

## A NOTE ON UNIFORM ORDERED SPACES

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### Abstract

We characterize the generalized ordered topological spaces  $X$  for which the uniformity  $\mathcal{U}(X)$  is convex. Moreover, we show that a uniform ordered space for which every compatible convex uniformity is totally bounded, need not be pseudocompact.

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We answer two questions of P. Fletcher and W. F. Lindgren. In [2, Problem F, page 94] they ask for necessary and sufficient conditions such that the uniformity  $\mathcal{U}(X)$  of a *GO* space  $(X, \mathcal{T}, \leq)$  is convex with respect to  $\leq$ . In this note we show that the uniformity  $\mathcal{U}(X)$  of a *GO* spaces  $X$  is convex if and only if each closed discrete subset of  $X$  is countable. In [2, Problem I, p. 95] they ask whether a uniform ordered space  $(X, \mathcal{T}, \leq)$  for which every convex uniformity compatible with  $\mathcal{T}$  is totally bounded, is necessarily pseudocompact. They observe that the answer is positive, if  $\leq$  is a linear order on  $X$ . In this note we show that the answer is negative in general. We will use the notation and the terminology of [2].

### 1.

In the first part of this note we will need the following lemma.

**LEMMA.** *Let  $(X, \leq)$  be an uncountable linearly ordered set. Then there exists an uncountable subset  $A$  of  $X$  such that, if  $a, b \in A$  and  $a \leq b$ , then there is a  $c \in X \setminus A$  with  $a \leq c \leq b$ .*

**PROOF.** Denote by  $[X]^3$  the set of the subsets of  $X$  with three elements. If  $B \in [X]^3$ , we denote the minimal element of  $B$  by  $L_B$  and the maximal element of  $B$  by  $R_B$ . Finally,  $M_B$  denotes the element of  $B$  such that  $L_B < M_B < R_B$ .

*Case 1.* There exists a countable subset  $D$  of  $X$  so that for each  $B \in [X]^3$  there is a  $d \in D$  such that  $L_B \leq d \leq R_B$ . Set  $C = X \setminus D$  and  $A = \{x \in X \mid x \text{ is the smallest element of a convexity-component of } C \text{ in } X\}$ . Since each convexity-component of  $C$  in  $X$  has at most two elements,  $A$  is an uncountable set that satisfies the condition of the lemma.

*Case 2.* For each countable subset  $D$  of  $X$  there is a  $B \in [X]^3$  such that  $D \cap \{x \in X \mid L_B \leq x \leq R_B\} = \emptyset$ . Define by transfinite induction for each  $\beta < \omega_1$  a set  $B(\beta) \in [X]^3$ : Suppose that  $B(\alpha)$  has been defined for each  $\alpha < \beta$ . There is a set  $B \in [X]^3$  such that  $\{x \in X \mid L_B \leq x \leq R_B\}$  contains no element of the countable set  $\cup\{B(\alpha) \mid \alpha < \beta\}$ . Set  $B(\beta) = B$ . Then  $A = \{M_{B(\beta)} \mid \beta < \omega_1\}$  is uncountable and satisfies the condition of the lemma.

It is known that each  $GO$  space is normal. In the next proof we will use the result that for a normal  $T_2$ -space  $X$  the uniformity  $\mathcal{U}(X)$  is the finest compatible uniformity on  $X$  if and only if each locally finite open cover of  $X$  has a countable open refinement of finite order [3, Remark following the proof of the theorem; compare 1 and 2, p. 190, §5.28]. Recall that a topological space  $X$  is called  $\omega_1$ -compact, if each closed discrete subset of  $X$  is countable.

**PROPOSITION.** *Let  $(X, \mathcal{T}, \leq)$  be a  $GO$  space. Then the following conditions are equivalent:*

- (a)  $X$  is  $\omega_1$ -compact.
- (b)  $\mathcal{U}(X)$  is the finest uniformity for  $(X, \mathcal{T})$ .
- (c)  $\mathcal{U}(X)$  is convex.

**PROOF.** (a)  $\rightarrow$  (b). Since  $X$  is  $\omega_1$ -compact and  $\dim X \leq 1$ , every locally finite open cover of  $X$  is refined by a countable open refinement of finite order. We conclude that  $\mathcal{U}(X)$  is the finest uniformity for  $(X, \mathcal{T})$ .

(b)  $\rightarrow$  (c). Since the finest uniformity for a  $GO$  space is convex [2, Theorem 4.33],  $\mathcal{U}(X)$  is convex.

(c)  $\rightarrow$  (a). Let  $\mathcal{U}(X)$  be convex. Assume that  $X$  is not  $\omega_1$ -compact. Then  $X$  has an uncountable closed discrete subset  $A$ . By the lemma there exists an uncountable subset  $A'$  of  $A$  such that every subset of  $A'$  that is convex in  $A$  contains at most one point. Define a function  $f: A \rightarrow \mathbb{R}$  by  $f(x) = 0$  if  $x \in A'$ , and  $f(x) = 1$  if  $x \in A \setminus A'$ . Let  $g: X \rightarrow \mathbb{R}$  be continuous such that  $g|_A = f$ . Let

$V \in \mathcal{C}(X)$  such that  $V \subset \{(x, y) \in X \times X: |g(x) - g(y)| < \frac{1}{2}\}$  and  $V(x)$  is convex in  $X$  for each  $x \in X$ . Since  $V \in \mathcal{C}(X)$ , there is a countable subset  $D$  of  $X$  such that  $X = \bigcup\{V(d) | d \in D\}$ . Clearly, each  $V(d)$  contains at most one element of  $A'$ —a contradiction. We conclude that  $X$  is  $\omega_1$ -compact.

**EXAMPLE 1.** Let  $R$  denote the set of the reals and let  $\leq$  denote the usual order on  $R$ . Consider the *GO* space  $(R, \mathcal{F}, \leq)$  where  $\mathcal{F}$  denotes the discrete topology on  $R$ . Clearly  $(R, \mathcal{F})$  is not  $\omega_1$ -compact. Hence  $\mathcal{C}(R)$  is not convex.

## 2.

We construct a uniform ordered space  $(X, \mathcal{F}, \leq)$  such that every convex uniformity compatible with  $(X, \mathcal{F}, \leq)$  is totally bounded, but  $X$  is not pseudocompact.

**EXAMPLE 2.** We use a modification of the well-known pseudocompact space  $\psi$  [see 4, 5I]. Let  $N$  be the set of the positive integers and let  $\Gamma$  be an infinite maximal almost disjoint family of infinite subsets of  $N$ . As usual set  $\psi = N \cup \Gamma$ . Let  $(a_{2n-1})_{n \in N}$  be a sequence of pairwise different elements of  $\Gamma$ . Set  $a_{2n} = -n$  for each  $n \in N$ ,  $A = \{a_{2n} | n \in N\}$ , and  $D = \{a_n | n \in N\}$ . Let  $X = N \cup \Gamma \cup A$ . Consider the following collection of subsets of  $X$ :

$$\mathcal{B} = \{\{n\} | n \in N\} \cup \{(E \setminus F) \cup \{E\} | E \in \Gamma \setminus D, F \text{ is a finite subset of } N\} \cup \{\bigcup_{n=1}^k G(a_n) | k \in N; G(a_{2m}) = \{a_{2m}\} \text{ (for each } m \in N \text{ such that } 2m \leq k); G(a_{2m-1}) = (a_{2m-1} \setminus F) \cup \{a_{2m-1}\} \text{ where } F \text{ is a finite subset of } N \text{ (for each } m \in N \text{ such that } 2m - 1 \leq k)\}\}.$$

Set  $\mathcal{S} = \{[X \times G] \cup [(X \setminus G) \times X] | G \in \mathcal{B}\}$ . Then  $\mathcal{S}$  is a subbase for a quasi-uniformity  $\mathcal{U}$  on  $X$ . Consider the topology  $\mathcal{T}(\mathcal{U}^*)$  on  $X$  where  $\mathcal{U}^*$  denotes the uniformity generated by  $\{V \cap V^{-1} | V \in \mathcal{U}\}$  on  $X$ . One easily checks that the points of the  $\mathcal{T}(\mathcal{U}^*)$ -open subspace  $\psi$  of  $X$  have their usual neighbourhoods. Moreover, each point of  $X \setminus \psi$  is isolated in  $X$ . Hence  $\mathcal{T}(\mathcal{U}^*)$  is not pseudocompact. Since  $\mathcal{T}(\mathcal{U}^*)$  is a Hausdorff topology,  $(X, \mathcal{T}(\mathcal{U}^*), \cap \mathcal{U})$  is a uniform ordered space [2, pages 81, 84]. Let  $\mathcal{W}$  be a convex uniformity compatible with  $(X, \mathcal{T}(\mathcal{U}^*), \cap \mathcal{U})$ . Let  $Z \in \mathcal{W}$ . Since  $\mathcal{W}$  is convex, there is a  $Z' \in \mathcal{W}$  such that  $Z' \subset Z$  and  $Z'(x)$  is convex in  $X$  for each  $x \in X$ . Since  $\psi$  is a pseudocompact subspace of  $X$ , there is a finite subset  $F$  of  $\psi$  such that  $\psi \subset Z'(F)$ . Hence there is an  $x \in F$  such that  $Z'(x)$  contains infinitely many points of  $D \setminus A$ . Note that, if

$k, n \in N$  and  $k \leq n$ , then  $(a_n, a_k) \in \cap \mathcal{U}$ . Since  $Z'(x)$  is convex in  $X$ ,  $Z'(x)$  contains all but finitely many points of  $D$ . We conclude that  $\mathcal{W}$  is totally bounded.

### References

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