ORDER THEORETIC TYPE DECOMPOSITION OF JBW*-TRIPLES[1]

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[Received 25 September 1989]

1. Introduction

THE collection of tripotents in a JBW*-triple together with the partial ordering defined by O. Loos in **[12]** is a poset without greatest element. Therefore this poset does not form a complete orthomodular lattice and there is no obvious way to generalize the type decomposition of W*-algebras, JW-algebras and JBW-algebras to JBW*-triples. By algebraic methods a type decomposition was found by G. Horn and E. Neher in **[8]**, **[9]** and **[10]**. To obtain an order theoretic type decomposition of JBW*-triples similar to Topping's type decomposition of JW-algebras more information is needed about the poset of tripotents. Therefore the study of this poset begun by C. M. Edwards and G. T. Rüttimann in **[4]** is continued in the first part of the paper. It is shown that the poset of tripotents of a JBW*-triple is a $\wedge$-semilattice satisfying properties similar to those of a complete orthomodular lattice. Specifically, the question of the existence of suprema and infima is answered and central tripotents are defined with the help of w*-closed ideals which were examined by G. Horn in **[8]**.

In a second part of the paper order theoretic properties of a tripotent are considered. It is shown that for a large collection of order theoretic properties, including atomicity, modularity and distributivity, there exists a uniquely determined decomposition into a direct sum of two w*-closed ideals in such a way that the first ideal is generated by a tripotent with this property and the second ideal contains no non-zero tripotent with this property. An analogous result is obtained for central tripotents. As a corollary a JBW*-triple is decomposed into an atomic ideal and an ideal without atoms. A JBW*-triple is also decomposed into six direct summands corresponding to the usual type decomposition of W*-algebras, JW-algebras and JBW-algebras into summands of type I, II and III.

The paper is organized as follows. Section 2 contains the preliminaries, Section 3 the results of the poset of tripotents in a JBW*-triple and Section 4 the type decompositions of JBW*-triples.

[1] Research supported by the Swiss National Science Foundation.

Quart. J. Math. Oxford (2), 42 (1991), 129–147

0033-5606/91 $3.00

The author wishes to thank G. T. Rüttimann for encouragement and valuable discussions and the Swiss National Science Foundation for supporting this research.

2. Preliminaries

A real Jordan algebra A with unit e which is a Banach space such that, for all elements a and b in A, $\|a \circ b\| \leq \|a\| \, \|b\|$ and $\|a\|^2 \leq \|a^2 + b^2\|$ and which is a Banach space dual is said to be a *JBW-algebra*. The set A^+ of all squares of elements in A forms a generating cone. With respect to the ordering induced by the cone A^+ and the orthocomplementation defined, for all idempotents p in A, by $p' := e - p$, the set of idempotents of a JBW-algebra A becomes a complete orthomodular lattice. Two idempotents p and q in a JBW-algebra A are *orthogonal* if and only if $p \circ q$ equals zero and, if p and q are idempotents in A, $p \leq q$ if and only if $p \circ q$ equals p.

An unital Jordan*-algebra $\mathscr{A}$ which is a Banach space such that, for all elements a, b and c in $\mathscr{A}$, $\|a^*\| = \|a\|$, $\|a \circ b\| \leq \|a\| \, \|b\|$ and $\|\{aaa\}\| = \|a\|^3$, where

$$\{abc\} = a \circ (b^* \circ c) - b^* \circ (c \circ a) + c \circ (a \circ b^*), \tag{1}$$

and which is a Banach space dual is said to be a *JBW*-algebra* or a *Jordan W*-algebra*. The self-adjoint part $\mathscr{A}_{sa}$ of a JBW*-algebra $\mathscr{A}$ is a JBW-algebra and each JBW-algebra is obtained in this manner. The correspondence so established is one to one.

For further details of the theory of JBW-algebras and JBW*-algebras the reader is referred to **[1]**, **[3]**, **[7]**, **[14]**, **[16]**.

A complex vector space $\mathscr{A}$ equipped with a mapping $\{\ldots\}$: $\mathscr{A}^3 \to \mathscr{A}$ which is symmetric and linear in the first and third variables, conjugate linear in the second variable and satisfies, for all elements a, b, c, d and e in $\mathscr{A}$, the *Jordan triple identity*

$$\{ab\{cde\}\} - \{cd\{abe\}\} = \{\{abc\}de\} - \{c\{dab\}e\} \tag{2}$$

is said to be a *Jordan triple* and the mapping $\{\ldots\}$: $\mathscr{A}^3 \to \mathscr{A}$ is called *Jordan triple product*. For all elements a and b in $\mathscr{A}$, the linear mapping $D(a, b)$: $\mathscr{A} \to \mathscr{A}$ and the conjugate linear mapping $Q(a, b)$: $\mathscr{A} \to \mathscr{A}$ are defined, for all c in $\mathscr{A}$, by $D(a, b)c := \{abc\}$ and $Q(a, b)c := \{acb\}$, respectively. We write $Q(a)$ for $Q(a, a)$. Let $\mathscr{A}$ be a Jordan triple and let a and b be elements in $\mathscr{A}$, then

$$Q(Q(a)b) = Q(a)Q(b)Q(a). \tag{3}$$

A linear subspace $\mathscr{I}$ of a Jordan triple $\mathscr{A}$ is called a *subtriple*, resp. an *inner ideal*, resp. an *ideal* in $\mathscr{A}$, provided that $\{\mathscr{I}\mathscr{I}\mathscr{I}\} \subseteq \mathscr{I}$, resp.

$\{\mathcal{I}\mathcal{A}\mathcal{I}\} \subseteq \mathcal{I}$, resp. $\{\mathcal{I}\mathcal{A}\mathcal{A}\} + \{\mathcal{A}\mathcal{I}\mathcal{A}\} \subseteq \mathcal{I}$. Two ideals $\mathcal{I}$ and $\mathcal{J}$ in a Jordan triple $\mathcal{A}$ are called *orthogonal* if $\mathcal{I} \cap \mathcal{J}$ equals $\{0\}$. If $\mathcal{I}$ and $\mathcal{J}$ are orthogonal ideals in the Jordan triple $\mathcal{A}$, then, for all b in $\mathcal{I}$ and c in $\mathcal{J}$,

$$Q(b, c) = D(b, c) = D(c, b) = Q(b)Q(c) = Q(c)Q(b) = 0. \tag{4}$$

A *JB*-triple* $\mathcal{A}$ is a Jordan triple which is also a Banach space such that the mapping D from $\mathcal{A}^2$ to the Banach space of bounded linear operators on $\mathcal{A}$, defined, for all pairs (a, b) of elements in $\mathcal{A}$ and all c in $\mathcal{A}$, by $D(a, b)c := \{abc\}$, is continuous and such that, for all elements a in $\mathcal{A}$, $D(a, a)$ is hermitean with non-negative spectrum and $\|D(a, a)\|$ is equal to $\|a\|^2$. For all elements a, b and c in a JB*-triple $\mathcal{A}$, $\|\{abc\| \leq \|a\| \|b\| \|c\|$ and $\|\{aaa\}\| = \|a\|^3$. A JB*-triple $\mathcal{A}$ which is a Banach space dual is called a *JBW*-triple*. In a JBW*-triple $\mathcal{A}$ the Jordan triple product is separately w*-continuous.

In a Jordan triple $\mathcal{A}$ the odd powers of an element a in $\mathcal{A}$ are defined inductively by $a^1 := a$ and $a^{2n+1} := \{aa^{2n-1}a\}$. An element u in a Jordan triple $\mathcal{A}$ is called a *tripotent* if u^3 equals u. The set of all tripotents in $\mathcal{A}$ is denoted by $U(\mathcal{A})$. In the theory of JBW*-triples the tripotents play a role analogous to that played by the idempotents in a JBW-algebra. Specifically, a JBW*-triple equals the closure of the linear span of its tripotents. For all tripotents u in a Jordan triple $\mathcal{A}$, the operators $P_j(u)$: $\mathcal{A} \to \mathcal{A}$ are defined, for $j \in \{0, 1, 2\}$, by $P_2(u) := Q(u)^2$, $P_1(u) := 2(D(u, u) - Q(u)^2)$ and $P_0(u) := I - 2D(u, u) + Q(u)^2$, respectively. For $j \in \{0, 1, 2\}$ and u a tripotent in the Jordan triple $\mathcal{A}$, the operators $P_j(u)$ are the projections onto the eigenspaces $\mathcal{A}_j(u)$ corresponding to the eigenvalues $\frac{j}{2}$ of the linear mapping $D(u, u)$ and $\mathcal{A}$ equals $\mathcal{A}_2(u) \oplus \mathcal{A}_1(u) \oplus \mathcal{A}_0(u)$, the *Peirce decomposition of* $\mathcal{A}$ *relative to* u. $\mathcal{A}_1(u)$ is a subtriple, $\mathcal{A}_2(u)$ and $\mathcal{A}_0(u)$ are inner ideals and if $\mathcal{A}$ is a JBW*-triple, then, for all $j \in \{0, 1, 2\}$, $\mathcal{A}_j(u)$ is w*-closed.

If $\mathcal{A}$ is a Jordan triple and u is a tripotent in $\mathcal{A}$ then $\mathcal{A}_2(u)$ together with the mappings $\circ_u$: $\mathcal{A}_2(u)^2 :\to \mathcal{A}_2(u)$ and $*^u$: $\mathcal{A}_2(u) \to \mathcal{A}_2(u)$ defined, for all a and b in $\mathcal{A}_2(u)$, by $a \circ_u b := \{aub\}$ and $a^{*^u} := \{uau\}$ is a Jordan*-algebra with unit u, denoted by $\mathcal{A}_2(u)^J$. Let $\mathcal{A}_2(u)^J_{sa}$ denote the set of elements in $\mathcal{A}_2(u)^J$ which are invariant under the involution $*^u$. If $\mathcal{A}$ is a JBW*-triple, then $\mathcal{A}_2(u)^J$ is a JBW*-algebra and $\mathcal{A}_2(u)^J_{sa}$ is a JBW-algebra.

A tripotent u in a Jordan triple $\mathcal{A}$ is called *unitary* if, for all a in $\mathcal{A}$, $\{uua\}$ equals a, i.e. $\mathcal{A}_2(u)$ equals $\mathcal{A}$. Let $\mathcal{A}$ be a Jordan triple and let u be a tripotent in $\mathcal{A}$ then u is an unitary tripotent in $\mathcal{A}_2(u)$. A Jordan triple $\mathcal{A}$ is called *abelian* if, for all a, b, c, d and e in $\mathcal{A}$, $\{ab\{cde\}\}$ equals $\{cd\{abe\}\}$. A Jordan triple $\mathcal{A}$ with unitary tripotent u is abelian if and only if $\mathcal{A}_2(u)^J$ is associative if and only if $\mathcal{A}_2(u)^J_{sa}$ is associative.

For further details of the theory of JBW*-triples the reader is referred to **[2]**, **[4]**, **[5]**, **[6]**, **[8]**, **[11]**, **[12]**, **[13]**, **[15]**.

3. The poset of tripotents in a JBW*-triple

A pair (u, v) of tripotents in a JBW*-triple $\mathcal{A}$ is called *orthogonal*, denoted by $u \perp v$, if v is an element in $\mathcal{A}_0(u)$.

LEMMA 3.1. *Let $\mathcal{A}$ be a JBW*-triple and let u and v be tripotents in $\mathcal{A}$. Then the following conditions are pairwise equivalent*:

(i) $u \perp v$ (ii) $v \perp u$ (iii) $D(u, v) = 0$

(iv) $D(v, u) = 0$ (v) $\{uuv\} = 0$ (vi) $\{vvu\} = 0$

Proof. This is proved in Lemma 3.9 **[12]**.

If u and v are tripotents in a JBW*-triple $\mathcal{A}$ then u is said to be *less than or equal to* v, denoted by $u \leqslant v$, if $v - u$ is a tripotent orthogonal to u.

LEMMA 3.2. *Let $\mathcal{A}$ be a JBW*-triple and let u and v be tripotents in $\mathcal{A}$. Then the following conditions are pairwise equivalent*:

(i) $u \leqslant v$ (ii) $P_2(u)v = u$ (iii) $\{uvu\} = u$

(iv) $\{uuv\} = u$ (v) $D(u, v) = D(u, u)$ (vi) $D(v, u) = D(u, u)$

Proof. By Corollary 1.7 **[5]**, it follows the equivalence of (i) and (ii). If $u \leqslant v$ then $u \perp v - u$ and, by (iii) and (iv) of Lemma 3.1, (i) implies (v) and (vi). It is clear that (v) implies (iii), that (iii) implies (ii) and that (vi) implies (iv). By (2), (ii) follows from (iv).

COROLLARY 3.3. *Let $\mathcal{A}$ be a JBW*-triple and let u and v be tripotents in $\mathcal{A}$ with $u \leqslant v$. Then $\{vuv\}$ equals $\{vvu\}$ equals u.*

Proof. Applying twice Lemma 3.2(vi), it follows that $\{vuv\}$ equals u. Therefore u is an element in $\mathcal{A}_2(v)$ and hence $\{vvu\}$ equals u.

Let $(L, \leqslant)$ be a poset and let M be a subset of L. Then the supremum, resp. the infimum, of M, provided it exists, is denoted by $\bigvee M$, resp. $\bigwedge M$. Let $(a_i)_{i \in I}$ be a family of elements in L. Then we denote the supremum, resp. the infimum, of the set $\{a_i; i \in I\}$ by $\bigvee_{i \in I} a_i$, resp. $\bigwedge_{i \in I} a_i$, provided it exists.

PROPOSITION 3.4. *Let $\mathcal{A}$ be a JBW*-triple and let $U(\mathcal{A})$ be the set of all tripotents in $\mathcal{A}$. Then*:

(i) *Orthogonality is a symmetric relation on $U(\mathcal{A})$ and, for all $u \in U(\mathcal{A})$, $0 \perp u$. Moreover, $u \perp u$ implies that u equals* 0.

(ii) *The pair $(U(\mathcal{A}), \leqslant)$ is a poset with* 0 *as the least element.*

(iii) *If u, v and w are elements in $U(\mathcal{A})$ with $u \leq v$ and $v \perp w$, then $u \perp w$.*

(iv) *If $(u_i)_{i=1}^n$ is a family of pairwise orthogonal elements in $U(\mathcal{A})$, then $\sum_{i=1}^n u_i$ is a tripotent and $\bigvee_{i=1}^n u_i$ exists and equals $\sum_{i=1}^n u_i$. Moreover, if v is an element in $U(\mathcal{A})$ such that, for all $i \in \{1, \ldots, n\}$, $v \perp u_i$ then $v \perp \bigvee_{i=1}^n u_i$.*

Proof. (i) is a consequence of (i), (ii) and (v) of Lemma 3.1.

(ii) Trivially, $u \leq u$ and $0 \leq u$, for all tripotents u in $\mathcal{A}$. If u and v are tripotents in $\mathcal{A}$ with $u \leq v$ and $v \leq u$, then u equals v, by Lemma 3.2(iii) and Corollary 3.3. If u, v and w are tripotents in $\mathcal{A}$ with $u \leq v$ and $v \leq w$ then Corollary 3.3, (3) and Lemma 3.2(iii) imply

$$\{uwu\} = \{\{vuv\}w\{vuv\}\} = \{v\{u\{vwv\}u\}v\} = u$$

and $u \leq w$ follows, by Lemma 3.2(iii).

(iii) By Lemma 3.2(v) and Lemma 3.1(iii), $\{uuw\}$ equals $\{uvw\}$ equals 0.

(iv) Obviously, $\sum_{i=1}^n u_i$ is a tripotent such that, for all $i \in \{1, \ldots, n\}$, $u_i \leq \sum_{i=1}^n u_i$. If w is an upper bound of the family $(u_i)_{i=1}^n$ then, by Lemma 3.2(v) and Lemma 3.1(v),

$$\left\{\left(\sum_{i=1}^n u_i\right)w\left(\sum_{j=1}^n u_j\right)\right\} = \sum_{i=1}^n \sum_{j=1}^n \{u_i w u_j\} = \sum_{i=1}^n u_i.$$

By Lemma 3.2(iii), $\sum_{i=1}^n u_i \leq w$. The last assertion is a consequence of Lemma 3.1(v).

Let $(L, \leq)$ be a poset and let a and b be elements in L with $a \leq b$. Then the set of all elements c in L with $a \leq c$ and $c \leq b$ is denoted by $[a, b]$ and is referred to as a *segment*. If L has a least element 0 and if a is an element in L then we denote with $\bigvee_a$, resp. $\bigwedge_a$, the supremum, resp. the infimum, in $([0, a], \leq)$, provided it exists.

Lemma 3.5. *Let $\mathcal{A}$ be a JBW*-triple and let w be a tripotent in $\mathcal{A}$. Then:*

(i) *An element u in $\mathcal{A}$ is a tripotent such that $u \leq w$ if and only if u is an idempotent in $\mathcal{A}_2(w)_{sa}^J$.*

(ii) *Let u and v be tripotents in $\mathcal{A}$ such that $u \leq w$ and $v \leq w$. Then $u \leq v$ if and only if $v - u$ lies in the cone of $\mathcal{A}_2(w)_{sa}^J$.*

(iii) *The segment* $[0, w]$ *equipped with the restricted partial ordering* $\leqslant$ *and the mapping* ${}^w: [0, w] \to [0, w]$ *such that, for all* u *in* $[0, w]$, u^w *is defined to be* $w - u$ *is a complete orthomodular lattice.*

(iv) *Let* u *and* v *be tripotents in* $\mathscr{A}$ *such that* $u \leqslant w$ *and* $v \leqslant w$. *Then* $u \perp v$ *if and only if* u *and* v *are orthogonal in* $\mathscr{A}_2(w)^J_{sa}$.

Proof. (i) and (ii) are proved in Lemma 2.4 [**4**]. (iii) and (iv) are consequences of (i), (ii), the theory of JBW-algebras, Lemma 3.1(v) and Lemma 3.2(v).

The poset of tripotents in a JBW*-triple $\mathscr{A}$ has a greatest element if and only if $\mathscr{A}$ equals $\{0\}$. Therefore, in non-trivial cases, this poset does not form a complete lattice but we get the following theorem:

THEOREM 3.6. *Let* $\mathscr{A}$ *be a JBW*-triple and let* $(U(\mathscr{A}), \leqslant)$ *be the poset of its tripotents. Then*:

(i) *If* F *is a non-empty set of tripotents in* $(U(\mathscr{A}), \leqslant)$ *then the infimum* $\bigwedge F$ *exists.*

(ii) *If* F *is a set of tripotents in* $(U(\mathscr{A}), \leqslant)$ *then the supremum* $\bigvee F$ *exists if and only if* F *has an upper bound.*

Proof. (i) is a consequence of Theorem 4.4 [**4**].

(ii) It follows from (i) that the infimum of the upper bounds of the set F exists if and only if an upper bound exists.

COROLLARY 3.7. *Let* $\mathscr{A}$ *be a JBW*-triple, let* w *be a tripotent in* $\mathscr{A}$ *and let* F *be a set of tripotents in* $[0, w]$. *Then*:

(i) $\bigwedge_w F$ *exists; and, if* F *is not empty, then* $\bigwedge F$ *exists and is equal to* $\bigwedge_w F$.

(ii) $\bigvee_w F$ *and* $\bigvee F$ *both exist and they are equal.*

Proof. The claims follow from Lemma 3.5(iii) and (i) and (ii) of Theorem 3.6.

PROPOSITION 3.8. *Let* $\mathscr{A}$ *be a JBW*-triple and let* $(u_i)_{i\in I}$ *be a family of tripotents in* $\mathscr{A}$. *Then*:

(i) *If* $(u_i)_{i\in I}$ *is an increasing net in* $(U(\mathscr{A}), \leqslant)$ *then the supremum* $\bigvee_{i\in I} u_i$ *exists and equals the* w^**-limit of the net* $(u_i)_{i\in I}$.

(ii) *If the supremum* $\bigvee_{i\in I} u_i$ *exists and if* $\mathscr{F}$ *is the directed set of finite subsets of* I *then* $\bigvee_{i\in I} u_i$ *equals the* w^**-limit of the increasing net* $\left(\bigvee_{i\in F} u_i\right)_{F\in\mathscr{F}}$.

(iii) *If* $(u_i)_{i\in I}$ *is a family of pairwise orthogonal tripotents and if* $\mathscr{F}$ *is the directed set of finite subsets of* I *then the supremum* $\bigvee_{i\in I} u_i$ *exists and equals the* w^**-limit of the increasing net* $\left(\sum_{i\in F} u_i\right)_{F\in\mathscr{F}}$.

Proof. (i) By Theorem 4.6 [4] and the fact that the unit ball of a JBW*-triple is w*-compact, it follows that there exists an upper bound of the family $(u_i)_{i\in I}$ and therefore, by Theorem 3.6(ii), the supremum $\bigvee_{i\in I} u_i$ exists. By (i) and (ii) of Lemma 3.5, $(u_i)_{i\in I}$ is an increasing net of idempotents in the JBW-algebra $\mathcal{A}_2\left(\bigvee_{i\in I} u_i\right)^J_{sa}$. By the theory of JBW-algebras, the fact that the w*-topology of the JBW-algebra $\mathcal{A}_2\left(\bigvee_{i\in I} u_i\right)^J_{sa}$ equals the restricted w*-topology of $\mathcal{A}$ and Corollary 3.7, it follows that $\bigvee_{i\in I} u_i$ equals the w*-limit of the net $(u_i)_{i\in I}$.

(ii) By Corollary 3.7, $\bigvee_{i\in I} u_i$ is also the supremum of the family $(u_i)_{i\in I}$ in the JBW-algebra $\mathcal{A}_2\left(\bigvee_{i\in I} u_i\right)^J_{sa}$ and there $\bigvee_{i\in I} u_i$ is the w*-limit of the net $\left(\bigvee_{i\in F} u_i\right)_{F\in\mathcal{F}}$.

(iii) is a consequence of (ii) and Proposition 3.4(iv).

PROPOSITION 3.9. *Let $\mathcal{A}$ be a JBW*-triple. Then:*

(i) *If $(u_i)_{i=1}^n$ is a family of tripotents in $\mathcal{A}$ and if v is a tripotent in $\mathcal{A}$ which is an upper bound of the family then $\bigvee_{i=1}^n u_i$ equals the support idempotent $r\left(\sum_{i=1}^n u_i\right)$ of $\sum_{i=1}^n u_i$ in the JBW-algebra $\mathcal{A}_2(v)^J_{sa}$.*

(ii) *If $(u_i)_{i=1}^n$ is a family of tripotents in $\mathcal{A}$ such that its supremum exists then $\bigvee_{i=1}^n u_i$ is an element in the smallest JBW*-subtriple of $\mathcal{A}$ containing $\sum_{i=1}^n u_i$.*

Proof. (i) For all $i\in I$, u_i is an idempotent in $\mathcal{A}_2(v)^J_{sa}$ and there $\bigvee_{i=1}^n u_i$ exists and equals $r\left(\sum_{i=1}^n u_i\right)$, by Lemma 4.2.8 [7]. The assertion now follows from Corollary 3.7(ii).

(ii) The support idempotent $r(a)$ of a positive element a in a JBW-algebra A is an element in the w*-closure of the linear span of the set of the odd powers of a. The claim now follows from (i), the fact that the odd powers of $\sum_{i=1}^n u_i$ in the JBW-algebra $\mathcal{A}_2\left(\bigvee_{i=1}^n u_i\right)^J_{sa}$ coincide with the odd powers of $\sum_{i=1}^n u_i$ in the JBW*-triple $\mathcal{A}$ and since the w*-topology of the JBW-algebra $\mathscr{A}_2\left(\bigvee_{i=1}^{n} u_i\right)_{sa}^{J}$ equals the w*-topology of $\mathscr{A}$ restricted to $\mathscr{A}_2\left(\bigvee_{i=1}^{n} u_i\right)_{sa}^{J}$.

COROLLARY 3.10. *Let $\mathscr{A}$ be a JBW*-triple. If $(w_i)_{i\in I}$ is a family of tripotents such that its supremum exists. Then*:

(i) *If v is a tripotent in $\mathscr{A}$ such that, for all $i \in I$, $w_i \perp v$ then $\left(\bigvee_{i\in I} w_i\right) \perp v$.*

(ii) *$\mathscr{A}_0\left(\bigvee_{i\in I} w_i\right)$ equals $\bigcap_{i\in I} \mathscr{A}_0(w_i)$.*

Proof. (i) By Proposition 3.8(ii), separate w*-continuity of the Jordan triple product and Lemma 3.1(v), it is sufficient to show that $\bigvee_{i=1}^{n} w_i \perp v$, for all finite families $(w_i)_{i=1}^{n}$ of tripotents in $\mathscr{A}$ such that $\bigvee_{i=1}^{n} w_i$ exists and such that $v \perp w_i$, for all $i \in \{1, \ldots, n\}$. With $u := \sum_{i=1}^{n} w_i$, $\{vvu\}$ equals zero. If also $\{vvu^{2n-1}\}$ equals zero it follows, by (2),

$$\{vv\{uu^{2n-1}u\}\} = \{\{vvu\}u^{2n-1}u\} + \{uu^{2n-1}\{vvu\}\} - \{u\{u^{2n-1}vv\}u\} = 0.$$

Therefore, for all natural numbers n, $\{vvu^{2n-1}\}$ equals zero. By the separate w*-continuity of the Jordan triple product and Proposition 3.9(ii), it follows that $\left\{vv\left(\bigvee_{i=1}^{n} w_i\right)\right\}$ equals zero and the assertion is proved.

(ii) A tripotent v is an element in $\mathscr{A}_0\left(\bigvee_{i\in I} w_i\right)$ if and only if $v \perp \left(\bigvee_{i\in I} w_i\right)$. By (i) and Proposition 3.4(iii), $v \perp \bigvee_{i\in I} w_i$ if and only if, for all $i \in I$, $v \perp w_i$ if and only if, for all $i \in I$, v is an element in $\mathscr{A}_0(w_i)$. The claim follows since a JBW*-triple is the closure of the linear span of its tripotents.

LEMMA 3.11. *Let $\mathscr{A}$ be a JBW*-triple and let $(U(\mathscr{A}), \leqslant)$ be the poset of its tripotents. Then*:

(i) *A tripotent u is maximal in $(U(\mathscr{A}), \leqslant)$ if and only if $\mathscr{A}_0(u)$ equals $\{0\}$.*
(ii) *Let $(u_i)_{i\in I}$ be a family of pairwise orthogonal tripotents in $\mathscr{A}$. Then there exists a maximal tripotent v such that, for all $i \in I$, $u_i \leqslant v$.*
(iii) *Let u be a tripotent in $\mathscr{A}$ and let $(v_i)_{i\in I}$ be the family of maximal tripotents in $\mathscr{A}$ such that, for all $i \in I$, $u \leqslant v_i$. Then u equals $\bigwedge_{i\in I} v_i$.*

Proof. (i) is proved in Corollary 4.8 **[4]** and (ii) is shown in (3.12) **[8]**.

(iii) Since the unit ball $\mathscr{A}_1$ of $\mathscr{A}$ is convex and w*-compact it is the w*-closure of the convex hull of its extreme points. By Theorem 4.6 **[4]** there is an anti-order isomorphism from $(U(\mathcal{A}), \leq)$ onto the set of all non-empty w*-closed faces of $\mathcal{A}_1$ ordered by set inclusion. The images of the maximal tripotents are precisely the extreme points of $\mathcal{A}_1$. The claim now follows since the supremum of the set of the extreme points of a w*-closed face of $\mathcal{A}_1$ is the w*-closed face itself.

By Lemma 3.11(i), the maximal tripotents are precisely the complete tripotents, in the sense of [**8**].

LEMMA 3.12. *Let $\mathcal{A}$ be a JBW*-triple, let $\mathcal{I}$ and $\mathcal{J}$ be w*-closed ideals and let u, resp. v, be a tripotent in $\mathcal{I}$, resp. $\mathcal{J}$. Then*:

(i) *If $\mathcal{I}$ and $\mathcal{J}$ are orthogonal then $u \perp v$.*
(ii) *$\mathcal{A}_2(u) \oplus \mathcal{A}_1(u)$ and the segment $[0, u]$ are subsets of $\mathcal{I}$.*
(iii) *There exists at least one maximal tripotent in $\mathcal{I}$ and if w is a maximal tripotent in $\mathcal{I}$ then $\mathcal{A}_2(w) \oplus \mathcal{A}_1(w)$ equals $\mathcal{I}$.*

Proof. (i) is a consequence of (4) and Lemma 3.1(v).

(ii) The first assertion holds, because $\mathcal{A}_2(u)$, resp. $\mathcal{A}_1(u)$, is the eigenspace of $D(u, u)$ corresponding to the eigenvalue 1, resp. $\frac{1}{2}$. The second claim is now a consequence of Lemma 3.5(i).

(iii) By Lemma 3.11(ii), there is a maximal tripotent in $\mathcal{I}$. If w is a maximal tripotent in $\mathcal{I}$ then $\mathcal{A}_2(w) \oplus \mathcal{A}_1(w)$ is a subset of $\mathcal{I}$, by (ii). Moreover, $\mathcal{I} \cap \mathcal{A}_0(w)$ equals $\{0\}$ because otherwise w would not be maximal in $\mathcal{I}$ and the assertion follows.

If $\mathcal{I}$, $\mathcal{J}$ is an orthogonal pair of w*-closed ideals in a JBW*-triple $\mathcal{A}$ then $\mathcal{I} \oplus \mathcal{J}$ is a w*-closed ideal and, for all a in $\mathcal{I}$ and all b in $\mathcal{J}$, $\|a + b\|$ equals $\|a\| \vee \|b\|$, by (4.4) [**8**].

LEMMA 3.13. *Let $\mathcal{A}$ be a JBW*-triple and let $\mathcal{I}$ be a w*-closed ideal in $\mathcal{A}$. Then*:

(i) *There exists a unique w*-closed ideal $\mathcal{I}'$ such that $\mathcal{A}$ equals $\mathcal{I} \oplus \mathcal{I}'$.*
(ii) *If $\mathcal{J}$ is also a w*-closed ideal in $\mathcal{A}$ then $\mathcal{J}$ equals $\mathcal{J} \cap \mathcal{I} \oplus \mathcal{J} \cap \mathcal{I}'$.*
(iii) *If $\mathcal{J}$ is a w*-closed ideal in $\mathcal{I}$ then it is a w*-closed ideal in $\mathcal{A}$.*

Proof. (i) is proved in (4.2)(4) [**8**] and (ii) in (4.2)(5) [**8**].

(iii) By (i), $\mathcal{A}$ equals $\mathcal{I} \oplus \mathcal{I}'$. By (4) and the fact that $\mathcal{J}$ is a w*-closed ideal in $\mathcal{I}$, it is easily verified that $\{\mathcal{J}(\mathcal{I} \oplus \mathcal{I}')(\mathcal{I} \oplus \mathcal{I}')\} + \{(\mathcal{I} \oplus \mathcal{I}')\mathcal{J}(\mathcal{I} \oplus \mathcal{I}')\} \subseteq \mathcal{J}$.

Let $\mathcal{A}$ be a JBW*-triple. For u a tripotent let $\mathcal{I}(u)$ denote the smallest w*-closed ideal containing u. A tripotent u in $\mathcal{A}$ is called *central* if u is maximal in $(U(\mathcal{I}(u)), \leq)$. Note that 0 and all maximal tripotents are central.

LEMMA 3.14. *Let $\mathcal{A}$ be a JBW*-triple. Then:*

(i) *A tripotent u in $\mathcal{A}$ is central if and only if $\mathcal{J}(u)$ equals $\mathcal{A}_2(u) \oplus \mathcal{A}_1(u)$ if and only if u is maximal in a w*-closed ideal if and only if $\mathcal{A}_0(u)$ is a w*-closed ideal if and only if $\mathcal{A}_0(u)$ equals $\mathcal{J}(u)'$.*

(ii) *If $(u_i)_{i\in I}$ is a family of central tripotents in $\mathcal{A}$ such that $\bigvee_{i\in I} u_i$ exists then $\bigvee_{i\in I} u_i$ is a central tripotent.*

(iii) *If v is a central tripotent in $\mathcal{A}$ and if u is a tripotent in $\mathcal{A}$ with $v \leqslant u$ then v is a central idempotent in the JBW-algebra $\mathcal{A}_2(u)^J_{sa}$.*

(iv) *If v is a central tripotent in $\mathcal{A}$ and if u is a central idempotent in the JBW-algebra $\mathcal{A}_2(v)^J_{sa}$ then u is a central tripotent in $\mathcal{A}$.*

(v) *If $(u_i)_{i\in I}$ is a non-empty family of central tripotents in $\mathcal{A}$ such that its supremum exists then $\bigwedge_{i\in I} u_i$ is a central tripotent.*

(vi) *If $\mathcal{J}$ is a w*-closed ideal in $\mathcal{A}$ and if u is a tripotent in $\mathcal{J}$. Then u is central in the JBW*-triple $\mathcal{A}$ if and only if it is central in the JBW*-triple $\mathcal{J}$.*

Proof. (i) follows from Lemma 3.12(iii) and Lemma 3.13(i).

(ii) is a consequence of (i) and Corollary 3.10(ii).

(iii) Since $\mathcal{J}(v)$ is a w*-closed ideal in the JBW*-triple $\mathcal{A}$, it follows that $\mathcal{J}(v) \cap \mathcal{A}_2(u)^J_{sa}$ is a w*-closed Jordan ideal in the JBW-algebra $\mathcal{A}_2(u)^J_{sa}$ and, by (i) and (ii) of Lemma 3.5, v is a maximal idempotent $\mathcal{J}(v) \cap \mathcal{A}_2(u)^J_{sa}$. By the theory of JBW-algebra, v is a central idempotent in $\mathcal{A}_2(u)^J_{sa}$.

(iv) By Lemma 3.5(i), u is a tripotent with $u \leqslant v$. Since v is maximal in $\mathcal{J}(v)$ it follows, by (4.2)(3) [8], that $\mathcal{J}(v)_2(u) \oplus \mathcal{J}(v)_1(u)$ is a w*-closed ideal in $\mathcal{J}(v)$. By Lemma 3.13(iii), $\mathcal{J}(v)_2(u) \oplus \mathcal{J}(v)_1(u)$ is a w*-closed ideal in $\mathcal{A}$ containing u as a maximal tripotent.

(v) The infimum of central idempotents in a JBW-algebra is again a central idempotent and the claim follows from (ii), (iii), (iv) and Corollary 3.7(i).

(vi) is a consequence of Lemma 3.13(iii).

By Lemma 3.11(iii), the infimum of a family of central tripotents need not be central.

LEMMA 3.15. *Let $\mathcal{A}$ be a JBW*-triple and let u be a tripotent in $\mathcal{A}$. Then:*

(i) *Let $\mathcal{J}$ be a w*-closed ideal in $\mathcal{A}$ and let $v + v'$ be the uniquely determined decomposition of u with respect to $\mathcal{J} \oplus \mathcal{J}'$. Then v and v' are tripotents such that $v \perp v'$, $v \leqslant u$ and $v' \leqslant u$. Moreover, v is central if u is central.*

(ii) *Let $\mathcal{J}$ be an ideal in $\mathcal{J}(u)$ and let $\mathcal{J}'$ be the w*-closed ideal such that $\mathcal{J} \oplus \mathcal{J}'$ equals $\mathcal{J}(u)$. If $v + v'$ is the uniquely determined decomposition of u with respect to $\mathcal{J} \oplus \mathcal{J}'$ then $\mathcal{J}(v)$ equals $\mathcal{J}$.*

Proof. (i) By (4), the uniqueness of the decomposition and

$$u = v + v' = \{(v+v')(v+v')(v+v')\} = \{vvv\} + \{v'v'v'\},$$

it follows that v and v' are tripotents with $v \leq u$, $v' \leq u$ and $v \perp v'$. Let now u be central. If w is a tripotent in $\mathcal{J}(v)$ with $v \leq w$, then $w - v$ is a tripotent in $\mathcal{J}(v)$ orthogonal to both v and v'. By $\mathcal{J}(v) \subseteq \mathcal{J}(u)$ and Proposition 3.4(iv), $u + (w - v)$ is a tripotent in $\mathcal{J}(u)$ and therefore $w - v$ equals 0.

(ii) Certainly, $\mathcal{J}(v) \subseteq \mathcal{J}$ and $\mathcal{J}(v') \subseteq \mathcal{J}'$. Because $\mathcal{J}(v) \oplus \mathcal{J}(v')$ is a w*-closed ideal containing u, $\mathcal{J}(v) \oplus \mathcal{J}(v')$ equals $\mathcal{J}(u)$. Hence $\mathcal{J}(v)$ equals $\mathcal{J}$.

A pair u, v of tripotents in a JBW*-triple $\mathcal{A}$ is called *unrelated*, denoted by $u \# v$, if $\mathcal{J}(u)$ and $\mathcal{J}(v)$ are orthogonal.

LEMMA 3.16. *Let $\mathcal{A}$ be a JBW*-triple and let u and v be tripotents in $\mathcal{A}$. Then*:

(i) *If $u \# v$ then $u \perp v$.*
(ii) *The following conditions are pairwise equivalent*:

$$(\alpha)\ u \# v, \qquad (\beta)\ v \# u, \qquad (\gamma)\ u \in \mathcal{J}(v)', \qquad (\delta)\ v \in \mathcal{J}(v)'.$$

(iii) *If u is central, then $u \# v$ if and only if $u \perp v$.*

Proof. (i) follows from Lemma 3.12(i).

(ii) Obviously, $\#$ is a symmetric relation and it is clear that (γ), resp. (δ), implies (α). Conversely, (α) implies (γ), resp. (δ) by Lemma 3.15(i) and Lemma 3.12(ii).

(iii) is a consequence of Lemma 3.14(i).

4. Type decomposition of JBW*-triples

This section contains the main results of the paper.

LEMMA 4.1. *Let $\mathcal{A}$ be a JBW*-triple, let $(u_i)_{i\in I}$ be a family of pairwise unrelated tripotents in $\mathcal{A}$ and let r be a tripotent in the segment $\left[0, \bigvee_{i\in I} u_i\right]$. Then, for all $i \in I$,*

$$r \wedge u_i = \{ru_ir\}.$$

Proof. For all $i \in I$, $\mathcal{A}$ equals $\mathcal{J}(u_i) \oplus \mathcal{J}(u_i)'$ and therefore r equals $r_i + r_i'$ where r_i, resp. r_i', is a tripotent in $\mathcal{J}(u_i)$, resp. in $\mathcal{J}(u_i)'$. Clearly, for all $i \in I$, $r_i \leq r \leq \bigvee_{j\in I} u_j$ and, for all $j \in I$ not equal to i, $r_i \perp u_j$. Let $\mathcal{F}$ denote the directed set of finite subsets of I. By Lemma 3.2(iii), Proposition 3.8(iii), separate w*-continuity of the Jordan triple product and Lemma 3.1(iii), for all $i \in I$,

$$r_i = \left\{r_i\left(w^* - \lim_{F\in\mathcal{F}} \sum_{j\in F} u_j\right)r_i\right\} = w^* - \lim_{F\in\mathcal{F}} \sum_{j\in F} \{r_iu_jr_i\} = \{r_iu_ir_i\}.$$

Lemma 3.2(iii) implies $r_i \leq u_i$, for all $i \in I$. Let now w be a tripotent such that $w \leq r$ and $w \leq u_i$ then w is an element in $\mathcal{I}(u_i)$ and therefore $w \perp r_i'$. By Lemma 3.2(iii) and Lemma 3.1(iii),

$$w = \{wrw\} = \{wr_iw\} + \{wr_i'w\} = \{wr_iw\}$$

and, by Lemma 3.2(iii), $w \leq r_i$. Therefore r_i equals $r \wedge u_i$ and the claim follows, by

$$\{ru_ir\} = \{(r_i + r_i')u_i(r_i + r_i')\} = \{r_iu_ir_i\} = r_i.$$

Let $\bigotimes_{i \in I} (P_i, \leq)$ denote the direct product of the posets $(P_i, \leq)$ and let $\leq_\otimes$ denote the (pointwise) ordering on $\bigotimes_{i \in I} (P_i, \leq)$.

Lemma 4.2. *Let $\mathcal{A}$ be a JBW*-triple and let $(u_i)_{i \in I}$ be a family of pairwise unrelated tripotents in $\mathcal{A}$. Then the mapping*

$$\phi\colon \bigotimes_{i \in I} ([0, u_i], \leq) \to \left(\left[0, \bigvee_{i \in I} u_i\right], \leq\right)$$

defined, for all families $(v_i)_{i \in I}$ of tripotents in $\bigotimes_{i \in I} ([0, u_i], \leq)$, by

$$\phi((v_i)_{i \in I}) := \bigvee_{i \in I} v_i$$

is an order isomorphism.

Proof. Let $(r_i)_{i \in I}$ be an element in the poset $\bigotimes_{i \in I} ([0, u_i], \leq)$ then $(r_i)_{i \in I}$ is a family of pairwise orthogonal tripotents. If $\mathcal{F}$ denotes the directed set of finite subsets of I then, by Proposition 3.8(iii), $\bigvee_{i \in I} r_i$ exists and equals $w^* - \lim_{F \in \mathcal{F}} \sum_{i \in F} r_i$. Clearly, if $(v_i)_{i \in I} \leq_\otimes (w_i)_{i \in I}$ then $\phi((v_i)_{i \in I}) \leq \phi((w_i)_{i \in I})$.

Conversely, $\phi((v_i)_{i \in I}) \leq \phi((w_i)_{i \in I})$ implies $\left[0, \bigvee_{i \in I} v_i\right] \subseteq \left[0, \bigvee_{i \in I} w_i\right]$ and therefore, for all $j \in I$, $\left[0, \bigvee_{i \in I} v_i\right] \cap U(\mathcal{I}(u_j)) \subseteq \left[0, \bigvee_{i \in I} w_i\right] \cap U(\mathcal{I}(u_j))$. In order to prove $v_j \leq w_j$, for all $j \in I$, it is sufficient to show that, if $(r_i)_{i \in I}$ in $\bigotimes_{i \in I} ([0, u_i], \leq)$, $\left[0, \bigvee_{i \in I} r_i\right] \cap U(\mathcal{I}(u_j))$ equals $[0, r_j]$. Certainly, $[0, r_j] \subseteq \left[0, \bigvee_{i \in I} r_i\right] \cap U(\mathcal{I}(u_j))$. Let now q be an element in $\left[0, \bigvee_{i \in I} r_i\right] \cap U(\mathcal{I}(u_j))$. Then, for all $i \in I$ not equal to j, $q \perp r_i$ and, by Lemma 3.2(iii), separate w*-continuity of the Jordan triple product and Lemma 3.1(iii),

$$q = \left\{q\left(\bigvee_{i\in I} r_i\right)q\right\} = \left\{q\left(w^* - \lim_{F\in\mathscr{F}} \sum_{i\in F} r_i\right)q\right\} = w^* - \lim_{F\in\mathscr{F}} \sum_{i\in F} \{qr_iq\} = \{qr_jq\}.$$

Again by Lemma 3.2(iii), $q \leqslant r_j$.

It remains to prove that ϕ is onto. Let r be an element in $\left[0, \bigvee_{i\in I} u_i\right]$ then $r \wedge u_i$ equals $\{ru_ir\}$, by Lemma 4.1. By Proposition 3.4(iii), $(r \wedge u_i)_{i\in I}$ is a family of pairwise orthogonal tripotents and, if $\mathscr{F}$ denotes the directed set of finite subsets of I, it follows, by Proposition 3.8(iii), that

$$\bigvee_{i\in I} (r \wedge u_i) = w^* - \lim_{F\in\mathscr{F}} \sum_{i\in F} \{ru_ir\} = \left\{r\left(w^* - \lim_{F\in\mathscr{F}} \sum_{i\in F} u_i\right)r\right\} = \left\{r \bigvee_{i\in I} u_ir\right\} = r.$$

Let $\mathscr{P}$ be the class of lattices with 0 and 1. Then $\mathscr{P}$ is made into a category by means of the order preserving maps as morphisms. A non-empty subclass P is called an *order property* if it is closed under direct products, if with $(L, \leqslant)$ all isomorphic images of $(L, \leqslant)$ belong to P and if $(L, \leqslant)$ in P and $p \in L$ implies that $([0, p], \leqslant)$ belongs to P. Let $\mathscr{A}$ be a JBW*-triple and let P be an order property. A tripotent u in $\mathscr{A}$ is called a *P-tripotent* if $([0, u], \leqslant)$ belongs to P. A JBW*-triple $\mathscr{A}$ is said to have *property P*, resp. *local-property P*, if there exists a central P-tripotent u, resp. a P-tripotent u, in $\mathscr{A}$ such that $\mathscr{I}(u)$ equals $\mathscr{A}$.

LEMMA 4.3. *Let $\mathscr{A}$ be a JBW*-triple and let P be an order property. Then*:

(i) *If $(u_i)_{i\in I}$ is a family of pairwise unrelated P-tripotents, resp. pairwise orthogonal central P-tripotents, then $\bigvee_{i\in I} u_i$ is a P-tripotent, resp. a central P-tripotent.*

(ii) *$\mathscr{A}$ has property P if and only if there exists a P-tripotent u in $\mathscr{A}$ which is maximal.*

(iii) *Let $\mathscr{A}$ be a JBW*-triple not equal to $\{0\}$. Then $\mathscr{A}$ has property P, resp. local-property P, if and only if every w*-closed ideal in $\mathscr{A}$ not equal to $\{0\}$ contains a non-zero central P-tripotent, resp. a non-zero P-tripotent.*

(iv) *Let $\mathscr{I}$ and $\mathscr{J}$ be w*-closed orthogonal ideals in $\mathscr{A}$ such that, considered as JBW*-triples, both have property P, resp. local-property P. Then $\mathscr{I} \oplus \mathscr{J}$ is a w*-closed ideal such that, considered as a JBW*-triple, it has property P, resp. local-property P.*

Proof. (i) Note that, by Proposition 3.8(iii), $\bigvee_{i\in I} u_i$ exists. The claim is now a consequence of Lemma 4.2 and Lemma 3.14(ii).

(ii) is implied by Lemma 3.14(i).

(iii Let $\mathscr{A}$ be a JBW*-triple not equal to $\{0\}$ such that every w*-closed ideal in $\mathscr{A}$ not equal to $\{0\}$ contains a non-zero P-tripotent, resp. a non-zero central P-tripotent, and let $(u_i)_{i\in I}$ be a maximal family of pairwise unrelated P-tripotents, resp. pairwise orthogonal central P-tripotents. By (i), $\bigvee_{i\in I} u_i$ is a P-tripotent, resp. a central P-tripotent. If $\mathscr{I}\left(\bigvee_{i\in I} u_i\right)$ is not equal to $\mathscr{A}$ then the ideal $\mathscr{I}\left(\bigvee_{i\in I} u_i\right)'$ is an ideal which contains, by the maximality of the family $(u_i)_{i\in I}$, no non-zero P-tripotent, resp. no non-zero central P-tripotent. This contradicts the assumption.

Conversely, if u is a non-zero P-tripotent, resp. a non-zero central P-tripotent, such that $\mathscr{I}(u)$ equals $\mathscr{A}$ then, by (i)-and (ii) of Lemma 3.15, it follows that every w*-closed ideal in $\mathscr{A}$ contains a non-zero P-tripotent, resp. a non-zero central P-tripotent.

(iv) Let u, resp. v, be P-tripotents in $\mathscr{I}$, resp. $\mathscr{J}$, such that $\mathscr{I}(u)$ equals $\mathscr{I}$, resp. $\mathscr{I}(v)$ equals $\mathscr{J}$. Since u and v are unrelated it follows, by (i), that $u+v$ is a P-tripotent. Because u and v are tripotents in $\mathscr{I}(u+v)$ it follows that $\mathscr{I}(u)\oplus\mathscr{I}(v)\subseteq\mathscr{I}(u+v)$. Conversely, $u+v$ is an element in $\mathscr{I}(u)\oplus\mathscr{I}(v)$ and therefore $\mathscr{I}(u+v)\subseteq\mathscr{I}(u)\oplus\mathscr{I}(v)$. Moreover, if u and v are central then, by Lemma 3.14(ii), also $u+v$ is central.

Let A be a JBW-algebra let p be an idempotent in A and let P be an order property, then p is called a *P-idempotent* if $([0,p],\leqslant)$ belongs to P. A JBW-algebra A is called to have *property P*, resp. *local-property P*, if its unit e is a P-idempotent, resp. if there is a P-idempotent q in A with central support e.

LEMMA 4.4. *Let $\mathscr{A}$ be a JBW*-triple and let P be an order property. Then:*

(i) *$\mathscr{A}$ has property P, resp. local-property P, if and only if there exists a maximal tripotent u in $\mathscr{A}$ such that the JBW-algebra $\mathscr{A}_2(u)^J_{sa}$ has property P, resp. local-property P.*

(ii) *$\mathscr{A}$ contains no non-zero P-tripotents, resp. no non-zero central P-tripotents, if and only if, for all maximal tripotents u in $\mathscr{A}$, the JBW-algebra $\mathscr{A}_2(u)^J_{sa}$ contains no non-zero P-idempotents, resp. no non-zero central P-idempotents.*

Proof. (i) By Lemma 4.3(ii), $\mathscr{A}$ has property P if and only if there exists a maximal P-tripotent u in $(U(\mathscr{A}),\leqslant)$. This is equivalent to the fact that the JBW-algebra $\mathscr{A}_2(u)^J_{sa}$ has property P.

Let now $\mathscr{A}$ be a JBW*-triple which has local-property P then there exists a P-tripotent v in $\mathscr{A}$ such that $\mathscr{I}(v)$ equals $\mathscr{A}$. Let u be a maximal tripotent in $\mathscr{A}$ such that $v \leq u$, then, by (i) and (ii) of Lemma 3.5, v is a P-idempotent in $\mathscr{A}_2(u)^J_{sa}$. By (4.2) [8] and [3], $\mathscr{A}_2(u)^J_{sa}$ is the smallest w*-closed ideal in the JBW-algebra $\mathscr{A}_2(u)^J_{sa}$ containing v and therefore, in the JBW-algebra $\mathscr{A}_2(u)^J_{sa}$, the unit u is the central support of v. Therefore the JBW-algebra $\mathscr{A}_2(u)^J_{sa}$ has local-property P.

Conversely, let u be a maximal tripotent in the JBW*-triple $\mathscr{A}$ such that the JBW-algebra $\mathscr{A}_2(u)^J_{sa}$ has local-property P. Then there exists a P-idempotent v in $\mathscr{A}_2(u)^J_{sa}$ with central support u. Hence $\mathscr{A}_2(u)^J_{sa}$ is the smallest w*-closed ideal in the JBW-algebra $\mathscr{A}_2(u)^J_{sa}$ containing v and, again by (4.2) [8] and [3], $\mathscr{A}$ is the smallest w*-closed ideal in the JBW*-triple $\mathscr{A}$ containing v. It follows that $\mathscr{A}$ has local-property P.

(ii) The assertion is proved by (i) and (ii) of Lemma 3.5 and (iii) and (iv) of Lemma 3.14.

Let A be a JBW-algebra which has property, P, resp. local-property P, and let $\mathscr{A}$ be the complexification of A. Then, by Lemma 4.4(i), $\mathscr{A}$ together with the Jordan triple product defined by (1) becomes an unitary JBW*-triple which has property P, resp. local-property P.

THEOREM 4.5. *Let $\mathscr{A}$ be a JBW*-triple and let P be an order property. Then:*

(i) *There exists a unique decomposition of $\mathscr{A}$ into the direct sum of two orthogonal w*-closed ideals $\mathscr{I}$ and $\mathscr{I}'$ such that $\mathscr{I}$ considered as a JBW*-triple has local-property P and $\mathscr{I}'$ contains no non-zero P-tripotents.*

(ii) *There exists a unique decomposition of $\mathscr{A}$ into the direct sum of two orthogonal w*-closed ideals $\mathscr{I}$ and $\mathscr{I}'$ such that $\mathscr{I}$ considered as a JBW*-triple has property P and $\mathscr{I}'$ contains no non-zero central P-tripotents.*

Proof. (i) Let $(u_i)_{i\in I}$ be a maximal family of pairwise unrelated P-tripotents in $\mathscr{A}$. By Lemma 4.3(i), $\bigvee_{i\in I} u_i$ is a P-tripotent and $\mathscr{I}\left(\bigvee_{i\in I} u_i\right)$ is therefore an ideal which has local-property P. If v is a non-zero P-tripotent in $\mathscr{I}\left(\bigvee_{i\in I} u_i\right)'$ then v would enlarge the family $(u_i)_{i\in I}$ which is a contradiction. Let $\mathscr{I} \oplus \mathscr{I}'$ and $\mathscr{J} \oplus \mathscr{J}'$ be such decompositions. Then $\mathscr{I}$ equals $\mathscr{I} \cap \mathscr{J} \oplus \mathscr{I} \cap \mathscr{J}'$, by Lemma 3.13(ii). Since $\mathscr{I} \cap \mathscr{J}'$ contains no non-zero P-tripotents it follows, by (i) and (ii) of Lemma 3.15, that $\mathscr{I} \cap \mathscr{J}'$ equals $\{0\}$ and therefore $\mathscr{I}$ equals $\mathscr{J}$ and $\mathscr{I}'$ equals $\mathscr{J}'$.

(ii) Let $(u_i)_{i\in I}$ be a maximal family of pairwise orthogonal central P-tripotents in $\mathscr{A}$. By Lemma 4.3(i), $\bigvee_{i\in I} u_i$ is a central P-tripotent and

$\mathcal{I}\left(\bigvee_{i \in I} u_i\right)$ is therefore an ideal which has property P. If v is a central P-tripotent in $\mathcal{I}\left(\bigvee_{i \in I} u_i\right)'$ then v would enlarge the family $(u_i)_{i \in I}$ which is a contradiction. Uniqueness follows as in (i).

A non-zero element p in a poset $(L, \leqslant)$ with 0 is called an *atom* if, for all q in L, $q \leqslant p$ implies that q equals p or 0. A poset $(L, \leqslant)$ is called *atomic* if for every non-zero element p in L there exists an atom q in L such that $q \leqslant p$. Considering lattices with 0 and 1 only, atomicity is an order property.

PROPOSITION 4.6. *A JBW*-triple $\mathcal{A}$ has a unique decomposition*

$$\mathcal{A} = \mathcal{I}_a \oplus \mathcal{I}_{na}$$

into a direct sum of two pairwise orthogonal w-closed ideals such that $\mathcal{I}_a$ is atomic and $\mathcal{I}_{na}$ contains no non-zero atomic tripotents.*

Proof. By (i) and (ii) of Theorem 4.5, there exists a unique decomposition

$$\mathcal{A} = \mathcal{I}_a \oplus \mathcal{I}_{la} \oplus \mathcal{I}_{na}$$

into a direct sum of three w*-closed ideals such that $\mathcal{I}_a$ is atomic, $\mathcal{I}_{la}$ is locally atomic and contains no non-zero central atomic tripotent and $\mathcal{I}_{na}$ contains no non-zero atomic tripotent. As in Proposition 20 [14], it can be shown in the JBW-algebra case that the supremum of all atoms in a JBW-algebra is a central atomic idempotent and therefore every locally atomic JBW-algebra is atomic. By Lemma 4.4(i), $\mathcal{I}_{la}$ equals $\{0\}$.

LEMMA 4.7. *Let $\mathcal{A}$ be a JBW*-triple and let u be a tripotent in $\mathcal{A}$. Then u is an atom if and only if $\mathcal{A}_2(u)$ equals* $\mathbb{C}u$.

Proof. This follows from the fact that $\mathcal{A}_2(u)$ is the closure of the linear span of the tripotents in $[0, u]$.

By Lemma 4.7, the atoms are precisely the minimal tripotents, in the sense of [5]. In Theorem 2 [5], a JBW*-triple is uniquely decomposed into a w*-closed ideal $\mathcal{I}$ which is the w*-closure of the linear span of the minimal tripotents and an ideal $\mathcal{I}'$ which contains no minimal tripotents. The relationship between this decomposition and the decomposition found in Proposition 4.6 is established in the following corollary:

COROLLARY 4.8. *Let $\mathcal{A}$ be a JBW*-triple. Then the w*-closure of the linear span of the atoms of $(U(\mathcal{A}), \leqslant)$ equals $\mathcal{I}_a$ where the notation is the same as in Proposition* 4.6.

Proof. Let $\mathcal{I}$ denote the w*-closure of the linear span of the atoms in $\mathcal{A}$. Since an atom is atomic, $\mathcal{I}$ is a w*-closed subset of $\mathcal{I}_a$. By Theorem 2

[5], $\mathcal{I}$ is an ideal and therefore $\mathcal{I}_a$ equals $\mathcal{I} \oplus \mathcal{I}'$, where $\mathcal{I}'$ is a w*-closed ideal which contains no atoms. By Proposition 4.6 and Lemma 4.3(iii), $\mathcal{I}'$ equals $\{0\}$.

A lattice $(L, \leq)$ is called *modular* if, for all p, q and r in L with $p \leq r$, $(p \vee q) \wedge r$ equals $p \vee (q \wedge r)$. A lattice $(L, \leq)$ is called *distributive* if, for all p, q and r in L, $p \wedge (q \vee r)$ equals $(p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r)$ equals $(p \vee q) \wedge (p \vee r)$. Note that distributivity implies modularity and that, considering lattices with 0 and 1 only, modularity and distributivity are order properties.

PROPOSITION 4.9. *A JBW*-triple $\mathcal{A}$ has a unique decomposition*

$$\mathcal{A} = \mathcal{I}_d \oplus \mathcal{I}_{mld} \oplus \mathcal{I}_{ld} \oplus \mathcal{I}_m \oplus \mathcal{I}_{lm} \oplus \mathcal{I}_n$$

into a direct sum of six pairwise orthogonal w-closed ideals such that $\mathcal{I}_d$ is distributive, $\mathcal{I}_{mld}$ is locally distributive and modular and contains no non-zero central distributive tripotents, $\mathcal{I}_{ld}$ is locally distributive and contains no non-zero central modular tripotents, $\mathcal{I}_m$ is modular and contains no non-zero distributive tripotents, $\mathcal{I}_{lm}$ is locally modular and contains no non-zero central modular tripotents and no non-zero distributive tripotents and $\mathcal{I}_n$ contains no non-zero modular tripotents. Moreover, $\mathcal{I}_d \oplus \mathcal{I}_{mld} \oplus \mathcal{I}_{ld}$ is locally distributive and $\mathcal{I}_m \oplus \mathcal{I}_{lm}$ is locally modular and contains no non-zero distributive tripotents.*

Proof. By (i) and (ii) of Theorem 4.5, there exists such a decomposition. Uniqueness is shown as in the proof of Theorem 4.5(i). The remaining assertions are consequences of Lemma 4.3(iv).

LEMMA 4.10. *Let $\mathcal{A}$ be a JBW*-triple and let u be a tripotent in $\mathcal{A}$. Then u is distributive if and only if $\mathcal{A}_2(u)$ is an abelian JBW*-triple.*

Proof. $\mathcal{A}_2(u)$ is an abelian JBW*-triple if and only if $\mathcal{A}_2(u)^J_{sa}$ is an associative JBW-algebra and a JBW-algebra is associative if and only if the poset of its idempotents is distributive. By (i) and (ii) of Lemma 3.5, the claim now follows.

By Lemma 4.10, the distributive tripotents are precisely the abelian tripotents, in the sense of [8]. Moreover, a JBW*-triple $\mathcal{A}$ is called a JBW*-*triple of type I*, in [8], if there exists an abelian tripotent v in $\mathcal{A}$ such that $\mathcal{I}(v)$ equals $\mathcal{A}$. In (4.13) [8], a JBW*-triple is uniquely decomposed into a direct sum of two w*-closed ideals such that the first one is of type I and the second one contains no non-zero abelian tripotents. In [9], a JBW*-triple $\mathcal{A}$ of type I is called a *JBW*-triple of type I_1* if there exists a central abelian tripotent v such that $\mathcal{I}(v)$ equals

$\mathcal{A}$. The relationship between this decomposition and the decomposition found in Proposition 4.9 is established in the following corollary:

COROLLARY 4.11. *Let $\mathcal{A}$ be a JBW*-triple and let $\mathcal{J} \oplus \mathcal{J}'$ be the decomposition of $\mathcal{A}$ into the direct sum of two w*-closed ideals such that $\mathcal{J}$ is of type I and $\mathcal{J}'$ contains no non-zero abelian tripotents. Then $\mathcal{J}$ equals $\mathcal{J}_d \oplus \mathcal{J}_{mld} \oplus \mathcal{J}_{ld}$ and $\mathcal{J}'$ equals $\mathcal{J}_m \oplus \mathcal{J}_{lm} \oplus \mathcal{J}_n$ where the notation is the same as in Proposition* 4.9. *Moreover, $\mathcal{J}_d$ is of type I_1 and $\mathcal{J}'_d$ contains no non-zero central abelian tripotents.*

Proof. By Proposition 4.9, Lemma 4.10 and Lemma 4.3(iv), the corollary follows

COROLLARY 4.12. *Let $\mathcal{A}$ be a JBW*-triple. Then, using the notation from Proposition* 4.9 *and the terminology of* [7], *$\mathcal{A}$ equals $\mathcal{J}_d$ ($\mathcal{J}_d \oplus \mathcal{J}_{mld} \oplus \mathcal{J}_{ld}$, $\mathcal{J}_m$, $\mathcal{J}_{lm}$, $\mathcal{J}_m \oplus \mathcal{J}_{lm}$, $\mathcal{J}_n$, respectively) if and only if there exists a maximal tripotent u in $\mathcal{A}$ such that $\mathcal{A}_2(u)^J_{sa}$ is a JBW-algebra of type I_1 (I, II_1, II_∞, II, III, respectively).*

Proof. The claim is a consequence of Proposition 4.9, Lemma 4.3(iv) and (i) and (ii) of Lemma 4.4.

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