# CLOSURE OPERATORS AND PROJECTIONS ON INVOLUTION POSETS

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## 1. Introduction

Investigations of closure operators on an involution poset T lead to a certain type of closure operators (so called *c*-closure operators) that are closely related to projections on T.

In terms of these operators we give a necessary and sufficient condition for an involution poset to be an orthomodular lattice. An involution poset is an orthomodular lattice if and only if it admits certain c-closure operators. In that case, if L is an orthomodular lattice, the set of c-closure operators, under the usual ordering of closure operators, is orderisomorphic to the set of projections of the Baer \*-semigroup B(L) of hemimorphisms on L [4]. In this sense, but working on the "opposite end", this treatment enlarges that given in [3] where a similar necessary and sufficient condition is represented but for orthocomplemented posets and for mappings which in the case of an orthomodular lattice are exactly the closed projections of B(L). C-closure operators appear as a natural generalization of symmetric closure operators [5].

## 2. C-closure operators

An involution poset T is a poset with largest element (1) and a mapping  $e \in T \rightarrow e' \in T$  such that e'' = e and  $e \leq f \Rightarrow f' \leq e'$ . For basic definitions see [1, 2].

A projection  $\phi$  on an involution poset T is a mapping  $\phi: T \to T$  with the following properties:

- i)  $e \leq f \Rightarrow e\phi \leq f\phi$ ,
- ii)  $(e\phi)\phi = e\phi$ ,
- iii)  $(e\phi)'\phi \leq e' \quad (e, f \in T)$ .

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The set of projections on T, denoted by P(T), is not empty since I defined by eI: = e is a projection.

LEMMA 1. Let  $\phi$  be a projection on T. Then  $((e\phi)'\phi)'\phi = e\phi$  is valid for all  $e \in T$ .

PROOF. Since  $(e\phi)' \leq e'$  for all  $e \in T$ , it follows that  $((e\phi)'\phi)'\phi \leq ((e\phi)')' = e\phi$ . Clearly  $e \leq ((e\phi)'\phi)'$ . Using monotony, we get from the latter inequality  $e\phi \leq ((e\phi)'\phi)'\phi$ . Hence  $e\phi = ((e\phi)'\phi)'\phi$ .

REMARK 1. Let L be an orthomodular lattice. A projection  $\phi \in P(L)$  is a join-homomorphism of L [2, Theorem 5.2, page 37]. On the other hand every join-homomorphism is monotone. From  $(1\phi)'\phi \leq 1'$  we get  $0\phi = 0$ , where 0: = 1'. Therefore P(L) coincides with the set of projections introduced by Foulis [4], namely the set of idempotent, self-adjoint hemimorphisms on L.

One verifies that in an involution poset T a closure operator  $\gamma$  satisfies

$$((e\gamma)'\gamma)'\gamma \leq e\gamma \quad (e \in T).$$

Those closure operators for which the equality

$$((e\gamma)'\gamma)'\gamma = e\gamma \quad (e \in T)$$

is valid are of special interest. As we will see below they are closely related to projections and determining for the lattice and orthomodular structure of T. We call these operators *c*-closure operators and denote with C(T) the set of all *c*-closure operators on an involution poset T. The mappings I and eJ := 1 are *c*-closure operators.

C(T) is partially ordered by means of the ordering relation

$$\gamma_1 \leq \gamma_2$$
:  $\Leftrightarrow e\gamma_2 \leq e\gamma_1 \qquad (e \in T).$ 

I is the largest and J the smallest element of C(T).

THEOREM 2. Let T be an involution poset. If  $\gamma$  is a c-closure operator, then  $((e\gamma)'\gamma)'$  is a projection on T. If  $\phi$  is a projection, then  $((e\phi)'\phi)'$  is a c-closure operator on T.

The mapping  $\gamma \in C(T) \to \phi \in P(T)$  where  $e\phi := ((e\gamma)'\gamma)'$  is one-to-one and maps the set of c-closure operators onto the set of projections on  $T. \phi \in P(T) \to \gamma \in C(T)$  where  $e\gamma := ((e\phi)'\phi)'$  is the corresponding inverse mapping.

**PROOF.** Clearly the mapping  $e \rightarrow ((e\gamma)'\gamma)'$  is monotone. Using properties of *c*-closure operators, we get

$$((((e\gamma)'\gamma)'\gamma)'\gamma)' = ((e\gamma)'\gamma)',$$

which proves idempotence of the mapping. Furthermore

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$$(((((e\gamma)'\gamma)')'\gamma)'\gamma)' = (((e\gamma)'\gamma)'\gamma)' = (e\gamma)' \leq e'.$$

Hence the mapping is a projection.

Let  $\phi$  be a projection. By i) and iii) of the definition of a projection, one easily sees that the mapping  $e\gamma$ : = ( $(e\phi)'\phi$ )' is monotone and majorizes the argument. By Lemma 1 and the basic properties of projections we get

$$(e\gamma)\gamma = ((((e\phi)'\phi)'\phi)'\phi)' = ((e\phi)'\phi)' = e\gamma$$

and

$$((e\gamma)'\gamma)'\gamma = ((((((e\phi)'\phi)\phi)'\phi)\phi)'\phi))' = ((((e\phi)'\phi)'\phi)'\phi)' = ((e\phi)'\phi)' = e\gamma.$$

Hence  $\gamma \in C(T)$ .

For all  $\phi \in P(T)$ ,  $\gamma \in C(T)$  and  $e \in T$ 

$$((((e\gamma)'\gamma)''\gamma)'\gamma)'' = ((e\gamma)'\gamma)'\gamma = e\gamma$$

and

$$((((e\phi)'\phi)''\phi)'\phi)'' = ((e\phi)'\phi)'\phi = e\phi$$

is valid. This proves the second part of the theorem.

REMARK 2. Because of the one-to-one correspondence between P(T) and C(T) the ordering in the set of *c*-closure operators induces an ordering in the set of projections as follows:

Let  $\phi_1, \phi_2$  be two projections and  $\gamma_1, \gamma_2$  the corresponding *c*-closure operators. The relation

$$\phi_1 \leqq \phi_2 : \Leftrightarrow \gamma_1 \leqq \gamma_2$$

is an ordering relation that makes P(T) into a partially ordered set. The mapping  $\gamma \to \phi$  where  $e\phi := ((e\gamma)'\gamma)'$  can then be interpreted as an order-isomorphism between the posets C(T) and P(T).

The next two lemmata lead us to the main result of this paper.

LEMMA 3. Let T be an orthocomplemented poset and  $\gamma \in C(T)$ . Then i)  $e\gamma \lor (e\gamma)'\gamma$  exists and is equal to 1, ii)  $e\gamma \land (e\gamma)'\gamma$  exists and is equal to 0 $\gamma$ .

**PROOF.** i) Of course  $e\gamma \leq 1$  and  $(e\gamma)'\gamma \leq 1$ . If there is an  $f \in T$  such that  $e\gamma \leq f$  and  $(e\gamma)'\gamma \leq f$ , then also  $(e\gamma)' \leq f$  since  $(e\gamma)' \leq (e\gamma)'\gamma$ . But  $e\gamma \lor (e\gamma)' = 1$ , hence  $1 \leq f$ . This proves that  $e\gamma \lor (e\gamma)'\gamma = 1$ . ii) By monotony  $e\gamma \leq e\gamma$  and  $e\gamma \leq (e\gamma)'\gamma$ . Let  $f \in T$  be an element such that  $f \leq e\gamma$  and  $f \leq (e\gamma)'\gamma$ . By monotony and idempotence of the closure operator we get

$$f\gamma \leq e\gamma$$
 and  $f\gamma \leq (e\gamma')\gamma$  or  $(e\gamma)' \leq (f\gamma)'$ 

and  $((e\gamma)'\gamma)' \leq (f\gamma)'$ . Again by monotony we have then  $(e\gamma)'\gamma \leq (f\gamma)'\gamma$  and

$$e\gamma = ((e\gamma)'\gamma)'\gamma \leq (f\gamma)'\gamma$$

According to part i) of this proof, this implies that  $(f\gamma)'\gamma = 1$  or  $((f\gamma)'\gamma)' = 0$ . Finally we get  $f \leq f\gamma = ((f\gamma)'\gamma)'\gamma = 0\gamma$ . Thus  $e\gamma \wedge (e\gamma)'\gamma = 0\gamma$ .

LEMMA 4. Let T be an involution poset and  $\gamma$  a c-closure operator, then  $(0\gamma)'\gamma = 1$ .

**PROOF.** By theorem 2 there is a projection  $\phi$  such that  $e\gamma = ((e\phi)'\phi)'$ . Since  $0\phi = 0$  and by lemma 1 we get

$$(0\gamma)'\gamma = ((((0\phi)'\phi)'\phi)'\phi)' = (((0\phi)'\phi)'\phi)' = (0\phi)' = 1.$$

THEOREM 5. Let T be an involution poset. T is an orthomodular lattice if and only if every interval [e, 1]  $(e \in T)$  is the range of a c-closure operator.

PROOF. Assume that T is an orthomodular lattice. One verifies that for a given interval [e,1] the mapping  $f \rightarrow e \lor f$  is a closure operator that maps T onto it. We show that this mapping has the characteristic property of c-closure operators.

Since  $e \leq e \lor f$ , there exists by orthomodularity of the lattice T an element  $g \in T$  such that  $e \lor g = e \lor f$  and  $e \leq g'$ . Now

$$e \lor (e \lor (e \lor f)')' = e \lor (e \lor (e \lor g)')' = e \lor (e' \land (e \lor g)) = e \lor (e' \land g) = e \lor g = e \lor f.$$

Conversely, we prove first that T must be a lattice. When  $e, f \in T$ , then there is a c-closure operator  $\gamma$  that maps T onto the interval [f, 1]. Clearly  $e \leq e\gamma$  and  $f = 0\gamma \leq e\gamma$ . Let  $g \in T$  be an element such that  $e \leq g$  and  $f \leq g$ . Since  $\gamma$  maps T onto [f, 1], it follows from the latter inequality that  $g\gamma = g$ . From  $e \leq g$ we then get  $e\gamma \leq g\gamma = g$ . Thus  $e \lor f$  exists in T and is equal to  $e\gamma$ .

Let  $\gamma \in C(T)$  with  $T\gamma = [e, 1]$ . By lemma 4 we get  $1 = (0\gamma)'\gamma = e'\gamma = e' \lor e$  for all  $e \in T$ . Therefore T is an orthocomplemented lattice.

Now we prove orthomodularity of the lattice T. Let  $e \leq f$  and  $y \in C(T)$  such that  $T\gamma = [e, 1]$ . We again have  $e = 0\gamma$  and  $f\gamma = f$ . By Lemma 3 (ii) and the result above we get  $e = 0\gamma = f\gamma \wedge (f\gamma)'\gamma = f \wedge f'\gamma = f \wedge (e \lor f')$ .

REMARK 3. Let L be an orthomodular lattice. By Theorem 2 and Remark 1 the mappings  $e \rightarrow e\phi$ : = (( $e\gamma$ )' $\gamma$ )' ( $\gamma \in C(L)$ ) are the projections in the Baer \*-semigroup of hemimorphisms on L. One can prove that

$$(e\phi_1)\phi_2 = e\phi_1 \ (\phi_1, \phi_2 \in P(L); e \in L) \Leftrightarrow \phi_1 \leq \phi_2,$$

thus the usual ordering of projections coincides with that induced by the poset C(L) (Remark 2). The closed projections, namely the Sasaki-projections, are given by  $((e\gamma_f)'\gamma_f)'$   $(f \in L)$  where  $\gamma_f \in C(L)$  and  $L\gamma_f = [f, 1]$ .

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Note that a mapping  $\gamma$  is a symmetric closure operator on L[5] if and only if  $\gamma$  is a *c*-closure operator for which  $0\gamma = 0$  is valid. Furthermore, the symmetric closure operators are the fixelements of the mappings exhibited in theorem 2.

#### References

[1] G. Birkhoff, Lattice Theory, (Amer. Math. Soc., 3rd ed. (1967)).

- [2] T. S. Blyth and M. F. Janowitz, Residuation Theory, (Pergamon Press (1972)).
- [3] P. D. Finch, 'Sasaki-Projections on Orthocomplemented Posets', Bull. Austral. Math. Soc. 1 (1969), 319-324.
- [4] D. J. Foulis, 'Baer \*-semigroups', Proc. Amer. Math. Soc. 11 (1960), 648-654.
- [5] M. F. Janowitz, 'Residuated Closure Operators', Portugal. Math. 26 (1967), 221-252.

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