

## HOMOTOPY GROUPS AND H-MAPS

by ALAIN JEANNERET

(Received 21st January 1994)

The first nonvanishing homotopy group of a finite H-space  $X$  whose mod 2 homology ring is associative occurs in degrees 1, 3 or 7. Generators of these groups can be represented by maps  $\alpha: S^n \rightarrow X$  for  $n=1, 3$  or  $7$ . In this note we prove that under some hypothesis on  $X$  there exists an H-structure on  $S^n$ ,  $n=1, 3$  or  $7$  such that  $\alpha$  is an H-map.

1991 *Mathematics subject classification*: 55P45, 55Q05, 55Q52.

### 1. Introduction

Recently, J. P. Lin and the author [7] have proven that the first nonvanishing homotopy group of a finite H-space whose mod 2 homology ring is associative occurs in degrees 1, 3 or 7. (Recall that a finite H-space is an H-space whose integral homology is finitely generated as a graded abelian group). This result improves Adams' famous theorem [1] saying that a sphere  $S^n$  admits an H-structure if and only if  $n=1, 3$  or  $7$ .

The goal of this paper is to discuss the following question. If  $(X; \mu)$  is a finite H-space, where  $\mu$  is the multiplication on  $X$ , does there exist a set of generators of the first nonvanishing homotopy group  $\{\alpha_i\} \subset \pi_n X$ ,  $n=1, 3$  or  $7$  and a multiplication  $m$  on  $S^n$  such that

$$\alpha_i: (S^n; m) \rightarrow (X; \mu)$$

are H-maps? (We will always identify a map with its homotopy class). The question can be easily answered in the case where  $n=1$ . The proofs are given in the next sections.

**Theorem 1.1.** *If  $X$  is a finite H-space, then there exists a set of generators  $\{\alpha_i\}_{i=1, \dots, n}$  of  $\pi_1 X$  such that*

$$\alpha_i: S^1 \rightarrow X$$

are H-maps for all  $i=1, \dots, n$ .

We do need to specify the H-structure on  $S^1$  because  $S^1$  admits only one multiplication.

The case where  $n=3$  was first investigated by J. Schiffmann [13]; more precisely he has proven the following theorem.

**Theorem 1.2.** *Let  $X$  be a finite H-space carrying an homotopy associative multiplication such that  $\pi_3 X \cong \mathbf{Z}$ . Let  $\mu$  be a multiplication on  $X$  and  $\alpha \in \pi_3 X$  a generator, then there exists a multiplication  $m$  on  $S^3$  such that*

$$\alpha : (S^3; m) \rightarrow (X; \mu)$$

is an H-map.

Note that if  $X$  is a finite H-space,  $\pi_3 X$  is always a free abelian group. Even if the associative multiplication does not appear in the conclusion of the theorem, it is strongly used to prove it.

We can improve Schiffmann’s result for a wider class of H-spaces, namely the non-associative ones.

**Theorem 1.3.** *Let  $(X; \mu)$  be a finite H-space with  $\pi_3 X \cong \mathbf{Z}$  and  $\alpha \in \pi_3 X$  a generator, then there exists a multiplication  $m$  on  $S^3$  such that*

$$\alpha : (S^3; m) \rightarrow (X; \mu)$$

is an H-map.

As an obvious corollary we can offer:

**Corollary 1.1.** *Let  $(X; \mu), (X_i; \mu_i), i=1, \dots, n$  be finite H-spaces with  $\pi_3 X_i \cong \mathbf{Z}$  and  $X \simeq \prod_{i=1}^n X_i$  as H-spaces. Then there exists a set of generators  $\{\alpha_i\}_{i=1, \dots, n}$  of  $\pi_3 X \cong \mathbf{Z}^n$  and multiplications  $m_i, i=1, \dots, n$  on  $S^3$  such that*

$$\alpha_i : (S^3; m_i) \rightarrow (X; \mu)$$

are H-maps.

Let us pause to comment:

- (1) Many H-spaces do not admit homotopy associative multiplication. For example if  $G$  is a one-connected simple Lie group different from  $G_2$  or  $Spin(7)$ , then  $G \times S^7$  does not carry a homotopy associative multiplication [4].
- (2) All finite H-spaces do not satisfy the splitting condition  $X \simeq \prod X_i$  with  $(X_i; \mu_i)$  a finite H-space, hence the corollary does not apply for all finite H-spaces.
- (3) Unlike  $S^1$  which has a unique H-structure,  $S^3$  has many [6], therefore the multiplications  $m_i$  on  $S^3$  depend on the generators  $\alpha_i$ .
- (4) The situation for compact Lie groups is much nicer because problems of type occurring in (2) and (3) do not appear, cf. below.

If  $G$  is a simply connected compact Lie group then  $G$  splits as  $G \simeq \prod_{k=1}^n G_k$  with  $G_k$  simple Lie groups. Let  $H$  be any simple Lie group. It is well known that for any root  $\rho$  of  $H$  there exists a subgroup  $SU(2) = S^3$  of  $H$  [12]. The inclusion  $i: S^3 \rightarrow H$  is not necessarily a generator of  $\pi_3 H \cong \mathbf{Z}$ , but if the root  $\rho$  is dominant [12], then  $i$  is a generator of  $\pi_3 H$ . If we come back to the simply connected Lie group  $G$  with decomposition  $G \cong \prod_{k=1}^n G_k$ , it is obvious then there exists a set of generators  $\{\alpha_k\}_{k=1, \dots, n}$  of  $\pi_3 G \cong \mathbf{Z}^n$  such that

$$\alpha_k: SU(2) = S^3 \rightarrow G$$

are group homomorphisms.

Before closing the introduction let us discuss the case of a 6-connected finite H-space. During many years the only known 6-connected finite H-spaces were products of 7-dimensional spheres. Recently Dwyer and Wilkerson have built up an “exotic” H-space at the prime 2 [3]. More precisely they constructed a mod 2 finite H-space  $DI(4)$  whose mod 2 cohomology ring satisfies

$$H^*DI(4) \cong \mathbf{F}_2[x_7]/(x_7^4) \otimes E(x_{11}, x_{13}).$$

See [3] for the details. Using Zabrodsky’s techniques of mixing homotopy types we can mix the 2-type of  $DI(4)$  with  $S^7 \times S^{11} \times S^{27}$  localised at the set of all odd primes to construct an H-space  $X(4)$  whose mod 2 cohomology ring satisfies

$$H^*X(4) \cong H^*DI(4)$$

and which has no  $p$ -torsion in integral homology for odd primes  $p$  (note that  $H^*(DI(4); \mathbf{Q}) \cong E(x_7, x_{11}, x_{27})$ ). In particular  $\pi_7 X(4) \cong \mathbf{Z}$ . We can state now our last proposition.

**Proposition 1.1.** *Let  $\mu$  be a multiplication on  $X(4)$  and  $\alpha$  a generator of  $\pi_7 X(4) \cong \mathbf{Z}$ . Then there exists a multiplication  $m$  on  $S^7$  such that*

$$\alpha: (S^7; m) \rightarrow (X(4); \mu)$$

is an H-map.

## 2. Proof of Theorem 1.1

The fundamental group of  $X$  is abelian and satisfies  $\pi_1 X \cong \bigoplus_{k=1}^n C_k$  with  $C_k$  a cyclic group. We start with the situation where  $\pi_1 X$  is an infinite cyclic group, i.e.  $\pi_1 X \cong \mathbf{Z}$ . If  $\pi_1 X \cong \mathbf{Z}$ , it is well known that  $X \simeq S^1 \times Y$  with  $\pi_1 Y = \pi_2 Y = 0$  cf. [11, page 63]. Therefore there exists a cellular decomposition of  $X$  such that its 2-skeleton,  $X^{(2)}$ , is just  $S^1$  [5]. Let  $\alpha$  be a generator of  $\pi_1 X$  that is identified with the inclusion  $S^1 = X^{(2)} \rightarrow X$ . The composition

$$S^1 \times S^1 \xrightarrow{\alpha \times \alpha} X \times X \xrightarrow{\mu} X$$

factors through  $X^{(2)} = S^1$ . We have thus constructed a map  $m: S^1 \times S^1 \rightarrow S^1$  such that the following diagram commutes

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\alpha \times \alpha} & X \times X \\ \downarrow m & & \downarrow \mu \\ S^1 & \xrightarrow{\alpha} & X. \end{array}$$

Hence  $\alpha$  is an H-map.

The second situation is when  $\pi_1 X$  is a finite cyclic group i.e.  $\pi_1 X \cong \mathbb{Z}/d$ . In this case standard techniques show that  $H_1(X; \mathbb{Z}) \cong \mathbb{Z}/d$ ,  $H^1(X; \mathbb{Z}) = 0$ ,  $H^2(X; \mathbb{Z}) \cong \mathbb{Z}/d$ . Let  $x$  be a generator of  $H^2(X; \mathbb{Z})$ . For dimensional reasons  $x$  is a primitive class, and so the map  $f: X \rightarrow K(\mathbb{Z}; 2)$  representing  $x$  is an H-map. The fiber  $F$  of  $f$  is then an H-space.

If  $\alpha$  is a generator of  $\pi_1 X$  then  $\alpha: S^1 \rightarrow X$  lifts to  $\tilde{\alpha}: S^1 \rightarrow F$ . It is sufficient to prove that  $\tilde{\alpha}$  is an H-map because the inclusion  $i: F \rightarrow X$  is an H-map and by construction  $\alpha = i \circ \tilde{\alpha}$ . Now observe that  $F$  is a finite H-space with  $\pi_1 F \cong \mathbb{Z}$  generated by  $\tilde{\alpha}$ . Using the first case discussed above we obtain that  $\tilde{\alpha}$  is an H-map.

Let us consider now the general case  $\pi_1 X \cong \bigoplus_{k=1}^n C_k$ ,  $C_k$  a cyclic group. Let  $p_k: X_k \rightarrow X$  be the covering space such that  $\pi_1 X_k \cong C_k$ . The covering map  $p_k$  is an H-map. Then there exists a generator  $\tilde{\alpha}_k$  of  $\pi_1 X_k$  which is an H-map. The set  $\{\alpha_k\}_{k=1, \dots, n}$  with  $\alpha_k = p_k \circ \tilde{\alpha}_k$  is a set of generators of  $\pi_1 X$  and all the  $\alpha_k$  are H-maps.

### 3. Proof of Theorem 1.3

First remark that  $X$  is not assumed to be 1-connected. The universal cover  $\tilde{X}$  of  $X$  satisfies  $\pi_3 \tilde{X} \cong \pi_3 X$  and  $\tilde{X}$  has the homotopy type of a finite H-space. We can therefore assume through all the proof that  $X$  is 2-connected and  $\pi_3 X \cong \mathbb{Z}$ .

A result of Lin [9] asserts that

$$QH^{even}(X; \mathbb{F}_p) \cong \sum_{i=1}^{\infty} \beta_1 \mathcal{P}^i H^{2n_i+1}(X; \mathbb{F}_p)$$

where  $p$  is an odd prime,  $QH^{even}$  is the module of indecomposables in even dimensions,  $\beta_1$  is the first Bockstein and  $\mathcal{P}^i$  is the  $i^{th}$  Steenrod power. In our situation this result implies that  $H^k(X; \mathbb{Z})$  has no  $p$ -torsion for  $p$  an odd prime and  $k \leq 6$ . Hence we are reduced to study the 2-torsion of the integral cohomology in low dimensions.

To simplify the notations,  $H^*X$  will stand for  $H^*(X; \mathbb{F}_2)$ . Recall two results due to Kane and Lin.

**Theorem 3.1 [8].** *Let  $X$  be a simply connected mod 2 finite H-space, then*

$$QH^{even} X = 0.$$

**Theorem 3.2 [10].** *Let  $X$  be a simply connected mod 2 finite H-space, then*

$$QH^{4k+1}X = Sq^{2k}QH^{2k+1}X, k \geq 1.$$

We apply these results in our particular case. The first result implies that  $H^4X = QH^4X = 0$  because  $H^2X = 0$ ,  $X$  being 2-connected. Moreover  $H^6X \cong \xi H^3X$  with  $\xi$  the cup square map. The second result implies that  $H^5X \cong Sq^2H^3X$ . Hence  $\xi H^3X \cong Sq^3H^3X \cong Sq^1Sq^2H^3X \cong Sq^1H^5X$ . So three different cases can occur:

1.  $H^6X \neq 0$  and so  $H^6X \cong H^5X \cong H^3X \cong F_2$ , if we set  $x_k$  a generator of  $H^kX$  then

$$Sq^2x_3 = x_5, Sq^1x_5 = x_6 = x_3^2.$$

2.  $H^6X = 0, H^5X \neq 0$  and so

$$Sq^2x_3 = x_5, Sq^1x_5 = 0.$$

3.  $H^5X = 0$  and so

$$Sq^2x_3 = 0.$$

We will prove the theorem under the conditions stated in (1), the cases (2) and (3) being similar or simpler.

Let  $K_0 = K(F_2; 5)$  and  $h_0: X \rightarrow K_0$  the classifying map of  $x_5$  the generator of  $H^5X$ . Define  $E_0$  to be the fiber of  $h_0$ . The Serre spectral sequence applied to the fibration

$$\begin{array}{ccc} \Omega K_0 & \xrightarrow{i_0} & E_0 \\ & & \downarrow p_0 \\ & & X \end{array}$$

implies that

$$H^j E_0 \cong \begin{cases} F_2 & \text{if } j = 3, 6 \\ 0 & \text{if } j = 1, 2, 4, 5 \end{cases}$$

and  $i_0^*: H^6 E_0 \cong H^6 \Omega K_0$ . Since the class  $x_5$  is primitive, the map  $h_0$  is an H-map, hence  $E_0$  is an H-space and  $i_0$  and  $p_0$  are H-maps too. We call  $a$  the generator of  $H^6 E_0 \cong F_2$ . It satisfies  $i_0^* a = Sq^2 \iota_4$ , where  $\iota_4$  is the fundamental class of  $\Omega K_0 = K(F_2; 4)$ . We will prove in the next section the central lemma.

**Lemma 3.1.** *The class  $a \in H^6 E_0$  is primitive.*

Now let  $K_1 = K(F_2; 6)$  and  $h_1: E_0 \rightarrow K_1$  be the classifying map for  $a$ . Let  $E_1$  be the fiber of  $h_1$ . The Serre spectral sequence of the fibration

$$\begin{array}{ccc} \Omega K_1 & \xrightarrow{i_1} & E_1 \\ & & \downarrow p_1 \\ & & E_0 \end{array}$$

implies that

$$H^j E_1 \cong \begin{cases} \mathbf{F}_2 & \text{if } j = 3 \\ 0 & \text{if } j = 1, 2, 4, 5, 6. \end{cases}$$

In particular  $E_1$  admits a cellular decomposition such that its 6-skeleton,  $E_1^{(6)}$ , is just  $S^3$  [5]. The map  $h_1$  is an H-map because the cohomology class  $a$  is primitive, therefore  $E_1$  is an H-space;  $i_1$  and  $p_1$  are H-maps.

Let  $\alpha$  be a generator of  $\pi_3 X$ , it is clear that  $\alpha$  lifts in the following way:

$$\begin{array}{ccccc} & \tilde{\alpha} & E_1 & & \\ & \nearrow & \downarrow p_1 & & \\ S^3 & \xrightarrow{\tilde{\alpha}} & E_0 & \xrightarrow{h_1} & K_1 \\ & \searrow & \downarrow p_0 & & \\ & \alpha & X & \xrightarrow{h_0} & K_0. \end{array}$$

The theorem will be proven as soon as we can exhibit a multiplication  $m$  on  $S^3$  such that  $\tilde{\alpha}$  is an H-map, because  $\alpha = p_0 \circ p_1 \circ \tilde{\alpha}$  and  $p_0, p_1$  are H-maps. The argument is the following:

As claimed above  $E_1$  admits a cellular decomposition such that  $E_1^{(6)} = S^3$ . We identify  $\tilde{\alpha}$  with the inclusion  $S^3 = E_1^{(6)} \rightarrow E_1$ . Let  $\tilde{\mu}$  be a cellular multiplication on  $E_1$  induced from the one on  $X$ . The composition

$$S^3 \times S^3 \xrightarrow{\tilde{\alpha} \times \tilde{\alpha}} E_1 \times E_1 \xrightarrow{\tilde{\mu}} E_1$$

factors through  $E_1^{(6)} = S^3$ . Therefore we have constructed a map  $m: S^3 \times S^3 \rightarrow S^3$  such that the following diagram is commutative

$$\begin{array}{ccc} S^3 \times S^3 & \xrightarrow{\tilde{\alpha} \times \tilde{\alpha}} & E_1 \times E_1 \\ \downarrow m & & \downarrow \tilde{\mu} \\ S^3 & \xrightarrow{\tilde{\alpha}} & E_1. \end{array}$$

The restriction of  $m$  to  $S^3 \vee S^3$  is homotopic to the folding map because the restriction of  $\tilde{\mu}$  to  $E_1 \vee E_1$  is homotopic to the folding map as  $\tilde{\mu}$  is a multiplication on  $E_1$ . So  $m$  is a multiplication on  $S^3$  and  $\tilde{\alpha}$  is an H-map. The theorem is proven.

#### 4. Proof of Lemma 3.1

The fibration defined in the previous section

$$\begin{array}{ccc}
 E_0 & & \\
 \downarrow p_0 & & \\
 X & \xrightarrow{h_0} & K_0
 \end{array} \tag{*}$$

induces an exact sequence in cohomology with  $F_2$  coefficients up to dimension 7. Recall that if  $Y$  is an H-space then the projective plane of  $Y$  [2] denoted by  $P_2Y$  fits into a cofibration

$$\Sigma Y \xrightarrow{i_Y} P_2 Y \xrightarrow{k_Y} \Sigma Y \wedge \Sigma Y.$$

The naturality of the cofibration allows us to construct the following commutative diagram:

$$\begin{array}{ccccc}
 \Sigma E_0 & \xrightarrow{i_{E_0}} & P_2 E_0 & \xrightarrow{k_{E_0}} & \Sigma E_0 \wedge \Sigma E_0 \\
 \downarrow \Sigma p_0 & & \downarrow P_2 p_0 & & \downarrow \Sigma p_0 \wedge \Sigma p_0 \\
 \Sigma X & \xrightarrow{i_X} & P_2 X & \xrightarrow{k_X} & \Sigma X \wedge \Sigma X \\
 \downarrow h & & \downarrow j & & \downarrow \\
 A & \xrightarrow{i} & B & \xrightarrow{k} & C.
 \end{array} \tag{**}$$

The spaces  $A, B$  and  $C$  are the cofibres of the maps  $\Sigma p_0, P_2 p_0, \Sigma p_0 \wedge \Sigma p_0$  respectively. The sequence

$$A \xrightarrow{i} B \xrightarrow{k} C$$

is again a cofibration.

As  $(\Sigma p_0)^* : H^n \Sigma E_0 \cong H^n \Sigma X$  for  $n \leq 5$ , we deduce that  $H^n C = 0$  for  $n \leq 10$ . In particular  $H^n A \cong H^n B$  for  $n \leq 9$ . The remark made after (\*\*) allows us to identify

$$H^n A \cong H^n B \cong H^n \Sigma K_0$$

for  $n \leq 8$ .

Let  $x_3 \in H^3 X$  be the generator and  $y_3 \in H^3 E_0$  be the class such that  $p_0^* x_3 = y_3$ . For dimensional reasons  $x_3$  and  $y_3$  are primitive, so there exist  $u_4 \in H^4 P_2 X$  and  $v_4 \in H^4 P_2 E_0$  satisfying  $i_X^* u_4 = \sigma x_3, i_{E_0}^* v_4 = \sigma y_3, \sigma$  standing as usual for the suspension isomorphism.

The top cofibration of (\*\*) can be written as

$$P_2 E_0 \xrightarrow{k_{E_0}} \Sigma E_0 \wedge \Sigma E_0 \xrightarrow{\Sigma^2 \mu} \Sigma^2 E_0 \tag{***}$$

with  $\mu$  the multiplication on  $E_0$ . Recall from [2] that  $\Sigma^2 \mu$  induces the reduced coproduct in cohomology. The only possible non-trivial reduced coproduct for  $a \in H^6 E_0$  is  $y_3 \otimes y_3$  so  $(\Sigma^2 \mu)^* \sigma^2 a = \sigma y_3 \otimes \sigma y_3$  or  $(\Sigma^2 \mu)^* \sigma^2 a = 0$ .

Browder and Thomas [2] have studied the cohomology ring structure of the projective plane. One of their results is that, in our situation,  $(k_{E_0})^* \sigma y_3 \otimes \sigma y_3 = v_4^2$ . In

particular we deduce from the exactness in cohomology of (\*\*\*) that  $a$  is primitive if and only if  $v_4^2 \neq 0$  in  $H^8 P_2 E_0$ .

As  $X$  is a finite H-space we already know [2] that  $u_4^2 \neq 0$  in  $H^8 P_2 X$ . The problem is that  $E_0$  is not a finite H-space and so  $v_4^2$  is not automatically non-trivial.

Using the exact sequence in cohomology induced from

$$P_2 E_0 \xrightarrow{p_2 p_0} P_2 X \xrightarrow{j} C$$

we therefore just need to prove that  $u_4^2$  is not in the image of  $j^*$  (recall that  $(P_2 p_0)^* u_4 = v_4$ ).

Under the identification  $H^n A \cong H^n \Sigma K_0$ ,  $n \leq 8$  the homomorphism  $h^*$  coincides with  $(\Sigma h_0)^*$  and  $j^*$  with  $(P_2 h_0)^*$  up to dimension 8.

By definition  $h_0^* i_5 = \text{Sq}^2 x_3 = x_5$ . So from commutativity of the diagram (\*\*) we get  $j^* \sigma i_5 = (P_2 h_0)^* \sigma i_5 = \text{Sq}^2 u_4$ , hence  $j^* \sigma \text{Sq}^2 i_5 = \text{Sq}^2 \text{Sq}^2 u_4 = 0$ . As  $\sigma \text{Sq}^2 i_5$  is the only non-trivial element of  $H^8 \Sigma K_0$ , we conclude that  $\text{Im}(P_2 h_0)^* = 0$  in dimension 8, which finishes the proof of the lemma.

**5. Proof of Proposition 1.1**

The proof of Proposition 1.1 is completely analogous to the one of Theorem 1.3. Let us just mention the 2 stage Postnikov tower needed:

$$\begin{array}{ccccc}
 \tilde{\alpha} & E_1 & & & \\
 \nearrow & \downarrow p_1 & & & \\
 S^7 & \xrightarrow{\tilde{\alpha}} E_0 & \xrightarrow{h_1} & K(\mathbb{F}_2; 14) & \\
 \searrow & \downarrow p_0 & & & \\
 \tilde{\alpha} & X & \xrightarrow{h_0} & K(\mathbb{Z}; 11). & 
 \end{array}$$

The map  $h_0$  is the classifying map for the class  $x_{11}$ ,  $h_1$  classifies the class  $a \in H^{14} E_0$  whose restriction to the fiber of  $p_0$ ,  $\Omega K(\mathbb{Z}; 11) = K(\mathbb{Z}; 10)$  is  $\text{Sq}^4 i_{10}$ . Again  $E_0$  and  $E_1$  are H-spaces, moreover the 14-skeleton of  $E_1$   $E_1^{(14)}$  satisfies  $E_1^{(14)} = S^7$ . As before we construct the following commutative diagram

$$\begin{array}{ccc}
 S^7 = S^{(7)} & \xrightarrow{\tilde{\alpha} \times \tilde{\alpha}} & E_1 \times E_1 \\
 \downarrow m & & \downarrow \tilde{\mu} \\
 S^7 = E_1^{(14)} & \longrightarrow & E_1.
 \end{array}$$

In this way  $S^7$  is endowed with a multiplication  $m$  such that  $\tilde{\alpha}$  is an H-map, so  $\alpha$  itself is an H-map.

REFERENCES

1. J. F. ADAMS, Vector fields on spheres, *Ann. of Math.* **75** (1962), 603–632.



2. W. BROWDER and E. THOMAS, On the projective plane of an H-space, *Illinois J. Math.* **7** (1963), 492–502.
3. W. G. DWYER and C. W. WILKERSON, A new finite loop space at the prime two, *J. Amer. Math. Soc.* **6** (1993), 37–64.
4. D. L. GONCALVES, Mod 2 homotopy associative H-spaces (*Lecture Notes in Mathematics*, **657**, 1978), 198–216.
5. P. HILTON, *Homotopy theory and duality* (Gordon and Breach, 1965).
6. I. M. JAMES, Multiplications on spheres III, *Trans. Amer. Math. Soc.* **84** (1957), 545–558.
7. A. JEANNERET et J. P. LIN, Connexité des H-spaces finis, *C. R. Acad. Sci. Paris* **315** (1992), 829–831.
8. R. M. KANE, Implications in Morava K-theory, *Mem. Amer. Math. Soc.* **340** (1986).
9. J. P. LIN, Torsion in H-spaces II, *Ann. of Math.* **107** (1978), 41–88.
10. J. P. LIN,  $4k+1$  dimensional generators of finite H-spaces, manuscript.
11. N. MAHAMED, R. PICCININI and U. SUTER, *Some Applications of Topological K-Theory* (*North-Holland Mathematics Studies* **45**, 1980).
12. M. MIMURA and H. TODA, *Topology of Lie groups I and II* (Amer. Math. Soc. Translations of Mathematical Monographs, 1991).
13. S. J. SCHIFFMAN, A Samelson product and homotopy associativity, *Proc. Amer. Math. Soc.* **70** (1978), 189–195.

UNIVERSITÄT BERN  
MATHEMATISCHES INSTITUT  
SIDLERSTRASSE 5  
CH-3012 BERN