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ON THE EULER CHARACTERISTIC OF SPHERICAL POLYHEDRA AND THE EULER RELATION

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Let E^{n+1} , for some integer $n \geq 0$, be the $(n+1)$ -dimensional Euclidean space, and denote by S^n the standard n -sphere in E^{n+1} , $S^n := \{x \in E^{n+1} : \|x\| = 1\}$. It is convenient to introduce the (-1) -dimensional sphere $S^{-1} := \emptyset$, where $\emptyset$ denotes the empty set. By an i -dimensional subsphere T of S^n , $i = 0, \dots, n$, we understand the intersection of S^n with some $(i+1)$ -dimensional subspace of E^{n+1} . The affine hull of T always contains, with this definition, the origin of E^{n+1} . $\emptyset$ is the unique (-1) -dimensional subsphere of S^n . By the spherical hull, $\text{sph } X$, of a set $X \subset S^n$, we understand the intersection of all subspheres of S^n containing X . Further we set $\dim X := \dim \text{sph } X$. The interior, the boundary and the complement of an arbitrary set $X \subset S^n$, with respect to S^n , shall be denoted by $\text{int } X$, $\text{bd } X$ and $\text{cpl } X$. Finally we define the relative interior $\text{rel int } X$ to be the interior of $X \subset S^n$ with respect to the usual topology of $\text{sph } X \subset S^n$. For $n \geq 1$ each $(n-1)$ -dimensional subsphere of S^n defines two closed hemispheres of S^n , whose common boundary it is. The two hemispheres of the sphere S^0 are defined to be the two one-pointed subsets of S^0 . A subset $P \subset S^n$ is called a closed (spherical) polytope, if it is the intersection of finitely many closed hemispheres, and, if, in addition, it does not contain a subsphere of S^n . $Q \subset S^n$ is called an i -dimensional, relatively open polytope, $i \geq 1$, or shortly an i -open polytope, if there exists a closed polytope $P \subset S^n$ such that $\dim P = i$ and $Q = \text{rel int } P$. $X \subset S^n$ is called a closed polyhedron, if it is a finite union of closed polytopes $P_1, \dots, P_r$. The empty set $\emptyset$ is the only (-1) -dimensional closed polyhedron of S^n . We denote by $\mathfrak{X}$ the set of all closed polyhedra of S^n . $Y \subset S^n$ is called an i -open polyhedron, for some $i \geq 1$, if there are finitely many i -open polytopes $Q_1, \dots, Q_r$ in S^n such that $Y = Q_1 \cup \dots \cup Q_r$, and $\dim Y = i$. By $\mathfrak{Y}_i$ we denote the set of all i -open polyhedra. Clearly $\emptyset \in \mathfrak{X}$, $\emptyset \notin \mathfrak{Y}_i$, for all $i \geq 1$, and each i -dimensional subsphere of S^n , $i \geq 1$, belongs to $\mathfrak{X}$ and to $\mathfrak{Y}_i$. For each i -dimensional subsphere T of S^n , set $\mathfrak{Y}_i(T) := \{T \in \mathfrak{Y}_i : Y \subset T\}$. A map $\varepsilon : \mathfrak{X} \cup \mathfrak{Y}_1 \cup \dots \cup \mathfrak{Y}_n \rightarrow \{0, 1\}$ is defined by $\varepsilon X := 0$, for all $X \in \mathfrak{X}$, and $\varepsilon Y := 1$, for all $Y \in \mathfrak{Y}_1 \cup \dots \cup \mathfrak{Y}_n$, $Y \notin \mathfrak{X}$.

DEFINITION 1. Let $\mathfrak{Z}$ be a ring of subsets of S^n , generated by some subset of $\mathfrak{X} \cup \mathfrak{Y}_1 \cup \dots \cup \mathfrak{Y}_n$. An Euler characteristic on $\mathfrak{Z}$ is a map $\psi : \mathfrak{Z} \rightarrow \mathbb{Z}$ (the ring of

integers) with the following properties:

- (1) If $\emptyset \in \mathfrak{Z}$, then $\psi\emptyset = 0$.
- (2) $\psi X = 1$, whenever X is a closed non-void polytope, or an i -open polytope ($i \geq 1$), contained in $\mathfrak{Z}$.
- (3) For all X, Y in $\mathfrak{Z}$, $\psi(X \cup Y) + \psi(X \cap Y) = \psi X + \psi Y$.

It is well known that there exists a unique Euler characteristic χ_0 on $\mathfrak{X}$, and, for each i -dimensional subsphere T of S^n , a unique Euler characteristic χ_T on $\mathfrak{Y}_i(T)$ (see [2], [3]). For notational convenience we denote all these characteristics by the same letter χ . Thus a mapping $\chi: \mathfrak{X} \cup \mathfrak{Y}_1 \cup \dots \cup \mathfrak{Y}_n \rightarrow \mathbb{Z}$ is defined, which satisfies (1) and (2), and which satisfies (3) for certain pairs of polyhedra. On the other hand we notice that there are rings $\mathfrak{Z}$ which admit no Euler characteristic, and others which admit more than one. For example there exists no Euler characteristic on the ring of sets generated by $\mathfrak{X} \cup \mathfrak{Y}_1 \cup \dots \cup \mathfrak{Y}_n$, $n \geq 1$. To see this, consider a 1-dimensional subsphere $S \subset S^n$, a set $X \subset S$ with two elements, and the complement $Y := S \sim X$. (3) would not hold for X and Y . Sometimes it is more convenient to study the map $\omega: \mathfrak{X} \cup \mathfrak{Y}_1 \cup \dots \cup \mathfrak{Y}_n \rightarrow \mathbb{Z}$ defined by $\omega(U) := (-1)^{\epsilon_U \dim U} \chi(U)$, rather than χ itself. For $n \geq 1$, let $S \subset S^n$ be a subsphere of dimension $n-2$, and denote by $\mathfrak{S}$ the set of all $(n-1)$ -dimensional subspheres of S^n containing S , together with the usual topology. $\mathfrak{S}$ is homeomorphic to the real projective line, and hence to S^1 . Each choice of an orientation of $\mathfrak{S}$ and of a fixed element $S_0 \in \mathfrak{S}$ determines, by means of the "angular parameter", a continuous and periodic map $p: \mathbb{R} \rightarrow \mathfrak{S}$ with $p(t) = p(t + \pi)$, for each real number t , and with the fundamental interval $I := [0, \pi)$. For the rest of this article we assume that a fixed choice of the covering projection p has been made, for every $(n-2)$ -dimensional subsphere $S \subset S^n$. The sphere $p(t) \in \mathfrak{S}$ will often be denoted by S_t . Given a map $f: \mathfrak{S} \rightarrow \mathbb{R}$ and an element $t \in I$, we define the right-hand limit $f^+(S_t)$ in the usual way. If there exists a real number x such that for each sequence of numbers t_n with $t_n \geq t$ and $t_n \rightarrow t$ ($n \rightarrow \infty$) we have $fp(t_n) \rightarrow x$ ($n \rightarrow \infty$), we set $f^+(S_t) := x$. We say that two subspheres S and T of S^n are in general position, if either $S \cap T = \emptyset$ or $\dim(S \cap T) = \dim S + \dim T - n$.

PROPOSITION 1. Let $X \subset S^n$, $n \geq 1$, be a spherical polyhedron,

$$X \in \mathfrak{X} \cup \mathfrak{Y}_1 \cup \dots \cup \mathfrak{Y}_n,$$

and let $S \subset S^n$ be an $(n-2)$ -dimensional subsphere. With the notation introduced above,

$$(i) \quad \omega X = \omega(X \cap S) + \sum_{t \in I} (\omega(X \cap S_t) - \omega^+(X \cap S_t)).$$

As above $I := [0, \pi)$ is the fundamental interval of the periodic map $p: \mathbb{R} \rightarrow \mathfrak{S}$, where $\mathfrak{S}$ stands for the set of all $(n-1)$ -spheres in S^n containing S . Before we proceed to prove Proposition 1, notice that the value $\omega(X \cap S_t) - \omega^+(X \cap S_t)$ vanishes for all but a single $t \in I$, whenever X is a closed polytope, or an i -open polytope, for some $i \geq 1$. Thus the sum to the right of the equality sign is in fact finite, for each polyhedron X . Proposition 1 is a spherical counterpart of a well

known recursion formula for the Euler characteristic for Euclidean polyhedra (see [1]).

Proof of Proposition 1. We assume $X \in \mathfrak{Y}_i$, for some $i \geq 1$. The case $X \in \mathfrak{X}$ may be treated by an obvious modification of the argument. Set $R := \text{sph } X$, and for each $Z \in \mathfrak{Y}_i(R)$,

$$\psi Z := (-1)^i \left(\omega(Z \cap S) + \sum_{t \in I} (\omega(Z \cap S_t) - \omega^+(Z \cap S_t)) \right).$$

It suffices to show that ψ is an Euler characteristic on $\mathfrak{Y}_i(R)$. The requirements (1) and (3) of Definition 1 are satisfied by ψ . Now suppose that Z is an i -open polytope in R . Let us first assume $Z \cap S \neq \emptyset$. We distinguish three cases. If the spheres S and R are in general position we have $i \geq 2$, $\dim(Z \cap S) = i - 2$, $\dim(Z \cap S_t) = i - 1$, for each t in the interval $I := [0, \pi)$, hence $\psi Z = \chi(Z \cap S) = 1$. In the case $R \subset S$ we find $Z \cap S_t = Z \cap S = Z$, for every $t \in I$. This again implies $\psi Z = \chi(Z \cap S) = 1$. If none of the above cases hold we see that $R \not\subset S$, but $R \subset S_{t'}$, for some number $t' \in I$. Hence $Z \cap S_{t'} = Z \cap S$ for all $t' \in I$, $t' \neq t$, and

$$\psi Z = (-1)^i (\omega(Z \cap S) + \omega(Z \cap S_t) - \omega(Z \cap S)) = 1.$$

Assume now $Z \cap S = \emptyset$. We are confronted with two cases. If $R \subset S_{t'}$, for some point $t' \in I$, we have $Z \cap S_t = Z$ and $Z \cap S_{t'} = \emptyset$, for every $t' \in I$, $t' \neq t$. Clearly $\psi Z = 1$. If R and S are in general position, let $A \subset I$ be the set of all points $t \in I$ such that $Z \cap S_t \neq \emptyset$. A is an open interval in I , denote its left end-point by x . Clearly

$$\omega(Z \cap S_x) - \omega^+(Z \cap S_x) = -(-1)^{i-1},$$

whereas $\omega(Z \cap S_t) - \omega^+(Z \cap S_t) = 0$, for all $t \neq x$. This shows again $\psi Z = 1$, and ψ is indeed an Euler characteristic on $\mathfrak{Y}_i(R)$. To prove (3) for ψ , notice that $\chi(X) = 0$, for each odd dimensional sphere X , hence for each $X \in \mathfrak{Y}_{2k+1} \cap \mathfrak{X}$.

DEFINITION 2. Let X be a spherical polyhedron, $X \in \mathfrak{X} \cup \mathfrak{Y}_1 \cup \dots \cup \mathfrak{Y}_n$. By a δ -decomposition of X we understand a finite set $\mathfrak{D} \subset X \cup \mathfrak{Y}_1 \cup \dots \cup \mathfrak{Y}_n$ such that $\bigcup \mathfrak{D} = X$, and, further, $U \cap V = \emptyset$ whenever U and V are two different members of $\mathfrak{D}$.

If, for example, $\mathfrak{C}$ is a complex, in the usual sense of the word, whose members are closed spherical simplices, then the relative interiors of the elements of $\mathfrak{C}$ form a δ -decomposition of $\bigcup \mathfrak{C}$.

PROPOSITION 2. For each spherical polyhedron $X \subset S^n$, $n \geq 1$,

$$X \in \mathfrak{X} \cup \mathfrak{Y}_1 \cup \dots \cup \mathfrak{Y}_n,$$

and for each δ -decomposition $\mathfrak{D}$ of X we have

$$(ii) \quad \omega X = \sum_{Y \in \mathfrak{D}} \omega Y.$$

Proof. We proceed by induction on the dimension n of the sphere S^n containing X , the case $n = 0$ being trivial. For given $n \geq 1$, $X \in \mathfrak{X} \cup \mathfrak{Y}_1 \cup \dots \cup \mathfrak{Y}_n$, and for a δ -decomposition $\mathfrak{D}$ of $X \subset S^n$, choose an $(n-2)$ -sphere $S \subset S^n$. With the notation of the section preceding Proposition 1 we find, by Proposition 1 and the inductive assumption of our statement

$$\begin{aligned}\omega X &= \omega(X \cap S) + \sum_{i \in I} (\omega(X \cap S_i) - \omega^+(X \cap S_i)) \\ &= \sum_{Y \in \mathfrak{D}} \omega(Y \cap S) + \sum_{i \in I} \sum_{Y \in \mathfrak{D}} (\omega(Y \cap S_i) - \omega^+(Y \cap S_i)) \\ &= \sum_{Y \in \mathfrak{D}} \left(\omega(Y \cap S) + \sum_{i \in I} (\omega(Y \cap S_i) - \omega^+(Y \cap S_i)) \right) \\ &= \sum_{Y \in \mathfrak{D}} \omega Y.\end{aligned}$$

As an application of the foregoing arguments let us derive some elementary relations involving the Euler characteristic.

PROPOSITION 3.

$$(iii) \chi(S^n) = 1 + (-1)^n$$

$$(iv) \chi X = \chi(\text{bd } X) + (-1)^n \chi(\text{int } X) \quad X \subset S^n, \quad X \in \mathfrak{X}$$

$$(v) \chi(\text{cpl } X) = 1 + (-1)^n - (-1)^n \chi X \quad X \subset S^n, \quad X \in \mathfrak{X}$$

$$(vi) \chi(\text{cpl } Y) = 1 + (-1)^n - (-1)^n \chi Y \quad Y \subset S^n, \quad Y \in \mathfrak{Y}_n.$$

Proof. (iii) We proceed by induction on n . The cases $n \leq 0$ are trivial. For $n \geq 1$ choose an arbitrary $(n-2)$ -dimensional subsphere S of S^n , and apply Proposition 1 to the polyhedron $S^n \in \mathfrak{X}$. By the inductive hypothesis,

$$\chi S^n = \chi S = 1 + (-1)^{n-2} = 1 + (-1)^n.$$

(iv) $\{\text{bd } X, \text{int } X\}$ is a δ -decomposition of the polyhedron $X \in \mathfrak{X}$. By Proposition 2, $\omega X = \omega(\text{bd } X) + \omega(\text{int } X)$. Since $\{X, \text{bd } X\} \subset \mathfrak{X}$ and $\text{int } X \in \mathfrak{Y}_n$, our assertion follows at once from the definition of ω .

(v) $\{X, \text{cpl } X\}$ is a δ -decomposition of the polyhedron $S^n \in \mathfrak{X}$. Our assertion follows immediately from Proposition 2 if we keep in mind that $\{X, S^n\} \subset \mathfrak{X}$ and $\text{cpl } X \in \mathfrak{Y}_n$.

(vi) The proof of this relation is quite analogous to that of (v).

References

1. H. Hadwiger, "Eine Schnittrekursion für die Eulersche Charakteristik euklidischer Polyeder mit Anwendungen innerhalb der kombinatorischen Geometrie", *El. Math.*, 23 (1968), 121–132.
2. V. Klee, "The Euler characteristic in combinatorial geometry", *Amer. Math. Monthly*, 70 (1963), 119–127.
3. H. Lenz, "Mengenalgebra und Eulersche Charakteristik," *Abh. Math. Seminar Univ. Hamburg*, 34 (1970), 135–147.

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