach, proposed by the Mainz group (see below), which has recently been adopted also by the RBC/UKQCD collaboration [31].

### 2.2 The Mainz approach

The approach followed by the Mainz group is based on the following formula which expresses the contribution as an integral in position (rather than in momentum) space [32]:

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int d^4y \Big[ \int d^4x \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\mathrm{QED}} \; \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\mathrm{QCD}} \Big] \; . \tag{2}$$

The advantage of such an approach is that the photon propagators are handled analytically and taken care of by the kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$, which has been calculated exactly. In this way all the problems related to the formulation of QED on the lattice, and in particular in finite volume, are completely overcome. Effectively one is using QED in infinite volume. Moreover it is much closer to what is actually calculated on the lattice, where it is position space which is discretized. The approach has been successfully tested numerically by inserting in $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$ the explicit expression for a muon loop and performing the calculation on the lattice. The outcome reproduced to percent accuracy the known result for the muonic light-by-light contribution [33]. A similar test has also been successfully performed by the RBC/UKQCD collaboration [31].

The Mainz group has also followed a different path, namely to calculate explictly the pion transition form factor on the lattice. This is only one of the contributions to the HLbL tensor (see below for a precise definition thereof), but arguably the most important one. Moreover, experimental data on this form factor are available only for the singly-virtual case, and efforts to measure the doubly-virtual configurations are plagued by serious difficulties. On the other hand this is not a particularly difficult calculation on the lattice, and the one done by the Mainz group shows that it can be performed with a very good accuracy:

$$a_{\mu,LMD+V}^{\mathrm{HLbL},\pi^0} = (6.50 \pm 0.83) \cdot 10^{-10} \tag{3}$$

of about 13% [34, 35]: investing more efforts and more computer time in such a calculation, has the potential to bring the accuracy of this contribution to well below 10%, which is sufficient for the present purpose. Note that although the calculation is done from first principles and aims to be model independent, in order to perform the full integral over the photon and muon loops one needs to model the $q_i^2$ dependence of the pion transition form factor even beyond the region where the lattice calculation has been made. Technically this means that one chooses a parametrization to describe the $q^2$ dependence and fixes the values of the parameters to fit the lattice data. In this way the result of the integration depends on the chosen parametrization — this explains the label "LMD+V" attached to the result, which indicates the kind of parametrization used. The final aim is to cover on the lattice a large enough region in $q^2$ such that the model-dependence due to the parametrization becomes negligible.

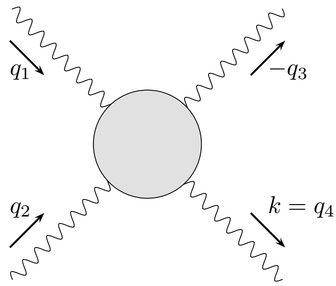



**Figure 1.** Kinematics of the light-by-light scattering amplitude.

## 3 Dispersive approach: preliminaries

In order to attack the calculation of the HLbL dispersively a few preliminary steps are necessary. For simpler objects, like the two-point function of the electromagnetic current, which is relevant for the HVP contribution, these steps are usually performed without even mentioning them, because they are almost trivial. For the two-point function of the electromagnetic current Lorentz invariance allows one to decompose the tensor into two independent structures. Gauge invariance reduces the two independent structures to a single one. Going from the two- to the four-point function the increase in complexity is baffling: the number of independent Lorentz structures jumps from 2 to 138 (136 in 4 dimensions, see [36]). Also the implementation of gauge invariance becomes significantly more complex – especially if one wants to obtain a basis which is free from kinematic singularities and zeros – but luckily a general procedure has been devised long ago by Bardeen and Tung [37], with an important addendum pointed out by Tarrach [38]. This can be applied also in this case without special difficulties other than those due to the inherent complexity of the problem.

### 3.1 Lorentz and gauge invariant decomposition

The HLbL tensor is the Green's function of four electromagnetic currents, evaluated in pure QCD:

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int d^4x\, d^4y\, d^4z\, e^{-i(q_1\cdot x + q_2\cdot y + q_3\cdot z)} \langle 0|T\{j^\mu_{\text{em}}(x) j^\nu_{\text{em}}(y) j^\lambda_{\text{em}}(z) j^\sigma_{\text{em}}(0)\}|0\rangle. \tag{4}$$

The electromagnetic current above is built out of the three lightest quarks only:

$$j^\mu_{\text{em}} := \bar{q} Q \gamma^\mu q, \tag{5}$$

where $q = (u, d, s)^T$ and $Q = \text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$. We define

$$q_4 := k = q_1 + q_2 + q_3, \tag{6}$$

and illustrate the kinematics in Fig. 1.

As invariant variable we adopt the usual Mandelstam variables:

$$s := (q_1 + q_2)^2, \quad t := (q_1 + q_3)^2, \quad u := (q_2 + q_3)^2, \qquad s + t + u = \sum_{i=1}^{4} q_i^2 =: \Sigma \tag{7}$$

(the limit $k^2 = 0$ will be considered later). The Ward–Takahashi identities implied by gauge invariance have the form

$$\{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = 0. \tag{8}$$

### 3.2 Tensor decomposition

As mentioned above the HLbL tensor can be decomposed into 138 Lorentz structures [39–41]:

$$
\begin{aligned}
\Pi^{\mu\nu\lambda\sigma} &= g^{\mu\nu}g^{\lambda\sigma}\,\Pi^1 + g^{\mu\lambda}g^{\nu\sigma}\,\Pi^2 + g^{\mu\sigma}g^{\nu\lambda}\,\Pi^3 + \sum_{\substack{i=2,3,4\\ j=1,3,4}}\sum_{\substack{k=1,2,4\\ l=1,2,3}} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma\,\Pi^4_{ijkl} \\
&\quad + \sum_{\substack{i=2,3,4\\ j=1,3,4}} g^{\lambda\sigma}q_i^\mu q_j^\nu\,\Pi^5_{ij} + \sum_{\substack{i=2,3,4\\ k=1,2,4}} g^{\nu\sigma}q_i^\mu q_k^\lambda\,\Pi^6_{ik} + \sum_{\substack{i=2,3,4\\ l=1,2,3}} g^{\nu\lambda}q_i^\mu q_l^\sigma\,\Pi^7_{il} \\
&\quad + \sum_{\substack{j=1,3,4\\ k=1,2,4}} g^{\mu\sigma}q_j^\nu q_k^\lambda\,\Pi^8_{jk} + \sum_{\substack{j=1,3,4\\ l=1,2,3}} g^{\mu\lambda}q_j^\nu q_l^\sigma\,\Pi^9_{jl} + \sum_{\substack{k=1,2,4\\ l=1,2,3}} g^{\mu\nu}q_k^\lambda q_l^\sigma\,\Pi^{10}_{kl} \quad . 
\end{aligned} \tag{9}
$$

The 138 scalar functions $\{\Pi^1, \Pi^2, \Pi^3, \Pi^4_{ijkl}, \Pi^5_{ij}, \Pi^6_{ik}, \Pi^7_{il}, \Pi^8_{jk}, \Pi^9_{jl}, \Pi^{10}_{kl}\}$ depend on six independent kinematic variables: the two Mandelstam variables $s$ and $t$ and the virtualities $q_1^2$, $q_2^2$, $q_3^2$, and $q_4^2$. They are free of kinematic singularities but since they have to fulfill kinematic constraints required by gauge invariance, they must have kinematic zeros. The Ward identities (8) impose 95 linearly independent relations on the scalar functions, reducing the set to 43 functions. To obtain these we apply the recipe devised by Bardeen and Tung [37], but this does not lead to a minimal basis free of kinematic singularities, as shown by Tarrach [38]. Following the latter we have constructed a redundant set of 54 structures, which is free of kinematic singularities and zeros.

The resulting representation of the HLbL tensor which we have obtained in this way reads

$$
\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\Pi_i, \tag{10}
$$

where

$$
\begin{aligned}
T_1^{\mu\nu\lambda\sigma} &= \epsilon^{\mu\nu\alpha\beta}\epsilon^{\lambda\sigma\gamma\delta}q_{1\alpha}q_{2\beta}q_{3\gamma}q_{4\delta}, \\
T_4^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_4^\lambda q_3^\sigma - q_3\cdot q_4 g^{\lambda\sigma}\right), \\
T_7^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_1\cdot q_4\left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_1^\sigma q_1\cdot q_3 - q_1^\lambda q_1^\sigma q_3\cdot q_4\right), \\
T_{19}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2\cdot q_4\left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_2^\sigma q_1\cdot q_3 - q_1^\lambda q_2^\sigma q_3\cdot q_4\right), \\
T_{31}^{\mu\nu\lambda\sigma} &= \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2^\lambda q_1\cdot q_3 - q_1^\lambda q_2\cdot q_3\right)\left(q_2^\sigma q_1\cdot q_4 - q_1^\sigma q_2\cdot q_4\right), \\
T_{37}^{\mu\nu\lambda\sigma} &= \left(q_3^\mu q_1\cdot q_4 - q_4^\mu q_1\cdot q_3\right)\Big(q_3^\nu q_4^\lambda q_2^\sigma - q_4^\nu q_2^\lambda q_3^\sigma + g^{\lambda\sigma}\left(q_4^\nu q_2\cdot q_3 - q_3^\nu q_2\cdot q_4\right) \\
&\qquad + g^{\nu\sigma}\left(q_2^\lambda q_3\cdot q_4 - q_4^\lambda q_2\cdot q_3\right) + g^{\lambda\nu}\left(q_3^\sigma q_2\cdot q_4 - q_2^\sigma q_3\cdot q_4\right)\Big), \\
T_{49}^{\mu\nu\lambda\sigma} &= q_3^\sigma\Big(q_1\cdot q_3 q_2\cdot q_4 q_4^\mu g^{\lambda\nu} - q_2\cdot q_3 q_1\cdot q_4 q_4^\nu g^{\lambda\mu} + q_4^\mu q_4^\nu\left(q_1^\lambda q_2\cdot q_3 - q_2^\lambda q_1\cdot q_3\right) \\
&\qquad + q_1\cdot q_4 q_3^\mu q_4^\nu q_2^\lambda - q_2\cdot q_4 q_4^\mu q_3^\nu q_1^\lambda + q_1\cdot q_4 q_2\cdot q_4\left(q_3^\nu g^{\lambda\mu} - q_3^\mu g^{\lambda\nu}\right)\Big) \\
&\quad - q_4^\lambda\Big(q_1\cdot q_4 q_2\cdot q_3 q_3^\mu g^{\nu\sigma} - q_2\cdot q_4 q_1\cdot q_3 q_3^\nu g^{\mu\sigma} + q_3^\mu q_3^\nu\left(q_1^\sigma q_2\cdot q_4 - q_2^\sigma q_1\cdot q_4\right) \\
&\qquad + q_1\cdot q_3 q_4^\mu q_3^\nu q_2^\sigma - q_2\cdot q_3 q_3^\mu q_4^\nu q_1^\sigma + q_1\cdot q_3 q_2\cdot q_3\left(q_4^\nu g^{\mu\sigma} - q_4^\mu g^{\nu\sigma}\right)\Big) \\
&\quad + q_3\cdot q_4\Big(\left(q_1^\lambda q_4^\mu - q_1\cdot q_4 g^{\lambda\mu}\right)\left(q_3^\nu q_2^\sigma - q_2\cdot q_3 g^{\nu\sigma}\right) - \left(q_2^\lambda q_4^\nu - q_2\cdot q_4 g^{\lambda\nu}\right)\left(q_3^\mu q_1^\sigma - q_1\cdot q_3 g^{\mu\sigma}\right)\Big).
\end{aligned} \tag{11}
$$

All the remaining structures are just crossed versions of the above seven structures, as shown in Ref. [14]. Since the HLbL tensor $\Pi^{\mu\nu\lambda\sigma}$ is totally crossing symmetric, the scalar functions $\Pi_i$ have to fulfill exactly the same crossing properties of the corresponding Lorentz structures, which we have not given here, but which can again be found in [14]. Therefore, only seven different scalar functions $\Pi_i$ appear, together with their crossed versions. These 54 scalar functions are free of kinematic singularities and zeros and hence fulfill a Mandelstam representation. They are suitable quantities for a dispersive description.

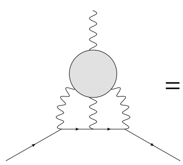
### 3.3 Master formula

As is well known, using projector techniques and angular averaging (see [42, 43]), the anomalous magnetic moment of the muon can be expressed as

$$a_\mu = \mathrm{Tr}\left(\left(\frac{1}{12}\gamma^\mu - \frac{1}{3}\left(\frac{p^\mu \not{p}}{m_\mu^2}\right) - \frac{1}{4}\frac{p^\mu}{m_\mu}\right)V_\mu(p)\right) - \frac{1}{48m_\mu}\mathrm{Tr}\left((\not{p}+m_\mu)[\gamma^\mu,\gamma^\rho](\not{p}+m_\mu)\Gamma_{\mu\rho}(p)\right), \tag{12}$$

where now $p^2 = m_\mu^2$.[1]

The contribution of the HLbL tensor to $a_\mu$, represented diagrammatically as



$$= (-ie)\bar{u}(p_2)\Gamma^\mu_{\mathrm{HLbL}}(p_1,p_2)u(p_1), \tag{13}$$

can be written as

$$\begin{aligned}\Gamma^\sigma_{\mathrm{HLbL}}(p_1,p_2) = -e^6 \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\gamma_\mu \frac{(\not{p}_2+\not{q}_1+m_\mu)}{(p_2+q_1)^2-m_\mu^2}\gamma_\lambda \frac{(\not{p}_1-\not{q}_2+m_\mu)}{(p_1-q_2)^2-m_\mu^2}\gamma_\nu \\ \times \frac{1}{q_1^2 q_2^2 (p_1-p_2-q_1-q_2)^2}\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,p_1-p_2-q_1-q_2).\end{aligned} \tag{14}$$

The HLbL tensor has been defined in (4). Differentiating the fourth Ward identity in (8) with respect to $k_\rho = (q_1+q_2+q_3)_\rho$ yields

$$\Pi_{\mu\nu\lambda\rho}(q_1,q_2,k-q_1-q_2) = -k^\sigma \frac{\partial}{\partial k^\rho}\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,k-q_1-q_2). \tag{15}$$

It was already argued in [44] that $\Pi_{\mu\nu\lambda\sigma}$ vanishes linearly with $k$ (*i.e.* the derivative contains no singularity), and so must $\Gamma^{\mathrm{HLbL}}_\sigma$. This is easily verified with our tensor decomposition (10). Therefore, the HLbL contribution to the anomalous magnetic moment is given by

$$a_\mu^{\mathrm{HLbL}} = -\frac{1}{48m_\mu}\mathrm{Tr}\left((\not{p}+m_\mu)[\gamma^\rho,\gamma^\sigma](\not{p}+m_\mu)\Gamma^{\mathrm{HLbL}}_{\rho\sigma}(p)\right), \tag{16}$$

where

$$\Gamma^{\mathrm{HLbL}}_{\rho\sigma}(p) = \frac{\partial}{\partial k^\sigma}\Gamma^{\mathrm{HLbL}}_\rho(p_1,p_2)\bigg|_{k=0}. \tag{17}$$

[1]Note that $k$ is defined as outgoing, resulting in the different sign of the second term with respect to [43].

We use the Ward identity (15) to write

$$\Gamma^{\mathrm{HLbL}}_{\rho\sigma}(p) = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \gamma^\mu \frac{(\not p + \not q_1 + m_\mu)}{(p+q_1)^2 - m_\mu^2} \gamma^\lambda \frac{(\not p - \not q_2 + m_\mu)}{(p-q_2)^2 - m_\mu^2} \gamma^\nu \times \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2} \frac{\partial}{\partial k^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)\Big|_{k=0}, \tag{18}$$

after taking the derivative and the limit $k_\mu \to 0$.

After a number of intermediate steps, which include performing five of the eight loop integrals by changing to spherical coordinates in four dimensions and applying Gegenbauer polynomial techniques we have obtained a master formula for the HLbL contribution to the anomalous magnetic moment of the muon:

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau), \tag{19}$$

where $Q_1 := |Q_1|$, $Q_2 := |Q_2|$. The hadronic scalar functions $\bar{\Pi}_i$ are linear combinations of the $\Pi_i$. They have to be evaluated for the reduced kinematics

$$\begin{aligned} s &= -Q_3^2 = -Q_1^2 - 2Q_1Q_2\tau - Q_2^2, \quad t = -Q_2^2, \quad u = -Q_1^2, \\ q_1^2 &= -Q_1^2, \quad q_2^2 = -Q_2^2, \quad q_3^2 = -Q_3^2 = -Q_1^2 - 2Q_1Q_2\tau - Q_2^2, \quad k^2 = q_4^2 = 0. \end{aligned} \tag{20}$$

The integral kernels $T_i$, provided in [16] are fully general for any light-by-light process, while the scalar functions $\Pi_i$ parametrize the hadronic content of the master formula. In particular, (19) can be considered a generalization of the three-dimensional integral formula for the pion-pole contribution [30]. It is valid for the whole HLbL contribution and completely generic, *i.e.* it can be used to compute the HLbL contribution to $(g-2)_\mu$ for any representation of the HLbL tensor, *i.e.* of the scalar functions.

Like in the case of the pion-pole contribution [42], the master formula (19) offers the great advantage of providing a representation of the HLbL contribution to the $(g-2)_\mu$ in terms of a three-dimensional integral, which is well-suited for a direct numerical implementation. In particular, the energy regions generating the bulk of the contribution can be identified by numerically integrating over $\tau$ and plotting the integrand as a function of $Q_1$ and $Q_2$ [18, 42, 45, 46].

### 3.4 An ordering principle

An important difference between the two-point function which is relevant for HVP and the four-point function of HLbL is that the dispersion relation for the former is in only one variable and that the discontinuity is given by the imaginary part which, thanks to unitarity, is related to an observable: the cross section $e^+e^- \to$ hadrons. Many intermediate states contribute to the discontinuity, but they all do with the same weight function inside the integral:

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\mathrm{thr}}}^\infty \frac{ds}{s^2} \hat{K}(s) R_{\mathrm{had}}(s) \quad \text{where} \quad R_{\mathrm{had}}(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{4\pi\alpha(s)^2/3s} \tag{21}$$

and $\hat{K}(s)$ is the integration kernel which grows monotonically from about 0.63 at the two-pion threshold up to 1 at $s = \infty$. The lower limit of integration $s_{\mathrm{thr}}$ is equal to $4M_\pi^2$ at $O(\alpha^0)$, but if one includes photons in the hadronic final state then it becomes $M_{\pi^0}^2$.

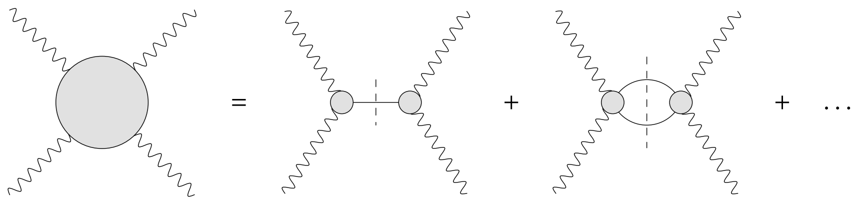



**Figure 2.** Intermediate states in the direct channel: pion pole and two-pion cut.

In the case of HLbL there is no such a simple formula, first of all because the independent kinematic variables are two instead of only one (we consider a dispersion relation at fixed $q_i^2$, so it is only two of the three Mandelstam variables which are independent). The situation is analogous to that of a scattering amplitude, which has been studied in depth and treated with different kinds of dispersion relations. The most general thing one can do in this case is to write down a Mandelstam representation (assuming that it holds). This has indeed be done in [14], but then the practical usefulness of such a representation is limited by the fact that the double spectral functions, which completely determine the scalar BTT functions, are not observables and cannot be directly measured in an inclusive manner, such that all intermediate states contributing are taken into account at the same time. The only possible way to make use of such a representation is to consider individual intermediate states and for each of these construct a relation between the double-spectral function and the relevant observable. By doing this one obtains a dispersive representation of the HLbL tensor as a sum of contributions of different, fully specified intermediate states, and for each term in the sum there is an explicit relation to the relevant observable.

Since it is impossible to include all possible intermediate states, the question arises whether one can find an ordering principle for these, such that one could concentrate on the most important ones, treat these explicitly and neglect the rest. This is the approach which has been adopted from the very start in this series of papers and has been described first in [12]. The basis of this approach is not an algebraically defined counting scheme, which so far has not been possible to derive, but simply the observation that in all model calculations, the importance of the contribution of an intermediate state decreases as the corresponding threshold increases. As it is well known, the pion-pole contribution is the dominating one overall and is more important than that of other single-particle intermediate states, like the $\eta$ or the $\eta'$ (with the former more important than the latter). The one-pion contribution is more important than the two-pion contribution, which in turn is more important than the two-kaon one, and so on. In order to set up the dispersive calculation and obtain the bulk of the total contribution it was argued in [12] that one could take into account only one- and two-pion intermediate states, as illustrated in Fig. 2. This allows one to break down the HLbL contribution as follows

$$\Pi_{\mu\nu\lambda\sigma}(s,t,u) = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma}(s,t,u) + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma}(s,t,u) + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma}(s,t,u) + \cdots \tag{22}$$

where the first term $\Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma}$ is the one generated by the exchange of a $\pi^0$ in one of the channels ($s$ or $t$ or $u$), the second one $\Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma}$ has two-pion discontinuities simultaneously in two channels ($s$ and $t$ or $t$ and $u$ or $s$ and $u$), (see first diagram after the equal sign in Fig. 3) whereas the third one $\Pi^{\pi\pi}_{\mu\nu\lambda\sigma}$ has a two-pion cut only in one of the three channels (second to fourth diagram after the equal sign in Fig. 3). The ellipsis stands for singularities with higher masses or thresholds. Note that even

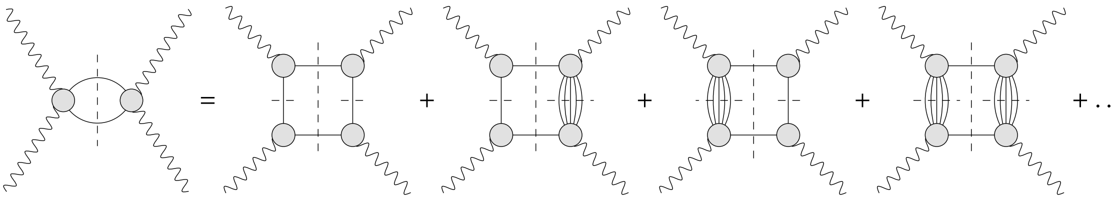



**Figure 3.** Two-pion contributions to HLbL. Further crossed diagrams are not shown explicitly.

though we restrict ourselves to pions, what really matters for the formalism is the number of particles: applying the formalism to $\eta$ or $\eta'$ exchange, or two-kaon intermediate states is a trivial extension.

### 3.5 Pion pole

In a dispersive calculation the key step is to identify the singularities: what defines the pion pole contribution is that the imaginary part is given by a $\delta$-function, with a tensorial structure and strength fully determined by the $\pi^0-\gamma\gamma$ vertex. This translates into the following expression for the imaginary part in the $s$ channel:

$$\begin{aligned}
&\mathrm{Im}_s^\pi \left(e^4(2\pi)^4\delta^{(4)}(q_1+q_2+q_3-q_4)H_{\lambda_1\lambda_2,\lambda_3\lambda_4}\right) \\
&\quad = \frac{1}{2}\int \widetilde{dp}\ \langle\gamma^*(-q_3,\lambda_3)\gamma^*(q_4,\lambda_4)|\pi^0(p)\rangle^*\langle\gamma^*(q_1,\lambda_1)\gamma^*(q_2,\lambda_2)|\pi^0(p)\rangle
\end{aligned} \tag{23}$$

which, after reducing the matrix elements and using the definition of the pion transition form factor

$$i\int d^4x\, e^{iqx}\langle 0|T\{j^\mu_{\mathrm{em}}(x)j^\nu_{\mathrm{em}}(0)\}|\pi^0(p)\rangle = \epsilon^{\mu\nu\alpha\beta}q_\alpha p_\beta \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q^2,(q-p)^2), \tag{24}$$

leads to

$$\begin{aligned}
\mathrm{Im}_s^\pi\Pi^{\mu\nu\lambda\sigma} &= -\frac{1}{2}\int\widetilde{dp}\ (2\pi)^4\delta^{(4)}(q_1+q_2-p)\epsilon^{\mu\nu\alpha\beta}\epsilon^{\lambda\sigma\gamma\delta}q_{1\alpha}q_{2\beta}q_{3\gamma}q_{4\delta}\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_3^2,q_4^2) \\
&= -\pi\delta(s-M_\pi^2)\epsilon^{\mu\nu\alpha\beta}\epsilon^{\lambda\sigma\gamma\delta}q_{1\alpha}q_{2\beta}q_{3\gamma}q_{4\delta}\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_3^2,q_4^2).
\end{aligned} \tag{25}$$

Finally we only need to project this expression onto the scalar functions $\Pi_i$, which leads to

$$\rho^t_{i;s} = \begin{cases}\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_3^2,q_4^2) & i=1,\\ 0 & i\neq 1,\end{cases} \tag{26}$$

and, analogously,

$$\rho^t_{i;u} = \begin{cases}\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_4^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2,q_3^2) & i=3,\\ 0 & i\neq 3\end{cases} \tag{27}$$

for the two contributions proportional to a $\delta$-function in the fixed-$t$ dispersion relation (as indicated by the superscript). By considering also fixed-$s$ and fixed-$u$ dispersion relations and symmetrizing the result, we obtain the following expression for the total pion-pole contribution:

$$\Pi_i^{\pi^0\text{-pole}}(s,t,u) = \frac{\rho_{i,s}}{s-M_\pi^2}+\frac{\rho_{i,t}}{t-M_\pi^2}+\frac{\rho_{i,u}}{u-M_\pi^2} \tag{28}$$

where

$$\begin{aligned}
\rho_{i,s} &= \delta_{i1}\ \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_3^2,q_4^2),\\
\rho_{i,t} &= \delta_{i2}\ \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_3^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2,q_4^2),\\
\rho_{i,u} &= \delta_{i3}\ \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_4^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2,q_3^2).
\end{aligned}$$

Only the first three of the 54 scalar functions receive a contribution from the pion pole.

This result can now be inserted into our master formula Eq. (19) which provides a fully explicit representation of the HLbL pion-pole contribution to $(g-2)_\mu$ as a three-dimensional integral with as hadronic matrix element in the integrand the pion transition form factor. Such a representation is not new and was first derived in Ref. [42], but the fact that our result agrees with it represents a welcome check on our master formula which is much more general than that.

We will not dwell on the numerics here, because we have nothing new to contribute or to report: the main difficulty in improving the numerical evaluation of this contribution is related to obtaining a reliable representation of the pion transition form factor for both photons off-shell. Efforts in this direction are being made both with a dispersive approach [47] (for related work providing essential input see [48, 49]) as well as on the lattice [34, 35], as already mentioned in Sect. 2.2.

## 4 Pion Box

The second contribution to the HLbL tensor and to $(g-2)_\mu$ which we consider is the one given by the so-called "pion box". By this we mean a contribution generated by a simultaneous cut due to two-pion intermediate states in two of the three Mandelstam channels. Since the singularity is completely determined by the configuration in which the intermediate states are on-shell, and since in this case, as illustrated in Fig. 3, all four pions in the diagram contribute to the singularity and have to be put on-shell, the only unknown hadronic matrix element in this contribution is the matrix element of an electromagnetic current between two on-shell pions, which is (if we neglect isospin breaking) nothing but the vector form factor of the pion:

$$\langle \pi^i(p_2)|\bar{q}\lambda_3\gamma_\mu q|\pi^j(p_1)\rangle = i\varepsilon^{i3j}(p_1+p_2)_\mu F_\pi^V((p_1-p_2)^2)\ . \tag{29}$$

Since the $q_i^2$ variables are completely independent of the (two independent) Mandelstam variables, and the singularities depend only on the latter, the $q_i^2$ dependence does not play a role when one reconstructs the full pion-box contribution in terms of its singularity. Since the latter singularity is completely identical to the one appearing in the scalar-QED (sQED) one-loop contribution to HLbL, we can express the contribution of the pion box as follows:

$$\Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma}(s,t,u) = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)F_\pi^V(q_4^2)\Pi^{\text{sQED}}_{\mu\nu\lambda\sigma}(s,t,u)\ . \tag{30}$$

The statement that the singularities of the pion box are identical to those of the one-loop sQED calculation may appear puzzling at first, especially if one considers the Feynman diagrams in sQED, which are shown in Fig. 4, but has been proven explicitly in [14]. There the sQED one-loop contribution was calculated explicitly and then projected onto the BTT basis functions: the corresponding Mandelstam representation obtained for the scalar functions showed only singularities of the box type. This means that the seagull vertex is only there to restore gauge invariance, and that if one works with a gauge-invariant set of structures (like the BTT one), there is no remnant of the seagull vertex in the singularity structure of the corresponding scalar function. An even simpler case where a similar

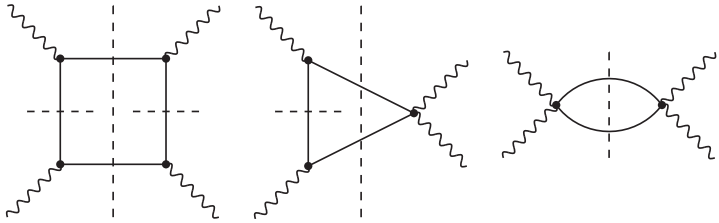


**Figure 4.** Scalar QED diagrams contributing to light-by-light scattering to one loop.

phenomenon is also visible, and probably in a clearer way, is the $\gamma^*\gamma^* \to \pi\pi$ process. As discussed in Sect. 2.6. of Ref. [14], when one projects the sQED tree-level contributions to this process, the corresponding scalar functions only have pole singularities and no nonsingular term, despite what one would be led to think by looking at the seagull vertex. Here also the latter only serves the purpose of restoring gauge invariance at the Feynman-diagram level, and when one works with a gauge-invariant basis there is no trace of it anymore.

Projecting representation (30) onto the BTT basis and taking the limit of $(g-2)_\mu$ kinematics we obtain a very practical representation of the pion box in terms of two-dimensional Feynman-parameter integrals

$$\bar{\Pi}_i^{\pi\text{-box}}(q_1^2, q_2^2, q_3^2) = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy I_i(x,y), \tag{31}$$

where the integrands $I_i(x,y)$ have very compact expressions which are given explicitly in appendix C of Ref. [16]. Inserting these expressions in the master formula, Eq. (19), we obtain

$$a_\mu^{\pi\text{-box}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3\, F_\pi^V(-Q_1^2)F_\pi^V(-Q_2^2)F_\pi^V(-Q_3^2) \times \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau)\bar{\Pi}_i^{\text{sQED}}(Q_1, Q_2, \tau). \tag{32}$$

For a numerical evaluation one needs an explicit representation of the vector form factor of the pion for spacelike momenta, and since about 95% of the final pion-box $(g-2)_\mu$ integral originate from virtualities below 1 GeV, it is essential that the low-energy properties be correctly reproduced. Experimentally, the available constraints derive from $e^+e^- \to \pi^+\pi^-$ data, which determine the time-like form factor [50–55], and space-like measurements by scattering pions off an electron target [56, 57]. We have also checked that our representation is consistent with extractions of the space-like form factor from $e^-p \to e^-\pi^+n$ data [58–61], although due to the remaining model dependence of extrapolating to the pion pole we do not use these data in our fits. To obtain a representation that allows us to simultaneously fit space- and time-like data, and thereby profit from the high-statistics form factor measurements motivated mainly by the two-pion contribution to HVP, we adopt the formalism suggested in [62, 63] (similar representations have been used in [64–69]). A brief description of the formalism can be found in Ref. [16]. Here we limit ourselves to a discussion of the results, which are best illustrated by the two plots in Fig. 5.

The dispersive representation of the pion form factor is fixed by fitting simultaneously the space-like data from [57] as well as one of the time-like data sets [50–55] (restricted to data points below 1 GeV). All input parameters are varied within reasonable bounds to check the dependence of the results

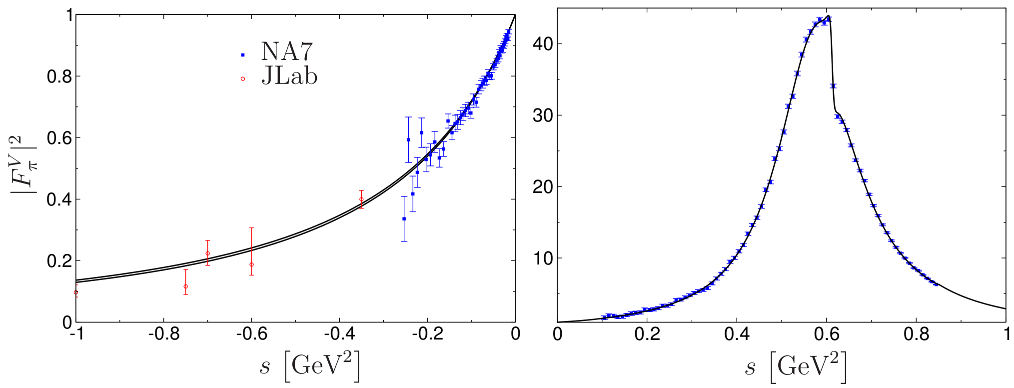



**Figure 5.** Left: space-like pion form factor from our dispersive fit in comparison to data from NA7 [57] and JLab [59–61] (the latter are not included in the fit). The error band represents the variation observed between different time-like data sets. Right: pion form factor in the time-like region from the combined fit to NA7 and [53], chosen here for illustrative purposes only. Fits to the other time-like data sets look very similar and lead to the same numerical results within the accuracy quoted in (33).

on these. We find that the results for the space-like form factor are extremely stable against all these variations, the largest effect being produced by the differences between the time-like data sets. For the accuracy required in HLbL scattering we can therefore simply take the largest variation among them as an uncertainty estimate, without having to perform a careful investigation of the statistical and systematic errors that are crucial when combining the different data sets for HVP. The result for the space-like form factor is shown in Fig. 5, leading to a numerical evaluation for the pion box of

$$a_\mu^{\pi\text{-box}} = -15.9(2) \times 10^{-11}. \tag{33}$$

While the central value is in the ballpark of what had been obtained in Ref. [17], and much larger (in absolute value) than the estimate provided in Ref. [19] which was based on the hidden gauge model, the greatest progress in our evaluation is in the error reduction. Indeed all previous estimates assigned an uncertainty estimate close to 100% to this contribution, essentially because the description of the photon off-shell dependence was considered to be pure model work. Within our dispersive approach we were able to prove that the photon off-shell dependence is rigorously described by the vector form factor of the pion, which is very well known experimentally. The uncertainty in Eq. (33) is similar to the one obtained in the HVP calculation, because it is essentially the same very precise data which constrain both contributions. Which means that at the level of precision needed for the HLbL contribution to $(g-2)_\mu$, the pion box is known essentially exactly.

## 5 Partial waves, pion rescattering contribution

The last contribution which we need to consider is the one coming from diagrams two, three and four after the equal sign in Fig. 3. These have two-pion discontinuities in one channel but discontinuities of higher mass in the other: the latter will not be treated explicitly, according to our approximation scheme, but will be projected onto partial waves. In order to treat these contributions we therefore need a formalism for dealing with two-pion discontinuities in partial waves for HLbL. The case of $S$ waves has been dealt with in Ref. [12] and is relatively simple, but going beyond that has turned out to be a formidable task, which has been solved and discussed in detail in Ref. [16]. We list below here in the form of bullet points some of the key reasons why this is so complicated:

- unitarity relations are diagonal in a helicity amplitude basis and the relation between the BTT (redundant, 54 elements) set and the helicity basis is neither unique nor invertible;
- the helicity basis relevant for $(g-2)_\mu$ is the one with one on-shell photon, which has only 27 elements (half of the BTT set);
- in the limit $q_4^2, q_4^\sigma \to 0$ of the HLbL tensor the number of independent elements of the BTT set drops from 41 to 27;
- there is freedom in the choice of this subset (which we call a singly-on-shell basis);
- the transformation from helicity amplitudes to the singly-on-shell basis is easy to derive, but what one needs is the inverse of that;
- inverting this relation (a $27 \times 27$ matrix) is a lot more complicated but was done (analytically) in [16];
- the arbitrariness in the choice of the 27 elements of the singly-on-shell basis would seem to affect the final result at first sight, but in fact it does not, because of sum rules;
- these sum rules follow from the assumption that the HLbL tensor has a uniform behavior at short distances.

Understanding all this has not been easy, and it has been very important— even mandatory— to put the formalism under test, and the case of the pion box has been invaluable for that. We have projected the pion box onto partial waves and checked numerically whether the resummation of the partial waves (carried out only up to a finite number of course) approached the total, which can be calculated in one go, as discussed above. The test was passed and the formalism is now ready to be used.

As a first numerical application we considered only $S$ waves. Their contribution to the scalar functions can be expressed as follows:

$$
\begin{aligned}
\hat{\Pi}_4^S &= \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{-2}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(4s'\mathrm{Im}h^0_{++,++}(s') - (s'+q_1^2-q_2^2)(s'-q_1^2+q_2^2)\mathrm{Im}h^0_{00,++}(s')\Big)\\
\hat{\Pi}_5^S &= \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{-2}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(4t'\mathrm{Im}h^0_{++,++}(t') - (t'+q_1^2-q_3^2)(t'-q_1^2+q_3^2)\mathrm{Im}h^0_{00,++}(t')\Big)\\
\hat{\Pi}_6^S &= \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{-2}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(4u'\mathrm{Im}h^0_{++,++}(u') - (u'+q_2^2-q_3^2)(u'-q_2^2+q_3^2)\mathrm{Im}h^0_{00,++}(u')\Big)\\
\hat{\Pi}_{11}^S &= \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{4}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(2\mathrm{Im}h^0_{++,++}(u') - (u'-q_2^2-q_3^2)\mathrm{Im}h^0_{00,++}(u')\Big)\\
\hat{\Pi}_{16}^S &= \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{4}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(2\mathrm{Im}h^0_{++,++}(t') - (t'-q_1^2-q_3^2)\mathrm{Im}h^0_{00,++}(t')\Big)\\
\hat{\Pi}_{17}^S &= \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{4}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(2\mathrm{Im}h^0_{++,++}(s') - (s'-q_1^2-q_2^2)\mathrm{Im}h^0_{00,++}(s')\Big)
\end{aligned}
\tag{34}
$$

where $\mathrm{Im}h^0_{00,++}$ and $\mathrm{Im}h^0_{++,++}$ are the imaginary parts of the $S$-wave helicity amplitudes (the subscripts indicate the helicities of the four photons) of the HLbL scattering amplitude, which unitarity relates to (products of) the $S$-wave helicity amplitudes of $\gamma^*\gamma^* \to \pi\pi$. Unfortunately, the latter amplitudes, though measurable in principle, have not been measured yet. There are very good data for on-shell photons and some for singly-on-shell, but none for both photons off-shell.

| cutoff | 1 GeV | 1.5 GeV | 2 GeV | $\infty$ |
|---|---|---|---|---|
| $I = 0$ | −9.2 | −9.5 | −9.3 | −8.8 |
| $I = 2$ | 2.0 | 1.3 | 1.1 | 0.9 |
| sum | −7.3 | −8.3 | −8.3 | −7.9 |

**Table 1.** $S$-wave rescattering corrections to $a_\mu^{\pi\text{-box}}$, in units of $10^{-11}$, for both isospin components and in total.

To obtain our first numerical estimate we therefore proceeded as follows: we considered the Roy-Steiner equations for $\gamma^*\gamma^* \to \pi\pi$ [12], which needs as input an explicit representation of the left-hand cut (as well as the $\pi\pi$ phase shifts, which are known, however [70–72]), for arbitrary photon virtualities. If one only considers the contribution to the left-hand cut coming from the pion pole, the photon $q^2$ dependence is again completely given by the pion vector form factor. Extensions to other contributions to the left-hand cut will require additional input and will be considered later.

For the concrete numerical implementation we have used the simplified representation of the $\pi\pi$ phase shifts based on the modified inverse-amplitude method [73], for the main reason that it has a simple analytic expression which is convenient to use in combination with Muskhelishvili–Omnès methods. This reproduces at the same time the low-energy properties of the phase shifts as well as pole position and couplings of the $f_0(500)$ resonance to a good accuracy. This phase shift departs from the correct one just below the $K\bar{K}$ threshold because it does not feature the sharp rise due to the $f_0(980)$ resonance but continues flat with a smooth high-energy behavior. A full-fledged evaluation of the $f_0(980)$ resonance would require a proper treatment of the $K\bar{K}$ channel, which is beyond the scope of this first estimate. To control the dependence on the high-energy input we have introduced a cutoff in the integral and have varied it from 1 GeV up to infinity.

The results for the rescattering contribution, summarized in Table 1, are stable over a wide range of cutoffs, indicating that our input for the $\gamma^*\gamma^* \to \pi\pi$ partial waves reliably unitarizes the Born-term left-hand cut (LHC), which should indeed dominate at low energies. In addition, we checked that the only sum rule that receives $S$-wave contributions is already saturated at better than 90%, completely in line with the expectation that the sum rules will be fulfilled only after partial-wave resummation. The isospin-0 part of the result can be interpreted as a model-independent implementation of the contribution from the $f_0(500)$ of about $-9 \times 10^{-11}$ to HLbL scattering in $(g-2)_\mu$. In total, we obtain for the $\pi\pi$-rescattering effects related to the pion-pole LHC

$$a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}, \tag{35}$$

where the error is dominated by the uncertainties related to the asymptotic parts of the integral.

## 6 Conclusions and outlook

After a quick overlook at ongoing lattice calculations of the HLbL contribution to the $(g-2)_\mu$ we have concentrated on the dispersive approach and briefly summarized its basic steps. We have then described the ingredients of the first numerical evaluation of the pion box and related rescattering corrections in the $S$ wave. Adding the two contributions together we have obtained

$$a_\mu^{\pi\text{-box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -24(1) \times 10^{-11}, \tag{36}$$

which represents the first precise and model-independent evaluation of these two contributions. Compared to other sources of uncertainty in HLbL the error in the estimate provided here is negligible.

This is of course only a first step and in order to complete the calculation based on the dispersive approach, there are other contributions which will need to be considered. The most important one is the one due to the pion-pole and for that it is the pion transition form factor which represents the crucial input quantity. Efforts to evaluate the latter on the basis of a dispersion relation are ongoing [47]. But also for what concerns the two-pion contribution there is still more work to be done, in particular (i) by adopting a more realistic representation of the left-hand cut, which in turn will require modeling the off-shell behavior of the photons; (ii) including higher partial waves, in particular to $D$ wave, which contains the very prominent $f_2(1270)$ resonance; (iii) even for the $S$ wave, it is important to include a realistic description of the region above 1 GeV, even if this will probably be subdominant; (iv) as mentioned above doing this will only be possible if simultaneously considering the contribution of two-kaon intermediate states; (v) finally, an estimate of contributions which go beyond two-pion intermediate states will be very important in order to assess the robustness of the final numerical evaluation. For example, three-pion intermediate states can be modeled by means of axial resonance contributions.

These comments show that there is still a lot of work ahead of us before being able to provide a complete estimate of the HLbL contribution to the $(g-2)_\mu$ based on a dispersive approach. We are confident, however, that this ambitious goal is now in sight, and that the availability of two model-independent approaches (the lattice and the dispersive one) to the calculation of this contribution is a very significant step ahead towards a deeper understanding of the $(g-2)_\mu$ puzzle.

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