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SOME THEORIES WITH POSITIVE INDUCTION OF ORDINAL STRENGTH $\varphi\omega 0$

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Abstract. *This paper deals with: (i) the theory* $\mathsf{ID}_1^{\#}$ *which results from* $\widehat{\mathsf{ID}}_1$ *by restricting induction on* the natural numbers to formulas which are positive in the fixed point constants, (ii) the theory $\mathsf{BON}(\mu)$ plus various forms of positive induction, and (iii) a subtheory of Peano arithmetic with ordinals in which induction on the natural numbers is restricted to formulas which are Σ in the ordinals. We show that these systems have proof-theoretic strength $\varphi\omega 0$.

§1. Introduction. Systems of explicit mathematics were introduced in Feferman [7, 9]. In particular, two families of theories were presented there: (i) the theory T_0 and its subsystems, (ii) extensions of these theories by the non-constructive minimum operator. The original work on systems of explicit mathematics was mainly concerned with the analysis of classification existence axioms. It turned out only recently that already the applicative basis of these theories is of significant proof-theoretic interest.

Feferman and Jäger [12] is concerned with the basic applicative theory of operations and numbers BON and especially with the theory $\mathsf{BON}(\mu)$ which results from BON by adding a natural axiomatization of the unbounded minimum operator. A proof-theoretic analysis is provided there for BON and $\mathsf{BON}(\mu)$ plus a very weak form of induction on the natural numbers, called set induction as well as induction for arbitrary formulas. Natural intermediate forms of induction like operation induction, N induction and positive formula induction (for the exact definitions see below) have not been studied in [12] and will be treated now.

The attempt of providing a proof-theoretic analysis of these extensions of $\mathsf{BON}(\mu)$ by forms of positive induction was the starting point for the present paper. Very soon it became clear that the analysis of such systems is conceptually similar to that of the theory $\mathsf{ID}_1^{\#}$, which results from the well-known fixed-point theory $\widehat{\mathsf{ID}}_1$ by restricting induction on the natural numbers to formulas positive in the fixed point constants. Moreover, all these systems can be easily reconstructed within the framework of Peano arithmetic with ordinals (cf. Jäger [17]), namely the theory $\mathsf{PA}_\Omega^r + (\Sigma^\Omega\text{-}\mathsf{I}_\mathsf{N})$: there these forms of positive induction correspond to induction on the natural numbers for formulas which are Σ in the ordinals.

Many formal systems are introduced in this article, often only to round off our results or for technical intermediate steps. The main emphasis, however, is on

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$\mathsf{BON}(\mu)$ plus operation induction, N induction, and positive formula induction, on $\mathsf{ID}_1^\#$, and on $\mathsf{PA}_\Omega^r + (\Sigma^\Omega\text{-}\mathsf{I}_\mathsf{N})$.

The plan of this paper is as follows. In Section 2 we introduce the theory $\mathsf{ID}_1^\#$ and show that the second order system $(\Pi_1^0\text{-}\mathsf{CA})_{<\omega^\omega}$ can be embedded into $\mathsf{ID}_1^\#$. Section 3 is centered around the theory $\mathsf{BON}(\mu)$ plus various forms of positive induction and provides a wellordering proof for all ordinals less than $\varphi\omega 0$. The core of Section 4 is to show that the proof-theoretic ordinal of Peano arithmetic with ordinals and positive induction is less than or equal to $\varphi\omega 0$. This theory $\mathsf{PA}_\Omega^r + (\Sigma^\Omega\text{-}\mathsf{I}_\mathsf{N})$ extends the system PA_Ω^r of Jäger [17] by Σ^Ω induction on the natural numbers so that the theories $\mathsf{ID}_1^\#$ and the relevant applicative theories with positive induction can be easily embedded. Thus the circle is closed and all theories are shown to be of proof-theoretic strength $\varphi\omega 0$. The paper ends with some conclusions concerning related topics.

§2. Fixed point theories with positive induction. The famous theory $\widehat{\mathsf{ID}}_1$ is an extension of Peano arithmetic PA by new relation symbols and axioms which claim that every inductive operator form has a fixed point. It is described and studied in detail for example in Feferman [10]. Here we consider the subsystem $\mathsf{ID}_1^\#$ of $\widehat{\mathsf{ID}}_1$ in which induction on the natural numbers is restricted to formulas positive in the fixed point constants.

2.1. The theory $\mathsf{ID}_1^\#$. Let L be any of the usual first order languages with number variables $a, b, c, x, y, z, \ldots$ (possibly with subscripts), the constant 0 as well as function and relation symbols for all primitive recursive functions and relations. We assume further that L contains a unary relation symbol U which will have no specific interpretation and whose role will become clear by Definition 1. The notation $\vec{e}$ is a shorthand for a finite string $e_1, \ldots, e_n$ of expressions whose length will be specified by the context. The terms $r, s, t, \ldots$ and formulas $A, B, C, \ldots$ (both possibly with subscripts) are defined as usual.

If P is a new n-ary relation symbol, then $\mathrm{L}(P)$ is the extension of L by P. An $\mathrm{L}(P)$ formula is said to be *P-positive* if each occurrence of P in this formula is positive. We call P-positive formulas which contain at most $x_1, \ldots, x_n$ free n-ary *inductive operator forms*, and let $\mathscr{A}(P, x_1, \ldots, x_n)$ range over such forms. Now we extend L to a language L_{FP} by adding a new n-ary relation symbol $\mathscr{P}_\mathscr{A}$ for each n-ary inductive operator form $\mathscr{A}(P, \vec{x})$. An L_{FP} formula is called *positive*, if it is positive in all relation symbols $\mathscr{P}_\mathscr{A}$; it is called *negative*, if it has negative occurrences of relation symbols $\mathscr{P}_\mathscr{A}$ only.

The theories $\widehat{\mathsf{ID}}_1$ and $\mathsf{ID}_1^\#$ are formulated in L_{FP}. The axioms of $\widehat{\mathsf{ID}}_1$ comprise the axioms of Peano arithmetic PA with the scheme of complete induction on the natural numbers for all L_{FP} formulas plus the fixed point axioms

$$(\forall \vec{x})(\mathscr{A}(\mathscr{P}_\mathscr{A}, \vec{x}) \leftrightarrow \mathscr{P}_\mathscr{A}(\vec{x}))$$

for all relation symbols $\mathscr{P}_\mathscr{A}$. The theory $\mathsf{ID}_1^\#$ is the restriction of $\widehat{\mathsf{ID}}_1$ which results from $\widehat{\mathsf{ID}}_1$ if we permit induction on the natural numbers for *positive* L_{FP} formulas only. We will see below that this restriction of induction has a great effect.

The proof-theoretic strength of formal systems is generally measured in terms of their proof-theoretic ordinals. To introduce this notion we proceed as usual and set for all primitive recursive relations $\sqsubset$ and all formulas A:

$$\begin{aligned} Prog(\sqsubset, A) &:= (\forall x)((\forall y)(y \sqsubset x \to A(y)) \to A(x)), \\ TI(\sqsubset, A) &:= Prog(\sqsubset, A) \to (\forall x)A(x). \end{aligned}$$

DEFINITION 1. Let Th be a theory formulated in a language containing L.

1. We say that the ordinal α is *provable in* Th if there exists a primitive recursive wellordering $\sqsubset$ of order type α so that $\mathsf{Th} \vdash TI(\sqsubset, U)$.
2. The *proof-theoretic ordinal of* Th, denoted by $|\mathsf{Th}|$, is the least ordinal which is not provable in Th.

It is well-known (cf. e.g., Feferman [10]) that $|\widehat{\mathsf{ID}}_1| = \varphi\varepsilon_0 0$, and we will show that the proof-theoretic ordinal of $\mathsf{ID}_1^\#$ is $\varphi\omega 0$. Hence, $\mathsf{ID}_1^\#$ is significantly weaker than $\widehat{\mathsf{ID}}_1$.

2.2. Embedding $(\Pi_1^0\text{-}\mathsf{CA})_{<\omega^\omega}$ into $\mathsf{ID}_1^\#$. In this article we assume familiarity with the standard ordinal notation system $(T, \prec)$ for the ordinals less than Γ_0 which is based on the Veblen functions φ_α. From now on we write $\prec$ for the corresponding primitive recursive standard wellordering on the natural numbers of order type Γ_0. We assume that the field of $\prec$ is $\mathbb{N}$ and its least element is 0. Furthermore, if n is a natural number, then we write $\prec_n$ for the restriction of $\prec$ to the numbers $m \prec n$. The reader is referred to Schütte [22] for all details concerning these ordinals, ordinal notations and wellorderings. In order to simplify the notation, we sometimes identify natural numbers with their codes in the notation system, but it should always be clear from the context what we mean.

$|\mathsf{ID}_1^\#| \leq \varphi\omega 0$ will be proved in Section 4. Now we show that $\varphi\omega 0$ is a lower bound for the proof-theoretic ordinal of $\mathsf{ID}_1^\#$ by embedding the second order system $(\Pi_1^0\text{-}\mathsf{CA})_{<\omega^\omega}$, which has proof-theoretic ordinal $\varphi\omega 0$, into $\mathsf{ID}_1^\#$.

Let $\mathbb{L}$ be the second order language which extends L by set variables $X, Y, Z, \dots$ (possibly with subscripts) and the binary $\in$ relation. In the following we make use of standard terminology and notations of first and second order arithmetic: $\langle \dots \rangle$ is a standard primitive recursive function for forming n-tuples $\langle t_1, \dots, t_n \rangle$; Seq is the primitive recursive set of sequence numbers; $lh(t)$ denotes the length of (the sequence number coded by) t; $(t)_i$ is the ith component of (the sequence coded by) t if $i < lh(t)$, i.e., $t = \langle (t)_0, \dots, (t)_{lh(t)\dot{-}1} \rangle$ if t is a sequence number; $s \in (X)_t$ stands for $\langle s, t \rangle \in X$.

An $\mathbb{L}$ formula is called *arithmetic*, if it contains no bound set variables; it is called Δ_0, if it contains no bound set variables and, in addition, every number quantifier is bounded. A Σ_1^0 formula is an $\mathbb{L}$ formula of the form $(\exists x)A$ with A in Δ_0; a Π_1^0 formula is an $\mathbb{L}$ formula of the form $(\forall x)A$ with A in Δ_0. *Relative recursive comprehension* is the scheme

$$(\mathsf{RCA}) \qquad (\forall x)(A(x) \leftrightarrow B(x)) \to (\exists X)(\forall x)(x \in X \leftrightarrow A(x))$$

for all Σ_1^0 formulas $A(x)$ and Π_1^0 formulas $B(x)$. Now let $\mathcal{J}(X, x)$ be a complete Π_1^0 formula with at most X and x free. The jump hierarchy along $\prec_n$ starting with X is defined by the following transfinite recursion:

$$(Y)_0 = X \quad \text{and} \quad (Y)_i = \{\langle m, j \rangle : j \prec i \wedge \mathcal{J}((Y)_j, m)\}$$

for all $0 \prec i \prec n$, and we write $Hier(X, Y, n)$ for the arithmetic formula which formalizes this definition.

If α is an ordinal less than Γ_0, then we write $(\Pi^0_1\text{-CA})_\alpha$ for the system of second order arithmetic which extends Peano arithmetic PA by relative recursive comprehension (RCA) plus the additional axioms

$$(\forall X)(\exists Y)Hier(X, Y, n) \quad \text{and} \quad TI(\prec_n, A)$$

for all $\mathbb{L}$ formulas $A(x)$ where n is chosen so that the order type of $\prec_n$ is α. The union of the theories $(\Pi^0_1\text{-CA})_\beta$ with $\beta < \alpha$ is called $(\Pi^0_1\text{-CA})_{<\alpha}$.

In the sequel we give an embedding of the system $(\Pi^0_1\text{-CA})_{<\omega^\omega}$ into $\mathsf{ID}^\#_1$. In particular, we show that L theorems of $(\Pi^0_1\text{-CA})_{<\omega^\omega}$ carry over to L theorems of our theory $\mathsf{ID}^\#_1$.

As a first observation we need the fact that $\mathsf{ID}^\#_1$ proves transfinite induction up to each $\alpha < \omega^\omega$ with respect to *positive* $\mathrm{L_{FP}}$ formulas. Although the proof of this fact is elementary, we give it in full length here; an adaptation of this argument will be used in the wellordering proof in Section 3.3, where things will be much more delicate. We adopt the standard notation

$$TI(\prec, s, A) := Prog(\prec, A) \rightarrow (\forall x \prec s)A(x),$$

and in the sequel we often write $Prog(A)$ and $TI(s, A)$ instead of $Prog(\prec, A)$ and $TI(\prec, s, A)$.

LEMMA 2. *We have for all $k < \omega$ and every positive $\mathrm{L_{FP}}$ formula A:*

$$\mathsf{ID}^\#_1 \vdash TI(\omega^k, A).$$

PROOF. We prove the claim by induction on k. The case $k = 0$ is trivial. For the induction step assume that the assertion is true for some $k < \omega$, and choose a positive $\mathrm{L_{FP}}$ formula $A(x)$. Let us work informally in $\mathsf{ID}^\#_1$ and show

$$B(y) := (\forall x \prec \omega^k \cdot y)A(x) \tag{1}$$

by induction on y, assuming $Prog(A)$. This will immediately yield the induction step. $B(0)$ is trivially satisfied. So assume $B(y)$ and show $B(y + 1)$. First, one easily verifies

$$(\forall a)[(\forall b \prec a)(\forall x \prec \omega^k \cdot y + b)A(x) \rightarrow (\forall x \prec \omega^k \cdot y + a)A(x)] \tag{2}$$

by making use of the assumptions $B(y)$ and $Prog(A)$. Furthermore, by applying the (meta) induction hypothesis to (2) we obtain

$$(\forall a \prec \omega^k)(\forall x \prec \omega^k \cdot y + a)A(x). \tag{3}$$

From (3) and $Prog(A)$ we can conclude

$$(\forall a \prec \omega^k)A(\omega^k \cdot y + a), \tag{4}$$

which together with $B(y)$ yields $B(y + 1)$ as desired. This finishes our proof. ⊣

Using a standard argument (cf. e.g., Sieg [23, Proposition 3.1]) it follows that $\mathsf{ID}^\#_1$ proves induction on the natural numbers for *negative* $\mathrm{L_{FP}}$ formulas, too. Hence, the above proof yields the following corollary.

COROLLARY 3. *We have for every $k < \omega$ and every negative L_{FP} formula A:*

$$\mathsf{ID}_1^\# \vdash TI(\omega^k, A).$$

The main idea for our embedding is to build the jump hierarchy along $\prec$ starting with the empty set by means of a fixed point $\mathscr{P}_{\mathscr{A}}$ of a certain inductive operator form $\mathscr{A}(P, x, y, z)$ to be described below. The elements of the fixed point $\mathscr{P}_{\mathscr{A}}$ will be triples (a, i, x), where a is a code for an ordinal in T and i equals 0 or 1, depending on whether x belongs to the ath stage of the jump hierarchy.

In the following let $\mathscr{J}^+(X, Y)$ and $\mathscr{J}^-(X, Y)$ be $\mathbb{L}$ formulas which are positive in X and Y, so that $\mathscr{J}(X)$ is logically equivalent to $\mathscr{J}^+(X, \neg X)$ and $\neg\mathscr{J}(X)$ is logically equivalent to $\mathscr{J}^-(X, \neg X)$. Here $\mathscr{J}^+(X, \neg X)$ is the formula $\mathscr{J}^+(X, Y)$, where each atom $t \in Y$ is replaced by $\neg(t \in X)$. The formula $\mathscr{J}^-(X, \neg X)$ is defined analogously. Furthermore, if P is a ternary relation symbol, then we write $P_{r,s}(t)$ for $P(r, s, t)$.

The ternary inductive operator form $\mathscr{A}(P, x, y, z)$ is defined to be the disjunction of the following three formulas:

(1) $x = 0 \wedge y = 1 \wedge z = z$,
(2) $0 \prec x \wedge y = 0 \wedge Seq_2(z) \wedge (z)_1 \prec x \wedge \mathscr{J}^+(P_{(z)_1,0}, P_{(z)_1,1}, (z)_0)$,
(3) $0 \prec x \wedge y = 1 \wedge [\neg Seq_2(z) \vee x \preceq (z)_1 \vee \mathscr{J}^-(P_{(z)_1,0}, P_{(z)_1,1}, (z)_0)]$.

From the fixed point axioms alone we are not able to prove that the membership and non-membership relation defined above are *complementary*, i.e., that we have

$$(*) \qquad (\forall x)(\mathscr{P}_{\mathscr{A}}(a, 0, x) \leftrightarrow \neg\mathscr{P}_{\mathscr{A}}(a, 1, x))$$

for all sets (coded by) a. However, observe that $(*)$ is equivalent to *completeness* $Comp(a)$ and *consistency* $Cons(a)$ of the membership and non-membership relation, where one sets

$$\begin{aligned} Comp(a) &:= (\forall x)(\mathscr{P}_{\mathscr{A}}(a, 0, x) \vee \mathscr{P}_{\mathscr{A}}(a, 1, x)), \\ Cons(a) &:= (\forall x)(\neg\mathscr{P}_{\mathscr{A}}(a, 0, x) \vee \neg\mathscr{P}_{\mathscr{A}}(a, 1, x)). \end{aligned}$$

Obviously, $Comp(a)$ is a positive L_{FP} formula and $Cons(a)$ a negative L_{FP} formula. The idea is to prove $Comp(a)$ and $Cons(a)$ separately by transfinite induction up to ω^k, which is available in $\mathsf{ID}_1^\#$ according to our previous discussion.

LEMMA 4. *We have for all $k < \omega$:*
1. $\mathsf{ID}_1^\# \vdash (\forall a \prec \omega^k) Comp(a)$.
2. $\mathsf{ID}_1^\# \vdash (\forall a \prec \omega^k) Cons(a)$.

PROOF. One verifies in a straightforward manner that

$$\mathsf{ID}_1^\# \vdash Prog(Comp) \quad \text{and} \quad \mathsf{ID}_1^\# \vdash Prog(Cons),$$

where essential use is made of the fixed point axioms for $\mathscr{P}_{\mathscr{A}}$, which are available in $\mathsf{ID}_1^\#$. Then the claim follows from Lemma 2 and its corollary, respectively. ⊣

In order to increase readability we write $\mathscr{H}_a(x)$ instead of $\mathscr{P}_{\mathscr{A}}(a, 0, x)$. According to the previous lemma, $(\mathscr{H}_a)_{a \prec \omega^k}$ is a well-defined hierarchy of sets for each $k < \omega$ in the sense that $\neg\mathscr{H}_a(x)$ is equivalent to $\mathscr{P}_{\mathscr{A}}(a, 1, x)$, provably in $\mathsf{ID}_1^\#$.

Now we are ready to give the embedding of $(\Pi^0_1\text{-CA})_{<\omega^\omega}$ into $\mathsf{ID}^{\#}_1$. More precisely, we establish an interpretation of $(\Pi^0_1\text{-CA})_{\omega^k}$ into $\mathsf{ID}^{\#}_1$ for each $k < \omega$. Therefore, let us fix some $k < \omega$. We can now give a translation of $\mathbb{L}$ by interpreting the set variables as (codes of) the sets recursive in $\mathcal{H}_a$ for some $a \prec \omega^{k+1}$, and leaving the first order part of $\mathbb{L}$ unchanged. More formally, a set is a pair $\langle a, e\rangle$, where $a \prec \omega^{k+1}$ and e is the index of a set which is recursive in $\mathcal{H}_a$. Let us denote this translation (depending on k) by $(\cdot)^\circ$.

REMARK 5. The translation A° of an $\mathbb{L}$ formula A is equivalent to a *positive* L_{FP} formula B, provably in $\mathsf{ID}^{\#}_1$. This is readily seen by making use of the complement property (*).

THEOREM 6. *We have for all* $\mathbb{L}$ *sentences* A*:*

$$(\Pi^0_1\text{-CA})_{\omega^k} \vdash A \quad \Rightarrow \quad \mathsf{ID}^{\#}_1 \vdash A^\circ.$$

PROOF (SKETCH). Let us first consider the axiom $(\forall X)(\exists Y)Hier(X, Y, \omega^k)$ together with relative recursive comprehension (RCA). Assume that x codes a set according to the translation $(\cdot)^\circ$, i.e., x is a pair $\langle a, e\rangle$, where $a \prec \omega^{k+1}$ and e is an index of a set which is recursive in $\mathcal{H}_a$. By formalized recursion theory and transfinite induction up to ω^k we can find a set $y = \langle b, f\rangle$ so that $Hier(x, y, \omega^k)$ holds, where $b = a + \omega^k$ and f is the index of a set which is recursive in $\mathcal{H}_b$. Since ω^{k+1} is an additive principal number, we have $b \prec \omega^{k+1}$ as desired. The verification of (RCA) is trivial due to the choice of our interpretation. Furthermore, if B is an $\mathbb{L}$ formula, then $TI(\omega^k, B)^\circ$ is provable in $\mathsf{ID}^{\#}_1$ by Lemma 2 and Remark 5. This finishes the interpretation of $(\Pi^0_1\text{-CA})_{\omega^k}$ into $\mathsf{ID}^{\#}_1$. ⊣

Observe that a $(\Pi^0_1\text{-CA})_{<\omega^\omega}$ proof is in fact already a $(\Pi^0_1\text{-CA})_{\omega^k}$ proof for some $k < \omega$. In addition, the translation $(\cdot)^\circ$ preserves L formulas. Hence, we have established the following corollary.

COROLLARY 7. *We have for all* L *sentences* A*:*

$$(\Pi^0_1\text{-CA})_{<\omega^\omega} \vdash A \quad \Rightarrow \quad \mathsf{ID}^{\#}_1 \vdash A.$$

By methods of Schütte [22] it is well-known that $|(\Pi^0_1\text{-CA})_{<\omega^\omega}| = \varphi\omega 0$. This yields the following ordinal-theoretic lower bound for $\mathsf{ID}^{\#}_1$.

COROLLARY 8. $\varphi\omega 0 \leq |\mathsf{ID}^{\#}_1|$.

Let us finish this section by mentioning that it is also possible to provide a direct wellordering proof up to each ordinal less than $\varphi\omega 0$ within $\mathsf{ID}^{\#}_1$. For a similar argument the reader is referred to Jäger and Strahm [18].

§3. $\mathsf{BON}(\mu)$ plus positive induction.

In this section we introduce other very natural formal systems of ordinal strength $\varphi\omega 0$, namely extensions of the basic theory of operations and numbers with non-constructive μ operator $\mathsf{BON}(\mu)$ by various forms of positive induction on the natural numbers.

Applicative theories of operations and numbers were introduced in Feferman [7, 9] as a basis for his systems of explicit mathematics, and they have become relevant as an elementary framework for many activities in (the foundations of) mathematics and computer science. Recently, theories with self-application have been proof-theoretically analyzed in the context of the non-constructive minimum operator (cf. Feferman and Jäger [12, 13], Jäger and Strahm [19], Glaß and Strahm [15]), and the following considerations can be viewed as a continuation of that work.

In the first paragraph of this section we describe the formal framework for applicative theories with the non-constructive minimum operator and the relevant induction principles. In the second paragraph we briefly mention some known proof-theoretic equivalences, and in the third paragraph we show that the theory $\mathsf{BON}(\mu)$ plus so-called N induction proves transfinite induction up to each ordinal less than $\varphi\omega 0$.

3.1. The formal framework for applicative theories. In this paragraph we introduce the basic theory BON of operations and numbers together with various forms of complete induction on the natural numbers, and we give the axioms of the non-constructive minimum operator.

The language L_p of the basic theory of partial operations and numbers is a first order language of partial terms with *individual variables* $a, b, c, x, y, z, f, g, h, \ldots$ (possibly with subscripts). In addition, L_p includes *individual constants* k, s (combinators), $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1$ (pairing and unpairing), 0 (zero), s_N (numerical successor), p_N (numerical predecessor), d_N (definition by numerical cases), r_N (primitive recursion) and $\boldsymbol{\mu}$ (unbounded minimum operator). L_p has a binary function symbol $\cdot$ for (partial) term application, unary relation symbols $\downarrow$ (defined) and N (natural numbers) as well as a binary relation symbol $=$ (equality).

In order to use the same definition of proof-theoretic ordinal as in the previous section (cf. Definition 1), we also assume that the language L_p contains a unary relation symbol U. The operation constant c_U acts as a characteristic function of U. Of course, all meaningful applicative theories formulated in the language L_p are conservative over the corresponding theories without U and c_U.

The *individual terms* $(r, s, t, r_1, s_1, t_1, \ldots)$ of L_p are inductively defined as follows:

1. The individual variables and individual constants are individual terms.
2. If s and t are individual terms, then so also is $(s \cdot t)$.

In the following we write (st) or just st instead of $(s \cdot t)$, and we adopt the convention of association to the left, i.e., $s_1 s_2 \ldots s_n$ stands for $(\ldots(s_1 s_2)\ldots s_n)$. We also write (t_1, t_2) for $\mathsf{p} t_1 t_2$ and $(t_1, t_2, \ldots, t_n)$ for $(t_1, (t_2, \ldots, t_n))$. Further we put $t' := \mathsf{s}_N t$ and $1 := 0'$.

The *formulas* $(A, B, C, A_1, B_1, C_1, \ldots)$ of L_p are inductively defined as follows:

1. Each atomic formula $N(t)$, $U(t)$, $t\downarrow$ and $(s = t)$ is a formula.
2. If A and B are formulas, then so also are $\neg A$, $(A \vee B)$, $(A \wedge B)$ and $(A \rightarrow B)$.
3. If A is a formula, then so also are $(\exists x)A$ and $(\forall x)A$.

Our applicative theories are based on *partial* term application. Hence, it is not guaranteed that terms have a value, and $t\downarrow$ is read as 't is defined' or 't has a value.' The *partial equality relation* $\simeq$ is introduced by

$$s \simeq t \ := \ (s\downarrow \vee\, t\downarrow) \rightarrow (s = t).$$

In addition, we write $(s \neq t)$ for $(s\downarrow \wedge\, t\downarrow \wedge\, \neg(s = t))$. Finally, we use the following abbreviations concerning the predicate N:

$$
\begin{aligned}
t \in N &:= N(t), \\
(\exists x \in N)A &:= (\exists x)(x \in N \wedge A), \\
(\forall x \in N)A &:= (\forall x)(x \in N \rightarrow A), \\
(t \in N \rightarrow N) &:= (\forall x \in N)(tx \in N), \\
(t \in N^{m+1} \rightarrow N) &:= (\forall x \in N)(tx \in N^m \rightarrow N).
\end{aligned}
$$

The *positive* and *negative* formulas of L_p are given by the following simultaneous inductive definition:

Definition 9 (F^+ and F^- formulas).

1. Each atomic formula $N(t)$, $U(t)$, $t\downarrow$ and $(s = t)$ is an F^+ formula.
2. If A is an F^+ formula [F^- formula], then $\neg A$ is an F^- formula [F^+ formula].
3. If A and B are F^+ formulas [F^- formulas], then $(A \vee B)$ and $(A \wedge B)$ are F^+ formulas [F^- formulas].
4. If A is an F^- formula [F^+ formula] and B is an F^+ formula [F^- formula], then $(A \rightarrow B)$ is an F^+ formula [F^- formula].
5. If A is an F^+ formula [F^- formula], then $(\exists x \in N)A$ and $(\forall x \in N)A$ are F^+ formulas [F^- formulas].

The underlying logic of BON is the (classical) *logic of partial terms* due to Beeson [1]; it corresponds to E^+ logic with strictness and equality of Troelstra and Van Dalen [25]. The non-logical axioms of BON are divided into the following five groups.

I. Partial Combinatory Algebra.

(1) $\mathsf{k}xy = x$,
(2) $\mathsf{s}xy\downarrow \wedge \mathsf{s}xyz \simeq xz(yz)$.

II. Pairing and Projection.

(3) $\mathsf{p}_0(x, y) = x \wedge \mathsf{p}_1(x, y) = y$.

III. Natural Numbers.

(4) $0 \in N \wedge (\forall x \in N)(x' \in N)$,
(5) $(\forall x \in N)(x' \neq 0 \wedge \mathsf{p}_N(x') = x)$,
(6) $(\forall x \in N)(x \neq 0 \rightarrow \mathsf{p}_N x \in N \wedge (\mathsf{p}_N x)' = x)$.

IV. Characteristic Function of U.

(7) $(\forall x \in N)(\mathsf{c}_U x = 0 \vee \mathsf{c}_U x = 1)$,
(8) $(\forall x \in N)(U(x) \leftrightarrow \mathsf{c}_U x = 0)$.

V. Definition by Numerical Cases.

(9) $a \in N \wedge b \in N \wedge a = b \rightarrow \mathsf{d}_N xyab = x$,
(10) $a \in N \wedge b \in N \wedge a \neq b \rightarrow \mathsf{d}_N xyab = y$.

VI. Primitive Recursion on N.

(11) $(f \in N \rightarrow N) \wedge (g \in N^3 \rightarrow N) \rightarrow (\mathsf{r}_N fg \in N^2 \rightarrow N)$,
(12) $(f \in N \rightarrow N) \wedge (g \in N^3 \rightarrow N) \wedge x \in N \wedge y \in N \wedge h = \mathsf{r}_N fg \rightarrow hx0 = fx \wedge hx(y') = gxy(hxy)$.

As usual the axioms of a partial combinatory algebra allow one to define lambda abstraction. More precisely, for each L_p term t there exists an L_p term $(\lambda x.t)$ whose free variables are those of t, excluding x, so that

$$\mathsf{BON} \vdash (\lambda x.t)\downarrow \wedge (\lambda x.t)x \simeq t.$$

In addition, it is well-known that BON proves a recursion theorem. For proofs of these two important results the reader is referred to [1, 7].

Let us recall the definition of a *subset of* N from [8, 12]. Sets of natural numbers are represented via their characteristic functions which are total on N. Accordingly, we define

$$f \in P(N) := (\forall x \in N)(fx = 0 \vee fx = 1),$$

with the intention that an object x belongs to the set $f \in P(N)$ if and only if $(fx = 0)$.

In the following we are interested in five forms of complete induction on the natural numbers, namely set induction, operation induction, N induction, positive formula induction and full formula induction.

Set Induction on N $(\mathsf{S\text{-}I_N})$.

$$f \in P(N) \wedge f0 = 0 \wedge (\forall x \in N)(fx = 0 \rightarrow f(x') = 0) \rightarrow (\forall x \in N)(fx = 0).$$

Operation Induction on N $(\mathsf{O\text{-}I_N})$.

$$f0 = 0 \wedge (\forall x \in N)(fx = 0 \rightarrow f(x') = 0) \rightarrow (\forall x \in N)(fx = 0).$$

N Induction on N $(\mathsf{N\text{-}I_N})$.

$$f0 \in N \wedge (\forall x \in N)(fx \in N \rightarrow f(x') \in N) \rightarrow (\forall x \in N)(fx \in N).$$

Positive Formula Induction on N $(\mathsf{F^+\text{-}I_N})$. For all F^+ formulas $A(x)$ of L_p:

$$A(0) \wedge (\forall x \in N)(A(x) \rightarrow A(x')) \rightarrow (\forall x \in N)A(x).$$

Formula Induction on N $(\mathsf{F\text{-}I_N})$. For arbitrary formulas $B(x)$ of L_p:

$$B(0) \wedge (\forall x \in N)(B(x) \rightarrow B(x')) \rightarrow (\forall x \in N)B(x).$$

Remark 10. Sometimes it will be convenient to work with a slightly more general notion of set. According to this generalization, a set is not necessarily an element of $P(N)$ but an element of $(N \rightarrow N)$, and as above, an object x belongs to a set $(f \in N \rightarrow N)$ if and only if $(fx = 0)$. It is easily seen that these to notions of a set are equivalent. In particular, $\mathsf{BON} + (\mathsf{S\text{-}I_N})$ proves set induction for 'extended sets,'

$$(f \in N \rightarrow N) \wedge f0 = 0 \wedge (\forall x \in N)(fx = 0 \rightarrow f(x') = 0) \rightarrow (\forall x \in N)(fx = 0).$$

Therefore, we will tacitly use both $P(N)$ and $(N \rightarrow N)$ as our notion of set, whichever is more convenient.

Now we turn to the non-constructive minimum operator. We follow its axiomatization according to Jäger and Strahm [19], which is a strengthening of the formulation in Feferman [8] and Feferman and Jäger [12, 13].

THE UNBOUNDED MINIMUM OPERATOR

(μ.1) $(f \in N \to N) \leftrightarrow \boldsymbol{\mu} f \in N$,

(μ.2) $(f \in N \to N) \wedge (\exists x \in N)(fx = 0) \to f(\boldsymbol{\mu} f) = 0$.

REMARK 11. In Feferman [8] and Feferman and Jäger [12, 13] a weaker form $\boldsymbol{\mu}_w$ of the minimum operator is considered, where the axiom $(\mu_w.1)$ reads as

$$(f \in N \to N) \to \boldsymbol{\mu}_w f \in N,$$

and the second axiom $(\mu_w.2)$ for $\boldsymbol{\mu}_w$ is identical to $(\mu.2)$. The above formulation of the axiom $(\mu.1)$ is stronger than the axiom $(\mu_w.1)$ of [12, 13] in the following sense: $\boldsymbol{\mu}$ is not only a functional on $(N \to N)$ which assigns to each $(f \in N \to N)$ an $x \in N$ with $fx = 0$, if there is any such x, and any y in N otherwise, but $\boldsymbol{\mu}$ also has the property that $\boldsymbol{\mu} f \in N$ already implies that f is an operation from N to N, i.e., $(f \in N \to N)$. It is easy to see, however, that the proof-theoretic strength of the theories in [8, 12, 13] is not affected by moving from $\boldsymbol{\mu}_w$ to $\boldsymbol{\mu}$.

In the sequel we write $\mathsf{BON}(\mu)$ for $\mathsf{BON} + (\mu.1, \mu.2)$, and we will determine the proof-theoretic strength of $\mathsf{BON}(\mu)$ extended by the forms of induction mentioned above.

Finally, we will be interested in two possible strengthenings of the applicative axioms, namely totality and extensionality. The *totality axiom* (Tot) expresses that application is always total, i.e.

$$(\mathsf{Tot}) \qquad (\forall x)(\forall y)(xy\downarrow).$$

The *extensionality axiom* (Ext) claims that operations are extensional in the following sense:

$$(\mathsf{Ext}) \qquad (\forall x)(fx \simeq gx) \to (f = g).$$

This finishes the description of the formal framework for those applicative theories which will be studied below.

3.2. Some known proof-theoretic equivalences. In this paragraph we briefly address some known proof-theoretic equivalences concerning applicative theories with and without the non-constructive μ operator in the presence of various induction principles.

The proof-theoretic strength of all relevant theories without the operator μ is well-known, cf. e.g., Feferman and Jäger [12]. The corresponding reductions make use of formalized (ordinary) recursion theory.

PROPOSITION 12. We have the following proof-theoretic equivalences:

1. $\mathsf{BON} + (\mathsf{S\text{-}I_N}) \equiv \mathsf{BON} + (\mathsf{O\text{-}I_N}) \equiv \mathsf{BON} + (\mathsf{N\text{-}I_N}) \equiv \mathsf{PRA}$.
2. $\mathsf{BON} + (\mathsf{F^+\text{-}I_N}) \equiv \mathsf{BON} + (\mathsf{F\text{-}I_N}) \equiv \mathsf{PA}$.

Following Cantini [3] or Jäger and Strahm [19], the above equivalences still hold in the presence of totality (Tot) and extensionality (Ext), where formalized term model constructions serve to determine proof-theoretic upper bounds.

The proof-theoretic strength of $\mathsf{BON}(\mu)$ with set and formula induction is due to Feferman and Jäger [12]. Here essential use is made of so-called fixed-point theories with ordinals (cf. Section 4), which have been introduced in Jäger [17].

These theories turned out to be an adequate framework for formalized Π^1_1 recursion theory, which is used to interpret the μ operator.

PROPOSITION 13. We have the following proof-theoretic equivalences:
1. $\mathsf{BON}(\mu) + (\mathsf{S\text{-}I_N}) \equiv \mathsf{PA}$.
2. $\mathsf{BON}(\mu) + (\mathsf{F\text{-}I_N}) \equiv (\Pi^0_1\text{-}\mathsf{CA})_{<\varepsilon_0}$.

As for the case without the μ operator, these results can be strengthened to include (Tot) and (Ext). This is due to Jäger and Strahm [19], where formalized *infinitary* term models and Church Rosser properties yield the desired upper bounds, again making use of fixed point theories with ordinals.

Let us finish this paragraph by mentioning some crucial relationships between set induction $(\mathsf{S\text{-}I_N})$, operation induction $(\mathsf{O\text{-}I_N})$ and N induction $(\mathsf{N\text{-}I_N})$, which have been established in Kahle [20]. Observe that $(\mathsf{S\text{-}I_N})$ is trivially contained in $(\mathsf{O\text{-}I_N})$.

PROPOSITION 14. We have the following relationships:
1. $(\mathsf{N\text{-}I_N})$ implies $(\mathsf{S\text{-}I_N})$ over BON.
2. $(\mathsf{N\text{-}I_N})$ and $(\mathsf{O\text{-}I_N})$ are equivalent over $\mathsf{BON}(\mu)$.

For the second assertion of this proposition the presence of the *strong* μ operator is crucial. Furthermore, it is not yet known whether $(\mathsf{N\text{-}I_N})$ is equivalent to $(\mathsf{O\text{-}I_N})$ over BON or $\mathsf{BON}(\mu_w)$, although $\mathsf{BON}+(\mathsf{N\text{-}I_N})$ and $\mathsf{BON}+(\mathsf{O\text{-}I_N})$ are *proof-theoretically* equivalent according to Proposition 12.

In the next paragraph we determine $\varphi\omega 0$ as a lower bound of $\mathsf{BON}(\mu) + (\mathsf{N\text{-}I_N})$. According to the proposition above, this will also yield a lower bound for the system $\mathsf{BON}(\mu) + (\mathsf{O\text{-}I_N})$. Both theories are contained in $\mathsf{BON}(\mu) + (\mathsf{F^+\text{-}I_N})$, and we will show in Section 4 of this paper that $\mathsf{BON}(\mu) + (\mathsf{F^+\text{-}I_N})$ does not go beyond $\varphi\omega 0$.

3.3. The wellordering proof for $\mathsf{BON}(\mu) + (\mathsf{N\text{-}I_N})$. In the sequel we show that $\mathsf{BON}(\mu)+(\mathsf{N\text{-}I_N})$ proves transfinite induction along each initial segment of $\varphi\omega 0$. We are implicitly working with a translation of $\mathbb{L}$ into L_p, where the number variables of $\mathbb{L}$ are interpreted as ranging over N, and the set variables as ranging over $(N \to N)$. Hence, an atomic formula $(x \in Y)$ of $\mathbb{L}$ is translated into $yx = 0$, where x and y are the variables of L_p which are associated to the variables x and Y of $\mathbb{L}$, respectively. Furthermore, using the recursion operator r_N, each primitive recursive function can be represented in BON by an individual term of L_p. Summarizing, the translation $(\cdot)^N$ from $\mathbb{L}$ into L_p is such that

$$\begin{aligned} ((\exists x)A(x))^N &= (\exists x \in N)A^N(x), \\ ((\exists X)A(X))^N &= (\exists x \in N \to N)A^N(x), \end{aligned}$$

and similarly for universal quantifiers. In order to simplify notation, we identify individual terms and formulas of $\mathbb{L}$ and their translations into L_p, when there is no danger of confusion. In addition, we freely use symbols for primitive recursive relations, which are introduced as usual via their characteristic functions.

This is the right place to mention a crucial application of the unbounded μ operator, namely elimination of number quantifiers (cf. [12]).

PROPOSITION 15. For every arithmetic $\mathbb{L}$ formula $A(\vec{X}, \vec{y})$ with at most $\vec{X}, \vec{y}$ free there exists an individual term t_A of L_p so that
1. $\mathsf{BON}(\mu) \vdash (\forall \vec{x} \in N \to N)(\forall \vec{y} \in N)(t_A\vec{x}\vec{y} \in N)$,

2. $\mathsf{BON}(\mu) \vdash (\forall \vec{x} \in N \to N)(\forall \vec{y} \in N)(A^N(\vec{x}, \vec{y}) \leftrightarrow t_A \vec{x}\vec{y} = 0)$.

Recall that $\prec$ is a primitive recursive standard wellordering of ordertype Γ_0 with field $\mathbb{N}$ and least element 0. We will show that

$$\mathsf{BON}(\mu) + (\mathsf{N\text{-}I_N}) \vdash (\forall f \in N \to N) TI(a, f)$$

for each $a \prec \varphi\omega 0$, where $TI(a, f)$ abbreviates $TI(a, fx = 0)$.

In order to make the wellordering proof work, we need a certain amount of transfinite induction with respect to formulas of the form $tx \in N$. More precisely, we have to extend N induction $(\mathsf{N\text{-}I_N})$ to N transfinite induction up to ω^k for each $k < \omega$. This can be established in the very same way as in Lemma 2, however, there is one point where attention is needed: In the proof of Lemma 2 we used the fact that the class of positive $\mathrm{L_{FP}}$ formulas is closed under universal quantifiers of the form $(\forall x \prec s)$, a closure property which is not obvious for the formulas $tx \in N$. Observe that in the proof of the following lemma we make essential use of the (strong) non-constructive μ operator for the first time.

Lemma 16. *For every* L_p *term* s *there exists an* L_p *term* t *so that*

$$\mathsf{BON}(\mu) \vdash (\forall x \in N)((\forall y \prec x) sy \in N \leftrightarrow tx \in N).$$

Proof. Let r be an L_p term for the characteristic function of $\prec$. For a given L_p term s choose the term t' of L_p given by

$$t' := \lambda y.\mathsf{d}_N(sy)0(ryx)0. \tag{1}$$

Then it is straightforward to verify that

$$\mathsf{BON} \vdash x \in N \to [(\forall y \in N)(ryx = 0 \to sy \in N) \leftrightarrow (t' \in N \to N)]. \tag{2}$$

Using the axiom $(\mu.1)$ for the non-constructive μ operator we have

$$\mathsf{BON}(\mu) \vdash (t' \in N \to N) \leftrightarrow \boldsymbol{\mu} t' \in N. \tag{3}$$

Hence, we can take $t := \lambda x.\boldsymbol{\mu} t'$ and read off our assertion from (2) and (3). ⊣

Now we can copy the proof of Lemma 2 to get the following important lemma.

Lemma 17. *We have for all* $k < \omega$:

$$\mathsf{BON}(\mu) + (\mathsf{N\text{-}I_N}) \vdash TI(\omega^k, fx \in N).$$

On the other hand, we already know that $\mathsf{BON}(\mu) + (\mathsf{N\text{-}I_N})$ proves transfinite induction up to each ordinal less than ε_0 with respect to *sets*: According to Proposition 14, $\mathsf{BON}(\mu) + (\mathsf{N\text{-}I_N})$ proves set induction $(\mathsf{S\text{-}I_N})$, and $\mathsf{BON}(\mu) + (\mathsf{S\text{-}I_N})$ in turn contains PA via the embedding described at the beginning of this paragraph (cf. also Proposition 13).

In the sequel we need primitive recursive auxiliary functions p and e on our ordinal notations, which satisfy

- $p(0) = e(0) = 0$; $p(\omega^a) = 0$ and $e(\omega^a) = a$;
- if $a = \omega^{a_1} + \cdots + \omega^{a_n}$ for more than one summand so that $a_n \preceq \cdots \preceq a_1$, then $p(a) = \omega^{a_1} + \cdots + \omega^{a_{n-1}}$ and $e(a) = a_n$.

In addition, let us define some sort of jump operator J, which is given by the following arithmetic definition:

$$J(X,a) := (\forall y)((\forall x \prec y)(x \in X) \rightarrow (\forall x \prec y+a)(x \in X)).$$

Let $(f \in N \rightarrow N)$ be a set. In order to prove $TI(a,f)$ for each $a \prec \varphi\omega 0$, we build up a hierarchy of sets $(H_b)_{b \prec \omega^k}$ for each $k < \omega$. The definition of the hierarchy corresponds to the formulas $\mathscr{R}(P,Q,t)$ of Schütte [22, p. 184ff]. More precisely,

$$\begin{aligned} H_0 &= f, \\ H_a &= \{y : (\forall z)\,(p(a) \preceq z \prec a \rightarrow J(H_z, \varphi(e(a),y)))\}, \quad (0 \prec a). \end{aligned}$$

In order to formalize $(H_b)_{b \prec \omega^k}$ in $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N})$, we need some preliminary considerations. The arithmetic $\mathbb{L}$ formula $A(X,a,y)$ is given by

$$A(X,a,y) := (\forall z)\,(p(a) \preceq z \prec a \rightarrow J((X)_z, \varphi(e(a),y))).$$

According to Proposition 15, there exists an L_p term t_A so that $\mathsf{BON}(\mu)$ proves:

$$(\forall x \in N \rightarrow N)(\forall a,y \in N)(t_A xay \in N),$$
$$(\forall x \in N \rightarrow N)(\forall a,y \in N)(A^N(x,a,y) \leftrightarrow t_A xay = 0).$$

An application of the same proposition provides us with a term s so that $\mathsf{BON}(\mu)$ proves:

$$(\forall x,y \in N)(sxy \in N),$$
$$(\forall x,y \in N)((x = \langle (x)_0,(x)_1\rangle \wedge (x)_1 \prec y) \leftrightarrow sxy = 0).$$

Finally, the operation g is given by

$$g := \lambda xyz.(\mathsf{d}_N(x(z)_1(z)_0)1(szy)0).$$

If x is assumed to be an operation which enumerates the sets xb, then gxa is a characteristic function of the disjoint union of the sets $(xb)_{b \prec a}$.

We have prepared the ground in order to introduce an operation h so that hfa represents the ath level of the H hierarchy with initial set f. It is given by the recursion theorem to satisfy

$$hfay \simeq \begin{cases} fy, & \text{if } a = 0, \\ t_A(g(hf)a)ay, & \text{otherwise.} \end{cases}$$

So far we do not know that hfa represents a set in $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N})$. This is the content of the following crucial lemma. Observe that the presence of the *strong* μ operator is again essential.

LEMMA 18. *We have for all $k < \omega$:*

$$\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N}) \vdash (\forall f \in N \rightarrow N)(\forall a \prec \omega^k)(hfa \in N \rightarrow N).$$

PROOF. Let us first fix a $k < \omega$ and an $(f \in N \to N)$. We work informally in $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N})$ and show that

(1) $$Prog(ra \in N),$$

where r is defined to be the term $\lambda a.\boldsymbol{\mu}(hfa)$. Then our assertion immediateley follows from (1), Lemma 17 and an application of the axiom $(\mu.1)$. In order to prove (1) let us assume

(2) $$(\forall b \prec a)(rb \in N),$$

i.e., $(\forall b \prec a)(\boldsymbol{\mu}(hfb) \in N)$. The equivalence $(\mu.1)$ yields

(3) $$(\forall b \prec a)(hfb \in N \to N).$$

It is our aim to show $(hfa \in N \to N)$, which by $(\mu.1)$ yields $ra \in N$ as desired. If $a = 0$, then $(hf0 \in N \to N)$ holds since it is $(f \in N \to N)$ by assumption. Otherwise, we have to show $(t_A(g(hf)a)a \in N \to N)$. But this is immediate, since (3) implies $(g(hf)a \in N \to N)$, and t_A maps sets and numbers into sets according to our discussion above. This finishes the proof of (1), and hence our assertion follows as shown. ⊣

We have established the existence of the hierarchy $(H_a)_{a \prec \omega^k}$ as a hierarchy of *sets* in $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N})$ for each $k < \omega$, and its defining properties can be proved there.

The next lemma is essential in the wellordering proof for $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N})$. It corresponds to Lemma 9 of Schütte [22], and its proof is very similar to the proof of Lemma 9. A careful but straightforward formalization of that proof only uses set induction $(\mathsf{S}\text{-}\mathsf{I}_\mathsf{N})$, which is available in $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N})$ by Proposition 14. For the details the reader is referred to [22].

LEMMA 19. *We have that* $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N})$ *proves for all* $k < \omega$*:*

$$(f \in N \to N) \wedge 0 \prec a \prec \omega^k \wedge (\forall b \prec a)Prog(hfb) \to Prog(hfa).$$

We are now able to show that $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N})$ proves transfinite induction up to $\varphi k0$ for each $k < \omega$. This will immediately yield the desired lower bound.

THEOREM 20. *We have for all* $k < \omega$*:*

$$\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N}) \vdash (\forall f \in N \to N)TI(\varphi k0, f).$$

PROOF. In the following we work informally in the theory $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I}_\mathsf{N})$. Let us choose $k < \omega$ and an arbitrary $(f \in N \to N)$. By Lemma 18 we have a hierarchy of sets $(hfa)_{a \prec w^{k+1}}$ with initial set $hf0 = f$. Hence, we trivially have

(1) $$Prog(f) \to Prog(hf0).$$

A combination of (1) and the previous lemma yields

(2) $$Prog(f) \wedge a \prec \omega^{k+1} \wedge (\forall b \prec a)Prog(hfb) \to Prog(hfa).$$

If we abbreviate $B(a) := a \prec \omega^{k+1} \to Prog(hfa)$, then (2) amounts to

(3) $$Prog(f) \to Prog(B).$$

Furthermore, it is easily seen that B can be represented as a set t_B, provably in $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I_N})$, for example, choose

$$t_B := \lambda a.\mathsf{d}_N(prog(hfa))1(ra\omega^{k+1})0, \tag{4}$$

where *prog* is the set corresponding to *Prog* according to Proposition 15, and r represents the characteristic function of $\prec$. Therefore, we can conclude from (3) and set transfinite induction up to ω^{k+1} that

$$Prog(f) \rightarrow Prog(hf\omega^k). \tag{5}$$

In addition, we trivially have

$$Prog(hf\omega^k) \rightarrow (hf\omega^k 0 = 0). \tag{6}$$

Since $p(\omega^k) = 0$ and $e(\omega^k) = k$ we get by the definition of $hf\omega^k$ that

$$(hf\omega^k 0 = 0) \rightarrow J(f, \varphi k0). \tag{7}$$

Furthermore, it is immediate from the definition of J that

$$J(f, \varphi k0) \rightarrow (\forall x \prec \varphi k0)(fx = 0). \tag{8}$$

If we combine (5)–(8) we obtain $TI(f, \varphi k0)$ as desired. ⊣

By replacing $(f \in N \rightarrow N)$ by the characteristic function c_U of U we have shown that $\varphi\omega0$ is a lower bound for the proof-theoretic ordinal of $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I_N})$ in the sense of Definition 1. Together with Proposition 14 we have established the following corollary.

Corollary 21. $\varphi\omega0 \leq |\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I_N})| = |\mathsf{BON}(\mu) + (\mathsf{O}\text{-}\mathsf{I_N})|$.

Instead of giving a wellordering proof for $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I_N})$, it would also have been possible to provide a direct embedding of the second order system $(\Pi^0_1\text{-}\mathsf{CA})_{<\omega^\omega}$ into $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I_N})$. This embedding is similar to the one of $(\Pi^0_1\text{-}\mathsf{CA})_{<\omega^\omega}$ into $\mathsf{ID}_1^\#$ of Section 2.2, formalized in the framework of $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I_N})$ and by making use of the same techniques for building hierarchies of sets as in the wellordering proof above.

§4. Peano arithmetic with ordinals and positive induction. Fixed point theories in Peano arithmetic with ordinals were first introduced in Jäger [17] and then used in Feferman and Jäger [12] in order to provide upper proof-theoretic bounds for several first order systems of explicit mathematics with non-constructive μ operator. The weakest theory of PA plus ordinals considered in [17] is the theory PA^r_Ω which is a conservative extension of PA.

Now we study the effect of adding to PA^r_Ω a form of positive induction on the natural numbers, namely induction for Σ^Ω formulas. We will show that $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N})$ contains $\mathsf{ID}_1^\#$ and $\mathsf{BON}(\mu) + (\mathsf{F}^+\text{-}\mathsf{I_N})$ and establish that its proof-theoretic strength is bounded by $\varphi\omega0$.

4.1. The theory $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N})$. The theory $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N})$ is formulated in the language L_Ω which extends L by adding a new sort of ordinal variables $\sigma, \tau, \eta, \xi, \dots$ (possibly with subscripts), a new binary relation symbol $<$ for the less relation on the ordinals[1] and an $(n+1)$-ary relation symbol $P_{\mathscr{A}}$ for each inductive operator form $\mathscr{A}(P, x_1, \dots, x_n)$.

The number terms of L_Ω are the number terms of L; the ordinal terms of L_Ω are the ordinal variables. The formulas $A, B, C, \dots$ (possibly with subscripts) of L_Ω are inductively generated as follows:

1. If R is an n-ary relation symbol of L, then $R(s_1, \dots, s_n)$ is an (atomic) formula of L_Ω.
2. $(\sigma < \tau)$, $(\sigma = \tau)$ and $P_{\mathscr{A}}(\sigma, \vec{s})$ are (atomic) formulas of L_Ω. We write $P^\sigma_{\mathscr{A}}(\vec{s})$ for $P_{\mathscr{A}}(\sigma, \vec{s})$.
3. If A and B are formulas of L_Ω, then $\neg A$, $(A \vee B)$, $(A \wedge B)$ and $(A \to B)$ are formulas of L_Ω.
4. If A is a formula of L_Ω, then $(\exists x)A$ and $(\forall x)A$ are formulas of L_Ω.
5. If A is a formula of L_Ω, then $(\exists \xi)A$, $(\forall \xi)A$, $(\exists \xi < \sigma)A$ and $(\forall \xi < \sigma)A$ are formulas of L_Ω.

For every L_Ω formula A we write A^σ to denote the L_Ω formula which is obtained by replacing all unbounded quantifiers $(Q\xi)$ in A by $(Q\xi < \sigma)$. Additional abbreviations are:

$$P^{<\sigma}_{\mathscr{A}}(\vec{s}) := (\exists \xi < \sigma) P^\xi_{\mathscr{A}}(\vec{s}) \quad \text{and} \quad P_{\mathscr{A}}(\vec{s}) := (\exists \xi) P^\xi_{\mathscr{A}}(\vec{s}).$$

Finally we introduce subclasses of L_Ω formulas which will be needed for formulating the axioms of the theory $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N})$.

DEFINITION 22 (Δ^Ω_0 formulas). The Δ^Ω_0 formulas of L_Ω are inductively defined as follows:

1. Every atomic formula of L_Ω is a Δ^Ω_0 formula.
2. If A and B are Δ^Ω_0 formulas, then $\neg A$, $(A \vee B)$, $(A \wedge B)$ and $(A \to B)$ are Δ^Ω_0 formulas.
3. If A is a Δ^Ω_0 formula, then $(\exists x)A$ and $(\forall x)A$ are Δ^Ω_0 formulas.
4. If A is a Δ^Ω_0 formula, then $(\exists \xi < \sigma)A$ and $(\forall \xi < \sigma)A$ are Δ^Ω_0 formulas.

DEFINITION 23 (Σ^Ω and Π^Ω formulas). The Σ^Ω and Π^Ω formulas are inductively generated as follows:

1. Every Δ^Ω_0 formula is a Σ^Ω and Π^Ω formula.
2. If A is a Σ^Ω formula [Π^Ω formula], then $\neg A$ is a Π^Ω formula [Σ^Ω formula].
3. If A and B are Σ^Ω formulas [Π^Ω formulas], then $(A \vee B)$ and $(A \wedge B)$ are Σ^Ω formulas [Π^Ω formulas].
4. If A is a Π^Ω formula [Σ^Ω formula] and B is a Σ^Ω formula [Π^Ω formula], then $(A \to B)$ is a Σ^Ω formula [Π^Ω formula].
5. If A is a Σ^Ω formula [Π^Ω formula], then $(\exists x)A$ and $(\forall x)A$ are Σ^Ω formulas [Π^Ω formulas].
6. If A is a Σ^Ω formula [Π^Ω formula], then $(\exists \xi < \sigma)A$ and $(\forall \xi < \sigma)A$ are Σ^Ω formulas [Π^Ω formulas].

[1] In general it will be clear from the context whether $<$ and $=$ denote the less and equality relation on the nonnegative integers or on the ordinals.

7. If A is a Σ^Ω formula, then $(\exists\xi)A$ is a Σ^Ω formula; if A is a Π^Ω formula, then $(\forall\xi)A$ is a Π^Ω formula.

In [17] three theories PA^r_Ω, PA^w_Ω and PA_Ω of Peano arithmetic with ordinals are considered. Now we restrict ourselves to repeating the axioms of PA^r_Ω. This system is the restriction of PA_Ω in the sense that induction on the natural numbers and on the ordinals is permitted for Δ^Ω_0 formulas only. PA^r_Ω comprises the usual logical axioms of two-sorted predicate logic plus the following non-logical axioms:

I. Number-theoretic Axioms. The axioms of Peano arithmetic PA with the exception of complete induction on the natural numbers.

II. Inductive Operator Axioms. For all inductive operator forms $\mathscr{A}(P,\vec{x})$:

$$P^\sigma_{\mathscr{A}}(\vec{s}) \leftrightarrow \mathscr{A}(P^{<\sigma}_{\mathscr{A}},\vec{s}).$$

III. Σ^Ω Reflection Axioms. For all Σ^Ω formulas A:

$$A \rightarrow (\exists\xi)A^\xi.$$

IV. Linearity Axioms.

$$\sigma \not< \sigma \wedge (\sigma < \tau \wedge \tau < \eta \rightarrow \sigma < \eta) \wedge (\sigma < \tau \vee \sigma = \tau \vee \tau < \sigma).$$

V. Δ^Ω_0 Induction on the Natural Numbers. For all Δ^Ω_0 formulas $A(x)$:

$$A(0) \wedge (\forall x)(A(x) \rightarrow A(x')) \rightarrow (\forall x)A(x).$$

VI. Δ^Ω_0 Induction on the Ordinals. For all Δ^Ω_0 formulas $A(\xi)$:

$$(\forall\xi)[(\forall\eta < \xi)A(\eta) \rightarrow A(\xi)] \rightarrow (\forall\xi)A(\xi).$$

From the inductive operator axioms and the Σ^Ω reflection axioms one can easily deduce that the Σ^Ω formulas $P_{\mathscr{A}}$ describe fixed points of the inductive operator form $\mathscr{A}(P,\vec{x})$.

Lemma 24. *We have for all inductive operator forms $\mathscr{A}(P,\vec{x})$:*

$$\mathsf{PA}^r_\Omega \vdash (\forall\vec{x})(P_{\mathscr{A}}(\vec{x}) \leftrightarrow \mathscr{A}(P_{\mathscr{A}},\vec{x})).$$

As mentioned before, PA^r_Ω is a conservative extension of PA. In view of Corollary 8 and Corollary 21 it is therefore impossible to embed $\mathsf{ID}^\#_1$ or $\mathsf{BON}(\mu) + (\mathsf{F}^+\text{-}\mathsf{I_N})$ into PA^r_Ω. In order to obtain a proper framework for such interpretations we have to strengthen induction on the natural numbers. The scheme of Σ^Ω induction on the natural numbers consists of all formulas

$$(\Sigma^\Omega\text{-}\mathsf{I_N}) \qquad A(0) \wedge (\forall x)(A(x) \rightarrow A(x')) \rightarrow (\forall x)A(x)$$

so that $A(x)$ is a Σ^Ω formula. In the following we write $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N})$ for the extension of PA^r_Ω by Σ^Ω induction on the natural numbers.

There is a natural interpretation of $\mathrm{L_{FP}}$ into L_Ω: For every relation symbol $\mathscr{P}_{\mathscr{A}}$ of $\mathrm{L_{FP}}$ translate the $\mathrm{L_{FP}}$ formula $\mathscr{P}_{\mathscr{A}}(\vec{s})$ by the Σ^Ω formula $P_{\mathscr{A}}(\vec{s})$ of L_Ω. This determines a translation of $\mathrm{L_{FP}}$ formulas A in L_Ω formulas $A^\diamond$ which leaves L unchanged and interprets $\mathrm{L_{FP}}$ formulas, which are positive in fixed point constants $\mathscr{P}_{\mathscr{A}}$, as Σ^Ω formulas. Hence, we have the following embedding.

THEOREM 25. *We have for all* $\mathrm{L_{FP}}$ *formulas* A:

$$\mathsf{ID}_1^{\#} \vdash A \quad \Rightarrow \quad \mathsf{PA}_\Omega^r + (\Sigma^\Omega\text{-}\mathsf{I_N}) \vdash A^\diamond.$$

The article Feferman and Jäger [12] contains an embedding of $\mathsf{BON}(\mu) + (\mathsf{S}\text{-}\mathsf{I_N})$ into PA_Ω^r; actually, this embedding is for the old form of the μ-operator (which is called $\boldsymbol{\mu}_w$ here) but an obvious modification works for the new form of μ as well. One crucial step in this embedding is to translate the L_p formula $(xy = z)$ as a suitable Σ^Ω formula $App(x, y, z)$ and to lift this interpretation to a natural translation of L_p formulas A into L_Ω formulas A^*.

Now we work exactly with the same translation and observe that each instance A of $(\mathsf{F}^+\text{-}\mathsf{I_N})$ goes over into an instance A^* of $(\Sigma^\Omega\text{-}\mathsf{I_N})$. Together with the results of [12] this translation therefore yields an embedding of $\mathsf{BON}(\mu) + (\mathsf{F}^+\text{-}\mathsf{I_N})$ into $\mathsf{PA}_\Omega^r + (\Sigma^\Omega\text{-}\mathsf{I_N})$. Since $(\mathsf{O}\text{-}\mathsf{I_N})$ and $(\mathsf{N}\text{-}\mathsf{I_N})$ are special cases of $(\mathsf{F}^+\text{-}\mathsf{I_N})$, it follows that the theories $\mathsf{BON}(\mu) + (\mathsf{O}\text{-}\mathsf{I_N})$ and $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I_N})$ are contained in $\mathsf{PA}_\Omega^r + (\Sigma^\Omega\text{-}\mathsf{I_N})$, too.

THEOREM 26. *We have for all* L_p *formulas* A:

$$\mathsf{BON}(\mu) + (\mathsf{F}^+\text{-}\mathsf{I_N}) \vdash A \quad \Rightarrow \quad \mathsf{PA}_\Omega^r + (\Sigma^\Omega\text{-}\mathsf{I_N}) \vdash A^*.$$

4.2. A Gentzen-style reformulation of $\mathsf{PA}_\Omega^r + (\Sigma^\Omega\text{-}\mathsf{I_N})$. The purpose of this subsection is to give a Gentzen-style reformulation G of the theory $\mathsf{PA}_\Omega^r + (\Sigma^\Omega\text{-}\mathsf{I_N})$. This step is essentially performed for obtaining a weak cut elimination theorem which then will be used for the final proof-theoretic analysis of $\mathsf{PA}_\Omega^r + (\Sigma^\Omega\text{-}\mathsf{I_N})$ in the following subsection.

The capital Greek letters $\Gamma, \Theta, \Phi, \Psi \ldots$ (possibly with subscripts) will be used to denote finite sets of L_Ω formulas, and sequents are formal expressions of the form $\Gamma \supset \Theta$. Often we write (for example) $\Gamma,\ A$ for the union of Γ and $\{A\}$.

The system G is an extension of the classical Gentzen calculus LK (cf. [14] or [24]), in which the structural rules are obsolete since we work with sets, and weakening is built in. G is formulated in the language L_Ω and comprises the following axioms and rules of inference.

I. AXIOMS OF G. For all finite sets Γ and Θ of L_Ω formulas, all Δ_0^Ω formulas A and all Δ_0^Ω formulas B which are axioms of $\mathsf{PA}_\Omega^r + (\Sigma^\Omega\text{-}\mathsf{I_N})$:

$$\Gamma,\ A \supset \Theta,\ A \quad \text{and} \quad \Gamma \supset \Theta,\ B.$$

II. PROPOSITIONAL AND QUANTIFIER RULES. These include the usual Gentzen-style inference rules for the propositional connectives and all sorts of quantifiers.

III. Σ^Ω REFLECTION. For all finite sets Γ and Θ of L_Ω formulas and for all Σ^Ω formulas A:

$$\frac{\Gamma \supset \Theta,\ A}{\Gamma \supset \Theta,\ (\exists \xi) A^\xi}.$$

IV. Σ^Ω INDUCTION ON THE NATURAL NUMBERS. For all finite sets Γ and Θ of L_Ω formulas, all Σ^Ω formulas $A(x)$ and all number variables u which do not occur in $\Gamma \supset \Theta,\ A(0)$:

$$\frac{\Gamma \supset \Theta,\ A(0) \qquad \Gamma,\ A(u) \supset \Theta,\ A(u')}{\Gamma \supset \Theta,\ (\forall x) A(x)}.$$

V. Δ_0^Ω Induction on the Ordinals. For all finite sets Γ and Θ of L_Ω formulas, all Δ_0^Ω formulas $A(\sigma)$ and all ordinals variables ξ which do not occur in $\Gamma \supset \Theta, A(\sigma)$:

$$\frac{\Gamma, (\forall \eta < \xi) A(\eta) \supset \Theta, A(\xi)}{\Gamma \supset \Theta, A(\sigma)}.$$

VI. Cuts of G. For all finite sets Γ and Θ of L_Ω formulas and all L_Ω formulas A:

$$\frac{\Gamma, A \supset \Theta \qquad \Gamma \supset \Theta, A}{\Gamma \supset \Theta}.$$

The notion $\mathsf{G} \mid\!\frac{n}{} \Gamma \supset \Theta$ is used to express that the sequent $\Gamma \supset \Theta$ is provable in G by a proof of depth less than or equal to n; we write $\mathsf{G} \mid\!\frac{n}{\star} \Gamma \supset \Theta$ if $\Gamma \supset \Theta$ is provable in G by a proof of depth less than or equal to n so that all its cut formulas are Σ^Ω formulas. In addition, $\mathsf{G} \vdash \Gamma \supset \Theta$ or $\mathsf{G} \vdash_\star \Gamma \supset \Theta$ means that there exists a natural number n so that $\mathsf{G} \mid\!\frac{n}{} \Gamma \supset \Theta$ or $\mathsf{G} \mid\!\frac{n}{\star} \Gamma \supset \Theta$, respectively.

One immediately observes that the main formulas of all axioms and rules of the systems G are Σ^Ω formulas. Hence, the following weak cut elimination theorem is a matter of routine.

Theorem 27 (Weak cut elimination). *We have for all finite sets Γ and Θ of L_Ω formulas:*

$$\mathsf{G} \vdash \Gamma \supset \Theta \quad \Rightarrow \quad \mathsf{G} \vdash_\star \Gamma \supset \Theta.$$

Furthermore, the axioms and rules of G are tailored so that the theory $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I}_\mathsf{N})$ can be easily embedded into G.

Theorem 28 (Embedding of $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I}_\mathsf{N})$). *We have for all L_Ω formulas A:*

$$\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I}_\mathsf{N}) \vdash A \quad \Rightarrow \quad \mathsf{G} \vdash \supset A.$$

Combining the previous two theorems yields the following corollary, which will be used for the proof of Theorem 35 below.

Corollary 29. *If the L_Ω formulas A is provable in $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I}_\mathsf{N})$, then there exists a natural number n so that $\mathsf{G} \mid\!\frac{n}{\star} \supset A$.*

4.3. The system G_∞. The system G_∞ is based on the language L_∞ which extends L_Ω by constants $\bar{\alpha}$ for all ordinals $\alpha < \omega^\omega$ (in the sense of the notation system). The *ordinal terms* $(\theta, \theta_0, \theta_1, \dots)$ of L_∞ are the ordinal variables and the ordinal constants of L_∞. The *atoms* of L_∞ are the atoms of L_Ω plus all expressions which result from the atoms of L_Ω by replacing some ordinal variables by ordinal constants. To simplify the notation we often write $A(\alpha)$ instead of $A(\bar{\alpha})$ if α is an ordinal less than ω^ω.

The *formulas* of L_∞ are inductively generated as follows:

1. Every atom of L_∞ is an L_∞ formula.
2. If A and B are L_∞ formulas, then $\neg A$, $(A \vee B)$, $(A \wedge B)$, and $(A \rightarrow B)$ are L_∞ formulas.
3. If A is an L_∞ formula, then $(\exists x)A$, $(\forall x)A$, $(\exists \xi < \theta)A$ and $(\forall \xi < \theta)A$ are L_∞ formulas.

Notice that L_∞ formulas do not contain unbounded ordinal quantifiers. The CL_∞ formulas are the L_∞ formulas which do not contain free number and free ordinal variables. Two L_∞ formulas A and B are called *numerically equivalent* if they differ in closed number terms with identical value only. Furthermore, an atom of CL_∞ is called *primitive* if it is not of the form $U(s)$ or $P^\alpha_{\mathscr{A}}(\vec{s})$. Obviously, every primitive atom of CL_∞ is either true or false, and in the following we write TRUE for the set of true primitive atoms and FALSE for the set of false primitive atoms.

In order to measure the complexity of cuts in G_∞ we assign a rank to each CL_∞ formula. This definition is tailored so that the process of building up stages of an inductive definition is reflected by the rank of the formulas $P^\alpha_{\mathscr{A}}(\vec{s})$.

DEFINITION 30. The *rank* $rn(A)$ of a CL_∞ formula A is inductively defined as follows:

1. If A is an atom of L or an atom $(\alpha < \beta)$ or $(\alpha = \beta)$ for some ordinals α and β, then $rn(A) := 0$.
2. If A is an atom $P^\alpha_{\mathscr{A}}(\vec{s})$ for some ordinal α, then $rn(A) := \omega(\alpha + 1)$.
3. If A is a formula $\neg B$ so that $rn(B) = \alpha$, then $rn(A) := \alpha + 1$.
4. If A is a formula $(B \vee C)$, $(B \wedge C)$ or $(B \rightarrow C)$ so that $rn(B) = \beta$ and $rn(C) = \gamma$, then $rn(A) := max(\beta, \gamma) + 1$.
5. If A is a formula $(\exists x)B(x)$ or $(\forall x)B(x)$ so that $rn(B(0)) = \alpha$, then $rn(A) := \alpha + 1$.
6. If A is a formula $(\exists \xi < \alpha)B(\xi)$ or $(\forall \xi < \alpha)B(\xi)$ for some ordinal α, then
$$rn(A) := sup\{rn(B(\beta)) + 1 : \beta < \alpha\}.$$

We write $oc(B)$ for the set of ordinal constants which occur in the L_∞ formula B. The proof of the following lemma is a matter of routine (cf. Jäger and Strahm [18]).

LEMMA 31. *We have for all inductive operator forms $\mathscr{A}(P, \vec{x})$, all CL_∞ formulas A and all oridnals $\alpha \in T$:*

1. $rn(\mathscr{A}(P^{<\alpha}_{\mathscr{A}}, \vec{t})) < rn(P^\alpha_{\mathscr{A}}(\vec{t}))$.
2. *If $\beta < \alpha$ for all $\beta \in oc(A)$, then $rn(A) < \omega\alpha + \omega$.*

The system G_∞ is formulated in the language CL_∞ and contains the following axioms and rules of inference.

I. AXIOMS OF G_∞. For all finite sets Γ and Θ of CL_∞ formulas, all CL_∞ formulas $U(s)$ and $U(t)$ which are numerically equivalent, all atoms B in TRUE and all atoms C in FALSE:

$$\Gamma,\, U(s) \supset \Theta,\, U(t) \quad \text{and} \quad \Gamma \supset \Theta,\, B \quad \text{and} \quad \Gamma,\, C \supset \Theta.$$

II. PROPOSITIONAL RULES. These include the usual Gentzen-style inference rules for the propositional connectives $\neg$, $\vee$, $\wedge$ and $\rightarrow$.

III. Number Quantifier Rules. For all finite sets Γ and Θ of CL_∞ formulas and all CL_∞ formulas $A(s)$:

$$\frac{\Gamma \supset \Theta,\ A(s)}{\Gamma \supset \Theta,\ (\exists x)A(x)}, \qquad \frac{\Gamma,\ A(t) \supset \Theta \text{ for all closed terms } t}{\Gamma,\ (\exists x)A(x) \supset \Theta},$$

$$\frac{\Gamma \supset \Theta,\ A(t) \text{ for all closed terms } t}{\Gamma \supset \Theta,\ (\forall x)A(x)}, \qquad \frac{\Gamma,\ A(s) \supset \Theta}{\Gamma,\ (\forall x)A(x) \supset \Theta}.$$

IV. Inductive Operator Rules. For all finite sets Γ and Θ of CL_∞ formulas, all inductive operator forms $\mathscr{A}(P, \vec{x})$, all closed number terms $\vec{s}$ and all ordinals α:

$$\frac{\Gamma \supset \Theta,\ \mathscr{A}(P_{\mathscr{A}}^{<\alpha}, \vec{s})}{\Gamma \supset \Theta,\ P_{\mathscr{A}}^{\alpha}(\vec{s})}, \qquad \frac{\Gamma,\ \mathscr{A}(P_{\mathscr{A}}^{<\alpha}, \vec{s}) \supset \Theta}{\Gamma,\ P_{\mathscr{A}}^{\alpha}(\vec{s}) \supset \Theta}.$$

V. Ordinal Quantifier Rules. For all finite sets Γ and Θ of CL_∞ formulas, all CL_∞ formulas $A(\alpha)$ and all ordinals β with $\alpha < \beta$:

$$\frac{\Gamma \supset \Theta,\ A(\alpha)}{\Gamma \supset \Theta,\ (\exists \xi < \beta)A(\xi)}, \qquad \frac{\Gamma,\ A(\gamma) \supset \Theta \quad \text{for all } \gamma < \beta}{\Gamma,\ (\exists \xi < \beta)A(\xi) \supset \Theta},$$

$$\frac{\Gamma \supset \Theta,\ A(\gamma) \quad \text{for all } \gamma < \beta}{\Gamma \supset \Theta,\ (\forall \xi < \beta)A(\xi)}, \qquad \frac{\Gamma,\ A(\alpha) \supset \Theta}{\Gamma,\ (\forall \xi < \beta)A(\xi) \supset \Theta}.$$

VI. Cuts of G_∞. For all finite sets Γ and Θ of CL_∞ formulas and all CL_∞ formulas A:

$$\frac{\Gamma,\ A \supset \Theta \qquad \Gamma \supset \Theta,\ A}{\Gamma \supset \Theta}.$$

The formula A is the *cut formula* of this cut; the rank of a cut is the rank of its cut formula.

Based on the axioms and rules of inference, derivability in the system G_∞ is defined as usual.

Definition 32. Let Γ and Θ be finite sets of CL_∞ formulas. Then $\mathrm{CL}_\infty \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \supset \Theta$ is defined for all ordinals α and ρ in T by induction on α.

1. If $\Gamma \supset \Theta$ is an axiom of G_∞, then we have $\mathrm{G}_\infty \overset{\alpha}{\underset{\rho}{\vdash}}$ for all ordinals α, ρ in T.
2. If $\mathrm{G}_\infty \overset{\alpha_i}{\underset{\rho}{\vdash}} \Gamma_i \supset \Theta_i$ and $\alpha_i < \alpha$ for every premise $\Gamma_i \supset \Theta_i$ of a propositional rule, a number quantifier rule, an inductive operator rule, an ordinal quantifier rule or a cut of G_∞ whose rank is less than ρ, then we have $\mathrm{G}_\infty \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \supset \Theta$ for the conclusion $\Gamma \supset \Theta$ of this rule.

It is easy to check that the assignment of ranks and the rules of inference are tailored so that the methods of predicative proof theory yield full cut elimination for G_∞. Therefore, we omit the proof fo the following theorem and refer to Pohlers [21] or Schütte [22].

Theorem 33. *We have for all finite sets* Γ *and* Θ *of* CL_∞ *formulas and all ordinals* $\alpha, \beta, \rho \in T$*:*

$$\mathrm{G}_\infty \overset{\alpha}{\underset{\beta+\omega^\rho}{\vdash}} \Gamma \supset \Theta \quad \Rightarrow \quad \mathrm{G}_\infty \overset{\varphi\rho\alpha}{\underset{\beta}{\vdash}} \Gamma \supset \Theta.$$

The next step is to reduce the theory G to the systems G_∞ via an asymmetric interpretation. For this purpose, it is useful to have the following persistency lemma, whose straightforward proof by induction on α will be omitted.

LEMMA 34. *We have for all finite sets Γ and Θ of CL_∞ formulas, all Σ^Ω formulas $A(\vec{\xi},\vec{x})$ of L_Ω with free variables as indicated, all closed number terms $\vec{r}$, and all ordinals $\alpha, \beta, \vec{\gamma}, \delta, \rho \in T$ so that $\beta \leq \delta < \omega^\omega$:*

1. $\mathsf{G}_\infty \vdash^{\alpha}_{\rho} \Gamma, A^\delta(\vec{\gamma},\vec{r}) \supset \Theta \quad \Rightarrow \quad \mathsf{G}_\infty \vdash^{\alpha}_{\rho} \Gamma, A^\beta(\vec{\gamma},\vec{r}) \supset \Theta.$
2. $\mathsf{G}_\infty \vdash^{\alpha}_{\rho} \Gamma \supset \Theta, A^\beta(\vec{\gamma},\vec{r}) \quad \Rightarrow \quad \mathsf{G}_\infty \vdash^{\alpha}_{\rho} \Gamma \supset \Theta, A^\delta(\vec{\gamma},\vec{r}).$

We proceed with introducing the notion of an (α,β) instance which will be needed in the proof of Theorem 35 below. Suppose that Γ and Θ are finite sets of L_Ω formulas and let Φ and Ψ be finite sets of CL_∞ formulas; assume further that α and β are ordinals less that ω^ω. Then the sequent $\Phi \supset \Psi$ is called an (α,β) *instance* of the sequent $\Gamma \supset \Theta$ provided the following conditions are satisfied:

(i) each free number variable is replaced by a closed number term and each free ordinal variable by an ordinal less than α.

(ii) each occurrence of an unbounded ordinal quantifier $(Q\xi)$ in the formulas of Γ is replaced by $(Q\xi < \alpha)$; each occurrence of an unbounded ordinal quantifier $(Q\xi)$ in the formulas of Θ is replaced by $(Q\xi < \beta)$.

(α,β) instances of a given sequent $\Gamma \supset \Theta$ may only vary in the interpretation of the free variables in the formulas of Γ and Θ. In particular, if A and B are closed L_Ω formulas, then the only (α,β) instance of $A \supset B$ is the sequent $A^\alpha \supset B^\beta$.

THEOREM 35 (Asymmetric interpretation). *Let Γ and Θ be finite sets of Σ^Ω formulas so that $\mathsf{G} \vdash^{n}_{\star} \Gamma \supset \Theta$ for some natural number n. Then we have for all ordinals $\alpha < \omega^\omega$ and all finite sets Φ and Ψ of CL_∞ formulas:*

$$\Phi \supset \Psi \textit{ is an } (\alpha, \alpha + \omega^n) \textit{ instance of } \Gamma \supset \Theta \quad \Rightarrow \quad \mathsf{G}_\infty \vdash^{\omega^{\alpha+\omega^n}}_{\omega(\alpha+\omega^n)} \Phi \supset \Psi.$$

PROOF. The theorem is proved by induction on n. Apart from Σ^Ω induction on the natural numbers all axioms and rules of inference are treated as in similar asymmetric interpretations, cf. Jäger [16, 17] and Schütte [22]. In the following argument we make tacitly use of Lemma 34.

Now suppose that $\Gamma \supset \Theta$ is the conclusion of the rule for Σ^Ω induction on the natural numbers. Then there exists a Σ^Ω formula $A(x)$ and $n_0, n_1 < n$ so that

$$\mathsf{G} \vdash^{n_0}_{\star} \Gamma \supset \Theta, A(0), \tag{1}$$

$$\mathsf{G} \vdash^{n_1}_{\star} \Gamma, A(u) \supset \Theta, A(u'). \tag{2}$$

Let m be the maximum of n_0 and n_1 and set $\beta_k := \alpha + \omega^m(k+1)$ for all natural numbers k. We show by side induction on k that

$$\mathsf{G}_\infty \vdash^{\omega^{\beta_k}+1}_{\omega(\alpha+\omega^n)} \Phi \supset \Psi, A^{\beta_k}(k).^2 \tag{3}$$

[2]To be more precise, we mean the instance of $A^{\beta_k}(k)$ where all free variables are replaced according to $\Phi \supset \Psi$.

If $k = 0$ then (3) follows from (1) and the main induction hypothesis. If $k > 0$ then the side induction hypothesis yields

(4) $$\mathsf{G}_\infty \mathrel{\Big|\!\frac{\omega^{\beta_{k-1}+1}}{\omega(\alpha+\omega^n)}} \Phi \supset \Psi,\, A^{\beta_{k-1}}(k-1).$$

Now we apply the main induction hypothesis to (2) with α replaced by β_{k-1} and obtain

(5) $$\mathsf{G}_\infty \mathrel{\Big|\!\frac{\omega^{\beta_k}}{\omega(\alpha+\omega^n)}} \Phi,\, A^{\beta_{k-1}}(k-1) \supset \Psi,\, A^{\beta_k}(k).$$

Hence (4), (5) and a cut imply

(6) $$\mathsf{G}_\infty \mathrel{\Big|\!\frac{\omega^{\beta_k}+1}{\omega(\alpha+\omega^n)}} \Phi \supset \Psi,\, A^{\beta_k}(k).$$

This finishes the proof of (3). A further application of Lemma 34 to (3) gives

(7) $$\mathsf{G}_\infty \mathrel{\Big|\!\frac{\omega^{\beta_k}+1}{\omega(\alpha+\omega^n)}} \Phi \supset \Psi,\, A^{\beta}(k)$$

for $\beta := \alpha + \omega^n$ and all natural numbers k. In (7) we can replace k by an arbitrary closed term with value k. Hence, we are in a position to apply the inference rule for numerical universal quantification on the right hand side and conclude

(8) $$\mathsf{G}_\infty \mathrel{\Big|\!\frac{\omega^{\alpha+\omega^n}}{\omega(\alpha+\omega^n)}} \Phi \supset \Psi,\, (\forall x)A^{\beta}(x).$$

Since the formula $(\forall x)A^\beta(x)$ is contained in Ψ the treatment of Σ^Ω induction on the natural numbers is completed. ⊣

Corollary 29, the above asymmetric interpretation and complete cut elimination for G_∞ provide a reduction of the Σ^Ω fragment of $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N})$ to the cut-free part of G_∞. This means that the following theorem is obtained from Corollary 29, Theorem 35 and Theorem 33.

THEOREM 36. *Let A be a closed Σ^Ω formula which is provable in $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N})$. Then there exists an $\alpha < \varphi\omega 0$ and a $\beta < \omega^\omega$ so that $\mathsf{G}_\infty \mathrel{|\!\frac{\alpha}{0}} \supset A^\beta$.*

As usual this result also gives an upper bound for the proof-theoretic ordinal of the theory $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N})$, cf. e.g., Schütte [22].

COROLLARY 37. $|\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N})| \le \varphi\omega 0$.

4.4. Summary. Let us now collect what we have achieved. From Corollary 7, Corollary 21, Theorem 25, Theorem 26 and Corollary 37 we obtain the following relationship between our theories:

(i) $(\Pi^0_1\text{-}\mathsf{CA})_{<\omega^\omega} \le \mathsf{ID}^\#_1 \le \mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N}) \le (\Pi^0_1\text{-}\mathsf{CA})_{<\omega^\omega}$,

(ii) $(\Pi^0_1\text{-}\mathsf{CA})_{<\omega^\omega} \le \mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I_N}) \equiv \mathsf{BON}(\mu) + (\mathsf{O}\text{-}\mathsf{I_N}) \le \mathsf{BON}(\mu) + (\mathsf{F}^+\text{-}\mathsf{I_N}) \le \mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N}) \le (\Pi^0_1\text{-}\mathsf{CA})_{<\omega^\omega}$.

Unfortunately, we do not know yet whether it is possible to get a direct reduction of $\mathsf{ID}^\#_1$ to $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I_N})$ or a reduction of $\mathsf{BON}(\mu) + (\mathsf{F}^+\text{-}\mathsf{I_N})$ to $\mathsf{ID}^\#_1$. However, the problem does not affect proof-theoretic equivalences of these systems, and we can formulate the following main theorem.

THEOREM 38.

1. *The five theories* $\mathsf{ID}_1^\#$, $\mathsf{BON}(\mu)+(\mathsf{N}\text{-}\mathsf{I_N})$, $\mathsf{BON}(\mu)+(\mathsf{O}\text{-}\mathsf{I_N})$, $\mathsf{BON}(\mu)+(\mathsf{F}^+\text{-}\mathsf{I_N})$ *and* $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N})$ *have proof-theoretic ordinal* $\varphi\omega 0$.
2. *All these theories are proof-theoretically equivalent to* $(\Pi^0_1\text{-}\mathsf{CA})_{<\omega^\omega}$.

This result remains true if we add the axioms of totality (Tot) and extensionality (Ext) to the theories $\mathsf{BON}(\mu) + (\mathsf{N}\text{-}\mathsf{I_N})$, $\mathsf{BON}(\mu) + (\mathsf{O}\text{-}\mathsf{I_N})$ and $\mathsf{BON}(\mu) + (\mathsf{F}^+\text{-}\mathsf{I_N})$. To see this one only has to follow the pattern of [19].

§5. Related systems of strength $\varphi\omega 0$. We conclude this article by making some comments on related theories of strength $\varphi\omega 0$. Of course we do not want to present a complete list of such systems, so that we confine ourselves to some typical candidates.

In Schütte [22] it is shown that the theory $(\Delta^1_1\text{-}\mathsf{CR})$, i.e., $(\Pi^0_1\text{-}\mathsf{CA})$ plus Δ^1_1 comprehension rule is of ordinal strength $\varphi\omega 0$. A system of explicit mathematics corresponding to $(\Delta^1_1\text{-}\mathsf{CR})$ is the theory EM_0 plus join rule (cf. [2]). Furthermore, it is shown in Cantini [4] that $(\Sigma^1_1\text{-}\mathsf{DC})\restriction$ has proof-theoretic ordinal $\varphi\omega 0$.

An additional subsystem of second order arithmetic of this strength is the theory $(\Sigma^1_1\text{-}\mathsf{AC})\restriction$ plus induction on the natural numbers for Σ^1_1 formulas. In order to establish the lower bound one just interprets $(\Pi^0_1\text{-}\mathsf{CA})_{<\omega^\omega}$ or $\mathsf{ID}_1^\#$ by standard methods. The upper bound is obtained by partial cut elimination and a straightforward asymmetric interpretation.

Feferman [11] introduces the general notion of reflective closure of a theory. In particular, he considers an extension $\mathsf{Ref(PA)}$ of Peano arithmetic which makes crucial use of two unary predicates $T(x)$ and $F(x)$ of partial and self-reflecting truth and falsity. The proof-theoretic ordinal of $\mathsf{Ref(PA)}$ is $\varphi\varepsilon_0 0$. However, if induction on the natural numbers is restricted in $\mathsf{Ref(PA)}$ to formulas positive in T and F, we end up with a system $\mathsf{Ref^\#(PA)}$ of strength $\varphi\omega 0$. Formal theories of truth are also considered in Cantini [5], and he presents, among other systems, a theory KF which is similar to $\mathsf{Ref^\#(PA)}$. All these theories contain $\mathsf{ID}_1^\#$ and are contained in $\mathsf{PA}^r_\Omega + (\Sigma^\Omega\text{-}\mathsf{I_N})$.

Finally, it should be mentioned that there are also interesting term rewriting systems whose termination ordering has order type $\varphi\omega 0$. We refer to Dershowitz and Jounannaud [6] for further reading.

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