SMOOTHNESS PROPERTIES OF THE UNIT BALL IN A JB*-TRIPLE

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Abstract

An element a of norm one in a JB*-triple A is said to be *smooth* if there exists a unique element x in the unit ball A_1^* of the dual A^* of A at which a attains its norm, and is said to be *Fréchet-smooth* if, in addition, any sequence (x_n) of elements in A_1^* for which $(x_n(a))$ converges to one necessarily converges in norm to x. The sequence (a^{2n+1}) of odd powers of a converges in the weak*-topology to a tripotent $u(a)$ in the JBW*-envelope A^{**} of A. It is shown that a is smooth if and only if $u(a)$ is a minimal tripotent in A^{**} and a is Fréchet-smooth if and only if, in addition, $u(a)$ lies in A.

1. *Introduction*

Over several years, the authors have studied the facial structure of the unit balls in a Banach space and its dual [**5**, **6**, **7**, **8**, **9**, **10**]. This note is concerned with two smoothness properties of the unit ball in a complex Banach space. In a recent paper [**19**], Taylor and Werner studied these properties for the unit ball of a C*-algebra and were able to give several algebraic conditions which were equivalent to smoothness. The purpose of this paper is to consider the same properties for the unit ball in a JB*-triple. The main result shows that both smoothness conditions are equivalent to algebraic conditions which, necessarily, are expressed in terms of the triple product. This leads to further equivalent new conditions for the case of a C*-algebra.

2. *Preliminaries*

Let V be a complex vector space, and let C be a convex subset of V. A convex subset E of C is said to be a *face* of C provided that, if $ta_1+(1-t)a_2$ is an element of E, where a_1 and a_2 lie in C and $0<t<1$, then a_1 and a_2 lie in E. A face E of C is said to be *proper* if it differs from C. An element a in C for which $\{a\}$ is a face is said to be an *extreme point* of C. Let τ be a locally convex Hausdorff topology on V, and let C be τ-closed. Let $\mathscr{F}_\tau(C)$ denote the set of τ-closed faces of C. Both $\varnothing$ and C are elements of $\mathscr{F}_\tau(C)$, and the intersection of an arbitrary family of elements of $\mathscr{F}_\tau(C)$ again lies in $\mathscr{F}_\tau(C)$. Hence, with respect to ordering by set inclusion, $\mathscr{F}_\tau(C)$ forms a complete lattice. A subset E of C is said to be a *τ-exposed face* of C if there exists a τ-continuous linear functional f on V and a real number t such that, for all elements a in $C\backslash E$, $\mathrm{Re}f(a)<t$ and, for all elements a in E, $\mathrm{Re}f(a)=t$. Let $\mathscr{E}_\tau(C)$ denote the set of τ-exposed faces of C. Clearly, $\mathscr{E}_\tau(C)$ is contained in $\mathscr{F}_\tau(C)$, and the intersection of a finite number of elements of $\mathscr{E}_\tau(C)$ again lies in $\mathscr{E}_\tau(C)$. The intersection of an arbitrary family of elements of $\mathscr{E}_\tau(C)$ is said to be a *τ-semi-exposed face* of C. Let $\mathscr{S}_\tau(C)$ denote the set of τ-semi-exposed faces of C. Clearly, $\mathscr{E}_\tau(C)$ is contained in $\mathscr{S}_\tau(C)$, and

Received 11 April 1994.

1991 *Mathematics Subject Classification* 46L10.

Research supported in part by a grant from the Schweizerische Nationalfonds/Fonds national suisse.

the intersection of an arbitrary family of elements of $\mathcal{S}_\tau(C)$ again lies in $\mathcal{S}_\tau(C)$. Hence, with respect to ordering by set inclusion, $\mathcal{S}_\tau(C)$ forms a complete lattice and the infimum of a family of elements of $\mathcal{S}_\tau(C)$ coincides with its infimum when taken in $\mathcal{F}_\tau(C)$.

When V is a complex Banach space with dual space V^*, the abbreviations n and w^* will be used for the norm topology of V and the weak*-topology of V^*. For each subset E of the unit ball V_1 in V and F of the unit ball V_1^* in V^*, let the subsets E' and $F_\prime$ be defined by

$$E' = \{x \in V_1^* : x(a) = 1 \text{ for all } a \in E\}, \quad F_\prime = \{a \in V_1 : x(a) = 1 \text{ for all } x \in F\}.$$

Notice that E lies in $\mathcal{S}_n(V_1)$ if and only if $(E')_\prime = E$, F lies in $\mathcal{S}_{w^*}(V_1^*)$ if and only if $(F_\prime)' = F$, and the mappings $E \to E'$ and $F \to F_\prime$ are anti-order isomorphisms between $\mathcal{S}_n(V_1)$ and $\mathcal{S}_{w^*}(V_1^*)$ and are inverses of each other. The reader is referred to **[8]** for details.

An element a in V_1 is said to be *smooth* if $\{a\}'$ coincides with $\{x\}$ for some x in V_1^*. It follows that the extreme point x of V_1^* is weak*-exposed. A smooth point a is said to be *Fréchet-smooth* if, for each sequence (x_n) in V_1^* such that the sequence $(x_n(a))$ converges to one, it follows that (x_n) converges in norm to x. For the equivalence of these definitions with the conventional ones, the reader is referred to **[18]** and **[19]**.

LEMMA 2.1. *Let V be a complex Banach space, let W be a closed subspace of V, and let a be an element of norm one in W. If a is smooth in V, then a is smooth in W, and if a is Fréchet-smooth in V, then a is Fréchet-smooth in W.*

Proof. Suppose that the element a in W of norm one is smooth in V. Then there exists a unique element x of norm one in V^* such that $x(a) = 1$. The restriction x_0 of x to W has the property that $x_0(a) = 1$, and is therefore of norm one. Let y be an element of W^* of norm one such that $y(a) = 1$. Then, by the Hahn–Banach theorem, y can be extended to an element z of V^* of norm one. Since a is smooth, z and x coincide, and therefore y is the restriction of x to W. It follows that a is smooth in W. Now suppose that a is Fréchet-smooth in V, and suppose that (y_n) is a sequence of elements of W_1^* such that $(y_n(a))$ converges to one. By the Hahn–Banach theorem, there is a sequence (x_n) in V_1^* such that y_n is the restriction of x_n to W. The sequence $(x_n(a))$ converges to one and, by the Fréchet-smoothness of a in V, it follows that (x_n) converges in norm to x. However,

$$\|y_n - x_0\| = \|x_n|_W - x|_W\| \leqslant \|x_n - x\|,$$

and therefore (y_n) converges in norm to x_0 as required.

The following result is well known, but a proof will be given for completeness.

LEMMA 2.2. *Let Ω be a locally compact Hausdorff space, and let $C_0(\Omega)$ denote the complex Banach space of continuous complex-valued functions on Ω vanishing at infinity, endowed with the supremum norm. Let a be a positive element of $C_0(\Omega)$ of norm one. Then a is smooth if and only if there exists a unique element ω_0 in Ω such that $a(\omega_0) = 1$, and a is Fréchet-smooth if and only if, in addition, ω_0 is an isolated point.*

Proof. The dual of $C_0(\Omega)$ may be identified with the space $M(\Omega)$ of complex regular Borel measures of finite total variation on Ω. Let a be a positive smooth point in the unit ball in $C_0(\Omega)$. There exists a point ω_0 in Ω such that $a(\omega_0) = 1$. Since a is smooth, it follows that the unit point measure δ_{ω_0} is the unique element of $M(\Omega)$ such that $\delta_{\omega_0}(a) = 1$.

Suppose that a is Fréchet-smooth and ω_0 is not isolated. Then, for each positive integer n, the open set $\{\omega: |a(\omega)-1| < 1/n\}$ contains a point ω_n not equal to ω_0. The sequence (δ_{ω_n}) of unit point measures is such that the sequence $(\delta_{\omega_n}(a))$ converges to one. However, for each n there exists a positive element b of $C_0(\Omega)$ such that $\delta_{\omega_n}(b) = 1$ and $\delta_{\omega_0}(b) = 0$. Hence (δ_{ω_n}) does not converge in norm to δ_{ω_0}. This yields a contradiction.

Recall that a complex vector space A equipped with a triple product $(a,b,c) \to \{a\,b\,c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate linear in the second variable and satisfies the identity

$$[D(a,b), D(c,d)] = D(\{a\,b\,c\}, d) - D(c, \{d\,a\,b\}),$$

where [,] denotes the commutator and D is the mapping from $A \times A$ to A defined by

$$D(a,b)\,c = \{a\,b\,c\},$$

is said to be a *Jordan*-triple*. When A is also a Banach space such that D is continuous from $A \times A$ to the Banach space $B(A)$ of bounded linear operators on A, and, for each element a in A, $D(a,a)$ is hermitian with non-negative spectrum and satisfies

$$\|D(a,a)\| = \|a\|^2,$$

then A is said to be a *JB*-triple*. A JB*-triple which is the dual of a Banach space is said to be a *JBW*-triple*.

Examples of JB*-triples are C*-algebras, and examples of JBW*-triples are W*-algebras. The second dual A^{**} of a JB*-triple A possesses a triple product with respect to which it is a JBW*-triple, the canonical mapping from A into A^{**} being an isomorphism. For details, the reader is referred to **[1, 2, 3, 4, 11, 12, 13, 14, 15, 16, 17, 20]**.

An element u in a JBW*-triple A is said to be a *tripotent* if $\{u\,u\,u\}$ is equal to u. The set of tripotents in A is denoted by $\mathscr{U}(A)$. A pair u, v of elements of $\mathscr{U}(A)$ is said to be *orthogonal* if $\{v\,v\,u\} = 0$. For two elements u and v of $\mathscr{U}(A)$, write $u \leqslant v$ if $\{u\,v\,u\} = u$ or, equivalently, if $v-u$ is a tripotent orthogonal to u. This defines a partial ordering on $\mathscr{U}(A)$ with respect to which $\mathscr{U}(A)$ with a greatest element adjoined forms a complete lattice denoted by $\mathscr{U}(A)^{\sim}$. When A is a W*-algebra, the set of tripotents in A coincides with the set of partial isometries in A.

The proof of the following result can be found in **[8]**.

LEMMA 2.3. *Let A be a JBW*-triple with predual A_* and let $\mathscr{U}(A)^{\sim}$ be the complete lattice of tripotents in A with a largest element adjoined.*

(i) *Every norm-closed face of the unit ball A_{*1} in A_* is norm-exposed and of the form $\{u\}$, for some u in $\mathscr{U}(A)^{\sim}$.*

(ii) *Every weak*-closed face of the unit ball A_1 in A is weak*-semi-exposed and of the form $(\{u\},)'$ for some u in $\mathscr{U}(A)^{\sim}$.*

(iii) *The map $u \mapsto \{u\}$, is an order isomorphism from the complete lattice $\mathscr{U}(A)^{\sim}$ onto the complete lattice $\mathscr{F}_n(A_{*1})$, and the map $u \mapsto (\{u\},)'$ is an anti-order isomorphism from $\mathscr{U}(A)^{\sim}$ onto the complete lattice $\mathscr{F}_{w^*}(A_1)$.*

For each element a in a JBW*-triple A, the odd powers a^{2n+1} are defined inductively by

$$a^1 = a, \quad a^{2n+1} = \{a\, a^{2n-1}\, a\}.$$

The proof of the following lemma can be found in [**8**].

LEMMA 2.4. *Let a be an element of norm one in a JBW*-triple A. Then the sequence (a^{2n+1}) converges in the weak*-topology to a tripotent $u(a)$ such that the norm-closed faces $\{a\}_{,}$ and $\{u(a)\}_{,}$ of the unit ball A_{*1} of the predual A_* of A coincide.*

3. *Main results*

The main result of the paper can now be given using those above. Let A be a JB*-triple that will be regarded as being canonically embedded in its JBW*-envelope A^{**}.

THEOREM 3.1. *Let A be a JB*-triple, and let a be an element of norm one in A.*
(i) *The point a is smooth if and only if $u(a)$ is a minimal non-zero element of the complete lattice $\mathscr{U}(A^{**})^{\sim}$.*
(ii) *The point a is Fréchet-smooth if and only if $u(a)$ is a minimal non-zero element of the complete lattice $\mathscr{U}(A^{**})^{\sim}$ and is contained in A.*

Proof. Suppose that a is smooth, in which case $\{a\}'$ consists of the set $\{x\}$, for some extreme point x of A_1^*. By [**11**, Proposition 4] and Lemma 2.4, it follows that $u(a)$ is a minimal non-zero element of $\mathscr{U}(A^{**})^{\sim}$. Conversely, if $u(a)$ is non-zero and minimal, again using [**11**, Proposition 4], there exists a unique extreme point x of A_1^* such that

$$\{a\}' = \{u(a)\}_{,} = \{x\}$$

and a is smooth.

Let $C(a)$ be the JB*-subtriple of A generated by the odd powers of a. By [**14**, Lemma 1.14], $C(a)$ is isomorphic to the commutative C*-algebra $C_0(\Omega)$ of continuous functions on a locally compact Hausdorff space Ω which vanish at infinity, and under this isomorphism a is mapped into a positive element. That a is Fréchet-smooth if and only if $u(a)$ is contained in A follows immediately from Lemmas 2.1 and 2.2.

When A is a C*-algebra, the partial isometry $u(a)$ corresponding to an element a in A of norm one is the weak*-limit of the sequence (a^{2n+1}), where

$$a^1 = a, \quad a^{2n+1} = aa^*(a^{2n-1}).$$

Theorem 3.1 leads to the following result for C*-algebras.

COROLLARY 3.2. *Let a be an element of norm one in the C*-algebra A.*
(i) *The point a is smooth if and only if the partial isometry $u(a)$ is such that*

$$u(a)\, u(a)^* A^{**} u(a)^* u(a) = \mathbb{C}u(a).$$

(ii) *The point a is Fréchet-smooth if and only if a is smooth and $u(a)$ is contained in A.*

For several equivalent conditions in the case of a C*-algebra, the reader is referred to [**19**].

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