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Compact tripotents in bi-dual JB*-triples[1]

BY C. MARTIN EDWARDS
The Queen's College, Oxford

AND GOTTFRIED T. RÜTTIMANN
University of Berne, Berne, Switzerland

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Abstract

The set $\mathscr{U}(C)^{\sim}$ consisting of the partially ordered set $\mathscr{U}(C)$ of tripotents in a JBW*-triple C with a greatest element adjoined forms a complete lattice. This paper is mainly concerned with the situation in which C is the second dual A^{**} of a complex Banach space A and, more particularly, when A is itself a JB*-triple. A subset $\mathscr{U}_c(A)^{\sim}$ of $\mathscr{U}(A^{**})^{\sim}$ consisting of the set $\mathscr{U}_c(A)$ of tripotents *compact relative to* A (defined in Section 4) with a greatest element adjoined is studied. It is shown to be an atomic complete lattice with the properties that the infimum of an arbitrary family of elements of $\mathscr{U}_c(A)^{\sim}$ is the same whether taken in $\mathscr{U}_c(A)^{\sim}$ or in $\mathscr{U}(A^{**})^{\sim}$ and that every decreasing net of non-zero elements of $\mathscr{U}_c(A)^{\sim}$ has a non-zero infimum. The relationship between the complete lattice $\mathscr{U}_c(A)^{\sim}$ and the complete lattice $\mathscr{U}_c(B)^{\sim}$, where B is a Banach space such that B^{**} is a weak*-closed subtriple of A^{**} is also investigated. When applied to the special case in which A is a C*-algebra the results provide information about the set of compact partial isometries relative to A and are closely related to those recently obtained by Akemann and Pedersen. In particular it is shown that a partial isometry is compact relative to A if and only if, in their terminology, it belongs locally to A. The main results are applied to this and other examples.

1. *Introduction*

This paper presents a further investigation into the structure of JB*-triples. The work of Kaup and Upmeier[**21–23**], [**30**], [**31**] and Vigué[**32–35**], shows how the holomorphic structure of the open unit ball in a complex Banach space A leads to the existence of a closed subspace A_s of A and a triple product $\{\ldots\}$ from $A \times A_s \times A$ to A. The purely algebraic properties of the triple product, namely the linearity and symmetry in the first and third variables, the conjugate linearity in the second variable and, most important of all, the existence of a Jordan triple identity, relate any complex Banach space to the Jordan triple systems studied by Koecher[**24**], Loos[**25**] and Meyberg[**26**]. When A_s exhausts A or, equivalently, when the open unit ball in A is a bounded symmetric domain, the complex Banach space A is said to be a JB*-triple, the properties of which have received much attention in recent years. See, for example, [**2**], [**6–13**], [**15–17**], [**19**], [**30**], [**31**].

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Examples of JB*-triples are C*-algebras and Jordan C*-algebras and a question often asked is when a certain property of C*-algebras can be generalized to JB*-triples. There are many obstructions to such a programme, the main one being that, unlike C*-algebras, JB*-triples do not in general possess a global linear order structure. However, many of those generalizations which have so far reached fruition show not only that generalization is possible but also that new results about C*-algebras can be obtained by considering their triple structure. See, for example, [**9**–**11**].

The observation that there is much to be known about a locally compact Hausdorff topological space Ω which cannot be determined by a study of the commutative C*-algebra $C_0(\Omega)$ of continuous functions on Ω vanishing at infinity led C*-algebraists to the study of the second dual of a C*-algebra. That the same technique is also available in the case of JB*-triples owes much to the work of Barton, Timoney, Dineen and Horn[**3**], [**4**], [**6**], [**19**]. A JB*-triple A which is the dual of a Banach space A_* is said to be a JBW*-triple and their results demonstrate that the predual of a JBW*-triple is unique and that the second dual A^{**} of a JB*-triple A is a JBW*-triple.

When studying the structure of a W*-algebra C, the complete orthomodular lattice $\mathscr{P}(C)$ of self-adjoint idempotents in C plays a vital role. In the case of a JBW*-triple A, the natural analogue is given by the set $\mathscr{U}(A)$ of tripotents. In [**8**] the authors showed that $\mathscr{U}(A)$ possesses a partial ordering and that, when a largest element is adjoined to $\mathscr{U}(A)$, the partially ordered set $\mathscr{U}(A)^\sim$ so formed is a complete lattice, further properties of which were discussed in [**5**]. It is the purpose of this paper to study the situation in which the JBW*-triple is the second dual Banach space A^{**} of a complex Banach space A. This of course includes the case in which A is a JB*-triple with second dual A^{**} and, in particular, that in which A is a C*-algebra. The properties of a subset $\mathscr{U}_c(A)$ of $\mathscr{U}(A^{**})$ consisting of tripotents in A^{**} which are compact relative to A are investigated. The terminology is motivated by the fact that, in the special case when A is the C*-algebra $C_0(\Omega)$ of complex-valued continuous functions on a locally compact Hausdorff space Ω which vanish at infinity, a self-adjoint tripotent in A^{**} is compact if and only if it is the characteristic function of a compact subset of Ω. It is shown that, with the partial ordering inherited from $\mathscr{U}(A^{**})$, the partially ordered set $\mathscr{U}_c(A)^\sim$ consisting of $\mathscr{U}_c(A)$ with a greatest element adjoined forms a complete lattice, and that the infimum of an arbitrary set of elements of $\mathscr{U}_c(A)^\sim$ is the same whether taken in $\mathscr{U}_c(A)^\sim$ or in $\mathscr{U}(A^{**})^\sim$. The most striking properties of the complete lattice $\mathscr{U}_c(A)^\sim$ are that it is atomic, that it is order isomorphic to the complete lattice of weak* semi-exposed faces of the unit ball A_1^* in the dual A^* of A and that, in common with characteristic functions of compact subsets of Ω, every decreasing net of non-zero elements of $\mathscr{U}_c(A)^\sim$ has a non-zero infimum.

The algebraically interesting weak* closed subspaces of a JBW*-triple are its subtriples and its inner ideals. An investigation is conducted into the relationship between compactness of elements of $\mathscr{U}(A^{**})^\sim$ relative to A, where A is a complex Banach space with second dual A^{**} a JB*-triple, and compactness relative to B, where B is a complex Banach space with second dual B^{**} a weak* closed subtriple or inner ideal in A^{**}.

The paper is organized as follows. In Section 2 definitions are given, notation is established and certain preliminary results are described. In Section 3 the properties of the complete lattice $\mathscr{U}(C)^{\sim}$, where C is a JBW*-triple, are described, some of which have not appeared elsewhere. The main results are stated and proved in Section 4 and in Section 5 these are applied to C*-algebras and other examples. In particular the connections between the main results of this paper and those of Akemann and Pedersen [1] are described.

2. *Preliminaries*

Recall that a partially ordered set $\mathscr{P}$ is said to be a *lattice* if, for each pair (e,f) of elements of $\mathscr{P}$, the supremum $e \vee f$ and the infimum $e \wedge f$ exist with respect to the partial ordering of $\mathscr{P}$. The partially ordered set $\mathscr{P}$ is said to be a *complete lattice* if, for any subset M of $\mathscr{P}$, the supremum $\vee M$ and the infimum $\wedge M$ exist. A complete lattice has a greatest element and a least element, denoted by 1 and 0 respectively. A complete lattice is said to be *atomic* if, for each non-zero element f in $\mathscr{P}$ there exists a minimal non-zero element e in $\mathscr{P}$ majorized by f. A complete lattice together with an anti-order automorphism $e \mapsto e^{\perp}$ on $\mathscr{P}$ such that, for all elements e and f in $\mathscr{P}$, $e \vee e^{\perp} = 1$, $e^{\perp\perp} = e$, and, if $e \leqslant f$, then $f = e \vee (f \wedge e^{\perp})$, is said to be *orthomodular*. An element z of $\mathscr{P}$ is said to be *central* if, for all elements e in $\mathscr{P}$, $z = (z \wedge e) \vee (z \wedge e^{\perp})$. The set $\mathscr{Z}(\mathscr{P})$ of central elements of the complete orthomodular lattice $\mathscr{P}$ contains 0 and 1, and if z is contained in $\mathscr{Z}(\mathscr{P})$ then so also is $z^{\perp}$. With the restricted order and orthocomplementation $\mathscr{Z}(\mathscr{P})$ forms a subcomplete complete Boolean orthomodular sublattice of $\mathscr{P}$ which is said to be the *centre* of $\mathscr{P}$. The *central support* $z(e)$ of an element e in $\mathscr{P}$ is the infimum of the set of elements in $\mathscr{Z}(\mathscr{P})$ which dominate e.

Let V be a complex vector space and let C be a convex subset of V. A convex subset E of C is said to be a *face* of C provided that, if $tx_1 + (1-t)x_2$ is an element in E, where x_1 and x_2 lie in C and $0 < t < 1$, then x_1 and x_2 lie in E. A face E of C is said to be *proper* if it differs from C. An element x in C for which $\{x\}$ is a face is said to be an *extreme point* of C. Let τ be a locally convex Hausdorff topology on V and let C be τ-closed. Let $\mathscr{F}_\tau(C)$ denote the set of τ-closed faces of C. Both $\varnothing$ and C are elements of $\mathscr{F}_\tau(C)$ and the intersection of an arbitrary family of elements of $\mathscr{F}_\tau(C)$ again lies in $\mathscr{F}_\tau(C)$. Hence, with respect to ordering by set inclusion, $\mathscr{F}_\tau(C)$ forms a complete lattice. A subset E of C is said to be a *τ-exposed face* of C if there exists a τ-continuous linear functional f on V and a real number t such that, for all elements x in $C \backslash E$, $\mathrm{Re} f(x) < t$ and, for all elements x in E, $\mathrm{Re} f(x) = t$. Let $\mathscr{E}_\tau(C)$ denote the set of τ-exposed faces of C. Clearly $\mathscr{E}_\tau(C)$ is contained in $\mathscr{F}_\tau(C)$ and the intersection of a finite number of elements of $\mathscr{E}_\tau(C)$ again lies in $\mathscr{E}_\tau(C)$. The intersection of an arbitrary family of elements of $\mathscr{E}_\tau(C)$ is said to be a *τ-semi-exposed face* of C. Let $\mathscr{S}_\tau(C)$ denote the set of τ-semi-exposed faces of C. Clearly $\mathscr{E}_\tau(C)$ is contained in $\mathscr{S}_\tau(C)$ and the intersection of an arbitrary family of elements of $\mathscr{S}_\tau(C)$ again lies in $\mathscr{S}_\tau(C)$. Hence, with respect to the ordering by set inclusion $\mathscr{S}_\tau(C)$ forms a complete lattice and the infimum of a family of elements of $\mathscr{S}_\tau(C)$ coincides with its infimum when taken in $\mathscr{F}_\tau(C)$.

When V is a complex Banach space with dual space V^* the abbreviations n and w^* will be used for the norm topology of V and the weak* topology of V^*. For each subset E of the unit ball V_1 in V and F of the unit ball V_1^* of V^* let the subsets E' and $F_,$ be defined by

$$E' = \{a \in V_1^* : a(x) = 1 \; \forall x \in E\}, \quad F_, = \{x \in V_1 : a(x) = 1 \; \forall a \in F\}.$$

Notice that E lies in $\mathscr{S}_n(V_1)$ if and only if $(E')_, = E$, F lies in $\mathscr{S}_{w^*}(V_1^*)$ if and only if $(F_,)' = F$ and the mappings $E \mapsto E'$ and $F \mapsto F_,$ are anti-order isomorphisms between $\mathscr{S}_n(V_1)$ and $\mathscr{S}_{w^*}(V_1^*)$ and are inverses of each other. The reader is referred to [**8**] for details. When the anti-order isomorphisms above are being considered relative to different Banach spaces the symbols $'^V$ and $_{,V}$ will be used. The following lemma will be required in the proof of the main results.

LEMMA 2·1. *Let E be a proper norm semi-exposed face of the unit ball V_1 in the complex Banach space V. Then there exists a maximal proper norm semi-exposed face of V_1 containing E.*

Proof. Observe that a Zorn's lemma argument shows that every proper face E of V_1 is contained in a maximal proper face G. The Hahn–Banach theorem shows that every convex subset of the boundary of V_1 is contained in a proper norm-exposed face. It follows that G is a maximal proper face which is norm exposed and is therefore a maximal norm semi-exposed face.

A Jordan *-algebra A which is also a complex Banach space such that, for all elements a and b in A, $\|a^*\| = \|a\|$, $\|a \circ b\| \leqslant \|a\|\,\|b\|$ and $\|\{a\ a\ a\}\| = \|a\|^3$, where

$$\{a\ b\ c\} = a \circ (b^* \circ c) + (a \circ b^*) \circ c - b^* \circ (a \circ c)$$

is the Jordan triple product on A, is said to be a *Jordan C*-algebra* [**36**] or *JB*-algebra* [**37**]. A Jordan C*-algebra which is the dual of a Banach space is said to be a *Jordan W*-algebra* [**7**] or a *JBW*-algebra* [**37**]. Examples of JB*-algebras are C*-algebras and examples of JBW*-algebras are W*-algebras in both cases equipped with the Jordan product

$$a \circ b = \tfrac{1}{2}(ab + ba).$$

The self-adjoint parts of JB*-algebras and JBW*-algebras are said to be JB-*algebras* and JBW-*algebras* respectively. For the properties of C*-algebras and W*-algebras the reader is referred to [**28**], [**29**] and for the algebraic properties of Jordan algebras to [**17**], [**18**], [**25**], [**27**].

The set $\mathscr{P}(A)$ of self-adjoint idempotents, the *projections*, in a JBW*-algebra A forms a complete orthomodular lattice with respect to the partial ordering defined, for elements e and f in $\mathscr{P}(A)$, by $e \leqslant f$ if $e \circ f = e$, and the mapping $e \mapsto e^{\perp}$ defined by $e^{\perp} = 1 - e$ where 1 is the unit in A.

Recall that a complex vector space A equipped with a triple product $(a,b,c) \mapsto \{a\ b\ c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate linear in the second variable and satisfies the identity

$$[D(a,b), D(c,d)] = D(\{a\ b\ c\}, d) - D(c, \{d\ a\ b\}),$$

where [,] denotes the commutator and D is the mapping from $A \times A$ to A defined by

$$D(a,b)\,c = \{a\ b\ c\},$$

is said to be a *Jordan*-triple*. A subspace B of a Jordan*-triple A is said to be a *subtriple* if $\{B\ B\ B\}$ is contained in B and is said to be an *inner ideal* if $\{B\ A\ B\}$ is contained in B. When A is also a Banach space such that D is continuous from $A \times A$ to the Banach space $B(A)$ of bounded linear operators on A, and, for each element a in A, $D(a,a)$ is hermitian with non-negative spectrum and satisfies

$$\|D(a,a)\| = \|a\|^2,$$

then A is said to be a JB*-*triple*. A JB*-triple which is the dual of a Banach space is called a JBW*-*triple*. Examples of JB*-triples are JB*-algebras and examples of JBW*-triples are JBW*-algebras. The second dual A^{**} of a JB*-triple A is a JBW*-triple. For details of these results the reader is referred to [**3**], [**4**], [**6**], [**19**], [**21**–**23**], [**30**], [**31**].

3. *Tripotents in JBW*-triples*

An element u in a JBW*-triple A is said to be a *tripotent* if $\{u\ u\ u\}$ is equal to u. The set of tripotents in A is denoted by $\mathscr{U}(A)$. For each tripotent u in the JBW*-triple A the weak* continuous conjugate linear operator $Q(u)$ and the weak* continuous linear operators, $P_j(u)$, $j = 0, 1, 2$, are defined by

$$Q(u)\,a = \{u\ a\ u\},\quad P_2(u) = Q(u)^2,$$
$$P_1(u) = 2(D(u,u) - Q(u)^2),\quad P_0(u) = I - 2D(u,u) + Q(u)^2.$$

By the results of [**3**], [**5**] the linear operators $P_j(u)$, $j = 0, 1, 2$, are weak* continuous projections onto the eigenspaces $A_j(u)$ of $D(u,u)$ corresponding to eigenvalues $j/2$. The corresponding decomposition

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u)$$

is said to be the *Peirce decomposition of* A relative to u. For $j, k, l = 0, 1, 2$, $A_j(u)$ is a sub-JBW*-triple such that $\{A_j(u)\,A_k(u)\,A_l(u)\}$ is contained in $A_{j-k+l}(u)$ when $j-k+l = 0$, 1, or 2, and $\{0\}$ otherwise. Moreover,

$$\{A_2(u)\,A_0(u)\,A\} = \{A_0(u)\,A_2(u)\,A\} = \{0\},$$

and $A_0(u)$ and $A_2(u)$ are inner ideals in A. With respect to the product $(a,b) \mapsto a \circ b = \{a\ u\ b\}$, unit u and involution $a \mapsto a_\dagger = \{u\ a\ u\}$, $A_2(u)$ is a JBW*-algebra. A pair u, v of elements of $\mathscr{U}(A)$ is said to be *orthogonal* if v is contained in $A_0(u)$. For two elements u and v of $\mathscr{U}(A)$, write $u \leqslant v$ if $\{u\ v\ u\} = u$ or, equivalently, if $v - u$ is a tripotent orthogonal to u. Let $\mathscr{U}(A)^\sim$ be the disjoint union of the set $\mathscr{U}(A)$ and a one point set $\{\omega_A\}$ and, define a relation on $\mathscr{U}(A)^\sim$ by writing $u \leqslant v$ if and only if both u and v lie in $\mathscr{U}(A)$ and $u \leqslant v$ in the ordering of $\mathscr{U}(A)$ or if u is an arbitrary element in $\mathscr{U}(A)^\sim$ and v and ω_A coincide. It is clear that this defines a partial ordering on $\mathscr{U}(A)^\sim$. Notice that if the supremum of a family (u_j) of elements in $\mathscr{U}(A)$ exists in $\mathscr{U}(A)$ then it is equal to the supremum as formed in $\mathscr{U}(A)^\sim$.

Observe that, when A is a W*-algebra, $\mathscr{U}(A)$ is the set of partial isometries in A.

Recall that, for each element u in $\mathscr{U}(A)$, the set $\{u\}_,$ is a norm-exposed face of $A_{*,1}$. Define $\{\omega_A\}_,$ to the set $A_{*,1}$. The following result was proved in [**8**].

Lemma 3·1. *Let A be a JBW*-triple with predual A_*.*

(i) *The mapping $u \mapsto \{u\}_,$ is an order isomorphism from the partially ordered set $\mathscr{U}(A)^\sim$ of tripotents in A with a largest element adjoined onto the complete lattice $\mathscr{F}_n(A_{*,1})$ of all norm closed faces of the closed unit ball $A_{*,1}$ in A_* and hence $\mathscr{U}(A)^\sim$ is a complete lattice.*

(ii) *The mapping $u \mapsto \{u\}_,{}'$ is an anti-order isomorphism from $\mathscr{U}(A)^\sim$ onto the complete lattice $\mathscr{F}_{w^*}(A_1)$ of weak* closed faces of the closed unit ball A_1 in A and, for u in $\mathscr{U}(A)$,*

$$\{u\}_,{}' = u + A_0(u)_1.$$

We shall make use of the following property of the complete lattice $\mathscr{U}(A)^\sim$ [**5**]. We provide a short proof.

LEMMA 3·2. *Let A be a JBW*-triple and let (u_j) be an increasing net in $\mathscr{U}(A)$. Then the supremum u of the net (u_j) in $\mathscr{U}(A)^\sim$ is contained in $\mathscr{U}(A)$ and (u_j) converges to u in the weak* topology.*

Proof. By Lemma 3·1 (ii), $(\{u_j\}_{,}')$ is a decreasing net of non-empty weak* closed faces of A_1. The intersection of the members of each finite subfamily of this net is non-empty. Since A_1 is weak* compact it follows that the net has a non-empty intersection which is a non-empty weak* closed face of A_1. A further appeal to Lemma 3·1 (ii) shows that the supremum u of the net (u_j) is not equal to ω_A. By [**8**], lemma 2·4, (u_j) is an increasing net of elements of the complete lattice $\mathscr{P}(A_2(u))$ of projections in the JBW*-algebra $A_2(u)$ with supremum u. Therefore by [**18**], 4·5·6, (u_j) converges to u in the weak* topology of $A_2(u)$ and, since the Peirce projection $P_2(u)$ is weak* continuous, also in the weak* topology of A.

The next result can be extracted from [**8**] and [**16**] but, for completeness, a proof is given.

LEMMA 3·3. *Let a be an element of norm one in the JBW*-triple A.*

(i) *The sequence (a^{2n+1}) defined recursively by*

$$a^1 = a, \quad a^{2n+1} = \{a \;\; a^{2n-1} \;\; a\}$$

converges in the weak topology to an element $u(a)$ in $\mathscr{U}(A)$. Moreover,*

$$\{u(a)\, a\, u(a)\} = u(a) = \{u(a)\, u(a)\, a\}$$

and the norm closed faces $\{u(a)\}_{,}$ and $\{a\}_{,}$ of $A_{*,1}$ coincide.

(ii) *There exists a smallest element $r(a)$ in $\mathscr{U}(A)$ such that a is positive in the JBW*-algebra $A_2(r(a))$ and, in $A_2(r(a))$,*

$$0 \leqslant u(a) \leqslant a^{2n+1} \leqslant a \leqslant r(a).$$

Proof. Let $A(a)$ denote the smallest weak* closed subtriple of A containing a. The results of [**4**] and [**21**] show that there exists a locally compact subset Ω of the set $\mathbf{R}^+$ of non-negative real numbers, containing 1 and a positive measure ν on Ω such that $A(a)$ is isometrically isomorphic to $L^\infty(\Omega, \nu)$. Under the isomorphism the element a^{2n+1} is mapped into the function f_n, where

$$f_n(z) = z^{2n+1}.$$

Since the sequence (f_n) converges pointwise on Ω to the characteristic function of the set $\{1\}$ it follows that the sequence (a^{2n+1}) converges in the weak* topology to an element $u(a)$ of $\mathscr{U}(A)$ and that

$$\{u(a)\, a\, u(a)\} = u(a) = \{u(a)\, u(a)\, a\}.$$

There exists a sequence of real odd polynomials that converges pointwise on Ω to the characteristic function of $\Omega\backslash\{0\}$. It follows that there exists a sequence (b_{2n+1}) of real odd polynomials in a which converges in the weak* topology to an element $r(a)$ in $\mathscr{U}(A)$ and that

$$\{r(a) \;\; a \;\; r(a)\} = a = \{r(a) \;\; r(a) \;\; a\}.$$

Therefore a is a self-adjoint element in the JBW*-algebra $A_2(r(a))$. Moreover, since the function $z \mapsto z^{\frac{1}{2}}$ is continuous on Ω it follows that there exists an element c in $A(a)$ such that

$$\{c \;\; r(a) \;\; c\} = a, \quad \{r(a) \;\; c \;\; r(a)\} = c.$$

Hence c is a self-adjoint element in $A_2(r(a))$ the square of which coincides with a. Then, using [**18**], 3·1·6, a is a positive element in $A_2(r(a))$. Notice that similar arguments show that, for $n = 0, 1, 2, \ldots$, in $A_2(r(a))$,

$$0 \leqslant u(a) \leqslant a^{2n+1} \leqslant a \leqslant r(a).$$

Suppose now that r is an element in $\mathscr{U}(A)$ such that a is a positive element in the JBW*-algebra $A_2(r)$. Then, since the Jordan triple product in $A_2(r)$ coincides with that in A it follows that $A(a)$ is contained in $A_2(r)$. Moreover, $\{r \;\; a \;\; r\} = a$, and there exists an element b in $A_2(r)$ such that $\{r \;\; b \;\; r\} = b$ and $\{b \;\; r \;\; b\} = a$. Using the standard Jordan triple identity, it follows that, for $n = 0, 1, 2, \ldots$,

$$\{r \;\; a^{2n+1} \;\; r\} = a^{2n+1}, \quad \{r \;\; b^{2n+1} \;\; r\} = b^{2n+1}, \quad a^{2n+1} = \{b^{2n+1} \;\; r \;\; b^{2n+1}\}.$$

and, by taking limits in the weak* topology,

$$\{r \;\; u(a) \;\; r\} = u(a), \quad \{r \;\; u(b) \;\; r\} = u(b), \quad \{r \;\; r(a) \;\; r\} = r(a), \quad \{u(b) \;\; r \;\; u(b)\} = u(a).$$

Therefore $u(a)$ and $r(a)$ are self-adjoint elements of $A_2(r)$ with $u(a)$ positive. Since $\|u(a)\| \leqslant 1$, it follows that $0 \leqslant u(a) \leqslant r$ and, by [**8**], lemma 2·4, $u(a)$ is a projection in $A_2(r)$. Since $r - a$ is also a positive element of $A_2(r)$ of norm one, it follows that $u(r-a)$ is also a projection in $A_2(r)$. Considering the associative JBW*-subalgebra of $A_2(r)$ generated by r and a, it can be seen that the sequence $(r-(r-a)^{2n+1})$ converges to $r(a)$ and hence $r(a) = r - u(r-a)$ and is therefore a projection in $A_2(r)$. It follows from [**8**], lemma 2·4 that, in $\mathscr{U}(A)$, $r(a)$ is majorized by r.

Finally, suppose that x lies in $\{u(a)\}_{\prime}$. Then, from [**15**], $P_2(u(a))^* x = x$ and $P_2(u(a))\, a = u(a)$. Therefore,

$$x(a) = x(P_2(u(a))\, a) = x(u(a)) = 1$$

and it follows that x lies in $\{a\}_{\prime}$. Hence $\{u(a)\}_{\prime}$ is contained in $\{a\}_{\prime}$. Also, by Lemma 3·1 (i), there exists an element w in $\mathscr{U}(A)$ such that $\{a\}_{\prime}$ and $\{w\}_{\prime}$ coincide and therefore $u(a) \leqslant w$. Observe that since a lies in $\{w\}'_{\prime}$, by Lemma 3·1 (ii),

$$a = w + P_0(w)\, a.$$

Using the orthogonality of $A_2(w)$ and $A_0(w)$ it can be seen that, for $n = 0, 1, 2, \ldots$,

$$a^{2n+1} = w + (P_0(w)\, a)^{2n+1}.$$

Therefore, for x in $\{w\}_{\prime}$,

$$x(a^{2n+1}) = 1$$

and, since the sequence (a^{2n+1}) converges in the weak* topology to $u(a)$, it follows that x is contained in $\{u(a)\}_{\prime}$. Hence, $\{a\}_{\prime}$ is contained in $\{u(a)\}_{\prime}$ and the proof is complete.

Let a be an element in the JBW*-triple A. The least tripotent $r(a)$, such that a is a positive element in the JBW*-algebra $A_2(r(a))$ is called the *support* of a. Notice that, for every element a in A,

$$\{r(a) \;\; a \;\; r(a)\} = \{r(a)\, r(a)\, a\} = a.$$

The proof of the following lemma can be found in [5].

Lemma 3·4. *Let A be a JBW*-triple, let $u_1, u_2, \ldots, u_r$ be elements of $\mathscr{U}(A)$ and suppose that there exists u in $\mathscr{U}(A)$ such that, for $j = 1, 2, \ldots, r$, $u_j \leqslant u$. Then the supremum in $\mathscr{U}(A)^{\sim}$ of the set $\{u_1, u_2, \ldots, u_r\}$ is equal to the support of $\sum_{j=1}^{r} u_j$.*

The rest of this section is concerned with the order-theoretic structure of the set of tripotents in subtriples of a JBW*-triple. Notice that a weak* closed subtriple of a JBW*-triple is also a JBW*-triple.

Lemma 3·5. *Let B be a weak* closed subtriple of a JBW*-triple and let i be the mapping from the complete lattice $\mathscr{U}(B)^{\sim}$ to the complete lattice $\mathscr{U}(A)^{\sim}$ defined by $i(u) = u$, for all u in $\mathscr{U}(B)$, and $i(\omega_B) = \omega_A$.*

(i) *The mapping i is an order isomorphism from $\mathscr{U}(B)^{\sim}$ into $\mathscr{U}(A)^{\sim}$; i.e., for elements u, v in $\mathscr{U}(B)^{\sim}$, $u \leqslant v$ if and only if $i(u) \leqslant i(v)$.*

(ii) *Let (u_j) be a family of elements of $\mathscr{U}(B)$ and let u_B and u_A be the suprema of (u_j) in $\mathscr{U}(B)^{\sim}$ and $\mathscr{U}(A)^{\sim}$, respectively. Then $i(u_B) = u_A$.*

(iii) *For each element u of $\mathscr{U}(B)^{\sim}$,*

$$\{u\}_{\prime_B}{}^{\prime_B} = \{u\}_{\prime_A}{}^{\prime_A} \cap B.$$

Proof. (i) Let u and v be elements of $\mathscr{U}(B)$ such that $u \leqslant v$. Then, $\{u\ \ v\ \ u\} = u$ and it follows that $i(u) \leqslant i(v)$ as required. The remaining part follows easily.

(ii) If $u_B \neq \omega_B$ then u_B is an upper bound of (u_j) in $\mathscr{U}(B)$. Then $u_A \leqslant u_B$ which shows that $u_A \neq \omega_A$. Therefore, if u_A equals ω_A then u_A is equal to $i(u_B)$.

Suppose now that $u_A \neq \omega_A$. Let Λ be the set of all finite non-empty subsets of the index set of the net (u_j). For each α in Λ let u_α be the supremum in $\mathscr{U}(A)^{\sim}$ of the subset $\{u_j : j \in \alpha\}$. Clearly, for every α in Λ, u_α lies in $\mathscr{U}(A)$ and $u_\alpha \leqslant u_A$. By Lemma 3·4, for every α in Λ, u_α is equal to the support of $\sum_{j\in\alpha} u_j$ which, by Lemma 3·3, is an element in $\mathscr{U}(B)$. It follows that $(u_\alpha)_{\alpha\in\Lambda}$ is an increasing net in $\mathscr{U}(B)$. Since the subspace B is weak* closed there exists, by Lemma 3·2, an element v in $\mathscr{U}(B)$ such that, for all elements α in Λ, $u_\alpha \leqslant v$ and $v \leqslant u_A$. It follows that, for all j, $u_j \leqslant v$ and hence $u_B \leqslant v \leqslant u_A$. Consequently, u_B lies in $\mathscr{U}(B)$ and therefore u_B equals u_A.

(iii) Using Lemma 3·1, observe that

$$\{u\}_{\prime_B}{}^{\prime_B} = u + B_0(u)_1, \quad \{u\}_{\prime_A}{}^{\prime_A} = u + A_0(u)_1.$$

Let a lie in $\{u\}_{\prime_B}{}^{\prime_B}$. Then there exists an element b_0 in $B_0(u)_1 \subseteq A_0(u)_1$ such that $a = u + b_0$. Hence a is contained in $\{u\}_{\prime_A}{}^{\prime_A} \cap B$. Conversely, let a be an element in $\{u\}_{\prime_A}{}^{\prime_A} \cap B$. Then there exists an element a_0 of $A_0(u)_1$ such that $a = u + a_0$. Since a and u lie in B it follows that a_0 lies in B. Moreover, $\{u\ \ u\ \ a_0\} = 0$ and therefore a_0 lies in $B_0(u)_1$. It follows that a is an element in $\{u\}_{\prime_B}{}^{\prime_B}$.

Recall that a norm non-increasing projection P from a Banach space V to itself is said to be *neutral* if, for each x in V for which $\|Px\| = \|x\|$, it follows that $Px = x$. Moreover, a subtriple B of JBW*-triple A is said to be *complemented* if A is the direct sum of B and the subspace $\{a : a \in A, \{B\ \ a\ \ B\} = 0\}$. A linear projection Q on the JBW*-triple A is said to be *structural* if, for all elements a, b and c in A,

$$\{Qa\ \ b\ \ Qc\} = Q\{a\ \ Qb\ \ c\}.$$

The relationship between these three notions was studied in [11], [13] and [14] where the proof of the following result may be found.

LEMMA 3·6. *Let A be a JBW*-triple with predual A_*.*

(i) *The mapping $P \mapsto P^*$ is a bijection from the set of neutral projections on A_* for which P^*A is a subtriple of A onto the set of structural projections on A.*

(ii) *The mapping $Q \mapsto QA$ is a bijection from the set of structural projections on A onto the set of complemented subtriples of A.*

(iii) *A subtriple B of A is complemented if and only if it is a weak*-closed inner ideal in A.*

The results of Lemma 3·5 can be considerably strengthened when B is a weak*-closed inner ideal in A.

LEMMA 3·7. *Let B be a weak*-closed inner ideal in the JBW*-triple A and let i be the order isomorphism from the complete lattice $\mathscr{U}(B)^\sim$ into the complete lattice $\mathscr{U}(A)^\sim$ defined in Lemma 3·6.*

(i) *Let (u_j) be a family of elements of $\mathscr{U}(B)^\sim$ and let v_B and v_A be the infima of (u_j) in $\mathscr{U}(B)^\sim$ and $\mathscr{U}(A)^\sim$ respectively. Then $i(v_B) = v_A$.*

(ii) *For each element u in $\mathscr{U}(B)$, $\{u\}_{\prime_B}$ coincides with $\{u\}_{\prime_A}$.*

Proof. (i) For each j, $v_A \leqslant i(u_j)$ and by ([**8**], lemma 2·4), v_A is a projection in the JBW*-algebra $A_2(u_j)$. However, since B is a weak* closed inner ideal in A and u_j lies in B it follows that $A_2(u_j)$ and $B_2(u_j)$ coincide. It follows that v_A is contained in B and hence $i(v_B) \leqslant v_A$. The reverse inequality clearly holds.

(ii) By Lemma 3·6(iii) and the results of [**11**] it can be seen that the predual B_* of B is given by

$$B_* = \bigcup_{u \in \mathscr{U}(B)} A_2(u)_*.$$

The assertion follows from ([**11**]; lemma 2·1).

The preceding results lead to the following theorem.

THEOREM 3·8. *Let B be a weak*-closed inner ideal in the JBW*-triple A and let $i: \mathscr{U}(B)^\sim \to \mathscr{U}(A)^\sim$ be defined as in Lemma 3·5. Then the range of the order isomorphism i is a sub-complete lattice of $\mathscr{U}(A)^\sim$.*

4. *Compact tripotents in JBW*-triples*

Let A be a complex Banach space with dual A^* and second dual A^{**}. In certain situations A will be identified with its canonical image in A^{**}. Suppose that A^{**} is a JB*-triple in which case, being a dual space, A^{**} is a JBW*-triple. In order to simplify notation in this situation the largest element adjoined to $\mathscr{U}(A^{**})$, to make it into a complete lattice, will be denoted by ω_A. The 'prime' maps between subsets of the unit balls in A, A^* and A^{**} will also be given the subscript A. An element u of $\mathscr{U}(A^{**})$ is said to be *compact-G_δ relative to A* if there exists an element a of A of norm one such that u coincides with $u(a)$, defined in Lemma 3·3.

LEMMA 4·1. *Let A be a complex Banach space the second dual A^{**} of which is a JB*-triple. Then every tripotent in A^{**} which is compact-G_δ relative to A is non-zero.*

Proof. Let a be an element in A of norm one. Then, by Lemma 3·3,

$$\{u(a)\}_{\prime_A} = \{x \in A_1^* : a(x) = 1\}.$$

However, a can be regarded as a weak* continuous complex-valued function on the weak* compact set A_1^* and therefore there exists an element y in A_1^* such that $|a(y)| = 1$. Choosing x equal to $y/a(y)$ it can be seen that x lies in $\{u(a)\}_{{}'A}$. Therefore $\{u(a)\}_{{}'A}$ is non-empty and, by Lemma 3·1 (i), $u(a)$ is non-zero.

It is now possible to make the central definition of the paper. An element u in the complete lattice $\mathscr{U}(A^{**})^{\sim}$ is said to be *compact relative to* A if there exists a decreasing net (u_j) of tripotents in A^{**} which are compact-G_δ relative to A with infimum u, or if u is zero.

Theorem 4·2. *Let A be a complex Banach space the second dual A^{**} of which is a JB*-triple and let u be a tripotent in A^{**}. Then u is compact relative to A if and only if $\{u\}_{{}'A}$ is a weak* semi-exposed face of the unit ball A_1^* of the dual A^* of A.*

Proof. The statement is trivially true if u is zero. Let u be a non-zero compact tripotent. Then there exists a family $(a_j)_{j\in\Lambda}$ of elements of A of unit norm such that u is the infimum in the complete lattice $\mathscr{U}(A^{**})^{\sim}$ of the decreasing net $(u(a_j))_{j\in\Lambda}$. It follows from ([**8**], lemma 2·4), that, for any j_0 in Λ and for all $j \geqslant j_0$, $u(a_j)$ and u may be regarded as self-adjoint idempotents in the JBW*-algebra $A_2^{**}(u(a_{j_0}))$ with u the infimum of the decreasing net $(u(a_j))_{j\geqslant j_0}$. It follows that this decreasing net converges to u in the weak* topology of $A_2^{**}(u(a_{j_0}))$. Since the Peirce projection $P_2(u(a_{j_0}))$ is weak* continuous it follows that the decreasing net $(u(a_j))_{j\in\Lambda}$ converges to u in the weak* topology of A^{**}.

It will be shown that the norm exposed face $\{u\}_{{}'A}$ of A_1^* coincides with the weak* semi-exposed face F of A_1^*, where

$$F = \bigcap_{j\in\Lambda} \{a_j\}'^{A}.$$

Let x be an element in F. Then, for all j in Λ, x lies in the weak* exposed face $\{a_j\}'^{A}$ of A_1^* and observe that, by Lemma 3·3, $\{a_j\}'^{A}$ coincides with $\{u(a_j)\}_{{}'A}$. Then $u(a_j)(x) = 1$ and, since the decreasing net $(u(a_j))_{j\in\Lambda}$ converges to u in the weak* topology, it follows that $u(x) = 1$ and x lies in $\{u\}_{{}'A}$. Therefore, F is contained in $\{u\}_{{}'A}$. Furthermore, for all j in Λ, $u \leqslant u(a_j)$, and, by Lemma 3.1,

$$\{u\}_{{}'A} \subseteq \{u(a_j)\}_{{}'A} = \{a_j\}'^{A}.$$

Hence $\{u\}_{{}'A}$ is contained in F and F coincides with $\{u\}_{{}'A}$ as required.

Conversely, suppose that u is a non-zero element in $\mathscr{U}(A^{**})$ such that $\{u\}_{{}'A}$ is a weak* semi-exposed face of A_1 and let the non-empty norm semi-exposed face $\{u\}_{{}'A{}'A}$ be denoted by G. Observe that G'^{A} coincides with $\{u\}_{{}'A}$. For each element a in G, let face(a) denote the smallest face of A_1 containing a. Clearly face(a) is contained in G. Consider the set Λ consisting of all faces of G of the form face(a). The set Λ is partially ordered by set inclusion and is directed upwards since, if face(a_1) and face(a_2) are elements of Λ, then both face(a_1) and face(a_2) are contained in the element face$(\frac{1}{2}a_1 + \frac{1}{2}a_2)$ of Λ. Moreover, if face(a_1) is contained in face(a_2), then

$$a_1 \in \text{face}(a_2) \subseteq \{a_2\}'^{A'A}$$

and therefore

$$\{a_1\}'^{A'A} \subseteq \{a_2\}'^{A'A}.$$

By Lemma 3·1 (i), $u(a_1) \geqslant u(a_2)$. Let γ be an element in Λ and let u_γ denote the tripotent $u(a)$ where a is an element in G such that $\gamma = \text{face}(a)$. Then $(u_\gamma)_{\gamma\in\Lambda}$ is a decreasing net in $\mathscr{U}(A^{**})^\sim$. For each element a in G,

$$\{u\}_{\prime_A} = G'^A \subseteq \{a\}'^A = \{u(a)\}_{\prime_A}.$$

By Lemma 3·1 (i), it follows that, for each γ in Λ, $u \leqslant u_\gamma$. An argument similar to that used in the first part of the proof shows that the decreasing net $(u_\gamma)_{\gamma\in\Lambda}$ in $\mathscr{U}(A^{**})^\sim$ converges in the weak* topology to its infimum v, a tripotent compact relative to A, where $u \leqslant v$. Consequently,

$$G'^A = \{u\}_{\prime_A} \subseteq \{v\}_{\prime_A}.$$

However, for each element a in G, since $v \leqslant u(a)$ it follows that $\{v\}_{\prime_A}$ is contained in $\{a\}'^A$ and hence that

$$\{v\}_{\prime_A} \subseteq G'^A = \{u\}_{\prime_A}.$$

Therefore u and v coincide and u is compact relative to A as required.

For a complex Banach space A the second dual A^{**} of which is a JB*-triple, let $\mathscr{U}_c(A)$ denote the subset of $\mathscr{U}(A^{**})$ consisting of tripotents compact relative to A and let $\mathscr{U}_c(A)^\sim$ be the subset of the complete lattice $\mathscr{U}(A^{**})^\sim$ that is the union of $\mathscr{U}_c(A)$ and the one-point set $\{\omega_A\}$. The properties of $\mathscr{U}_c(A)^\sim$ are described in the following theorem.

THEOREM 4·3. *Let A be a complex Banach space the second dual A^{**} of which is a JB*-triple and let $\mathscr{U}_c(A)^\sim$ be the partially ordered set consisting of those tripotents in A^{**} which are compact relative to A and the greatest element ω_A. Then, with respect to the partial ordering inherited from $\mathscr{U}(A^{**})^\sim$, $\mathscr{U}_c(A)^\sim$ is a complete lattice in which the infimum of an arbitrary family of elements coincides with its infimum in the complete lattice $\mathscr{U}(A^{**})^\sim$.*

Proof. The complete lattice $\mathscr{S}_{w^*}(A_1^*)$ of weak* semi-exposed faces of A_1^* is contained in the complete lattice $\mathscr{F}_n(A_1^*)$ of norm closed faces of A_1^*, and clearly the infimum of an arbitrary family of elements of $\mathscr{S}_{w^*}(A_1^*)$ is the same whether taken in $\mathscr{S}_{w^*}(A_1^*)$ or in $\mathscr{F}_n(A_1^*)$, being their intersection. By Lemma 3·1 (i) the mapping $u \mapsto \{u\}_{\prime_A}$ is an order isomorphism from the complete lattice $\mathscr{U}(A^{**})^\sim$ onto the complete lattice $\mathscr{F}_n(A_1^*)$. It follows from Theorem 4·2 and the fact that A_1^* is a weak* semi-exposed face of itself that the restriction of the mapping $u \mapsto \{u\}_{\prime_A}$ to $\mathscr{U}_c(A)^\sim$ is an order isomorphism onto the complete lattice $\mathscr{S}_{w^*}(A_1^*)$.

The results below follow immediately from the proof above and Lemma 3·1.

COROLLARY 4·4. *Let A be a complex Banach space the second dual A^{**} of which is a JB*-triple and let $\mathscr{U}_c(A)^\sim$ be the complete lattice consisting of those tripotents in A^{**} which are compact relative to A and the greatest element ω_A.*

(i) *The mapping $u \mapsto \{u\}_{\prime_A}$ is an order isomorphism from $\mathscr{U}_c(A)^\sim$ onto the complete lattice $\mathscr{S}_{w^*}(A_1^*)$ of weak* semi-exposed faces of the unit ball A_1^* of the dual space A^* of A.*

(ii) *The mapping $u \mapsto \{u\}_{\prime_{A}\prime_A}$ is an anti-order isomorphism from $\mathscr{U}_c(A)^\sim$ onto the complete lattice $\mathscr{S}_n(A_1)$ of norm semi-exposed faces of the unit ball A_1 in A and, for each element u in $\mathscr{U}_c(A)$,*

$$\{u\}_{\prime_{A}\prime_A} = (u + A_0^{**}(u)_1) \cap A.$$

The following result goes some way towards justifying the use of the word compact for elements of $\mathscr{U}_c(A)^\sim$.

Theorem 4·5. *Let A be a complex Banach space the second dual A^{**} of which is a JB^*-triple and let $\mathscr{U}_c(A)^\sim$ be the complete lattice consisting of those tripotents in A^{**} which are compact relative to A and the greatest element ω_A.*

(i) *A decreasing net of non-zero elements in $\mathscr{U}_c(A)^\sim$ has a non-zero infimum.*

(ii) *The complete lattice $\mathscr{U}_c(A)^\sim$ is atomic.*

Proof. (i) Let (u_j) be a decreasing net in $\mathscr{U}_c(A)^\sim$ consisting of non-zero elements. By Corollary 4·4(i), $(\{u_j\}_{,_A})$ is a decreasing net of non-empty weak* semi-exposed faces of the weak* compact set A_1^*. Therefore, the intersection of the net $\{(u_j\}_{,_A})$, itself a weak* semi-exposed face of A_1^*, is also non-empty. It follows from Corollary 4·4(i) that the infimum of the decreasing net (u_j) is non-zero.

(ii) This is immediate from Corollary 4·4(ii) and Lemma 2·1.

This completes the discussion of the intrinsic properties of the complete lattice $\mathscr{U}_c(A)^\sim$. The rest of this section is concerned with the relationship between the complete lattices $\mathscr{U}_c(A)^\sim$ and $\mathscr{U}_c(B)^\sim$ where B is a complex Banach space the second dual of which is a weak* closed subtriple of A^{**}. In the first place the situation in which B is a norm closed subspace of A will be discussed. In this case the second dual B^{**} of B can be identified with the second annihilator $B^{\circ\circ}$ *of* B in A^{**}.

Lemma 4·6. *Let A be a complex Banach space the second dual A^{**} of which is a JB^*-triple, let B be a norm closed subspace of A such that the second annihilator $B^{\circ\circ}$ of B is a subtriple of A^{**}, let $\mathscr{U}_c(A)^\sim$ be the complete lattice of tripotents in A^{**} which are compact relative to A and the greatest element ω_A and let $\mathscr{U}_c(B)^\sim$ be the corresponding complete lattice for B. Let i be the order-isomorphism of Lemma 3·5 from $\mathscr{U}(B^{\circ\circ})^\sim$ into $\mathscr{U}(A^{**})^\sim$.*

(i) *For each element u in $\mathscr{U}(B^{\circ\circ})^\sim$ the norm semi-exposed face $\{u\}_{,_{B'_B}}$ of the unit ball B_1 in B coincides with $\{i(u)\}_{,_{A'_A}} \cap B$.*

(ii) *For each element u in $\mathscr{U}_c(B)^\sim$ there exists an element v in $\mathscr{U}_c(A)^\sim$ such the norm semi-exposed faces $\{u\}_{,_{A'_A}}$ and $\{v\}_{,_{A'_A}}$ of the unit ball A_1 in A coincide and $u \leqslant v$.*

Proof. (i) We may assume that u is an element in $\mathscr{U}(B^{\circ\circ})$. Then, by Lemma 3·5(iii),

$$\{u\}_{,_{B'_B}} = \{u\}_{,_B}{}^{'_B} \cap B = \{u\}_{,_A}{}^{'_A} \cap B^{\circ\circ} \cap B = \{u\}_{,_{A'_A}} \cap B.$$

(ii) Provided that $u \neq \omega_B$, the norm semi-exposed face $\{u\}_{,_{B'_B}}$ of B_1 is non-empty and, by (i), the norm semi-exposed face $\{u\}_{,_{A'_A}}$ of A_1 is non-empty. By Corollary 4·4(ii), there exists a non-zero element v of $\mathscr{U}_c(A)^\sim$ such that $\{u\}_{,_{A'_A}}$ and $\{v\}_{,_{A'_A}}$ coincide. Moreover,

$$\{u\}_{,_A} \subseteq \{u\}_{,_{A'_A}}{}^{'_A} = \{v\}_{,_{A'_A}}{}^{'_A} = \{v\}_{,_A},$$

and, by Lemma 3·1(i), $u \leqslant v$.

That $B^{\circ\circ}$ is merely a subtriple of A^{**} does not appear to be enough to ensure that $i(\mathscr{U}_c(B)^\sim)$ is contained in $\mathscr{U}_c(A)^\sim$. However, if $B^{\circ\circ}$ is an inner ideal in A^{**} then this is the case, as the next result shows.

Theorem 4·7. *Let A be a complex Banach space the second dual A^{**} of which is a JB^*-triple, let B be a norm closed subspace of A such that the second annihilator $B^{\circ\circ}$ of B is an inner ideal in A^{**}, let $\mathscr{U}_c(A)^\sim$ be the complete lattice of tripotents in A^{**} which are* *compact relative to A and the greatest element ω_A and let $\mathscr{U}_c(B)^\sim$ be the corresponding complete lattice for B.*

(i) *The set $\mathscr{U}_c(B)$ of tripotents in $B^{\circ\circ}$ compact relative to B is contained in $\mathscr{U}_c(A)$. For each element u in $\mathscr{U}_c(B)$, the weak* semi-exposed face $\{u\}_{,A}$ of A_1^* coincides with $(\{u\}_{,A',A} \cap B)'^{A}$ and u is the greatest in the set of elements w of $\mathscr{U}_c(A)^\sim$ for which $\{u\}_{,A',A} \cap B$ and $\{w\}_{,A',A} \cap B$ coincide.*

(ii) *The infimum of a family of elements of $\mathscr{U}_c(B)$ is the same whether taken in $\mathscr{U}_c(B)^\sim$ or in $\mathscr{U}_c(A)^\sim$.*

Proof. (i) Let u be an element in $\mathscr{U}_c(B)$. Then, by Lemma 4·6,

$$\{u\}_{,B',B} = \{u\}_{,A',A} \cap B$$

and there exists an element v in $\mathscr{U}_c(A)$ with $u \leqslant v$ such that $\{u\}_{,A',A}$ and $\{v\}_{,A',A}$ coincide. Let x be an element in $(\{u\}_{,A',A} \cap B)'^{A}$. Then the restriction of x to B is a bounded linear functional in $\{u\}_{,B',B}{}'^{B}$ which, since u lies in $\mathscr{U}_c(B)$, coincides with $\{u\}_{,B}$. It follows from Lemma 3·7 (ii) that x lies in $\{u\}_{,A}$. Hence, $(\{u\}_{,A',A} \cap B)'^{A}$ is contained in $\{u\}_{,A}$. Moreover, since

$$\{u\}_{,A',A} \cap B \subseteq \{u\}_{,A',A} = \{v\}_{,A',A}$$

it follows that

$$\{v\}_{,A} = \{v\}_{,A',A}{}'^{A} \subseteq (\{u\}_{,A',A} \cap B)'^{A} \subseteq \{u\}_{,A}$$

and, by Lemma 3·1 (i), $v \leqslant u$. Therefore, u and v are equal. Finally, suppose that w is an element in $\mathscr{U}_c(A)$ such that $\{u\}_{,A',A} \cap B$ and $\{w\}_{,A',A} \cap B$ coincide. Then,

$$\{w\}_{,A} = \{w\}_{,A',A}{}'^{A} \subseteq (\{w\}_{,A',A} \cap B)'^{A} = (\{u\}_{,A',A} \cap B)'^{A} = \{u\}_{,A}$$

and therefore $w \leqslant u$.

(ii) This is immediate from Theorems 3·8 and 4·3.

As an application of the results above observe that, when A is a complex Banach space the second dual of which is a JB*-triple, an element u of $\mathscr{U}(A^{**})^\sim$ is said to be *open relative to A* if the weak* closure of the subspace $A_2^{**}(u) \cap A$ of A coincides with the JBW*-algebra $A_2^{**}(u)$. Since $A_2^{**}(u)$ is an inner ideal in A^{**} the result below follows immediately from Theorem 4·7.

Corollary 4·8. *Let A be a complex Banach space the second dual of which is a JB*-triple and let u be a tripotent in A^{**} open relative to A. Then every tripotent compact relative to $A_2^{**}(u) \cap A$ is compact relative to A.*

The final results of this section are concerned with those second dual spaces in A^{**} which are complemented subtriples. These are in some sense dual to those above. First the following certainly well-known result is needed.

Lemma 4·9. *Let A be a complex Banach space with dual A^* and second dual A^{**} and let P be a norm-non-increasing linear projection from A^* to A^* the range PA^* of which is weak* closed. Then the dual of the Banach space PA^* is P^*A^{**} in the natural duality. The Banach space P^*A is a predual of PA^* and the weak* topology of A^* restricted to PA^* coincides with the topology $\sigma(A^*, P^*A)$ restricted to PA^*.*

It is now possible to prove the main result about the situation under consideration.

Theorem 4·10. *Let A be a complex Banach space the second dual A^{**} of which is a JB*-triple and let P be a neutral projection on the dual A^* of A such that PA^* is weak** *closed and P^*A^{**} is a subtriple of A^{**}. Let B denote the predual P^*A of PA^*. Let $\mathscr{U}_c(A)^\sim$ be the complete lattice of tripotents in A^{**} which are compact relative to A and the greatest element ω_A and let $\mathscr{U}_c(B)^\sim$ be the corresponding complete lattice for B.*

(i) *The set $\mathscr{U}_c(A) \cap B^{**}$ of tripotents in the complemented subtriple B^{**} of A^{**} is contained in $\mathscr{U}_c(B)$ and for each element u of $\mathscr{U}_c(A) \cap B^{**}$,*

$$\{u\}_{\prime_B} = \{u\}_{\prime_A} = (P^*(\{u\}_{\prime_{A'_A}}))'^B.$$

(ii) *With respect to the ordering inherited from $\mathscr{U}_c(B)$, the set $(\mathscr{U}_c(A) \cap B^{**})^\sim$ consisting of the union of $\mathscr{U}_c(A) \cap B^{**}$ and $\{\omega_B\}$ is a complete lattice in which the infimum of a family of elements coincides with its infimum in $\mathscr{U}_c(B)^\sim$.*

Proof. (i) Let u be an element in $\mathscr{U}_c(A) \cap B^{**}$. Then, by Lemma 3·7 (ii), the faces $\{u\}_{\prime_B}$ and $\{u\}_{\prime_A}$ coincide. Let b be an element in the set $P^*(\{u\}_{\prime_{A'_A}})$. In this case there exists an element a in $\{u\}_{\prime_{A'_A}}$ such that $P^*a = b$. Then, for all x in $\{u\}_{\prime_B}$,

$$b(x) = P^*a(x) = a(Px) = a(x) = 1,$$

and therefore b lies in $\{u\}_{\prime_{B'_B}}$. It follows that $P^*(\{u\}_{\prime_{A'_A}})$ is contained in $\{u\}_{\prime_{B'_B}}$. Therefore,

$$\begin{aligned}
\{u\}_{\prime_{B'_B}}{}'^B &\subseteq (P^*(\{u\}_{\prime_{A'_A}}))'^B \\
&= \{x : x \in B_1^*, x(P^*a) = 1, \forall a \in \{u\}_{\prime_{A'_A}}\} \\
&= \{x : x \in B_1^*, Px(a) = 1, \forall a \in \{u\}_{\prime_{A'_A}}\} \\
&= \{x : x \in B_1^*, x(a) = 1, \forall a \in \{u\}_{\prime_{A'_A}}\} \\
&= \{u\}_{\prime_{A'_A}}{}'^A \cap B_1^* \\
&= \{u\}_{\prime_A} \cap B_1^* = \{u\}_{\prime_A} \\
&= \{u\}_{\prime_B} \subseteq \{u\}_{\prime_{B'_B}}{}'^B.
\end{aligned}$$

Therefore, $\{u\}_{\prime_B}$ and $\{u\}_{\prime_{B'_B}}{}'^B$ coincide and, by Theorem 4·2, u lies in $\mathscr{U}_c(B)$.

(ii) Let (u_j) be a family of elements of $U_c(A) \cap B^{**}$. By Lemma 3·7 (i) their infima in $\mathscr{U}(A^{**})^\sim$ and $\mathscr{U}(B^{**})^\sim$ coincide and, by Theorem 4·3, these in turn coincide with their infima in the complete lattices $\mathscr{U}_c(A)^\sim$ and $\mathscr{U}_c(B)^\sim$. It follows that $(\mathscr{U}_c(A) \cap B^{**})^\sim$ is a complete lattice and that the infimum of a family of elements of $(U_c(A) \cap B^{**})^\sim$ coincides with that in $\mathscr{U}_c(B)$.

A particularly interesting application of this result concerns a fixed tripotent in the JB*-triple A^{**} compact relative to the complex Banach space A.

Theorem 4·11. *Let A be a complex Banach space the second dual A^{**} of which is a JB*-triple, let u be a non-zero tripotent in A^{**} compact relative to A and let $P_2(u)$ be the Peirce projection on A^{**} corresponding to u. Then, the second dual of the Banach space $P_2(u)A$ is the JBW*-algebra $A_2^{**}(u)$, u is contained in $P_2(u)A$ and $(\mathscr{U}_c(A) \cap A_2^{**}(u))^\sim$ is a complete lattice contained in $\mathscr{U}_c(P_2(u)A)^\sim$ in which the infimum of a family of elements coincides with its infimum in $\mathscr{U}_c(P_2(u)A)^\sim$.*

Proof. Observe that the Peirce projection $P_2(u)$ is a structural projection on A^{**} and therefore, in order to apply Theorem 4·10, it remains to show that $P_2(u)^*A^*$, the predual $A_2^*(u)$ of the JBW*-algebra $A_2^{**}(u)$, is weak* closed in A^*. Theorems 3·2 and 3·6 of [7] show that the unit ball $A_2^*(u)_{sa,1}$ in the self-adjoint part $A_2^*(u)_{sa}$ of $A_2^*(u)$ coincides with the unit ball in the predual of the JBW-algebra $A_2^{**}(u)_{sa}$. Since $\{u\}_{\prime_A}$

is the normal state space of the JBW-algebra $A_2^{**}(u)_{sa}$ it can be seen that the unit ball $A_2^*(u)_{sa,1}$ is the convex hull of the set $\{u\}_{\prime_A} \cup (-\{u\}_{\prime_A})$. Since u is compact $\{u\}_{\prime_A}$ is weak* compact and therefore so also is $A_2^*(u)_{sa,1}$. Let (x_j) be a net in the unit ball $A^*(u)_1$ converging in the weak* topology to an element x in A_1^*. Using lemma 2·1 and theorem 3·2 of [7], it is clear that the nets $((1/2)(x_j+x_j^\dagger))$ and $((1/2i)(x_j-x_j^\dagger))$ lie in the weak* compact set $A_2^*(u)_{sa,1}$. Therefore there exists a subnet $(x_{j'})$ of the net (x_j) such that the nets $((1/2)(x_{j'}+x_{j'}^\dagger))$ and $((1/2i)(x_{j'}-x_{j'}^\dagger))$ converge in the weak* topology to elements y and z of $A_2^*(u)_{sa,1}$. Clearly, x and $y+iz$ coincide and x lies in $A_2^*(u)_1$ which is therefore weak* compact. By the Krein–Šmulian theorem it can be concluded that $A_2^*(u)$ is a weak* closed subspace of A^*.

It remains to show that u is contained in $P_2(u)A$. It is clear that the mapping Ψ from the quotient space $A/A_2^*(u)_\bigcirc$ to $A_2^{**}(u)$ defined by

$$\Psi(a+A_2^*(u)_\bigcirc) = P_2(u)\,a$$

is a linear isometry onto $P_2(u)A$. Since $\{u\}_{\prime_A}$ is a non-empty weak* semi-exposed face of A_1^* it follows that $\{u\}_{\prime_{A}\prime_{A}}$ is a non-empty norm semi-exposed face of A_1. Let e be an element in $\{u\}_{\prime_{A}\prime_{A}}$. Then, by Lemma 3·1 (ii),

$$e = u+P_0(u)\,e$$

and therefore,

$$\Psi(e+A_2^*(u)_\bigcirc) = P_2(u)\,e = u.$$

It follows that u is contained in $P_2(u)A$ and the proof is complete.

It should be noted that all the results of this section apply to the case in which A is a JB*-triple in which case the second dual A^{**} is automatically a JB*-triple. However, little more can be said in the special situation than has already been stated. Moreover, Theorems 4·10 and 4·11 applied in the special case would already require the more general results.

5. *Applications*

Observe that an example of a JBW*-triple is a Jordan W*-algebra and, in particular, a W*-algebra. Let C be a W*-algebra and recall that the set of tripotents in C coincides with the set of partial isometries in C. Moreover the complete orthomodular lattice $\mathscr{P}(C)$ of projections in C is a sub-complete lattice of $\mathscr{U}(C)^\sim$. A pair (e,f) of elements in $\mathscr{P}(C)$ are said to be *centrally equivalent* if their central supports $z(e)$ and $z(f)$ coincide. In [9] it is shown that the set $\mathscr{C}(\mathscr{P}(A))$ of centrally equivalent pairs of projections in C forms a complete lattice and that the mapping $(e,f)\mapsto eCf$ is an order isomorphism onto the complete lattice of weak* closed inner ideals in C ordered by set inclusion. It follows from this result that every weak* closed inner ideal in C is complemented and, by Lemma 3·8 and Corollary 3·9, that every structural projection on C is of the form $a\mapsto eaf$ for some centrally equivalent pair (e,f) of projections in C.

Now suppose that the W*-algebra C is the second dual A^{**} of a Banach space A, a situation which, in particular, arises when C is the second dual of a C*-algebra. Let B be a complex Banach space the second dual of which is a weak* closed inner ideal in A^{**}. Slightly extending the definition given in [1], a partial isometry u in A^{**} is said to *belong locally to* B if there exists an element a in B_1 such that $u = au^*u$.

Recall that, according to [10], the set of centrally equivalent pairs (e,f) of open projections in the second dual A^{**} of a C*-algebra A forms a complete lattice

$\mathscr{C}_0(\mathscr{P}(A^{**}))$ and that the mapping $(e,f) \mapsto eA^{**}f \cap A$ is an order isomorphism from $\mathscr{C}_0(\mathscr{P}(A^{**}))$ onto the complete lattice of norm closed inner ideals in A. The following result connects the notions of compactness and local containment for the case in which B is a norm closed subspace of the C^*-algebra A such that its second annihilator $B^{\circ\circ}$ is an inner ideal in A^{**}. It should be said that, in common with all of the results concerning C*-algebras, an alternative proof using the full force of [**1**], could be given. However, the aim is to show the more general methods required for JBW*-triples apply to the particular case of C*-algebras.

Theorem 5·1. *Let A be a C^*-algebra with second dual A^{**}, let B be a norm closed subspace of A whose double annihilator $B^{\circ\circ}$ is an inner ideal in A^{**} and let u be a partial isometry in A^{**}. Then, the following are equivalent:*

(i) *u is compact relative to B;*

(ii) *u is contained in $B^{\circ\circ}$ and u is compact relative to A;*

(iii) *u belongs locally to B.*

Proof. We may assume that u is non-zero. Observe that B is necessarily a norm closed inner ideal in A. By [**10**], theorem 3·10, there exists a unique pair (e,f) of centrally equivalent open projections in A^{**} such that B coincides with $eA^{**}f \cap A$ and $B^{\circ\circ}$ coincides with the weak* closed inner ideal $eA^{**}f$ of A^{**}.

(i) ⇒ (ii). This follows from Theorem 4·7.

(ii) ⇒ (iii). Since u is contained in $eA^{**}f$ it follows that $u^*u \leqslant f$ and $uu^* \leqslant e$. Since u is compact relative to A, Corollary 4·4(i) shows that $\{u\}_{\prime_A\prime_A}$ is a non-empty subset of A_1. Therefore, by Lemma 3·1(ii), there exists an element a in A_1 such that

$$a = u+(1-uu^*)\,a(1-u^*u).$$

It follows that $u = au^*u$ and u belongs locally to A. By [**1**], 6·4, it follows that u belongs locally to B.

(iii) ⇒ (i). There exists an element a in B_1 such that $u = au^*u$. Using [**1**], 6·4, it follows that u lies in $B^{\circ\circ}$ and $u^*u \leqslant a^*a \leqslant f$. By [**1**], lemma 2·7, there is a decreasing net (a_j) of positive elements of A_1 of norm one with infimum u^*u such that $u^*u \leqslant a_j \leqslant f$. Therefore, since $a = ea = af = eaf$ and $a_j = fa_j = a_jf$, it can be seen that $eaa_jf = aa_j$ and therefore (aa_j) is a net in B_1 converging to u in the weak* topology. It follows that

$$\bigcap_j \{aa_j\}'_B \subseteq \{u\}_{\prime_B}.$$

Moreover, since $u^*u \leqslant a_j$,

$$uu^*aa_j\,u^*u = uu^*u = u.$$

It follows that aa_j is an element in $\{u\}_{\prime_B\prime_B}$ which implies that

$$\{u\}_{\prime_B} \subseteq \{u\}_{\prime_B\prime_B}{}'_B \subseteq \bigcap_j \{aa_j\}'_B.$$

Hence $\{u\}_{\prime_B}$ is a weak* semi-exposed face of B_1 and therefore, by Theorem 4·2, u is compact relative to B.

This result shows that something more can be said about the complete lattices $\mathscr{U}_c(B)^\sim$ and $\mathscr{U}_c(A)^\sim$ in this example.

Theorem 5·2. *Let A be a C^*-algebra with second dual A^{**}, let B be a norm closed subspace of A whose double annihilator $B^{\circ\circ}$ is an inner ideal in A^{**}, let $\mathscr{U}_c(A)^\sim$ be the* *complete lattice of partial isometries in A^{**} which are compact relative to A and the greatest element ω_A and let $\mathcal{U}_c(B)^{\sim}$ be the corresponding complete lattice for B. Then $\mathcal{U}_c(B)$ coincides with $\mathcal{U}_c(A) \cap B^{\circ\circ}$ and $i(\mathcal{U}_c(B)^{\sim})$ is a sub-complete lattice of $\mathcal{U}_c(A)^{\sim}$.*

Proof. By Theorem 4·7 (ii) and Theorem 5·1, it remains to show that the supremum of a family (u_j) of elements of $\mathcal{U}_c(B)$ is the same whether taken in $\mathcal{U}_c(B)^{\sim}$ or in $\mathcal{U}_c(A)^{\sim}$. Suppose that there exists an element u in $\mathcal{U}_c(A)$ such that $u_j \leqslant u$. Retaining the notation used in the proof of Theorem 5·1, by Lemma 3·6 and Corollary 3·7 (ii), for all j, $\{u_j\}_{\prime_A}$ is contained in $eA^*f \cap \{u\}_{\prime_A}$. When eA^*f is regarded as the dual space B^* of B it is clear that $eA^*f \cap \{u\}_{\prime_A}$ is a weak* closed face of B_1^*. By Theorem 5·1 and [1], theorem 6·7, there exists an element v in $\mathcal{U}_c(B)$ such that $eA^*f \cap \{u\}_{\prime_A}$ coincides with $\{v\}_{\prime_B}$ which, since $B^{\circ\circ}$ is an inner ideal coincides with $\{v\}_{\prime_A}$. Therefore, by Lemma 3·1 (i), for all j, $u_j \leqslant v \leqslant u$. This completes the proof of the theorem.

When Theorem 4·10 is applied to the case of a C*-algebra the following stronger result is obtained.

THEOREM 5·3. *Let A be a C*-algebra with second dual A^{**} and let P be a neutral projection on the dual A^* of A such that PA^* is weak* closed and P^*A^{**} is a subtriple of A^{**}. Let B denote the predual P^*A of PA^*. Let $\mathcal{U}_c(A)^{\sim}$ be the complete lattice of tripotents in A^{**} which are compact relative to A and the greatest element ω_A and let $\mathcal{U}_c(B)^{\sim}$ be the corresponding complete lattice for B. Then $\mathcal{U}_c(B)$ coincides with $\mathcal{U}_c(A) \cap B^{**}$ and $i(\mathcal{U}_c(B)^{\sim})$ is a sub-complete lattice of $\mathcal{U}_c(A)^{\sim}$.*

Proof. As in Theorem 4·10, B^{**} is a weak* closed inner ideal in A^{**} and therefore there exists a unique pair (e,f) of centrally equivalent projections in A^{**} such that B^{**} coincides with $eA^{**}f$. It follows that, for all elements x in A^*, $Px = exf$ and, for all elements a in A^{**}, $P^*a = eaf$. Moreover, B coincides with eAf and B^* with eA^*f. Let u be an element of $\mathcal{U}_c(B)$. Then, $\{u\}_{\prime_B \prime_B}$ is a proper norm semi-exposed face of B_1. Let F be the set of elements a of A_1 for which P^*a is contained in $\{u\}_{\prime_B \prime_B}$. Then, F is a norm closed face of A_1 and, by Theorem 5·1 and [1], theorems 4·10 and 6·9, there exists an element v in $\mathcal{U}_c(A) \cap B^{**}$ such that F coincides with $\{v\}_{\prime_A \prime_A}$. Therefore, $\{u\}_{\prime_B \prime_B}$ coincides with $P^*(\{v\}_{\prime_A \prime_A})$. Using Lemma 3·7 and Theorem 4·10,

$$\{u\}_{\prime_A} = \{u\}_{\prime_B} = (P^*\{v\}_{\prime_A \prime_A})^{\prime_B} = \{v\}_{\prime_A} \cap PA^* = \{v\}_{\prime_A}.$$

It follows that u and v are equal and hence that $\mathcal{U}_c(B)$ and $\mathcal{U}_c(A) \cap B^{**}$ coincide.

Finally, suppose that (u_j) is a family of elements of $\mathcal{U}_c(B)$ and that u is an element in $\mathcal{U}_c(A)$ such that, for all j, $u_j \leqslant u$. Then, by Lemma 3·7, for all j, $\{u_j\}_{\prime_A}$ is contained in $\{u\}_{\prime_A} \cap PA^*$. However, $\{u\}_{\prime_A} \cap PA^*$ is a weak* closed face of B_1^* and, by [1], theorem 6·11, there exists an element v of $\mathcal{U}_c(A) \cap B^{**}$ such that $\{u\}_{\prime_A} \cap PA^*$ coincides with $\{v\}_{\prime_A}$. Therefore, for all j, $u_j \leqslant v \leqslant u$ and it follows that the supremum of the family (u_j) is the same, whether taken in $\mathcal{U}_c(B)^{\sim}$ or in $\mathcal{U}_c(A)^{\sim}$.

The case discussed above fails by a long way to exhaust all possible examples. An extreme example of a JBW*-triple is a complex Hilbert space A with triple product defined by

$$\{a\ b\ c\} = \tfrac{1}{2}(\langle a,b\rangle c + \langle c,b\rangle a).$$

Then, of course, A^{**} can be identified with A and $\mathcal{U}(A)$ consists of the elements u of A of norm one along with zero. It is clear that in this case $\mathcal{U}_c(A)$ and $\mathcal{U}(A)$ coincide.

A slightly more interesting case is that of a spin triple A which is a complex Hilbert space along with a conjugation J and triple product defined by

$$\{a\ b\ c\} = \tfrac{1}{2}(\langle a,b\rangle c - \langle a,Jc\rangle Jb + \langle c,b\rangle a).$$

Notice that in this case A is a JBW*-triple but the norm does not coincide with the Hilbert space norm. An element u is a tripotent in A if and only if either $\langle u,u\rangle = 2$ and $Ju = \lambda u$, where λ is a complex number of unit modulus, or $\langle u,u\rangle = 1$ and $\langle u, Ju\rangle = 0$. In the first case u is unitary, that is to say that $A_2(u) = A$, and in the second case u is a minimal non-zero tripotent with $A_2(u) = \mathbf{C}u$, $A_0(u) = \mathbf{C}Ju$ and $A_1(u) = \{u, Ju\}^{\perp}$. It is clear that $\mathscr{U}_c(A)$ and $\mathscr{U}(A)$ again coincide.

A rather more non-trivial example is the following. Let C be a C*-algebra and let α be a $*$-anti-automorphism of C of order two. Then α extends to a $*$-anti-automorphism of C^{**} of order two which will be denoted by the same symbol. Let A be Jordan C*-algebra $H(C,\alpha)$ consisting of elements of C left invariant by α. Then the Jordan W*-algebra A^{**} may be identified with $H(C^{**},\alpha)$. It is clear that $\mathscr{U}(A^{**})$ consists of the partial isometries in C^{**} left invariant by α. Moreover, it is not too difficult to see that an element in $\mathscr{U}(A^{**})$ is compact relative to A if and only if it is compact relative to C. Consequently, many, though not all, of the results in this case follow closely those for C*-algebras. In the interest of brevity they will not be explicitly stated.

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