

MODEL COMPANIONS OF DISTRIBUTIVE p -ALGEBRAS

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§0. Introduction. Let B_n , $0 \leq n \leq \omega$, be the equational classes of distributive p -algebras (precise definitions are given in §1). It has been known for some time that the elementary theories T_n of B_n possess model companions T_n^* ; see, e.g., [6] and [14] and the references given there. However, no axiomatizations of T_n^* were given, with the exception of $n = 0$ (Boolean case) and $n = 1$ (Stonian case). While the first case belongs to the folklore of the subject (see [6], also [11]), the second case presented considerable difficulties (see Schmitt [13]). Schmitt's use of methods characteristic for Stone algebras seems to prevent a ready adaptation of his results to the cases $n \geq 2$.

The natural way to get a hold on T_n^* is to determine the class $E(B_n)$ of existentially complete members of B_n : Since T_n^* exists, it equals the elementary theory of $E(B_n)$. The present author succeeded [12] in solving the simpler problem of determining the classes $A(B_n)$ of algebraically closed algebras in B_n (exact definitions of $A(B_n)$ and $E(B_n)$ are given in §1) for all $0 \leq n \leq \omega$. $A(B_n)$ is easier to handle since it contains sufficiently many "small" algebras—viz. finite direct products of certain subdirectly irreducibles—in terms of which the members of $A(B_n)$ may be analyzed (in contrast, all members of $E(B_n)$ are infinite and \aleph_0 -homogeneous). As it turns out, $A(B_n)$ is finitely axiomatizable for all n , and comparing the theories of $A(B_0)$, $A(B_1)$ with the explicitly known theories of $E(B_0)$, $E(B_1)$ —viz. T_0^* , T_1^* —, a reasonable conjecture for T_n^* , $2 \leq n \leq \omega$, is immediate. The main part of this paper is concerned with verifying that the conditions formalized by T_n^* suffice to describe the algebras in $E(B_n)$ (necessity is easy). This verification rests on the same combinatorial techniques as used in [12] to describe the members of $A(B_n)$.

§1 gives the pertinent definitions. For anything not found there, the reader is referred to Grätzer [3, Chapter III, in particular] for the algebraic part and to Hirschfeld and Wheeler [6] for the model-theoretic side. In §2, we summarize the results on $A(B_n)$ from [12] and characterize the members of $E(B_n)$ within $A(B_n)$ by four conditions, EC1 through EC4. Combination of these results yields the desired description of existentially complete algebras in B_n for $0 \leq n \leq \omega$. Formalizing these descriptions accordingly, T_n^* may be written down explicitly (§3) and shown to be \aleph_0 -categorical and complete for all n , whereas only T_0^* , T_1^* , T_2^* and T_ω^* are even model completions of their respective T_i .

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§1. Definitions and notation. A (*distributive*) p -algebra L is an algebra $L(\wedge, \vee, *, 0, 1)$ such that $L(\wedge, \vee, 0, 1)$ is a distributive lattice with universal bounds 0 and 1 and the unary operation $*$ satisfies $x \leq a^*$ iff $x \wedge a = 0$. Since only distributive p -algebras are considered in this paper, “distributive” will be omitted in the sequel. The class of all p -algebras is equational and will be denoted by \underline{B}_ω . The nontrivial equational subclasses of \underline{B}_ω may be enumerated in a sequence $\underline{B}_0 \subseteq \underline{B}_1 \subseteq \dots \subseteq \underline{B}_n \subseteq \dots \subseteq \underline{B}_\omega$ ($n \in \omega$) (see Lee [10]). \underline{B}_0 is the class of all Boolean algebras, \underline{B}_1 that of all Stone algebras (those satisfying the identity $x^* \vee x^{**} = 1$). The easiest way to describe the classes \underline{B}_n is by listing their subdirectly irreducible members. We need some notation. Let $\underline{2}$ be the two-element Boolean algebra, and put $F_n = \underline{2}_n$ for $n \in \omega$. For later reference, we agree to write \underline{C} for the countable atomless Boolean algebra. If L is any lattice, \hat{L} denotes the lattice obtained from L by adjoining a new greatest element to L . Now the subdirectly irreducible algebras in \underline{B}_n ($n < \omega$) are exactly $\hat{F}_0 \cong \underline{2}, \hat{F}_1, \dots, \hat{F}_n$, while an algebra is subdirectly irreducible in \underline{B}_ω iff it is of the form \hat{B} for some Boolean algebra B . For details, compare Chapter III of [3].

On the model-theoretic side, we use a first-order language \mathcal{L} with equality. \mathcal{L} has variables x_1, x_2, \dots and as nonlogical symbols two binary function symbols \wedge, \vee , a unary function symbol $*$ and two constants 0, 1 with the obvious intended interpretations. We define \mathcal{L} -theories T_n for $0 \leq n \leq \omega$ as follows: T_ω consists of any convenient set of \mathcal{L} -sentences axiomatizing distributive lattices with 0, 1 together with $(\forall x_1, x_2)(x_1 \wedge (x_1 \wedge x_2)^* = x_1 \wedge x_2^*)$. For $n \geq 1$, let θ_n be the sentence

$$(\forall x_1, \dots, x_n)((x_1 \wedge \dots \wedge x_n)^* \vee (x_1^* \wedge x_2 \wedge \dots \wedge x_n)^* \\ \vee (x_1 \wedge x_2^* \wedge \dots \wedge x_n)^* \vee \dots \vee (x_1 \wedge x_2 \wedge \dots \wedge x_n^*)^* = 1).$$

Let θ_0 be $(\forall x_1)(x_1 \vee x_1^* = 1)$. Now define $T_n = T_\omega \cup \{\theta_n\}$. Then \underline{B}_n is exactly the class of models of T_n for $0 \leq n \leq \omega$ (see [10]).

Consider now any fixed $\underline{B}_n, 0 \leq n \leq \omega$. $L \in \underline{B}_n$ is called *existentially complete* (abbreviated e.c.) iff for any \exists_1 -sentence θ from $\mathcal{L}(L)$ and for any extension $L' \in \underline{B}_n$ of $L, L' \models \theta$ implies $L \models \theta$. L is called *algebraically closed* (abbreviated a.c.) if the same holds for *positive* \exists_1 -sentences. We put $E(\underline{B}_n) = \{L \in \underline{B}_n; L \text{ is e.c.}\}$ and $A(\underline{B}_n) = \{L \in \underline{B}_n; L \text{ is a.c.}\}$. Hence $E(\underline{B}_n) \subseteq A(\underline{B}_n) \subseteq \underline{B}_n$ and all inclusions are strict, as we shall see. Given $L \in \underline{B}_\omega, SL = \{x \in L; x = x^{**}\}$ is the *skeleton* of $L, CL = \{x \in L; x \vee x^* = 1\}$ is the *center* of L and $DL = \{x \in L; x^* = 0\}$ is the filter of *dense* elements of L . Any $L \in \underline{B}_\omega$ contains a largest subalgebra which is *Stonian*, i.e., which belongs to \underline{B}_1 . This is the subalgebra of L generated by $CL \cup DL$, and we denote it by $\text{Ston } L$. This definition is due to Katriňák; see, e.g., [7] for more details. Alternatively, $\text{Ston } L = \{x \in L; x^* \vee x^{**} = 1\}$. The following definition is adopted for technical convenience: Let $L \in \underline{B}_\omega, s \in \text{Ston } L$. Define $B_L(s) = \{b \in SL; b \leq s \text{ and } b \vee b^* = s \vee s^*\} \cup \{0, s\}$. The subscript L will be omitted when there is no danger of confusion. In general, $B(s)$ is not closed under \wedge or \vee ; however $x \in B(s)$ implies $x^* \wedge s \in B(s)$ (since $x \vee (x^* \wedge s) = s$ and $x \wedge (x^* \wedge s) = 0$ this defines a relativized complementation on $B(s)$). Details may be found in [12]. Given $x \in L$ and a finite subset $\{y_1, \dots, y_n\} \subseteq L$, we say that $\{y_1, \dots, y_n\}$ is a *partition* of x provided $y_1 \vee \dots \vee y_n = x$ and $y_i \wedge y_k = 0$ for $1 \leq i < k \leq n$. A partition is called *proper* iff it does not contain 0. Finally,

given two p -algebras L_1, L_2 , we allow ourselves to write $L_1 \subseteq L_2$ whenever L_2 contains an isomorphic copy of L_1 as a p -subalgebra.

§2. Existentially complete distributive p -algebras. The members of $A(\underline{B}_n)$ for $0 \leq n \leq \omega$ were determined in [12]. The following two theorems summarize the situation.

THEOREM 1 ($n < \omega$). *Let $L \in \underline{B}_n$, $0 \leq n < \omega$. The following are equivalent:*

(i) $L \in A(\underline{B}_n)$.

(ii) L satisfies the following four conditions:

AC1 DL is relatively complemented.

AC2 For all $d_1, d_2 \in DL$ satisfying $d_1 \vee d_2 = 1$, there exists $c \in CL$ such that $c \leq d_1, c^* \leq d_2$.

AC3 $_n$ Assume $n \geq 2$. For all $s \in \text{Ston } L \setminus CL$, there exists a proper partition $\{b_1, \dots, b_n\} \subseteq B_L(s)$ of s .

AC4 $_n$ Assume $n \geq 2$. Put $N = 2^n + 1$. For all $s \in \text{Ston } L \setminus CL$, every proper partition $\{b_1, \dots, b_n\} \subseteq B_L(s)$ of s and every $0, s \neq b \in B_L(s)$, there exists a partition $\{c_1, \dots, c_N\} \subseteq CL$ of $s^{**} \in CL$ such that $b = \bigvee \{(b_i \wedge c_j); 1 \leq i \leq n, 1 \leq j \leq N\}$.

(iii) If L_0 is a finite subalgebra of L , there exists a p -algebra L_1 such that $L_0 \subseteq L_1 \subseteq L$ and $L_1 \cong \hat{2}^i \times \hat{F}_j^i$ for some $i, j \in \omega$.

THEOREM 1 $_\omega$. *Let $L \in \underline{B}_\omega$. The following are equivalent:*

(i) $L \in A(\underline{B}_\omega)$.

(ii) L satisfies the following four conditions:

AC1 as above.

AC2 as above.

AC3 $_\omega$ For all $s \in \text{Ston } L \setminus CL$, $B_L(s) \supsetneq \{0, s\}$.

AC4 $_\omega$ For all $s \in \text{Ston } L \setminus CL$ and all $0, s \neq b \in B_L(s)$, there exists a proper partition $\{b_1, b_2\} \subseteq B_L(s)$ of b .

(iii) If L_0 is a finite subalgebra of L , there exists a p -algebra L_1 such that $L_0 \subseteq L_1 \subseteq L$ and $L_1 \cong \hat{2}^i \times \hat{C}^j$ for some $i, j \in \omega$.

The following definition lists the conditions necessary and sufficient to characterize the members of $E(\underline{B}_n)$ within $A(\underline{B}_n)$ for $0 \leq n \leq \omega$:

DEFINITION. Let $L \in \underline{B}_n$, $0 \leq n \leq \omega$. L will be said to satisfy.

EC1 iff CL has no atoms.

EC2 iff DL has no antiatoms;

EC3 iff for any $1 \neq c \in CL$ the set $\{d \in DL; 1 \geq d \geq c\}$ has no least element;

EC4 iff for any $0 \neq b \in SL$ there exists $0 \neq c \in CL$ such that $c \leq b$.

The following theorem contains the main result of this paper.

THEOREM 2. *An algebra is existentially complete in \underline{B}_n ($0 \leq n \leq \omega$) iff it is algebraically closed and satisfies EC1–EC4. Alternatively, $L \in E(\underline{B}_n)$ iff L satisfies AC1–AC4 $_n$ and EC1–EC4.*

Theorem 2 may be rephrased in a more compact form using $\text{Ston } L$ and Theorem 3.2 of [13]:

COROLLARY 3. *$L \in E(\underline{B}_n)$ ($0 \leq n \leq \omega$) iff $\text{Ston } L \in E(\underline{B}_1)$ and L satisfies AC3 $_n$, AC4 $_n$ and EC4.*

PROOF. Schmitt [13] proved that $L \in E(\underline{B}_1)$ iff L satisfies AC1, AC2 and EC1–EC3. ■

The proof of Theorem 2 will be broken down into a series of lemmata. We begin with the easy half of the theorem.

LEMMA 4. *Let $L \in E(\underline{B}_n)$, $0 \leq n \leq \omega$. Then L satisfies EC1–EC4.*

PROOF. Consider the following \exists_1 -sentences from $\mathcal{L}(L)$:

- $\theta_1(c) : (\exists x_1)(x_1 \vee x_1^* = 1 \ \& \ 0 < x_1 < c), 0 \neq c \in CL,$
- $\theta_2(d) : (\exists x_1)(x_1^* = 0 \ \& \ d < x_1 < 1), 1 \neq d \in DL,$
- $\theta_3(c, d) : (\exists x_1)(x_1^* = 0 \ \& \ c \leq x_1 < d), 1 \neq c \in CL \text{ and } c < d \leq 1, d \in DL,$
- $\theta_4(b) : (\exists x_1)(x_1 \vee x_1^* = 1 \ \& \ x_1 \neq 0 \ \& \ x_1 \leq b), 0 \neq b \in SL.$

Each one of these sentences may be satisfied in some direct product $L' \supseteq L$ of suitably many subdirectly irreducibles from \underline{B}_n , so they must hold in L . ■

LEMMA 5. *Let $L \in \underline{B}_n$, $0 \leq n \leq \omega$, and assume $L \cong \prod(L_i, i \in I)$. Then any of EC1, . . . , EC4 holds in L iff it holds in every $L_i, i \in I$.*

PROOF. Straightforward. ■

We use the following notation: If $L \in \underline{B}_\omega$ and $a_1, \dots, a_n \in L$, $\langle a_1, \dots, a_n \rangle_L$ denotes the subalgebra of L generated by $\{a_1, \dots, a_n\}$. A homomorphism f is over an algebra L iff $L \subseteq \text{dom } f$ and f fixes L pointwise.

The following two lemmata take care of the essential cases of the sufficiency half of Theorem 2.

LEMMA 6. *Assume $L \in \underline{B}_n$ ($2 \leq n < \omega$) satisfies AC1–AC4n and EC1–EC4. Let $L \subseteq L'$ and $L' \cong \hat{F}_n^T$ (T some index set); $L_0 \subseteq L$ and $L_0 \cong F_r$ for some $1 \leq r \leq n$ or $L_0 \cong \hat{F}_n$; $L_1 \subseteq L'$ and $L_1 \cong F_s$ for some $1 \leq s \leq n$ or $L_1 \cong \hat{F}_n$. Then there exists $L_2 \subseteq L, L_2 \cong L_1$ such that $\langle L_0 \cup L_1 \rangle$ and $\langle L_0 \cup L_2 \rangle$ are isomorphic over L_0 .*

PROOF. Assume L, L', L_0 and L_1 are given as described. Let p_1, \dots, p_r be the atoms of L_0, q_1, \dots, q_s those of L_1 ; $d = p_1 \vee \dots \vee p_r, \delta = q_1 \vee \dots \vee q_s$; denote by S_n the group of permutations of an n -element set.

We define u_j ($1 \leq j \leq s$), v_h ($h \in S_n$), x_i ($1 \leq i \leq r$) and y_{ij} ($1 \leq i \leq r, 1 \leq j \leq s$) in CL' as follows by listing their components ($t \in T$):

$$u_{jt} = \begin{cases} 1, & d_t \neq 1 \text{ and } q_{jt} = 1, \\ 0, & \text{otherwise} \end{cases} \quad (L_0 \cong \hat{F}_n, r = n);$$

$$v_{ht} = \begin{cases} 1, & d_t = \delta_t \neq 1 \text{ and } q_{jt} = P_{h(j)t} \text{ for } 1 \leq j \leq s, \\ 0, & \text{otherwise} \end{cases} \quad (L_0 \cong \hat{F}_n \cong L_1, r = s = n);$$

$$x_{it} = \begin{cases} 1, & \delta_t \neq 1 \text{ and } p_{it} = 1, \\ 0, & \text{otherwise} \end{cases} \quad (L_1 \cong \hat{F}_n, s = n);$$

$$y_{ijt} = \begin{cases} 1, & p_{it} = 1 = q_{jt}, \\ 0, & \text{otherwise.} \end{cases}$$

It is fairly obvious that u_j, v_h, x_i, y_{ij} are central in L' , pairwise disjoint and have join 1. Note that if $u_j \neq 0$, then $u_j \not\leq d < 1$ and similarly for v_h , whereas $x_i, y_{ij} \leq p_i \leq d$. We will now use AC1 through EC4 to “simulate” these members of CL' within CL .

Suppose $d = 1$. Hence $p_i \in CL$ for $1 \leq i \leq r$. Use EC1 to find $\bar{x}_i, \bar{y}_{ij} \in CL$, pairwise disjoint, such that $p_i = \bar{x}_i \vee \bigvee_j \bar{y}_{ij}$ and $\bar{x}_i = 0$ iff $x_i = 0, \bar{y}_{ij} = 0$ iff $y_{ij} = 0$. Put $\bar{u}_j = \bar{v}_h = 0$ for $1 \leq j \leq s, h \in S_n$.

Suppose $d < 1$. Use EC4 to find $c_i \in CL, c_i \neq 0$, such that $c_i \leq p_i$ for $1 \leq i \leq r$. Use EC1 to find $\bar{x}_i, \bar{y}_{ij} \in CL$, pairwise disjoint, such that $c_i = \bar{x}_i \vee \bigvee_j \bar{y}_{ij}$ for $1 \leq i \leq r$, and $\bar{x}_i = 0$ iff $x_i = 0, \bar{y}_{ij} = 0$ iff $y_{ij} = 0$. Let $c = (c_1 \vee \dots \vee c_r)^*$. It follows that $c \not\leq d$. We want to construct $\bar{u}_j, \bar{v}_h \in CL$ for $1 \leq j \leq s, h \in S_n$, pairwise disjoint, $\not\leq d, c = \bigvee_j \bar{u}_j \vee \bigvee_h \bar{v}_h, \bar{u}_j = 0$ iff $u_j = 0, \bar{v}_h = 0$ iff $v_h = 0$. It suffices to show that if $c \not\leq d$, there exist nonzero $c_1, c_2 \in CL$ satisfying $c_1 \vee c_2 = c, c_1 \wedge c_2 = 0, c_1, c_2 \not\leq d$. Now $c \not\leq d$ implies $c^* \vee d < 1$. By EC3, we find $d_1 \in DL$ such that $c^* \vee d < d_1 < 1$. By AC1, there exists $d_2 \in DL$ such that $d_2 \wedge d_1 = c^* \vee d, d_2 \vee d_1 = 1$. By AC2, there exists $c_0 \in CL$ such that $c_0 \leq d_1, c_0^* \leq d_2$. It is easy to check that $c_1 = c \wedge c_0, c_2 = c \wedge c_0^*$ have the required properties.

L_2 will now be constructed by describing its atoms Q_1, \dots, Q_s , "piecewise", that is, by listing the meets of Q_1, \dots, Q_s with $\bar{u}_j, \bar{v}_h, \bar{x}_i, \bar{y}_{ij}$. This is sufficient since all algebras of type F_r or \hat{F}_r are generated, as p -algebras, by their atoms. Only $Q_j \wedge \bar{x}_i$ requires some preliminary work. Assume $\bar{x}_i \neq 0$. Apply EC2 to \bar{x}_i^* in order to produce $d_i \in DL$ satisfying $\bar{x}_i \wedge d_i < \bar{x}_i$. Obviously, $\bar{x}_i \wedge d_i \in \text{Ston}L \setminus CL$, so by AC3n there exists a proper partition $\{\beta_{i1}, \dots, \beta_{in}\} \subseteq B(\bar{x}_i \wedge d_i)$ of $\bar{x}_i \wedge d_i$ (as observed above, $\bar{x}_i \neq 0$ implies $L_1 \cong \hat{F}_n$ and thus $s = n$). Now put

$$Q_j = \bar{u}_j \vee \bigvee_{h \in S_n} (p_{h(j)} \wedge \bar{v}_h) \vee \bigvee_{\bar{x}_i \neq 0} \beta_{ij} \vee \bigvee_{i=1}^r (p_i \wedge \bar{y}_{ij})$$

and let $L_2 = \langle Q_1, \dots, Q_s \rangle_L$. It is fairly obvious from the construction that $\langle L_0 \cup L_1 \rangle$ and $\langle L_0 \cup L_2 \rangle$ are isomorphic over L_0 . ■

LEMMA 7. Assume $L \in \mathcal{B}_\omega$ satisfies AC1–AC4 and EC1–EC4. Let $L \subseteq L'$ and $L \cong \prod (A_t, t \in T)$, where A_t is an atomless Boolean algebra for each $t \in T; L_0 \subseteq L$ and $L_0 \cong F_r$ or $L_0 \cong \hat{F}_r$ for some $r \in \omega, r \geq 1; L_1 \subseteq L'$ and $L_1 \cong F_s$ or $L_1 \cong \hat{F}_s$ for some $s \in \omega, s \geq 1$. Then there exists $L_2 \subseteq L, L_2 \cong L_1$, such that $\langle L_0 \cup L_1 \rangle$ and $\langle L_0 \cup L_2 \rangle$ are isomorphic over L_0 .

PROOF. Assume L, L', L_0, L_1 are given as specified; let p_1, \dots, p_r be the atoms of L_0, q_1, \dots, q_s those of $L_1; d = p_1 \vee \dots \vee p_r, \delta = q_1 \vee \dots \vee q_s$. Let M be the set of all $r \times s$ (0, 1)-matrices having at least one 1 in each row and each column.

We define $u_j (1 \leq j \leq s), v_A (A \in M), x_i (1 \leq i \leq r)$ and $y_{ijt} (1 \leq i \leq r, 1 \leq j \leq s)$ in CL' as follows by listing their components ($t \in T$):

$$\begin{aligned} u_{jt} &= \begin{cases} 1, & d_t \neq 1 \text{ and } q_{jt} = 1, \\ 0, & \text{otherwise;} \end{cases} \\ v_{At} &= \begin{cases} 1, & d_t = \delta_t \neq 1 \text{ and } p_{it} \wedge q_{jt} = 0 \text{ iff } \alpha_{ij} = 0, \text{ where } A = (\alpha_{ij}), \\ 0, & \text{otherwise;} \end{cases} \\ x_{it} &= \begin{cases} 1, & \delta_t \neq 1 \text{ and } p_{it} = 1, \\ 0, & \text{otherwise;} \end{cases} \\ y_{ijt} &= \begin{cases} 1, & p_{it} = 1 = q_{jt}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Again, u_j, v_A, x_i, y_{ij} are central in CL' , pairwise disjoint and have join 1; $u_j \neq 0$ implies $u_j \not\leq d < 1$, and similarly for v_A .

Proceed as in the proof of Lemma 6 to obtain $\bar{u}_j, \bar{v}_A, \bar{x}_i, \bar{y}_{ij} \in CL$ which are 0 exactly if their counterparts in CL' are such, which are pairwise disjoint, have join 1, and satisfy $\bar{x}_i, \bar{y}_{ij} \leq p_i, \bar{u}_j, \bar{v}_A \not\leq d$ provided $d \leq 1$.

We construct Q_1, \dots, Q_s in the same way as in the proof of Lemma 6. Let $A = (\alpha_{ij}) \in M$. To obtain $Q_j \wedge \bar{v}_A$, consider $p_i \wedge \bar{v}_A$ for $i \leq 1 \leq r$: $p_i \wedge \bar{v}_A \in B(d \wedge \bar{v}_A)$, so use AC3 ω and AC4 ω to get a representation $p_i \wedge \bar{v}_A = \bigvee_{j=1}^s p_{ijA}$, where the p_{ijA} belong to $B(d_i \wedge \bar{v}_A)$, are disjoint and $p_{ijA} = 0$ iff $\alpha_{ij} = 0$. Put $Q_j \wedge \bar{v}_A = \bigvee_{i=1}^r p_{ijA}$. Next, assume $\bar{x}_i \neq 0$. Applying EC2 to \bar{x}_i^* yields $d_i \in DL$ satisfying $\bar{x}_i \wedge d_i < \bar{x}_i$. By AC3 ω , $B(\bar{x}_i \wedge d_i) \neq \{0, \bar{x}_i \wedge d_i\}$, so using AC4 ω suitably often one finds $\beta_{i1}, \dots, \beta_{is} \in B(\bar{x}_i \wedge d_i)$, nonzero, pairwise disjoint and satisfying $\bar{x}_i \wedge d_i = \beta_{i1} \vee \dots \vee \beta_{is}$. Put $Q_j \wedge \bar{x}_i = \beta_{ij}$. Finally, let

$$Q_j = \bar{u}_j \vee \bigvee_{A \in M} \bigvee_{i=1}^r p_{ijA} \vee \bigvee_{x_i \neq 0} \beta_{ij} \vee \bigvee_{i=1}^r (p_i \wedge \bar{y}_{ij}).$$

Define $L_2 = \langle Q_1, \dots, Q_s \rangle_L$. By construction, $\langle L_0 \cup L_1 \rangle$ and $\langle L_0 \cup L_2 \rangle$ are isomorphic over L_0 . ■

PROOF OF THEOREM 2, SUFFICIENCY PART. Let $L \in \mathcal{B}_n$ ($2 \leq n \leq \omega$) and assume L satisfies AC1–AC4 n and EC1–EC4. Consider $L_1 \in \mathcal{B}_n$, $L_1 \supseteq L$; $a_1, \dots, a_l \in L$, $v_1, \dots, v_m \in L_1$. Proving $L \in E(\mathcal{B}_n)$ amounts to constructing $u_1, \dots, u_m \in L$ such that $\langle a_1, \dots, a_l, v_1, \dots, v_m \rangle$ and $\langle a_1, \dots, a_l, u_1, \dots, u_m \rangle$ are isomorphic over $\langle a_1, \dots, a_l \rangle$: If $L_1 \models (\exists x_1, \dots, x_m) \phi(x_1, \dots, x_m, a_1, \dots, a_l)$ with ϕ quantifier-free from $\mathcal{L}(L)$, say $L_1 \models \phi(v_1, \dots, v_m, a_1, \dots, a_l)$, then by isomorphism over $\langle a_1, \dots, a_l \rangle$ we have $L \models \phi(u_1, \dots, u_m, a_1, \dots, a_l)$, that is, $L \models (\exists x_1, \dots, x_m) \phi(x_1, \dots, x_m, a_1, \dots, a_l)$. Using subdirect representation and the fact that every Boolean algebra may be embedded into an atomless one, it clearly suffices to assume that

$$(*) \quad L_1 \cong \hat{F}_n^T \ (n < \omega) \quad \text{or} \quad L_1 \cong \prod (\hat{A}_t, t \in T) \ (n = \omega),$$

where A_t is an atomless Boolean algebra for each $t \in T$ (T any suitable index set).

Next, we may assume w.l.o.g. that $\{a_1, \dots, a_l\}$ actually is a subalgebra of L . L is a.c. since it satisfies AC1–AC4 n , so by Theorem 1(iii) there exists a subalgebra L' of L such that $\{a_1, \dots, a_l\} \subseteq L'$ and $L' \cong 2^i \times \hat{F}_n^j$ ($n < \omega$) or $L' \cong 2^i \times \hat{C}^j$ ($n = \omega$) for suitable $i, j \in \omega$. In the second case, we may conclude that $\{a_1, \dots, a_l\} \subseteq 2^i \times \hat{F}_r^j$ for some $r \geq 1$. Hence, we may assume w.l.o.g. that $\{a_1, \dots, a_l\}$ is isomorphic to $2^i \times \hat{F}_n^j$ ($n < \omega$) or to $2^i \times \hat{F}_r^j$ ($n = \omega$) for some i, j, r . The centers of these finite subalgebras of L contain $i + j$ atoms $c_1, \dots, c_{i+j} \in CL \subseteq CL_1$. Divide L_1 by the canonical congruences $\theta(c_k)$, $1 \leq k \leq i + j$. $L_1/\theta(c_k)$ is still a direct product of type (*), and $L/\theta(c_k)$ still satisfies EC1–EC4 by Lemma 5 and AC1–AC4 n by Lemma 2.2 of [12]. Since $L \cong \prod (L/\theta(c_k), 1 \leq k \leq i + j)$, it will obviously suffice to construct the desired u_1, \dots, u_m modulo $\theta(c_k)$ for each k . Summing up, the problem reduces to the case where L_1 is a direct product of type (*), and $\{a_1, \dots, a_l\}$ is a subalgebra of L isomorphic to 2 or \hat{F}_n ($n < \omega$) or to 2 or \hat{F}_r for some $r \geq 1$ ($n = \omega$).

We turn to v_1, \dots, v_m . Observe L_1 is a.c. as a direct product of a.c. factors [12, Lemma 2.2]. So we may proceed as above and replace $\{v_1, \dots, v_m\}$ by a subalgebra of L_1 isomorphic to $2^p \times \hat{F}_n^q$ ($n < \omega$) or to $2^p \times \hat{F}_r^q$ ($n = \omega$) for suitable p, q, r . The centers of these finite algebras contain $p + q$ atoms $c'_1, \dots, c'_{p+q} \in CL_1$.

To carry out the same factorization as above, we have to find $c_1, \dots, c_{p+q} \in CL$ (nonzero, pairwise disjoint, with join 1) and $v'_1, \dots, v'_m \in L_1$ such that

$$\begin{aligned} j_k(a_1/\theta(c'_k)) &= a_1/\theta(c_k) \\ &\vdots \\ j_k(a_i/\theta(c'_k)) &= a_i/\theta(c_k) \\ j_k(v_1/\theta(c'_k)) &= v'_1/\theta(c_k) \\ &\vdots \\ j_k(v_m/\theta(c'_k)) &= v'_m/\theta(c_k) \end{aligned}$$

induces an isomorphism

$$j_k: \langle a_1, \dots, a_i, v_1, \dots, v_m \rangle / \theta(c'_k) \cong \langle a_1, \dots, a_i, v'_1, \dots, v'_m \rangle / \theta(c_k).$$

If $\{a_1, \dots, a_i\}$ is 2, let $\{c_1, \dots, c_{p+q}\}$ be an arbitrary proper central partition of 1 in L . If $\{a_1, \dots, a_i\}$ is \hat{F}_r for some $r \in \omega$, let β_1, \dots, β_r be the atoms of \hat{F}_r and d their join. Proceed as in the proof of Lemma 6 to produce $c_k \in CL$, $1 \leq k \leq p + q$ (nonzero, disjoint, with join 1) such that $c_k \leq d$ iff $c'_k \leq d$ and $c_k \wedge \beta_i \neq 0$ iff $c'_k \wedge \beta_i \neq 0$ for $1 \leq i \leq r$. Obviously, then, j_k as defined above will induce an isomorphism $\langle a_1, \dots, a_i \rangle / \theta(c'_k) \cong \langle a_1, \dots, a_i \rangle / \theta(c_k)$. $v'_1, \dots, v'_m \in L_1$ will be constructed in the same way as Q_1, \dots, Q_s were obtained in the proof of Lemma 7; the difference being that all the auxiliary elements used in that construction live trivially within the direct product L_1 so we need not appeal to the EC and AC conditions at this point (except for EC1 which guarantees the existence of arbitrarily fine central partitions of c_k within L , thus within L_1).

Now the problem of finding the required $u_1, \dots, u_m \in L$ may be factorized again since $L \cong \prod(L/\theta(c_k), 1 \leq k \leq p + q)$. Observing that every nontrivial homomorphic image of \hat{F}_r ($r \in \omega$) is some F_s ($s \leq r$), we are reduced to considering the two cases dealt with in Lemmata 6 and 7. In view of Lemma 4, the proof of Theorem 2 is now complete, since the cases B_0, B_1 are known [11], [13]. ■

§3. Model companions for T_n . The existence of T_n^* , the model companion of T_n , for $0 \leq n \leq \omega$ has been known for some time. As far as the author knows, it appeared in print first in Burris [1]. However, as noted there, no description of the theories T_n^* was known then. T_0^* belongs to the folklore of the subject: It is the elementary theory of atomless Boolean algebras, see, e.g., [6] or, for an elementary account, [11]. An explicit description of T_1^* appeared in Schmitt [13]. Schmitt's constructions are based on some specific features of Stone algebras: The availability of a workable "triple" characterization of Stone algebras, and the coincidence between skeleton and center in such algebras. While the second property fails for $n \geq 2$, triple constructions for algebras in B_n exist for $n \geq 2$; see Katriňák [8] and [9]. Their technical complexity seems, however, to prevent a ready adaptation of Schmitt's techniques to the cases $n \geq 2$. A further existence proof for T_0^* , T_1^* and T_2^* was given by Weispfenning in [14]. No axiomatization of T_2^* is provided there, however, and the absence of the amalgamation property in B_n for $2 < n < \omega$ prevents a direct extension of Weispfenning's results to T_n for these values of n .

It is well known that if $E(K)$ is a generalized elementary class for K the class of all models of some universal theory T , then $T^* = \text{Th}(E(K))$ is the model companion of T . The T_n are obviously universal for $0 \leq n \leq \omega$. Now, for $L \in \underline{B}_n$, $0 \leq n \leq \omega$, the sets CL , SL , DL , $\text{Ston } L$, $B_L(s)$ are clearly definable by formulae from \mathcal{L} ($\mathcal{L}(L)$ for the last); and so is the concept of a partition (proper partition) of fixed length. It follows that conditions AC1–AC4 n and EC1–EC4 may be formalized by \forall_2 -sentences from \mathcal{L} . Let $\phi_1, \phi_2, \phi_3(n), \phi_4(n)$ be such formalizations of AC1, AC2, AC3 n , AC4 n for $2 \leq n \leq \omega$, and similarly θ_i for EC i ($1 \leq i \leq 4$). We may now rephrase Theorem 2 as follows:

THEOREM 8. *The model companions T_n^* of T_n for $0 \leq n \leq \omega$ are given by:*

$$T_0^* = T_0 \cup \{\phi_1\},$$

$$T_1^* = T_1 \cup \{\phi_1, \phi_2, \theta_1, \theta_2, \theta_3\},$$

$$T_n^* = T_n \cup \{\phi_1, \phi_2, \phi_3(n), \phi_4(n), \theta_1, \theta_2, \theta_3, \theta_4\} \text{ for } 2 \leq n \leq \omega.$$

Hence, T_n^ is finitely axiomatizable for all n .*

COROLLARY 9. *T_n^* is \aleph_0 -categorical for all n .*

PROOF. See BURRIS [1]. ■

COROLLARY 10. *T_n^* is a model completion of T_n precisely for $n = 0, 1, 2, \omega$.*

PROOF. $\underline{B}_n = \text{Mod}(T_n)$ has the amalgamation property exactly for these values of n (see [4]). The result follows (see [2]). ■

COROLLARY 11. *T_n^* is a complete theory for all n .*

PROOF. T_n^* is complete iff $\underline{B}_n = \text{Mod}(T_n)$ has the joint embedding property (see [6]). Now $\underline{2}$ is an absolute subretract in \underline{B}_n for all n (see [5]), hence $L_1, L_2 \in \underline{B}_n$ may be embedded into $L_1 \times L_2 \in \underline{B}_n$. ■

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