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Annihilators in JB-algebras

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Abstract

Orthogonality is defined for all elements in a JB-algebra and Topping's results on annihilators in JW-algebras are generalized to the context of JB- and JBW-algebras. A pair (a, b) of elements in a JB-algebra A is said to be orthogonal provided that $a^2 \circ b$ equals zero. It is shown that this relation is symmetric. The annihilator $S^{\perp}$ of a subset S of A is defined to be the set of elements a in A such that, for all elements s in S, the pair (s, a) is orthogonal. It is shown that the annihilators are closed quadratic ideals and, if A is a JBW-algebra, a subset I of A is a w*-closed quadratic ideal if and only if I coincides with its biannihilator $I^{\perp\perp}$. Moreover, in a JBW-algebra A the formation of the annihilator of a w*-closed quadratic ideal is an orthocomplementation on the complete lattice of w*-closed quadratic ideals which makes it into a complete orthomodular lattice. Further results establish a connection between ideals, central idempotents and annihilators in JBW-algebras.

1. *Introduction*

Topping[7] studied the w*-closed Jordan algebras of bounded self-adjoint operators, the so-called JW-algebras. A generalization of these objects was given by Alfsen, Shultz and Størmer[1] by introducing JB- and JBW-algebras.

It is the aim of the present paper to generalize Topping's theory on annihilators in JW-algebras to the context of JB- and JBW-algebras. Annihilators in JW-algebras are defined in terms of the underlying associative operator product and therefore there is no obvious extension of this definition to JB-algebras.

Let A be a JB-algebra. A pair (a, b) of elements in A is said to be orthogonal provided that $a^2 \circ b$ equals zero. The relation of orthogonality is symmetric: this is shown in Section 3.

In Section 4 the annihilator $S^{\perp}$ of a subset S of the JB-algebra A is defined as the set of elements a in A such that, for all elements s in S, the pair (s, a) is orthogonal. It is shown that the annihilator of a general subset S of a JB-algebra A is a closed quadratic ideal.

In Section 5 it is shown that a subset I of a JBW-algebra A is a w*-closed quadratic ideal if and only if I coincides with its biannihilator $I^{\perp\perp}$. It also follows that the formation of the annihilator of a w*-closed quadratic ideal in a JBW-algebra A is an orthocomplementation on the complete lattice of w*-closed quadratic ideals which makes it into a complete orthomodular lattice. Further results establish a connection between ideals, central idempotents and annihilators in JBW-algebras.

2. *Preliminaries*

A (real) *Jordan algebra* A is a real vector space together with a bilinear mapping $\circ: A^2 \rightarrow A$, called the *Jordan product*, satisfying, for all elements a and b in A,

$$\text{(i)}\ a\circ b = b\circ a, \quad \text{(ii)}\ (a^2\circ b)\circ a = a^2\circ(b\circ a).$$

In a real associative algebra $(A, .)$ a canonical Jordan product can be defined, for all elements a and b in A, by

$$a\circ b := \tfrac{1}{2}(a.b+b.a).$$

A Jordan algebra which is isomorphic to a subalgebra of a Jordan algebra formed in this way is said to be special.

Let A be a Jordan algebra and let a be an element in A. Then a linear mapping $U_a: A \rightarrow A$ is defined, for all elements b in A, by

$$U_a b := 2(a\circ b)\circ a - a^2\circ b.$$

A subspace I of a Jordan algebra A is said to be an ideal, resp. a quadratic ideal, provided that $I\circ A \subseteq I$, resp. $U_I A \subseteq I$. An element a in a Jordan algebra A is called central provided that, for all elements b and c in A, $(a\circ b)\circ c$ equals $a\circ(b\circ c)$. The set $Z(A)$ of all central elements in A is called the centre of A. Notice that $Z(A)$ is a subalgebra.

A Jordan algebra A with unit e together with a complete norm $\|\cdot\|$ such that, for all elements a and b in A,

$$\text{(i)}\ \|a\circ b\| \leqslant \|a\|\,\|b\|, \quad \text{(ii)}\ \|a\|^2 \leqslant \|a^2+b^2\|$$

is called a *JB-algebra*. Notice that in a JB-algebra the Jordan product is jointly continuous. The set A^2 of all squares of elements in a JB-algebra A is a proper convex cone. Therefore A^2 induces an ordering $\leqslant$ on A. For all elements a in a JB-algebra A, the mapping $U_a: A \rightarrow A$ is *positive*, i.e. $U_a A^2 \subseteq A^2$.

A JB-algebra A which is a Banach space dual is said to be a *JBW-algebra*. Notice that in a JBW-algebra the Jordan product is separately w*-continuous and that a w*-closed subalgebra of a JBW-algebra has a unit.

An element p in a JB-algebra A is called *idempotent* if $p^2 = p$. Let $U(A)$ denote the collection of idempotents in A. With respect to the ordering $\leqslant$ and the orthocomplementation defined, for all idempotents p in A, by $p' := e-p, U(A)$ becomes an orthomodular poset with 0 as the least and e as the greatest element. Let p and q be idempotents in A. Then $p \leqslant q'$ if and only if $p\circ q$ equals zero and $p \leqslant q$ if and only if $p\circ q$ equals p if and only if $U_q U_p$ equals U_p. In a JBW-algebra A the triple $(U(A), \leqslant, ')$ is a complete orthomodular lattice.

Let a be an element in a JBW-algebra A. Then there exists a least idempotent $r(a)$ in A such that $r(a)\circ a$ equals a. The element $r(a)$ is called the *support idempotent* of a. Notice that $r(a)$ is the unit in the smallest w*-closed subalgebra $W(a)$ of A containing a. For every idempotent p in a JBW-algebra A there exists a least central idempotent $c(p)$ in A majorizing p. The element $c(p)$ is called the *central support* of p.

Edwards [2] has shown that every w*-closed quadratic ideal in a JBW-algebra A is of the form $U_p A$, for a suitable idempotent p in A, and conversely, for an idempotent p in A, $U_p A$ is a w*-closed quadratic ideal. Moreover, $U_p A$ is an ideal if and only if p is central.

For the details of the theory of JB- and JBW-algebras the reader is referred to [**1**, **2**, **3**, **4**, **6**].

3. *Orthogonality in JB-algebras*

Let A be a JB-algebra. A pair (a, b) of elements in A is said to be *orthogonal*, denoted by $a \perp b$, if $a^2 \circ b = 0$. Hence a pair (p, q) of idempotents is orthogonal if and only if $p \leqslant q'$.

Notice that, for elements a and b in a JB-algebra A, $a \circ b = 0$ does not imply $a^2 \circ b = 0$. In the JB-algebra M_3^8, i.e. the Jordan algebra of hermitian 3×3 matrices with Cayley number entries, consider the elements

$$a = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & -k & 0 \\ k & 0 & k \\ 0 & -k & 0 \end{pmatrix}.$$

Since $ik = -j$, $ki = j$ and $i^2 = -1$, $a \circ b = 0$ but $a^2 \circ b = 2b \neq 0$.

It is the aim of this section to show that orthogonality is a symmetric relation and that $a \perp b$ entails $a^n \circ b = 0$ for all natural numbers n.

LEMMA 3·1. *Let A be a JB-algebra. Let (a, b) be an orthogonal pair of elements in A. Then $a^2 \circ b^2$ is a positive element in A.*

Proof. By [**5**], theorem I·10, the smallest subalgebra B of A containing a^2 and b is special, i.e. there is an isomorphism j from B into an associative algebra $(J, .)$ equipped with the canonical Jordan product. Then

$$j(a)^2 . j(b) + j(b) . j(a)^2 = 0.$$

Multiplying the equation on the left and right, respectively, by $j(b)$ in the underlying associative product we obtain

$$j(a)^2 . j(b)^2 = j(b)^2 . j(a)^2.$$

Therefore a^2 and b^2 generate an associative subalgebra in A. By the continuity of the Jordan product, the smallest closed subalgebra C of A containing a^2 and b^2 is associative as well. Since a^2 and b^2 are positive elements in A, there exist, by [**2**], lemma 2·1, elements c and d in C such that $a^2 = c^2$ and $b^2 = d^2$. By the associativity of C, we have $(c \circ d)^2 = a^2 \circ b^2$.

LEMMA 3·2. *Let A be a JB-algebra. Let (a, b) be an orthogonal pair of elements in A. Then*

$$a^2 \circ b^2 = U_b a^2 = U_a b^2 = U_a b = a \circ b = 0.$$

Proof. From $a \perp b$ it follows that $U_b a^2 = -a^2 \circ b^2$. Then, by Lemma 3·1 and the positivity of the operator U_b, both $U_b a^2$ and $a^2 \circ b^2$ are zero. The remaining assertions follow from the identities

$$(U_a b^2)^2 = U_a U_b U_b a^2, \quad (U_a b)^2 = U_a U_b a^2,$$

$$(a \circ b)^2 = \tfrac{1}{2} b \circ U_a b + \tfrac{1}{4} U_a b^2 + \tfrac{1}{4} U_b a^2.$$

COROLLARY 3·3. *Let A be a JB-algebra and let a and b be elements in A. Then the following conditions are equivalent*:
(i) $a \perp b$; (ii) $a \circ b = 0$ *and* $a^2 \circ b = 0$; (iii) $a^n \circ b = 0$ *for all* $n \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii). See Lemma 3·2.
(ii) $\Rightarrow$ (iii). By [5], theorem I·10, the smallest subalgebra B of A containing a and b is special, i.e. there is an isomorphism j from B into an associative algebra $(J, .)$ equipped with the canonical Jordan product. Suppose now that $a^{n-1} \circ b = 0$. Then

$$j(a)^{n-1} . j(b) + j(b) . j(a)^{n-1} = 0.$$

Multiplying the equation on the left and right, respectively, by $j(a)$ in the underlying asociative product and adding the results we get

$$j(a)^n . j(b) + j(b) . j(a)^n + j(a)^{n-1} . j(b) . j(a) + j(a) . j(b) . j(a)^{n-1} = 0.$$

From this and the assumption it follows that $a^n \circ b = 0$.
(iii) $\Rightarrow$ (i). This is obvious.

We now proceed to the main result of this section.

PROPOSITION 3·4. *Let A be a JB-algebra. Let a and b be elements in A. Then $a \perp b$ if and only if $b \perp a$.*

Proof. Suppose that $a \perp b$. By Lemma 3·2 and the positivity of U_a and U_b,

$$0 \leqslant U_a b^4 \leqslant \|b^2\| \, U_a b^2 = 0.$$

Therefore $U_a b^4$ equals zero and the assertion now follows from Lemma 3·2 and the identity

$$(a \circ b^2)^2 = \tfrac{1}{4} U_a b^4 + \tfrac{1}{4} U_{b^2} a^2 + \tfrac{1}{2} b^2 \circ U_a b^2.$$

4. *Annihilators in JB-algebras*

Let A be a JB-algebra and let S be a subset of A. The *annihilator* $S^\perp$ of S is defined to be

$$S^\perp := \{a \in A ; s \perp a \quad \text{for all } s \in S\}.$$

The set $S^{\perp\perp}$ is referred to as the *biannihilator* of S.

LEMMA 4·1. *Let A be a JB-algebra, let S and T be subsets of A and let $(S_i)_{i \in I}$ be a family of subsets of A. Then*
(i) $S \subseteq T$ *implies* $T^\perp \subseteq S^\perp$; (ii) $S \subseteq S^{\perp\perp}$;
(iii) $(\bigcup_{i \in I} S_i)^\perp = \bigcap_{i \in I} (S_i)^\perp$; (iv) $S^{\perp\perp\perp} = S^\perp$;
(v) $S^\perp$ *is a closed subspace of* A; (vi) $S^\perp = (\mathrm{cl}\, S)^\perp = (\mathrm{lin}\, S)^\perp$.

Proof. (i), (ii), (iii) and (iv). These are immediate.
(v) By bilinearity and continuity of the Jordan product, $S^\perp$ is a closed subspace.
(vi) By (v), $\mathrm{cl}\, S$ and $\mathrm{lin}\, S$ all are contained in $S^{\perp\perp}$ and the assertion follows, by (i).

THEOREM 4·2. *Let A be a JB-algebra and let S be a subset of A. Then the annihilator $S^\perp$ of S is a closed quadratic ideal.*

Proof. Let a, b and c be elements in A such that the pair (a, b) is orthogonal. By the positivity of the operators U_{a^2}, U_b and U_c and Lemma 3·2, it follows that

$$0 \leqslant U_{a^2} U_b U_c b^2 \leqslant \|b^2\| \, U_{a^2} U_b c^2 \leqslant \|b^2\| \, \|c^2\| \, U_{a^2} b^2 = 0.$$

Therefore $U_{a^2} U_b U_c b^2 = 0$. Since $0 \leqslant a^4 \leqslant \|a^2\| \, a^2$, we conclude, by Lemma 3·2, that

$$0 \leqslant U_b a^4 \leqslant \|a^2\| \, U_b a^2 = 0.$$

Therefore $U_b a^4 = 0$. By Lemma 3·2 and the identity

$$(a^2 \circ U_b c)^2 = \tfrac{1}{4} U_{a^2} U_b U_c b^2 + \tfrac{1}{4} U_b U_c U_b a^4 + \tfrac{1}{2} a^2 \circ U_b U_c U_b a^2,$$

$a^2 \circ U_b c$ equals zero. Therefore by Lemma 4·1 (v), the assertion follows.

Proposition 4·3. *Let A be a JB-algebra and let p and q be idempotents in A. Then* (i) $\{p\}^\perp = U_{p'} A = (U_p A)^\perp$, *and* (ii) $p \leqslant q$ *if and only if* $\{p\}^\perp \supseteq \{q\}^\perp$.

Proof. (i) Let a be an element in $U_p A$ and let b be an element in $U_{p'} A$. Then, by (2·66) and (2·69) from [**4**], we have $a^2 \circ b = 0$. Therefore $U_{p'} A \subseteq (U_p A)^\perp$. By Lemma 4·1 (i), we have $(U_p A)^\perp \subseteq \{p\}^\perp$. Now let $b \in \{p\}^\perp$; then $p' \circ b = b$. Therefore $b \in U_{p'} A$.

(ii) Since $p \leqslant q$ if and only if $q' \leqslant p'$, (ii) follows from (i) and $U_{q'} A \subseteq U_{p'} A$.

5. *Annihilators in JBW-algebras*

In JBW-algebras Lemma 4·1 (vi) and Theorem 4·2 can be improved.

Lemma 5·1. *Let A be a JBW-algebra and let S be a subset of A. Then*
(i) $S^\perp$ *is a* w*-*closed quadratic ideal in* A, *and*
(ii) $S^\perp = (\mathrm{cl}\, S)^\perp = (\mathrm{w^*\text{-}cl}\, S)^\perp = (\mathrm{lin}\, S)^\perp$.

Proof. By Theorem 4·2 and separate w*-continuity of the Jordan product, (i) follows. The proof of (ii) is similar to that of Lemma 4·1 (vi).

We are now in a position to identify the subsets S of a JBW-algebra which coincide with their biannihilator $S^{\perp\perp}$.

Theorem 5·2. *Let A be a JBW-algebra. A subset S of A is a* w*-*closed quadratic ideal if and only if S coincides with its biannihilator* $S^{\perp\perp}$.

Proof. By Lemma 5·1, a subset which coincides with its biannihilator is a w*-closed quadratic ideal. Since every w*-closed quadratic ideal is of the form $U_p A$ for a suitable idempotent p in A, by [**2**], theorem 2·3, it follows from Proposition 4·3 that it coincides with its biannihilator.

Corollary 5·3. *Let A be a JBW-algebra, let $U(A)$ be the collection of idempotents in A and let $\mathscr{T}(A)$ be the collection of* w*-*closed quadratic ideals in A. Let p be an idempotent in A and define $\phi(p) := U_p A$. Then the mapping $\phi: U(A) \to \mathscr{T}(A)$ is an ortho-order isomorphism from the complete orthomodular lattice* $(U(A), \leqslant, ')$ *onto* $(\mathscr{T}(A), \subseteq, {}^\perp)$.

Proof. By Theorem 5·2, Proposition 4·3 and [**2**], corollary 2·5, the assertion follows.

In order to identify the annihilator of a general subset of a JBW-algebra we begin by examining annihilators of singleton sets.

Lemma 5·4. *Let A be a JBW-algebra and let a and b be elements in A. Then the following conditions are pairwise equivalent*: (i) $a \perp b$; (ii) $a \perp r(b)$; (iii) $r(a) \perp r(b)$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Since $W(b)$ coincides with the w*-closure of the linear hull of the set $\{b^n; n \in \mathbb{N}\}$ and since the Jordan product is separately w*-continuous, this follows from Corollary 3·3.

(iii) $\Rightarrow$ (i). If $r(a) \circ r(b)$ equals zero then $r(a) \leqslant r(b)'$. Therefore $p \leqslant r(b)'$ for all idempotents p in $W(a)$, and it follows that $r(b) \leqslant p'$. Hence $p \circ q = 0$ for all idempotents q in $W(b)$ and all idempotents p in $W(a)$, and, by spectral theory, $a \perp b$.

COROLLARY 5·5. *Let A be a JBW-algebra and let a be an element in A. Then $r(a)$ is the unique idempotent in A such that $\{a\}^\perp = \{r(a)\}^\perp$.*

Proof. By Lemma 5·4, $a \perp b$ if and only if $r(a) \perp b$. Hence $\{a\}^\perp = \{r(a)\}^\perp$. Uniqueness follows from Corollary 5·3.

THEOREM 5·6. *Let A be a JBW-algebra and let S be a non-empty subset of A. Let $W(S)$ denote the smallest* w*-*closed subalgebra of A containing S and let p be the unit of $W(S)$. Then $S^\perp = U_{p'}A$.*

Proof. By Lemma 4·1 (iii), Corollary 5·5, Proposition 4·3 and Corollary 5·3,

$$S^\perp = \bigcap_{a \in S} \{a\}^\perp = \bigcap_{a \in S} \{r(a)\}^\perp = \bigcap_{a \in S} U_{r(a)'} A = U_{r'} A,$$

where r denotes $\bigvee_{a \in S} r(a)$. It remains to prove that $r = p$. Clearly $r \leqslant p$. Conversely, $r(a) \leqslant r$, for all elements a in S. Therefore $W(S) \subseteq U_r A$ and $p \leqslant r$ follows.

We present two applications of the previously developed theory.

COROLLARY 5·7. *Let A be a JBW-algebra, let S be a subset of A and let $W(S)$ denote the smallest* w*-*closed subalgebra of A containing S. Then*

$$S^\perp = \{a \in A; W(S) \circ a = \{0\}\}.$$

Proof. Let p be the unit of $W(S)$. Then $W(S) \subseteq U_p A$ and, by Theorem 5·6, $S^\perp$ equals $U_{p'}A$. Hence, by [**4**], theorem 2·6·5, we have $W(S) \circ S^\perp = \{0\}$. Conversely, if $W(S) \circ a = \{0\}$ then $b^2 \circ a = 0$ for all b in S.

COROLLARY 5·8. *Let A be a JBW-algebra and let a and b be elements in A. Then $r(a+b) \leqslant r(a) \vee r(b)$.*

Proof. From $r(a) \leqslant r(a) \vee r(b)$, $r(b) \leqslant r(a) \vee r(b)$, Proposition 4·3 and Corollary 5·5 it follows that $\{r(a) \vee r(b)\}^\perp \subseteq \{a\}^\perp \cap \{b\}^\perp$. Therefore, for all z in $\{r(a) \vee r(b)\}^\perp$, both $a \circ z^2$ and $b \circ z^2$ equal zero. Hence $(a+b) \circ z^2 = 0$ and

$$z \in \{a+b\}^\perp = \{r(a+b)\}^\perp,$$

by Corollary 5·5. Therefore $\{r(a) \vee r(b)\}^\perp \subseteq \{r(a+b)\}^\perp$ and an application of Proposition 4·3 proves the claim.

The final results generalize [**7**], proposition 5 and section 7.

PROPOSITION 5·9. *Let A be a JBW-algebra and let I be an ideal in A. Then there exists a central idempotent f in A such that $I^\perp = U_f A$.*

Proof. By the separate w*-continuity of the Jordan product, w*-cl I is an ideal. By [**2**], theorem 2·3, there is a central idempotent f in A such that w*-cl $I = U_f A$. Then f' is a central idempotent in A and $I^\perp = U_{f'} A$, by Theorem 5·6 and Lemma 5·1.

PROPOSITION 5·10. *Let A be a JBW-algebra and let a be an element in A. Let (a) be the smallest ideal containing a and let $r(a)$ be the support idempotent of a. Then the central support $c(r(a))$ of $r(a)$ is the unique (central) idempotent such that $\{c(r(a))\}^{\perp} = (a)^{\perp}$.*

Proof. By Proposition 5·9, there is a central idempotent f in A such that

$$(a)^{\perp} = \{f\}^{\perp} = U_{f'}A.$$

Hence $f \circ a = a$. Therefore $c(r(a)) \leqslant f$. Since $U_{c(r(a))}A$ is an ideal containing a, it follows that

$$U_{c(r(a))'}A = (U_{c(r(a))}A)^{\perp} \subseteq (a)^{\perp} = U_{f'}A.$$

Therefore $c(r(a))' \leqslant f'$.

COROLLARY 5·11. *Let A be a JBW-algebra and let p and q be idempotents in A. Let $c(p)$ and $c(q)$ be the central supports of p and q, respectively. Let (p) and (q) be the smallest ideals containing p and q, respectively. Then the following conditions are equivalent:*
(i) $c(p) \perp c(q)$; (ii) $c(p) \perp q$; (iii) $p \perp c(q)$; (iv) $q \in (p)^{\perp}$; (v) $p \in (q)^{\perp}$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). These are immediate.

(ii) $\Leftrightarrow$ (iv). Since $U_{c(p)}A$ is an ideal containing p, $c(p) \perp q$ entails

$$q \in U_{c(p)'}A = (U_{c(p)}A)^{\perp} \subseteq (p)^{\perp}.$$

Conversely, let q be an element in $(p)^{\perp}$. Then, by Proposition 5·10, q is contained in $\{c(p)\}^{\perp}$. Therefore $c(p) \perp q$.

(iii) $\Leftrightarrow$ (v). The proof is similar to the one above.

PROPOSITION 5·12. *Let A be a JBW-algebra with centre $Z(A)$ and let S be a subset of $Z(A)$. Then $S^{\perp}$ is a w*-closed ideal in A.*

Proof. By Lemma 5·1, $S^{\perp}$ is a w*-closed subspace of A. Let $a \in A$ and $b \in S^{\perp}$. It is enough to show $c \perp a \circ b$, for each element c in S. Since c is central, c^2 also is central and it follows that

$$c^2 \circ (b \circ a) = (c^2 \circ b) \circ a = 0.$$

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