Let $E^{n+1}$, for some integer $n \geq 0$, be the $(n+1)$-dimensional Euclidean space, and denote by $S^n$ the standard $n$-sphere in $E^{n+1}$, $S^n := \{x \in E^{n+1} : \|x\| = 1\}$. It is convenient to introduce the $(-1)$-dimensional sphere $S^{-1} := \emptyset$, where $\emptyset$ denotes the empty set. By an $i$-dimensional subsphere $T$ of $S^n$, $i = 0, ..., n$, we understand the intersection of $S^n$ with some $(i+1)$-dimensional subspace of $E^{n+1}$. The affine hull of $T$ always contains, with this definition, the origin of $E^{n+1}$. $\emptyset$ is the unique $(-1)$-dimensional subsphere of $S^n$. By the spherical hull, $\text{sph} X$, of a set $X \subset S^n$, we understand the intersection of all subspheres of $S^n$ containing $X$. Further we set $\dim X := \dim \text{sph} X$. The interior, the boundary and the complement of an arbitrary set $X \subset S^n$, with respect to $S^n$, shall be denoted by $\text{int} X$, $\text{bd} X$ and $\text{cpl} X$. Finally we define the relative interior $\text{rel int} X$ to be the interior of $X \subset S^n$ with respect to the usual topology of $\text{sph} X \subset S^n$. For $n \geq 1$ each $(n-1)$-dimensional subsphere of $S^n$ defines two closed hemispheres of $S^n$, whose common boundary it is. The two hemispheres of the sphere $S^0$ are defined to be the two one-pointed subsets of $S^0$. A subset $P \subset S^n$ is called a closed (spherical) polytope, if it is the intersection of finitely many closed hemispheres, and, if, in addition, it does not contain a subsphere of $S^n$. $Q \subset S^n$ is called an $i$-dimensional, relatively open polytope, $i \geq 1$, or shortly an $i$-open polytope, if there exists a closed polytope $P \subset S^n$ such that $\dim P = i$ and $Q = \text{rel int} P$. $X \subset S^n$ is called a closed polyhedron, if it is a finite union of closed polytopes $P_1, ..., P_r$. The empty set $\emptyset$ is the only $(-1)$-dimensional closed polyhedron of $S^n$. We denote by $\mathcal{X}$ the set of all closed polyhedra of $S^n$. $Y \subset S^n$ is called an $i$-open polyhedron, for some $i \geq 1$, if there are finitely many $i$-open polytopes $Q_1, ..., Q_r$ in $S^n$ such that $Y = Q_1 \cup ... \cup Q_r$, and $\dim Y = i$. By $\mathcal{Y}_i$ we denote the set of all $i$-open polyhedra. Clearly $\emptyset \in \mathcal{X}$, $\emptyset \notin \mathcal{Y}_i$, for all $i \geq 1$, and each $i$-dimensional subsphere of $S^n$, $i \geq 1$, belongs to $\mathcal{X}$ and to $\mathcal{Y}_i$. For each $i$-dimensional subsphere $T$ of $S^n$, set $\mathcal{Y}_i(T) := \{T \in \mathcal{Y}_i : Y \subset T\}$. A map $\varepsilon : \mathcal{X} \cup \mathcal{Y}_1 \cup ... \cup \mathcal{Y}_n \rightarrow \{0, 1\}$ is defined by $\varepsilon X := 0$, for all $X \in \mathcal{X}$, and $\varepsilon Y := 1$, for all $Y \in \mathcal{Y}_1 \cup ... \cup \mathcal{Y}_n$, $Y \notin \mathcal{X}$.

**Definition 1.** Let $\mathcal{Z}$ be a ring of subsets of $S^n$, generated by some subset of $\mathcal{X} \cup \mathcal{Y}_1 \cup ... \cup \mathcal{Y}_n$. An Euler characteristic on $\mathcal{Z}$ is a map $\psi : \mathcal{Z} \rightarrow \mathbb{Z}$ (the ring of
integers) with the following properties:

(1) If $\emptyset \in \mathcal{Z}$ then $\psi \emptyset = 0$.

(2) $\psi X = 1$, whenever $X$ is a closed non-void polytope, or an $i$-open polytope ($i \geq 1$), contained in $\mathcal{Z}$.

(3) For all $X, Y$ in $\mathcal{Z}$, $\psi (X \cup Y) + \psi (X \cap Y) = \psi X + \psi Y$.

It is well known that there exists a unique Euler characteristic $\chi_0$ on $\mathfrak{X}$, and, for each $i$-dimensional subsphere $T$ of $S^n$, a unique Euler characteristic $\chi_T$ on $\mathfrak{Y}_i(T)$ (see [2], [3]). For notational convenience we denote all these characteristics by the same letter $\chi$. Thus a mapping $\chi: \mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n \to \mathbb{Z}$ is defined, which satisfies (1) and (2), and which satisfies (3) for certain pairs of polyhedra. On the other hand we notice that there are rings $\mathfrak{Z}$ which admit no Euler characteristic, and others which admit more than one. For example there exists no Euler characteristic on the ring of sets generated by $\mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n$, $n \geq 1$. To see this, consider a 1-dimensional subsphere $S \subseteq S^n$, a set $X \subseteq S$ with two elements, and the complement $Y := S \sim X$. (3) would not hold for $X$ and $Y$. Sometimes it is more convenient to study the map $\omega: \mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n \to \mathbb{Z}$ defined by $\omega(U) := (-1)^{\text{dim} U} \chi(U)$, rather than $\chi$ itself. For $n \geq 1$, let $S \subseteq S^n$ be a subsphere of dimension $n - 2$, and denote by $\mathcal{S}$ the set of all $(n - 1)$-dimensional subspheres of $S^n$ containing $S$, together with the usual topology. $\mathcal{S}$ is homeomorphic to the real projective line, and hence to $S^1$. Each choice of an orientation of $\mathcal{S}$ and of a fixed element $S_0 \in \mathcal{S}$ determines, by means of the "angular parameter", a continuous and periodic map $p: \mathbb{R} \to \mathcal{S}$ with $p(t) = p(t + \pi)$, for each real number $t$, and with the fundamental interval $I := [0, \pi)$. For the rest of this article we assume that a fixed choice of the covering projection $p$ has been made, for every $(n - 2)$-dimensional subsphere $S \subseteq S^n$. The sphere $p(t) \in \mathcal{S}$ will often be denoted by $S_t$. Given a map $f: \mathcal{S} \to \mathbb{R}$ and an element $t \in I$, we define the right-hand limit $f^+(S_t)$ in the usual way. If there exists a real number $x$ such that for each sequence of numbers $t_n$ with $t_n \geq t$ and $t_n \to t$ ($n \to \infty$) we have $f(S_{t_n}) \to x$ ($n \to \infty$), we set $f^+(S_t) := x$. We say that two subspheres $S$ and $T$ of $S^n$ are in general position, if either $S \cap T = \emptyset$ or $\dim (S \cap T) = \dim S + \dim T - n$.

**Proposition 1.** Let $X \subseteq S^n$, $n \geq 1$, be a spherical polyhedron, 

$$X \in \mathfrak{X} \cup \mathfrak{Y}_1 \cup \ldots \cup \mathfrak{Y}_n,$$

and let $S \subseteq S^n$ be an $(n - 2)$-dimensional subsphere. With the notation introduced above,

(i) $\omega X = \omega (X \cap S) + \sum_{t \in I} (\omega (X \cap S_t) - \omega^+ (X \cap S_t)).$

As above $I := [0, \pi)$ is the fundamental interval of the periodic map $p: \mathbb{R} \to \mathcal{S}$, where $\mathcal{S}$ stands for the set of all $(n - 1)$-spheres in $S^n$ containing $S$. Before we proceed to prove Proposition 1, notice that the value $\omega (X \cap S_i) - \omega^+ (X \cap S_i)$ vanishes for all but a single $t \in I$, whenever $X$ is a closed polytope, or an $i$-open polytope, for some $i \geq 1$. Thus the sum to the right of the equality sign is in fact finite, for each polyhedron $X$. Proposition 1 is a spherical counterpart of a well
known recursion formula for the Euler characteristic for Euclidean polyhedra (see [1]).

Proof of Proposition 1. We assume \( X \in \mathfrak{j}_i \) for some \( i > 1 \). The case \( X \in \mathfrak{x} \) may be treated by an obvious modification of the argument. Set \( R := \text{sph} X \), and for each \( Z \in \mathfrak{j}_i(R) \),

\[
\psi Z := (-1)^i \left( \omega(Z \cap S) + \sum_{t \in I} \left( \omega(Z \cap S_t) - \omega^+(Z \cap S_t) \right) \right).
\]

It suffices to show that \( \psi \) is an Euler characteristic on \( \mathfrak{j}_i(R) \). The requirements (1) and (3) of Definition 1 are satisfied by \( \psi \). Now suppose that \( Z \) is an i-open polytope in \( R \). Let us first assume \( Z \cap S \not= \emptyset \). We distinguish three cases. If the spheres \( S \) and \( R \) are in general position we have \( i > 2 \), \( \dim(Z \cap S) = i - 2 \), \( \dim(Z \cap S_t) = i - 1 \), for each \( t \) in the interval \( I := [0, \pi) \), hence \( \psi Z = \chi(Z \cap S) = 1 \). In the case \( R \subset S \) we find \( Z \cap S_t = Z \cap S = Z \), for every \( t \in I \). This again implies \( \psi Z = \chi(Z \cap S) = 1 \). If none of the above cases hold we see that \( R \not= S \), but \( R \subset S_t \), for some number \( t \in I \). Hence \( Z \cap S_t = Z \cap S \) for all \( t' \in I, t' \not= t \), and

\[
\psi Z = (-1)^i \left( \omega(Z \cap S) + \omega(Z \cap S_t) - \omega(Z \cap S) \right) = 1.
\]

Assume now \( Z \cap S = \emptyset \). We are confronted with two cases. If \( R \subset S_t \), for some point \( t \in I \), we have \( Z \cap S_t = Z \) and \( Z \cap S_t' = \emptyset \), for every \( t' \in I, t' \not= t \). Clearly \( \psi Z = 1 \). If \( R \) and \( S \) are in general position, let \( A \in I \) be the set of all points \( t \in I \), such that \( Z \cap S_t \not= \emptyset \). \( A \) is an open interval in \( I \), denote its left end-point by \( x \). Clearly

\[
\omega(Z \cap S_x) - \omega^+(Z \cap S_x) = -(-1)^{i-1},
\]

whereas \( \omega(Z \cap S_t) - \omega^+(Z \cap S_t) = 0 \), for all \( t \not= x \). This shows again \( \psi Z = 1 \), and \( \psi \) is indeed an Euler characteristic on \( \mathfrak{j}_i(R) \). To prove (3) for \( \psi \), notice that \( \chi(X) = 0 \), for each odd dimensional sphere \( X \), hence for each \( X \in \mathfrak{j}_{2k+1} \cap \mathfrak{x} \).

Definition 2. Let \( X \) be a spherical polyhedron, \( X \in \mathfrak{x} \cup \mathfrak{j}_1 \cup \ldots \cup \mathfrak{j}_n \). By a \( \delta \)-decomposition of \( X \) we understand a finite set \( \mathfrak{D} \subset X \cup \mathfrak{j}_1 \cup \ldots \cup \mathfrak{j}_n \) such that \( \bigcup \mathfrak{D} = X \), and, further, \( U \cap V = \emptyset \) whenever \( U \) and \( V \) are two different members of \( \mathfrak{D} \).

If, for example, \( \mathfrak{C} \) is a complex, in the usual sense of the word, whose members are closed spherical simplices, then the relative interiors of the elements of \( \mathfrak{C} \) form a \( \delta \)-decomposition of \( \bigcup \mathfrak{C} \).

Proposition 2. For each spherical polyhedron \( X \subset S^n, n \geq 1 \),

\[
X \in \mathfrak{x} \cup \mathfrak{j}_1 \cup \ldots \cup \mathfrak{j}_n,
\]

and for each \( \delta \)-decomposition \( \mathfrak{D} \) of \( X \) we have

\[
(\text{ii}) \quad \omega X = \sum_{T \in \mathfrak{D}} \omega Y.
\]
Proof. We proceed by induction on the dimension $n$ of the sphere $S^n$ containing $X$, the case $n = 0$ being trivial. For given $n \geq 1$, $X \in \mathcal{X} \cup \mathcal{Y}_1 \cup \ldots \cup \mathcal{Y}_n$, and for a $\delta$-decomposition $\mathcal{D}$ of $X \in S^n$, choose an $(n-2)$-sphere $S \subset S^n$. With the notation of the section preceding Proposition 1 we find, by Proposition 1 and the inductive assumption of our statement

$$\omega X = \omega(X \cap S) + \sum_{t \in \mathcal{I}} (\omega(X \cap S_t) - \omega^+(X \cap S_t))$$

$$= \sum_{Y \in \mathcal{D}} \omega(Y \cap S) + \sum_{t \in \mathcal{I}} \sum_{Y \in \mathcal{D}} (\omega(Y \cap S_t) - \omega^+(Y \cap S_t))$$

$$= \sum_{Y \in \mathcal{D}} \left(\omega(Y \cap S) + \sum_{t \in \mathcal{I}} (\omega(Y \cap S_t) - \omega^+(Y \cap S_t))\right)$$

$$= \sum_{Y \in \mathcal{I}} \omega Y.$$

As an application of the foregoing arguments let us derive some elementary relations involving the Euler characteristic.

**Proposition 3.**

(iii) $\chi(S^n) = 1 + (-1)^n$

(iv) $\chi X = \chi(bd X) + (-1)^n \chi(int X)$

(v) $\chi(cpl X) = 1 + (-1)^n - (-1)^n \chi X$

(vi) $\chi(cpl Y) = 1 + (-1)^n - (-1)^n \chi Y$

Proof. (iii) We proceed by induction on $n$. The cases $n \leq 0$ are trivial. For $n \geq 1$ choose an arbitrary $(n-2)$-dimensional subsphere $S$ of $S^n$, and apply Proposition 1 to the polyhedron $S^n \in \mathcal{X}$. By the inductive hypothesis,

$$\chi S^n = \chi S = 1 + (-1)^{n-2} = 1 + (-1)^n.$$

(iv) $\{bd X, int X\}$ is a $\delta$-decomposition of the polyhedron $X \in \mathcal{X}$. By Proposition 2, $\omega X = \omega(bd X) + \omega(int X)$. Since $\{X, bd X\} \subset \mathcal{X}$ and $int X \in \mathcal{Y}_n$, our assertion follows at once from the definition of $\omega$.

(v) $\{X, cpl X\}$ is a $\delta$-decomposition of the polyhedron $S^n \in \mathcal{X}$. Our assertion follows immediately from Proposition 2 if we keep in mind that $\{X, S^n\} \subset \mathcal{X}$ and $cpl X \in \mathcal{Y}_n$.

(vi) The proof of this relation is quite analogous to that of (v).
References


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