
Relations between Propositional Normal Modal Logics: an Overview

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Abstract

The modal logic literature is notorious for multiple axiomatizations of the same logic and for conflicting overloading of axiom names. Many of the interesting interderivability results are still scattered over the often hard to obtain classics. We catalogue the most interesting axioms, their numerous variants, and explore the relationships between them in terms of interderivability as both axiom (schema) and as simple formulae. In doing so we introduce the Logics Workbench (LWB, see <http://lwbwww.unibe.ch:8080/LWBinfo.html>), a versatile tool for proving theorems in numerous propositional (nonclassical) logics. As a side-effect we fulfill a call from the modal theorem proving community for a database of known theorems.

Keywords: Modal logic, automated theorem proving, sequent calculi, Logics Workbench.

1 Introduction

The modal logic literature is quite diverse but most beginners turn to the introductory works by Hughes and Cresswell [9, 10], Bull and Segerberg [1], Lemmon and Scott [11], or Chellas [2]. The classic by Segerberg [13] is indispensable but is probably the hardest to get hold of (legally). Unfortunately the nomenclature used in these works is not uniform, and the proof techniques range from semantic, to proof-theoretic, to algebraic. Consequently, it is difficult to form a clear picture of the many interesting results contained in these works, and even harder to keep track of the many different versions of the basic axioms, their different names, and even the overloading of names. In what follows we try to give a general picture of this nomenclature, the interesting interconnections between some of these logics, and the interderivability of different axiomatizations of the same logic.

Our basic tool is the Logics Workbench (LWB) [8], a program developed at the University of Bern that contains automated proof procedures based on modal Gentzen systems for numerous propositional (nonclassical) logics. Although the class of *modal* logics which have been automated in the LWB is relatively small, namely **K**, **KT**, **KT4**, **KT45** and **KW**, we can work with extensions of these logics by using a finite number of appropriate formula instances of the desired axioms, as explained below.

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2 Definitions and notational conventions

We assume that **formulae** are built as usual from the classical connectives $\wedge, \vee, \rightarrow, \neg, \top, \perp$ and the modal connectives \Box and \Diamond . The binding strengths of the connectives from strongest to weakest is: $\neg, \Box, \Diamond, \wedge, \vee, \rightarrow$. We also assume that \rightarrow associates so that $A \rightarrow B \rightarrow C$ is to be read as $A \rightarrow (B \rightarrow C)$. We use \Rightarrow to separate the two halves of Gentzen's sequents, and use \vdash to denote derivability in a Hilbert system for a particular modal logic. This notation has been chosen carefully to avoid the confusion that may arise from the fact that the 'deduction theorem' for modal logics is not the same as in classical logic.

We assume that our modal logics are formulated by taking all classical propositional tautologies as axioms, and then adding a finite set of formulae (called *the axioms*) together with the inference rules of modus ponens, universal substitution, and necessitation. The formula $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ is always one of the axioms (hence we deal only with normal modal logics). The name of the logic is obtained by concatenating the names of the axioms to the prefix **K**. A formula A is **derivable** in a modal logic L axiomatized in this manner iff there is a finite sequence of formulae A_1, A_2, \dots, A_n such that $A_n = A$ and each member of the sequence is either a classical tautology, an axiom, or is obtained from previous members of the sequence by one of the inference rules. If the formula A is derivable in L we write $L \vdash A$ and call A a **theorem** of L . **Substitutions** σ are denoted as follows: $A\sigma := A\{p := B\}$ means that in A all occurrences of the propositional variable p are replaced simultaneously with B . $A\{p := B\}$ is called an **instance** of A . In general, just using *instances* of axioms is not sufficient for our purposes, so let $\Box^0 A = A$ and $\Box^{n+1} A = \Box(\Box^n A)$ for $n \geq 0$. For any given $m \geq 0$, the formula $\Box^m(A\sigma)$ is a **modalized instance** of A . We use A^1, A^2, A^3, \dots to denote modalized instances of A and use $\bigwedge_{i=1}^k A^i$ to denote the formula $A^1 \wedge A^2 \wedge \dots \wedge A^k$ obtained by conjoining k modalized instances of A , for some finite $k \geq 1$.

LEMMA 2.1

Let L be a normal modal logic, and A, B formulae. Then there exist a finite number (say k) of modalized instances of A such that $L \vdash \bigwedge_{i=1}^k A^i \rightarrow B$ iff $LA \vdash B$.

Now suppose that we wish to use the LWB to prove that a particular formula P is a theorem of the logic **KT4A** obtained by adding A as an axiom to **KT4**. Since the LWB does not have a proof procedure for **KT4A** we must use the one for **KT4** by finding enough modalized instances of A such that $\mathbf{KT4} \vdash \bigwedge_{i=1}^k A^i \rightarrow P$. Lemma 2.1 then allows us to conclude that $\mathbf{KT4A} \vdash P$.

In this way, we can 'prove' theorems in the logic **KT4A** even though the LWB itself is capable of proving theorems only in the sublogic **KT4** (say). It is easy to generalize this process to use only the logic **K** by looking for modalized instances of the axioms T and 4 as well, and attempting to prove that:

$$\mathbf{K} \vdash A^1 \wedge A^2 \wedge \dots \wedge A^i \wedge T^1 \wedge T^2 \wedge \dots \wedge T^j \wedge 4^1 \wedge 4^2 \wedge \dots \wedge 4^k \rightarrow P.$$

For many logics, the literature contains more than one axiomatization, say **KAB** and **KAC** where the axiom A is common but the axioms B and C differ. We can show the equivalence between these axiomatizations using the technique described above by proving that each logic contains the axioms of the other as theorems.

In the rest of this paper we do the following: in Section 3 we list the axiom names, associated formula and some variants that we use throughout our paper. In Section 4 we cover the relations between some of these axioms, namely the basic axioms, the axioms of convergence,

axioms related to McKinsey's axiom, the axioms of provability logics, and finally some axioms from logics of time. The appendix contains a useful cross-reference to names used in the basic texts.

As a side-effect we provide a database of known theorems and their relationships for many propositional modal logics, thus answering a call from the modal theorem proving community (where testing an implementation calls for such a database).

3 Formulae

The tables below list all the formulae we consider and the names we use for them.

name	formula
D	$\Box p \rightarrow \Diamond p$
D_2	$\Diamond \top$
T	$\Box p \rightarrow p$
4	$\Box p \rightarrow \Box \Box p$
4_M	$\Box p \wedge \Diamond q \rightarrow \Diamond(\Box p \wedge q)$
5	$\Diamond p \rightarrow \Box \Diamond p$
5_M	$\Diamond p \wedge \Diamond q \rightarrow \Diamond(\Diamond p \wedge q)$
B	$p \rightarrow \Box \Diamond p$
B_M	$p \wedge \Diamond q \rightarrow \Diamond(\Diamond p \wedge q)$
G	$\Diamond \Box p \rightarrow \Box \Diamond p$
G_0	$\Diamond(p \wedge \Box q) \rightarrow \Box(p \vee \Diamond q)$
H	$\Box(p \vee q) \wedge \Box(\Box p \vee q) \wedge \Box(p \vee \Box q) \rightarrow \Box p \vee \Box q$
H^*	$\Box(\Box p \vee q) \wedge \Box(p \vee \Box q) \rightarrow \Box p \vee \Box q$
L	$\Box(p \wedge \Box p \rightarrow q) \vee \Box(q \wedge \Box q \rightarrow p)$
L^*	$\Box(\Box p \rightarrow q) \vee \Box(\Box q \rightarrow p)$
L^{**}	$\Box(\Box p \rightarrow \Box q) \vee \Box(\Box q \rightarrow \Box p)$
M	$\Box \Diamond p \rightarrow \Diamond \Box p$
M_2	$\Diamond \Box(p \rightarrow \Box p)$
M_3	$\Box \Diamond p \wedge \Box \Diamond q \rightarrow \Diamond(p \wedge q)$
Pt	$\Box(p \vee \Diamond p) \rightarrow \Diamond(p \wedge \Box p)$
W	$\Box(\Box p \rightarrow p) \rightarrow \Box p$
W_0	$\Box \Diamond \top \rightarrow \Box \perp$
Z	$\Box(\Box p \rightarrow p) \rightarrow (\Diamond \Box p \rightarrow \Box p)$

<i>Dum</i>	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow (\Diamond\Box p \rightarrow p)$
<i>Dum</i> ₁	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow (\Diamond\Box p \rightarrow \Box p)$
<i>Dum</i> ₂	$\Box(\Box(p \rightarrow \Box p) \rightarrow \Box p) \rightarrow (\Diamond\Box p \rightarrow p)$
<i>Dum</i> ₃	$\Box(\Box(p \rightarrow \Box p) \rightarrow \Box p) \rightarrow (\Diamond\Box p \rightarrow \Box p)$
<i>Dum</i> ₄	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow (\Diamond\Box p \rightarrow p \vee \Box p)$
<i>Grz</i>	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$
<i>Grz</i> ₁	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow \Box p$
<i>Grz</i> ₂	$\Box(\Box(p \rightarrow \Box p) \rightarrow \Box p) \rightarrow p$
<i>Grz</i> ₃	$\Box(\Box(p \rightarrow \Box p) \rightarrow \Box p) \rightarrow \Box p$
<i>Grz</i> ₄	$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p \vee \Box p$
<i>Grz</i> ₅	$\Box(\Box(p \rightarrow \Box q) \rightarrow \Box q) \wedge \Box(\Box(\neg p \rightarrow \Box q) \rightarrow \Box q) \rightarrow \Box q$
<i>F</i>	$(\Diamond\Box p \rightarrow q) \vee \Box(\Box q \rightarrow p)$
<i>H</i> _s	$p \rightarrow \Box(\Diamond p \rightarrow p)$
<i>P</i>	$\Diamond\Box\Diamond p \rightarrow (p \rightarrow \Box p)$
<i>R</i>	$\Diamond\Box p \rightarrow (p \rightarrow \Box p)$
<i>X</i>	$\Box\Box p \rightarrow \Box p$
<i>Zem</i>	$\Box\Diamond\Box p \rightarrow (p \rightarrow \Box p)$

4 Tables and diagrams

In the diagrams, $L_1 \rightarrow L_2$ means every theorem of L_1 is also a theorem of L_2 .

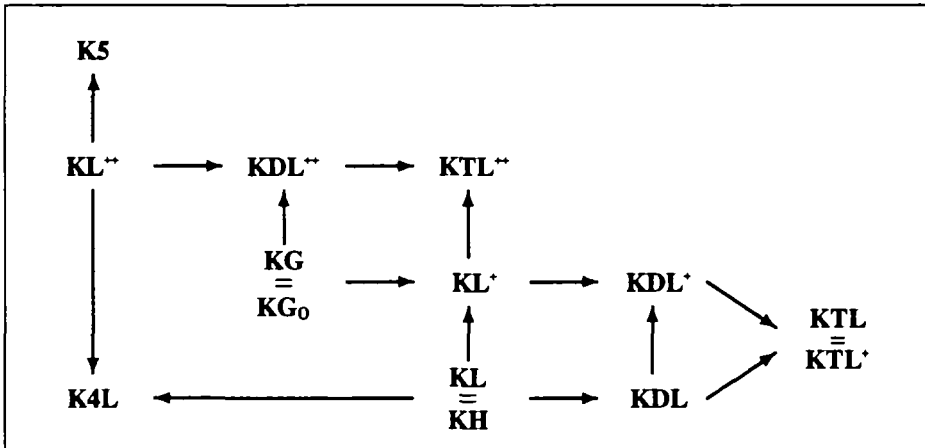
4.1 Relations concerning *D*, *T*, *4*, *5*, *B*

The most basic normal modal logics are obtained by adding combinations of the formulae *T*, *D*, *4*, *5* and *B* as axioms to the basic normal modal logic **K**. There are fifteen distinct logics obtained in this way and the relationships between them are quite well known (see [11] or [2] for a complete diagram). In the table, we simply confirm some of these relationships, and show the equivalence between different versions of these axioms.

$\mathbf{KA}_1 \dots \mathbf{A}_k \vdash C$		
A_1, \dots, A_k	<i>C</i>	modalized instances of A_1, \dots, A_k
<i>4</i>	<i>4</i> _M	<i>4</i>
<i>4</i> _M	<i>4</i>	<i>4</i> _M { $q := \neg\Box p$ }
<i>5</i>	<i>5</i> _M	<i>5</i>
<i>5</i> _M	<i>5</i>	<i>5</i> _M { $q := \neg\Diamond p$ }
<i>B</i>	<i>B</i> _M	<i>B</i>
<i>B</i> _M	<i>B</i>	<i>B</i> _M { $q := \neg\Diamond p$ }

$KA_1 \dots A_k \vdash C$		
A_1, \dots, A_k	C	modalized instances of A_1, \dots, A_k
D	D_2	D
D_2	D	D_2
T	D	$T\{p := \perp\}$
$T, 5$	4	$T\{p := \Diamond \neg p\}, 5\{p := \Box p\}, \Box 5\{p := \neg p\}$
$T, 5$	B	$T\{p := \neg p\}, 5$
$4, B$	5	$\Box 4\{p := \neg p\}, B\{p := \Diamond p\}$
$5, B$	4	$\Box 5\{p := \neg p\}, B\{p := \Box p\}$
$D, 4, B$	T	$D, 4, B\{p := \neg p\}$

4.2 Relations concerning $G, G_0, H, H', L, L', L''$

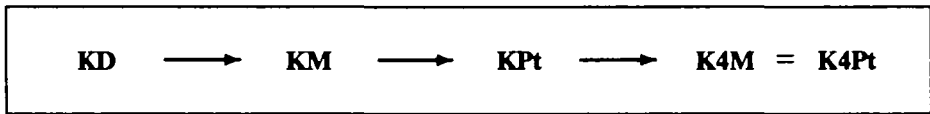


$KA_1 \dots A_k \vdash C$		
A_1, \dots, A_k	C	modalized instances of A_1, \dots, A_k
G	G_0	$G\{p := q\}$
D, G_0	G	$\Box D, G_0\{p := \Box p, q := p\}$
L	H	L
H	L	$H\{p := p \wedge \Box p \rightarrow q, q := q \wedge \Box q \rightarrow p\}$
L'	H'	L'
H'	L'	$H'\{p := \Box p \rightarrow q, q := \Box q \rightarrow p\}$

$\mathbf{KA}_1 \dots \mathbf{A}_k \vdash C$		
A_1, \dots, A_k	C	modalized instances of A_1, \dots, A_k
L^*	L	L^*
T, L	L^*	$\Box T, \Box T\{p := q\}, L$
T, L^*	L^*	$\Box T, \Box T\{p := q\}, L^*$
$4, L$	L^*	$4\{p := p \wedge \Box p \rightarrow q\}, 4\{p := q \wedge \Box q \rightarrow p\}, \Box 4, \Box 4\{p := q\}, L$
5	L^*	$5\{p := \Box p\}, \Box 5\{p := \neg p\}$
D, L^*	G	$\Box D\{p := \neg p\}, L^*\{q := \neg p\}$
5	G	$5, 5\{p := \neg p\}$

4.3 Relations concerning M, Pt

The axioms discussed in this section are loosely grouped around the famed McKinsey-axiom M . Variants of M are discussed in [14]. The formula M appears to be one of the simplest axioms defining a pure second-order property (M is not canonical), thus falling out of the class of Sahlqvist axioms [12]. In the presence of 4 , however, M defines a first-order property: every point in the frame has a successor which is a (reflexive) dead end point.



$\mathbf{KA}_1 \dots \mathbf{A}_k \vdash C$		
A_1, \dots, A_k	C	modalized instances of A_1, \dots, A_k
$T, 4, M$	M_2	$T\{p := M\}, \Box 4, \Box M$
$T, 4, M_2$	M	$T\{p := M_2\}, 4\{p := \Diamond \neg p\}, M_2$
$4, M$	M_3	$4\{p := \Diamond q \wedge (\neg p \vee \neg q)\}, M$
M_3	M	$M_3\{q := \neg p\}$
M	D	M
$4, M$	Pt	$4\{p := (p \vee \Diamond p) \wedge (\neg p \vee \Diamond \neg p)\},$ $\Box 4\{p := p \wedge \Diamond \neg p\},$ $\Box 4\{p := \neg p \wedge \Diamond p\},$ M
Pt	M	Pt

4.4 Relations concerning W, W_0, Z

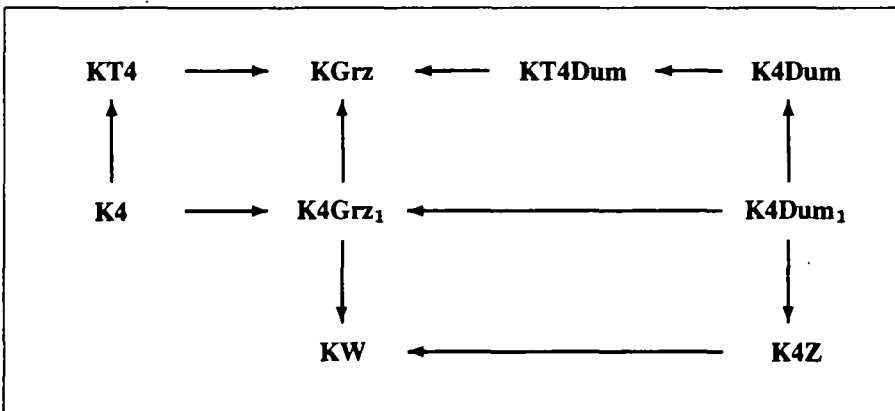
The formula we call W (and sometimes called G) has only one variant but it is related to the formula Z , used by Goldblatt [6] to describe the (temporal) logic of the frame (ω, \leq) where ω is the set of natural numbers and \leq is the usual ordering on the natural numbers. (See also the figure in Section 4.5.)

$\mathbf{KA}_1 \dots \mathbf{A}_k \vdash C$		
A_1, \dots, A_k	C	modalized instances of A_1, \dots, A_k
W	4	$W\{p := p \wedge \Box p\}$
W	W_0	$W\{p := \perp\}$
W	Z	W
W_0, Z	W	W_0, Z
W	Grz_1	$W\{p := p \wedge \Box(\Box p \rightarrow p)\}, \Box\Box W$
$4, Z$	Dum_1	$\Box 4, Z$

4.5 Relations concerning *Dum*, *Grz*

The formulae known as *Dum* and *Grz* are two of the most bizarre formulae that occur in the literature. They each have numerous variants and the relationships between these variants has attracted considerable attention in the literature. For example, Segerberg [13] proves (using semantic means) that these variants are ‘deductively equivalent’ [13, p. 108] in **KT4**. We not only show that $\mathbf{KT4Dum} = \mathbf{KT4Dum}_1$ but we give the exact substitutions necessary to prove (using LWB) that ‘ Dum_1 is deductively equivalent to *Dum* in **KT4**’. A detailed syntactical analysis of these axioms based on **K4** (explicitly benefitting from the usage of modal rules), has been undertaken in [7].

$\mathbf{KT4Dum} = \mathbf{KTDum}_1 = \mathbf{KTDum}_2 = \mathbf{KTDum}_3 = \mathbf{KTDum}_4$
$\mathbf{KT4Grz} = \mathbf{KTGrz}_1 = \mathbf{KTGrz}_2 = \mathbf{KTGrz}_3 = \mathbf{KTGrz}_4 = \mathbf{KTGrz}_5$
$\mathbf{KGrz} = \mathbf{KT4MDum}$



$\mathbf{KA}_1 \dots \mathbf{A}_k \vdash C$		
A_1, \dots, A_k	C	modalized instances of A_1, \dots, A_k
$4, Dum$	Dum_1	$4\{p := \Box(p \rightarrow \Box p) \rightarrow p\}, \Box 4, Dum, Dum\{p := p \rightarrow \Box p\}$
T, Dum_1	Dum	T, Dum_1
T, Dum	Dum_2	$\Box T, Dum$
T, Dum_2	Dum	$T, \Box T\{p := p \rightarrow \Box p\}, Dum_2$
$T, 4, Dum$	Dum_3	$\Box T, 4\{p := \Box(p \rightarrow \Box p) \rightarrow p\}, \Box 4, Dum, Dum\{p := p \rightarrow \Box p\}$
T, Dum_3	Dum	$T, \Box T\{p := p \rightarrow \Box p\}, Dum_3$
Dum	Dum_4	Dum
T, Dum_4	Dum	T, Dum_4
Grz	Grz_1	$Grz\{p := (\Box(p \rightarrow \Box p) \rightarrow p) \wedge 4\{p := \Box(p \rightarrow \Box p) \rightarrow p\}\}, \Box Grz$
T, Grz_1	Grz	$T, T\{p := Grz_1\}, \Box Grz_1$
Grz	Grz_2	$Grz, \Box Grz$
T, Grz_2	Grz	$\Box T\{p := p \rightarrow \Box p\}, T\{p := \neg \Box(\Box(p \rightarrow \Box p) \rightarrow p)\}, Grz_2$
Grz	Grz_3	$Grz\{p := (\Box(p \rightarrow \Box p) \rightarrow \Box p) \wedge Grz \wedge 4\{p := (\Box(p \rightarrow \Box p) \rightarrow \Box p) \wedge Grz\}\}, \Box Grz$
T, Grz_3	Grz	$T, \Box T\{p := p \rightarrow \Box p\}, Grz_3$
Grz	Grz_4	$Grz\{p := (\Box(p \rightarrow \Box p) \rightarrow p) \wedge 4\{p := \Box(p \rightarrow \Box p) \rightarrow p\}\}, Grz$
T, Grz_4	Grz	T, Grz_4
Grz	Grz_5	$Grz\{p := (\Box(p \rightarrow \Box q) \rightarrow \Box q) \wedge 4\{p := \Box(p \rightarrow \Box q) \rightarrow \Box q\}\}, Grz\{p := (\Box((p \rightarrow \Box q) \rightarrow \Box(p \rightarrow \Box q)) \rightarrow (p \rightarrow \Box q)) \wedge 4\{p := \Box((p \rightarrow \Box q) \rightarrow \Box(p \rightarrow \Box q)) \rightarrow (p \rightarrow \Box q)\}\}, \Box Grz\{p := p \rightarrow \Box q\}$
$T, 4, Grz_5$	Grz	$T\{p := \Box(p \rightarrow \Box p) \rightarrow p\}, \Box \Box T\{p := p \rightarrow \Box p\}, 4\{p := \Box(p \rightarrow \Box p) \rightarrow p\}, \Box 4, Grz_5\{q := p \rightarrow \Box p\}$
Grz	T	Grz
Grz	4	$Grz\{p := p \wedge 4\}$
Grz	M	$\Box Grz, Grz\{p := \Diamond \neg p\}, Grz\{p := \Diamond \neg p \wedge 4\{p := \Diamond \neg p\}\}, Grz\{p := \neg \Box(p \rightarrow \Box p) \wedge 4\{p := \neg \Box(p \rightarrow \Box p)\}\}$
Grz	Dum	Grz
Grz_1	Dum_1	Grz_1
T, M, Dum	Grz	$\Box T\{p := \neg p\}, M, Dum$

4.6 Miscellaneous relations

$\mathbf{KA}_1 \dots \mathbf{A}_k \vdash C$		
A_1, \dots, A_k	C	modalized instances of A_1, \dots, A_k
R	L^*	$R\{p := \Box p \rightarrow q\}$
T, R	Zem	$T\{p := \Diamond \Box p\}, R$
$T, 4, L^*, Zem$	R	$\Box \Box T\{p := 4\}, \Box \Box T\{p := \neg(\Box \Diamond \Box p \wedge \Box p)\}, \Box 4,$ $L^*\{p := \Diamond \Box p, q := \neg \Box p\}, Zem$
$T, 4, P$	M	$T\{p := \Diamond p\}, \Box T\{p := \neg \Box \Diamond p\}, 4\{p := \Diamond p\}, \Box P$
T, P	R	$\Box \Box T\{p := \neg p\}, P$
$4, M, R$	P	$4\{p := \neg \Box p\}, \Box M, R$
T, P	H_s	$\Box \Box \Box T, P, \Box P\{p := \neg p\}$
$T, 4, R$	Dum	$T\{p := \Box(p \rightarrow \Box p) \rightarrow p\}, \Box 4\{p := p \wedge (\Box(p \rightarrow \Box p) \rightarrow p)\},$ $R\{p := p \rightarrow \Box p\}$

5 Checking the results and obtaining purely syntactic proofs

Since we give the instances required for the proofs, it is easy to check all the results using a theorem prover for \mathbf{K} .

One possibility is the Logics Workbench (see [8]). You can use it via the Internet. Open <http://lwbwww.unibe.ch:8080/LWBinfo.html> in your WWW browser, choose the item `run a session` and type in your request.

It is also possible to check the results by hand (although it is a bit tiring in some cases). One can for example make a backward search in the usual sequent calculus. (See [3] or [4] for more information on such calculi.) Such proofs can then be converted into Hilbert-style proofs (see [5] for details).

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References

- [1] R. A. Bull and K. Segerberg. Basic modal logic. In *Handbook of Philosophical Logic. Volume 2: Extensions of Classical Logic*. D. Gabbay and F. Guentner, eds. Reidel, Dordrecht, 1986.
- [2] B. F. Chellas. *Modal Logic: An Introduction*. Cambridge University Press, 1980.
- [3] M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Reidel, Dordrecht, 1983.
- [4] R. Goré. Tableau methods for modal and temporal logics. Technical Report TR-ARP-15-95, Automated Reasoning Project, Australian National University, <http://arp.anu.edu.au/>, 1995.
- [5] R. Goré, W. Heinle and A. Heuerding. Relations between propositional modal logics: an overview. Technical Report TR-ARP-16-95, Automated Reasoning Project, Australian National University, 1995.
- [6] R. Goldblatt. *Logics of Time and Computation*. CSLI Stanford, 1987.
- [7] W. Heinle. *Expressivity and Definability in Extended Modal Languages*. PhD thesis, Fakultät für Informatik der TU-München, Germany, 1995.

- [8] A. Heuerding, G. Jäger, S. Schwendimann and S. Seyfried. Propositional logics on the computer. In *Theorem Proving with Analytic Tableaux and Related Methods. TABLEAUX 95*, P. Baumgartner, R. Hähnle and J. Posegga, eds. pp. 310–323. Vol. 918 of *Lecture Notes in Computer Science*, Springer-Verlag, Berlin, 1995.
- [9] G. E. Hughes and M. J. Cresswell. *An Introduction to Modal Logic*. Methuen, London, 1968.
- [10] G. E. Hughes and M. J. Cresswell. *A Companion to Modal Logic*. Methuen, London, 1984.
- [11] E. J. Lemmon and D. Scott. *An Introduction to Modal Logic*. Blackwell, Oxford, 1977.
- [12] H. Sahlqvist. Completeness and correspondence in the first and second order semantics for modal logic. In *Proceedings of the 3rd Scandinavian Logic Symposium*, S. Kanger, ed. pp. 110–142. North Holland, Amsterdam, 1973.
- [13] K. Segerberg. An essay in classical modal logic. Technical report, Filosofiska Studier, Uppsala University, 1971.
- [14] B. Sobociński. Family K of the non-Lewis modal systems. *Notre Dame Journal of Formal Logic*, 5, 313–318, 1964.

Appendix

The following table shows the names used in the literature. Note that the formulae are only equal modulo substitutions of the form $\{p := \neg p\}$.

here	[1]	[13]	[11]	[9]	[10]	[2]	[6]
<i>D</i>	<i>D</i>		<i>D</i> (p. 50)		<i>D</i>	<i>D</i>	<i>D</i>
<i>D</i> ₂		<i>D</i> (p. 47)					
<i>T</i>	<i>T</i>	<i>T</i> (p. 47)	<i>T</i> (p. 50)		<i>T</i>	<i>T</i>	<i>T</i>
<i>4</i>	<i>4</i>	<i>4</i> (p. 47)	<i>4</i> (p. 50)		<i>4</i>	<i>4</i>	<i>4</i>
<i>5</i>	<i>E</i>	<i>E</i> (p. 47)	<i>E</i> (p. 50)		<i>5</i>	<i>5</i>	<i>5</i>
<i>B</i>	<i>B</i>	<i>B</i> (p. 47)	<i>B</i> (p. 50)	<i>B</i>	<i>B</i>	<i>B</i>	<i>B</i>
<i>G</i>	<i>G</i>	<i>G</i> (p. 47)	<i>G</i> (p. 50)	<i>G</i> ₁	<i>G</i> ₁	<i>G</i>	
<i>G</i> ₀		<i>G</i> ₀ (p. 47)					
<i>H</i>	<i>H</i>		<i>H</i> (p. 69)			<i>H</i>	
<i>H</i> *			<i>H</i> *			<i>H</i> *	
<i>L</i>		<i>Lem</i> ₀ (p. 47)	<i>H</i> ₀ * (p. 80)		<i>D</i> ₁₀		<i>L</i>
<i>L</i> *		<i>Lem</i> (p. 47)	<i>H</i> ₀ (p. 80)		<i>D</i> ₁	<i>L</i> *	<i>L</i> ₁
<i>L</i> **				<i>D</i> ₂		<i>L</i> **	
<i>M</i>	<i>M</i>	<i>M</i> (p. 107)	<i>M</i> (p. 47)	<i>K</i> _b	<i>M</i>	<i>G</i> _c	
<i>M</i> ₃				<i>K</i> _a			
<i>W</i>	<i>W</i>	<i>W</i> (p. 84)			<i>W</i>	<i>Gr</i>	<i>W</i>
<i>W</i> ₀		<i>W</i> ₀ (p. 93)			<i>W</i> ₀		
<i>Z</i>		<i>Z</i> (p. 84)					<i>Z</i>
<i>Dum</i>	<i>Dum</i>			<i>N</i> ₁	<i>N</i> ₁		
<i>Dum</i> ₁		<i>Dum</i> ₁ (p. 107)					<i>Dum</i>
<i>Grz</i>	<i>Grz</i>		<i>Grz</i> (p. 81)	<i>J</i> ₁	<i>J</i> ₁		
<i>Grz</i> ₁		<i>Grz</i> ₁ (p. 107)					
<i>Grz</i> ₅		(p. 108)					
<i>F</i>		<i>F</i> (p. 161)					
<i>H</i> ₁		<i>H</i> (p. 148)		<i>H</i> ₁			
<i>P</i>		<i>P</i> (p. 152)					
<i>R</i>		<i>R</i> (p. 160)		<i>R</i> ₁			
<i>X</i>							<i>X</i>
<i>Zem</i>		<i>Zem</i> (p. 152)					