

DEGENERATIONS FOR SELF-INJECTIVE ALGEBRAS OF TREECLASS D_n

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ABSTRACT

Let Λ be a connected representation finite selfinjective algebra. According to G. Zwara the partial orders \leq_{ext} and \leq_{deg} on the isomorphism classes of d -dimensional Λ -modules are equivalent if and only if the stable Auslander–Reiten quiver Γ_Λ of Λ is not isomorphic to $\mathbb{Z}D_{3m}/\tau^{2m-1}$ for all $m \geq 2$. The paper describes all minimal degenerations $M \leq_{\text{deg}} N$ with $M \not\leq_{\text{ext}} N$ in the case when $\Gamma_\Lambda \cong \mathbb{Z}D_{3m}/\tau^{2m-1}$ for some $m \geq 2$.

1. Introduction

1.1. The affine variety $\text{mod}_d \Lambda$

Let k be an algebraically closed field and Λ be a finite dimensional associative k -algebra with unit. We denote by $\text{mod} \Lambda$ the category of finitely generated Λ -left-modules. A d -dimensional Λ -module M is the vectorspace k^d together with a multiplication by Λ from the left.

Now let $\lambda_1 = 1, \lambda_2, \dots, \lambda_n$ be a k -basis of Λ . Then $\lambda_i \lambda_j = \sum_l a_{ij}^l \lambda_l$ for $i, j = 1, \dots, n$ with the structure constants $a_{ij}^l \in k$. The multiplication of M by λ_i induces an endomorphism of k^d which we can represent by a $d \times d$ matrix over k with respect to the standard basis of k^d . Thus M corresponds to a unique n -tuple of matrices $m = (E, m_2, \dots, m_n) \in (\text{Mat}_{d \times d}(k))^n$, where E denotes the identity matrix, and such an n -tuple m with $m_1 = E$ corresponds to a d -dimensional Λ -module if and only if it satisfies the equations $m_i m_j = \sum_l a_{ij}^l m_l$ for $i, j = 1, \dots, n$. We denote the set of all n -tuples corresponding to a d -dimensional Λ -module by $\text{mod}_d \Lambda$ and we will identify the module with its n -tuple. For each i with $1 \leq i \leq n$ let X^i denote the matrix $(x_{\mu\nu}^i)_{\mu, \nu=1, \dots, d}$. Then $\text{mod}_d \Lambda$ is the zero set of the ideal $I \subset k[x_{\mu\nu}^\xi]$ ($\mu, \nu = 1, \dots, d; \xi = 1, \dots, n$), where I is generated by the components of the matrices $X^i X^j - \sum_l a_{ij}^l X^l$ for $i, j = 1, \dots, n$. This gives $\text{mod}_d \Lambda$ the structure of an affine variety, which does not have to be irreducible.

The general linear group $\text{GL}_d(k)$ acts on $\text{mod}_d \Lambda$ by conjugation, that is to say $g \cdot (m_1, \dots, m_n) = (gm_1 g^{-1}, \dots, gm_n g^{-1})$ for $g \in \text{GL}_d(k)$ and $(m_1, \dots, m_n) \in \text{mod}_d \Lambda$. The orbits under this action are the isomorphism classes of d -dimensional Λ -modules (see [7]). This definition of $\text{mod}_d \Lambda$ depends on the chosen basis of Λ only up to a $\text{GL}_d(k)$ equivariant isomorphism of affine varieties.

1.2. Partial orders on isomorphism classes of $\text{mod}_d \Lambda$

A module N is called a degeneration of M (in symbols $M \leq_{\text{deg}} N$) if N belongs to the Zariski closure of the $\text{GL}_d(k)$ -orbit of M in $\text{mod}_d \Lambda$. Since orbits are irreducible

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and open in their closure, this defines a partial order on the set of isomorphism classes of d -dimensional Λ -modules. It is an interesting problem to express the partial order \leq_{\deg} in algebraic terms. There are several articles in this direction, including works by S. Abeasis and A. del Fra [1], K. Bongartz [4, 5], C. Riedtmann [11] and G. Zwara [12, 14], connecting \leq_{\deg} to other partial orders on the isomorphism classes of d -dimensional Λ -modules.

In [15] Zwara gives an alternative description of \leq_{\deg} , that is to say $M \leq_{\deg} N$ if and only if there exists a short exact sequence

$$0 \longrightarrow S \longrightarrow S \oplus M \longrightarrow N \longrightarrow 0 \quad (1)$$

for some Λ -module S .

We are concerned with two other partial orders on the isomorphism classes of d -dimensional Λ -modules. The partial order \leq_{ext} is the transitive closure of the relation $M \leq_{\text{ext}} N$ if there exists a short exact sequence

$$0 \longrightarrow N_1 \longrightarrow M \longrightarrow N_2 \longrightarrow 0 \quad (2)$$

with $N \cong N_1 \oplus N_2$. We take the pullback of the sequence (2) with the canonical projection $N \longrightarrow N_2$ according to the isomorphism $N \cong N_1 \oplus N_2$. This results in a sequence as in (1) with $S = N_1$, so \leq_{ext} implies \leq_{\deg} .

The hom order \leq is the partial order given by $M \leq N$ if and only if

$$[M, X] \leq [N, X]$$

for every Λ -module X , where $[U, V] := \dim_k \text{Hom}_{\Lambda}(U, V)$ for Λ -modules U and V . It follows immediately from (1) and the left-exactness of $\text{Hom}_{\Lambda}(_, X)$ that \leq_{\deg} implies \leq . The reverse implication is not true in general. However it holds for representation finite algebras (see [14]) and tame concealed algebras (see [4]).

1.3. Statement of the theorem

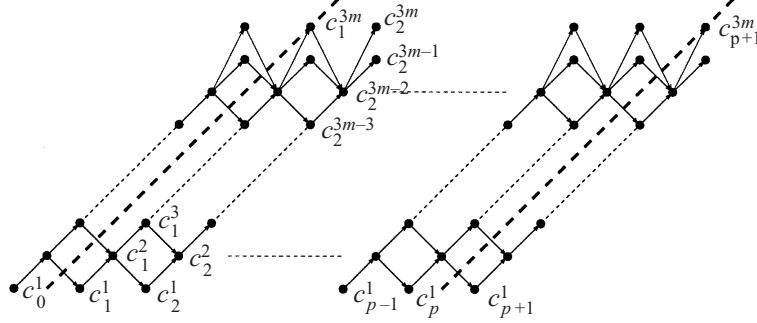
We define the Auslander–Reiten quiver Γ_{Λ} of Λ as the quiver whose vertices are representatives of the isomorphism classes of indecomposable Λ -modules. There is an arrow $x \longrightarrow y$ between the vertices x and y if there exists an irreducible morphism from a Λ -module represented by x to one of y . This definition coincides with the usual one in the representation finite case (see [3]) and is appropriate for our consideration.

We denote by τ the Auslander–Reiten translation. It is a bijection from the isomorphism classes of indecomposable non-projective Λ -modules to the isomorphism classes of indecomposable non-injective Λ -modules.

The stable Auslander–Reiten quiver Γ_{Λ}^s of Λ is the full subquiver of Γ_{Λ} containing all the vertices x for which $\tau^n(x)$ is defined for all $n \in \mathbb{Z}$.

Let Λ be connected and selfinjective of finite representation type. C. Riedtmann showed in [8] that the stable Auslander–Reiten quiver Γ_{Λ}^s of Λ is isomorphic to $\mathbb{Z}\Delta/G$ where Δ is one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8 and G is an admissible automorphism group of $\mathbb{Z}\Delta$. Using the results in [10], [9] and [6] about the category of modules over representation finite selfinjective algebras, G. Zwara showed in [13] that the partial orders \leq_{ext} and \leq_{\deg} coincide if and only if $\Gamma_{\Lambda}^s \not\cong \mathbb{Z}D_{3m}/\tau^{2m-1}$ for all $m \geq 2$.

We want to investigate the difference between the partial orders \leq_{\deg} and \leq_{ext} in those exceptional cases. In particular, we want to describe the minimal degenerations

FIGURE 1. $\mathbb{Z}D_{3m}/\tau^{2m-1}$.

$M \leq_{\text{deg}} N$ with $M \not\leq_{\text{ext}} N$ for a connected selfinjective algebra of finite representation type with stable Auslander–Reiten quiver $\mathbb{Z}D_{3m}/\tau^{2m-1}$. A degeneration $M \leq_{\text{deg}} N$ is called minimal if it is a proper degeneration, that is to say $M \not\cong N$, and if there exists no module P with $M \not\cong P \not\cong N$ and $M \leq_{\text{deg}} P \leq_{\text{deg}} N$. It is an interesting question how complicated minimal degenerations are. Some results concerning the complexity of degenerations can be found in [2].

G. Zwara proved in [15, Theorem 4] that for a minimal degeneration $M \leq_{\text{deg}} N$ with $M \not\leq_{\text{ext}} N$ there exist decompositions $M \cong M' \oplus W$ and $N \cong N' \oplus W$ such that N' is indecomposable and $M' \leq_{\text{deg}} N'$ is a minimal degeneration. Therefore it is enough to concentrate on degenerations to indecomposables.

The stable translation quiver $\mathbb{Z}D_{3m}$ has the vertices c_j^i where $i \in \{1, \dots, 3m\}$ and $j \in \mathbb{Z}$. There are arrows $c_j^i \rightarrow c_{j+1}^i$ and $c_{j+1}^i \rightarrow c_j^{i+1}$ for $1 \leq i \leq 3m-2$ and arrows $c_j^{3m-2} \rightarrow c_j^{3m}$ and $c_j^{3m} \rightarrow c_{j+1}^{3m-2}$. The translation is given by $\tau(c_j^i) = c_{j-1}^i$. Thus the vertices c_j^i and c_{j+2m-1}^i are identified in the quotient $\mathbb{Z}D_{3m}/\tau^{2m-1}$. In Figure 1 the stable Auslander–Reiten quiver $\mathbb{Z}D_{3m}/\tau^{2m-1}$ is drawn for m even. Every letter refers to the vertex at its left and the thick diagonal lines indicate the $(2m-1)$ -period of the translation τ .

THEOREM 1.1. *Let Λ be a connected and selfinjective algebra of finite representation type whose stable Auslander–Reiten quiver is isomorphic to $\mathbb{Z}D_{3m}/\tau^{2m-1}$. There exists a proper degeneration $M \leq_{\text{deg}} N$ to the indecomposable Λ -module N if and only if N corresponds to a vertex c_l^s with $m+1 \leq s \leq 2m$. Moreover, the module M is determined by N up to isomorphism.*

2. Preliminaries

We want to represent pairs of modules (M, N) in terms of \mathbb{Z} -valued difference-functions on the set of isomorphism classes of indecomposable modules and to characterize those functions corresponding to pairs (M, N) with $M \leq_{\text{deg}} N$. This will enable us to give a combinatorial proof of the theorem.

We say that the modules M and N are disjoint if they have no common direct summand. We denote by \bar{M} the isomorphism class of the module M and by \mathcal{S} the set of ordered pairs (\bar{M}, \bar{N}) such that M and N are disjoint. To every pair (\bar{M}, \bar{N}) we associate the function $\delta_{M,N}$ given by $\delta_{M,N}(X) = [N, X] - [M, X]$.

Let $\mu(X, A)$ be the multiplicity of the indecomposable direct summand X in the direct sum decomposition of A . In particular $A \cong \bigoplus_{\bar{X}: X \text{ indec}} X^{\mu(X,A)}$.

$$\begin{array}{ccc}
& \oplus_{\bar{X}; X \text{ indec.}} \mathbb{Z} \bar{X} & \\
\alpha \nearrow & & \searrow \beta \\
\mathcal{S} & \xrightarrow{\gamma} & (\oplus_{\bar{Y}; Y \text{ indec.}} \mathbb{Z} \bar{Y})^*
\end{array}$$

FIGURE 2.

Let $(\oplus_{\bar{Y}; Y \text{ indec.}} \mathbb{Z} \bar{Y})^* = \text{Hom}_{\mathbb{Z}}(\oplus_{\bar{Y}; Y \text{ indec.}} \mathbb{Z} \bar{Y}, \mathbb{Z})$. We consider the diagram in Figure 2 where α, β and γ are given by

$$\begin{aligned}
\alpha(\bar{M}, \bar{N}) &= \sum_{\bar{X}; X \text{ indec.}} (\mu(X, N) - \mu(X, M)) \bar{X}, \\
\gamma(\bar{M}, \bar{N}) &= \delta_{M, N},
\end{aligned}$$

$$\beta \left(\sum_{\bar{X}; X \text{ indec.}} \lambda_{\bar{X}} \bar{X} \right) = \sum_{\bar{X}; X \text{ indec.}} \lambda_{\bar{X}} [X, \quad].$$

The diagram commutes since

$$\beta \circ \alpha(\bar{M}, \bar{N}) = \sum_{\bar{X}} (\mu(X, N) - \mu(X, M)) [X, \quad] = [N, \quad] - [M, \quad] = \delta_{M, N}.$$

Obviously α is a bijection and β is \mathbb{Z} -linear.

LEMMA 2.1. *If Λ is of finite representation type then β is an isomorphism.*

Proof. The map β is \mathbb{Z} -linear between free \mathbb{Z} -modules of the same finite rank. Thus it suffices to show that β is surjective. For each indecomposable module X we consider the exact sequence

$$X \longrightarrow E'_X \longrightarrow \tau^{-1}X \longrightarrow 0$$

which is the Auslander–Reiten sequence starting in X if X is not injective. Otherwise we set $E'_X = X/\text{soc}(X)$ and $\tau^{-1}X = 0$. The functor $\text{Hom}_{\Lambda}(\quad, Y)$ induces the exact sequence

$$0 \longrightarrow \text{Hom}_{\Lambda}(\tau^{-1}X, Y) \longrightarrow \text{Hom}_{\Lambda}(E'_X, Y) \longrightarrow \text{Hom}_{\Lambda}(X, Y) \longrightarrow k^{\mu(X, Y)} \longrightarrow 0$$

of k -vectorspaces. Thus for every indecomposable module Y we have

$$([X, \quad] + [\tau^{-1}X, \quad] - [E'_X, \quad])(Y) = \begin{cases} 1 & \text{if } Y \cong X \\ 0 & \text{otherwise,} \end{cases}$$

showing that β is surjective. □

We want to describe the inverse of β . For each indecomposable module X we consider the exact sequence

$$0 \longrightarrow \tau X \longrightarrow E_X \longrightarrow X$$

which is the Auslander–Reiten sequence ending in X if X is not projective. Otherwise we set $E_X = \text{rad}(X)$ and $\tau X = 0$. Then β^{-1} is given by

$$\beta^{-1}(\delta) = \sum_{\bar{X}; X \text{ indec.}} (\delta(X) + \delta(\tau X) - \delta(E_X)) \bar{X}.$$

Thus, if Λ is of finite representation type, γ is bijective and we write $\gamma^{-1}(\delta) = (\bar{M}_\delta, \bar{N}_\delta) \in \mathcal{S}$. Then

$$\begin{aligned} \sum_{\bar{X}; X \text{ indec}} (\delta(X) + \delta(\tau X) - \delta(E_X)) \bar{X} &= \beta^{-1}(\delta) = \alpha \circ \gamma^{-1}(\delta) \\ &= \sum_{\bar{X}; X \text{ indec}} (\mu(X, N_\delta) - \mu(X, M_\delta)) \bar{X} \end{aligned}$$

and in consequence

$$\delta(X) + \delta(\tau X) - \delta(E_X) = \mu(X, N_\delta) - \mu(X, M_\delta) \quad (3)$$

for every indecomposable module X and every $\delta \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$.

Let $\mathcal{S}' \subset \mathcal{S}$ be the subset containing all pairs (\bar{M}, \bar{N}) with $M \leq_{\deg} N$.

LEMMA 2.2. *If Λ is of finite representation type then γ restricts to a bijection between \mathcal{S}' and the set of non-negative functions $\delta \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$ such that $\delta(I) = 0$ for every injective module I .*

Proof. Zwara showed in [14] that the partial orders \leq_{\deg} and \leq coincide for representation finite algebras. Hence $\gamma(\bar{M}, \bar{N})$ is a non-negative function for every $(\bar{M}, \bar{N}) \in \mathcal{S}'$. If I is an injective module then $[N, I] = [M, I]$ holds in consequence of the exactness of $\text{Hom}_\Lambda(-, I)$ and (1). On the other hand, let $\delta \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$ be non-negative such that $\delta(I) = 0$ for every injective module. We have to show that $\dim_k N_\delta = \dim_k M_\delta$ holds. We consider the injective module $\text{Hom}_k(\Lambda_\Lambda, k)$, where Λ_Λ denotes Λ as Λ -right module. Then the adjoint isomorphism gives $[A, \text{Hom}_k(\Lambda_\Lambda, k)] = \dim_k \text{Hom}_k(A, k) = \dim_k A$ for every Λ -module A . In particular $\dim_k N_\delta = \dim_k M_\delta$. \square

Let $\mathcal{S}'_N = \{(\bar{X}, \bar{Y}) \in \mathcal{S}' \mid \bar{Y} = \bar{N}\}$. As a consequence of Lemma 2.2 and (3) we can describe all isomorphism classes of modules degenerating to an indecomposable.

LEMMA 2.3. *Let Λ be of finite representation type and N be indecomposable. Then \mathcal{S}'_N is mapped bijectively by γ to the set of non-negative functions $\delta \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$ with $\delta(I) = 0$ for every injective module I , and satisfying*

$$\delta(X) + \delta(\tau X) - \delta(E_X) \begin{cases} = 1 & \text{if } X \cong N \\ \leq 0 & \text{otherwise} \end{cases} \quad (4)$$

for every indecomposable X .

Note that for $(\bar{M}, \bar{N}) = \gamma^{-1}(\delta)$ we have then

$$M \cong \bigoplus_{\bar{X}; X \not\cong N} X^{-(\delta(X) + \delta(\tau X) - \delta(E_X))}, \quad (5)$$

where the direct sum is taken over all isomorphism classes of indecomposable Λ -modules except that of N .

3. Proof of Theorem 1.1

Let Λ be a selfinjective finite dimensional k -algebra of finite representation type with stable Auslander–Reiten quiver isomorphic to $\mathbb{Z}D_{3m}/\tau^{2m-1}$. If $M \leq_{\deg} N$ is a

proper degeneration to the indecomposable N , then M and N are disjoint. If N were projective then the sequence (1) would split in contradiction to $M \not\cong N$. Thus N corresponds to a vertex c_l^s in the stable Auslander–Reiten quiver of Λ .

In subsection 3.1 we will characterize a function $\delta \in \gamma(\mathcal{S}'_N)$ describing a proper degeneration to N as the unique solution of a linear system depending on two vertices. One of these vertices is the vertex corresponding to N . In Subsection 3.2 we analyse this linear system. We will show that if this linear system has a solution in the natural numbers, then this solution is uniquely determined by the vertex c_l^s and $m+1 \leq s \leq 2m$ holds. It follows then from Lemma 2.3 that there exists up to isomorphism at most one module M degenerating to N . Finally we give, for the indecomposable module N corresponding to the vertex c_l^s with $m+1 \leq s \leq 2m$, a non-negative function δ which satisfies (4) for every indecomposable. Thus by Lemma 2.3 there exists a proper degeneration $M \leq_{\text{deg}} N$.

By reindexing the stable Auslander–Reiten quiver $\mathbb{Z}D_{3m}/\tau^{2m-1}$ we can assume that $l = 1$. From now on N always corresponds to the vertex c_1^s .

3.1. Characterization of δ by a linear system

We denote by $p := 2m - 1$ the period of the Auslander–Reiten translation τ and by $h := 3m - 1$ the ‘height’ of the Dynkin diagram D_{3m} .

Let us fix an element $\delta \in \gamma(\mathcal{S}'_N)$. We set

$$a_j^i := \begin{cases} 0 & \text{if } i = 0 \\ \delta(c_j^i) & \text{if } 1 \leq i \leq h-1 \\ b_j^h + b_j^{h+1} & \text{if } i = h, \end{cases}$$

where $b_j^i := \delta(c_j^i)$ for $h \leq i \leq h+1$.

Note that all the integers a_j^i and b_j^i are non-negative. We consider for each vertex c_j^i the Auslander–Reiten sequence ending in c_j^i . Since $\delta(I) = 0$ if I is an injective Λ -module we obtain the following set of inequalities from Lemma 2.3.

If $1 \leq i \leq h-1$

$$a_j^i + a_{j-1}^i - a_{j-1}^{i+1} - a_j^{i-1} \begin{cases} = 1 & \text{if } c_j^i = c_1^s \\ \leq 0 & \text{otherwise.} \end{cases} \quad (6)$$

If $h \leq i \leq h+1$

$$b_j^i + b_{j-1}^i - a_j^{h-1} \begin{cases} = 1 & \text{if } c_j^i = c_1^s \\ \leq 0 & \text{otherwise.} \end{cases} \quad (7)$$

These inequalities are the key to proving the theorem. First we derive some information on the τ -orbits in $\mathbb{Z}D_{3m}/\tau^{2m-1}$. We sum up the δ -values along each τ -orbit and set

$$a_i = \sum_{j=1}^p a_j^i \quad \text{for } 0 \leq i \leq h,$$

$$b_i = \sum_{j=1}^p b_j^i \quad \text{for } h \leq i \leq h+1.$$

Then by definition $a_h = b_h + b_{h+1}$ and $a_0 = 0$. For each fixed i we add up the inequalities of (6) and (7) respectively and we obtain

$$2a_i \leq a_{i+1} + a_{i-1} + \delta_{i,s} \quad \text{for } 1 \leq i \leq h-1, \quad (8)$$

$$2b_i \leq a_{h-1} + \delta_{i,s} \quad \text{for } h \leq i \leq h+1. \quad (9)$$

Here $\delta_{i,s}$ denotes the Kronecker symbol. By definition and the inequality (9) we get $2a_h = 2b_h + 2b_{h+1} \leq 2a_{h-1} + 1$. Hence

$$a_h \leq a_{h-1}. \quad (10)$$

REMARK 3.1. From $\delta(N) + \delta(\tau N) - \delta(E_N) = 1$ it follows immediately that $a_s \geq \delta(N) + \delta(\tau N) > 0$ if $s \leq h-1$ and $a_h > 0$ if $s \geq h$.

The following lemma implies that the case $s \geq h$ does not occur. In view of Figure 1 this means that the vertex c_1^s is not one of the somehow exceptional vertices on the upper boundary.

LEMMA 3.2. *It holds that $s \leq h-1$ and there exists an integer t with $2 \leq t \leq s$ such that*

$$t = 2b_h = 2b_{h+1} \quad \text{and} \quad a_i = \begin{cases} 0 & \text{if } 0 \leq i \leq s-t \\ i - (s-t) & \text{if } s-t \leq i \leq s \\ t & \text{if } s \leq i \leq h. \end{cases}$$

In particular $a_s = t$ is an even integer.

Proof. The inequalities in (8) are equivalent to

$$a_i - a_{i-1} \leq a_{i+1} - a_i + \delta_{i,s} \quad (11)$$

for $1 \leq i \leq h-1$. Suppose that $s \geq h$. It follows from (11) and (10) that

$$0 \leq a_1 = a_1 - a_0 \leq \dots \leq a_h - a_{h-1} \leq 0.$$

This implies that $a_i = 0$ for all $i \in \{1, \dots, h\}$ in contradiction to $a_h > 0$ by Remark 3.1. Hence $s \leq h-1$.

Again by (11) and (10) we obtain the following chain of inequalities:

$$\begin{aligned} 0 \leq a_1 = a_1 - a_0 &\leq a_2 - a_1 \leq \dots \leq a_s - a_{s-1} \\ &\leq a_{s+1} - a_s + 1 \leq \dots \leq a_h - a_{h-1} + 1 \leq 1. \end{aligned}$$

If $a_s - a_{s-1} = 0$ then $a_1 = a_2 = \dots = a_s = 0$ in contradiction to $a_s > 0$ by Remark 3.1. Hence there is an integer t with $0 < t \leq s$ such that

$$\begin{aligned} 0 &= a_1 - a_0 = \dots = a_{s-t} - a_{s-t-1}, \\ 1 &= a_{s-t+1} - a_{s-t} = \dots = a_s - a_{s-1} \\ &= a_{s+1} - a_s + 1 = \dots = a_h - a_{h-1} + 1. \end{aligned}$$

Our claim for the a_i is an easy consequence. In particular we have $a_{h-1} = a_h = b_h + b_{h+1}$, but $2b_{h+1} \leq a_{h-1}$ and $2b_h \leq a_{h-1}$ by (9). Hence we see that $t = a_{h-1} = 2b_h = 2b_{h+1}$ is an even integer. \square

As an immediate consequence of Lemma 3.2 we note that

$$2a_i - a_{i+1} - a_{i-1} = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = s - t \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

for $1 \leq i \leq h - 1$.

We want to describe the non-negative integers a_j^i and b_j^i and hence the function δ as the unique solution of a linear system. By Lemma 3.2 we have $1 = a_{s-t+1} = \sum_j \delta(c_j^{s-t+1})$, so there exists exactly one vertex c_{u-1}^{s-t+1} with $\delta(c_{u-1}^{s-t+1}) = a_{u-1}^{s-t+1} = 1$. Note that the index u is only determined modulo p . In the sequel let \tilde{u} be the representative of u with $1 \leq \tilde{u} \leq p$.

If $s > t$ the Auslander–Reiten sequence ending in c_u^{s-t} gives rise to the equation

$$a_u^{s-t} + a_{u-1}^{s-t} - a_{u-1}^{s-t+1} - a_u^{s-t-1} = -1 \quad (13)$$

because $a_i = 0$ for $i \leq s - t$, by Lemma 3.2.

We consider the following linear system which depends on the positions of the vertices c_1^s and c_{u-1}^{s-t+1} or equivalently on the integers s, t and \tilde{u} . The lower index is taken to be in $\mathbb{Z}/p\mathbb{Z}$.

$$x_j^0 = 0, \quad x_j^1 = \begin{cases} 1 & \text{if } c_j^1 = c_{u-1}^{s-t+1} \\ 0 & \text{otherwise,} \end{cases} \quad x_j^h = y_j^h + y_j^{h+1}. \quad (14)$$

If $1 \leq i \leq h - 1$

$$x_j^i + x_{j-1}^i - x_{j-1}^{i+1} - x_j^{i-1} = \begin{cases} 1 & \text{if } c_j^i = c_1^s \\ -1 & \text{if } c_j^i = c_u^{s-t} \\ 0 & \text{otherwise.} \end{cases} \quad (15)$$

If $h \leq i \leq h + 1$

$$y_j^i + y_{j-1}^i = x_j^{h-1}. \quad (16)$$

LEMMA 3.3. *If x_j^i, y_j^i is a rational solution of the linear system (14)–(16) then $y_j^{h+1} = y_j^h$ holds for all j .*

Proof. Suppose that there is j_0 with $y_{j_0}^{h+1} > y_{j_0}^h$. Since $y_j^{h+1} + y_{j+1}^{h+1} = x_{j+1}^{h-1} = y_j^h + y_{j+1}^h$ by (16) we have $-y_{j_0+1}^{h+1} > -y_{j_0+1}^h$ and successively

$$-y_{j_0}^{h+1} = (-1)^p y_{j_0+p}^{h+1} > (-1)^p y_{j_0+p}^h = -y_{j_0}^h, \quad \text{as } p \text{ is odd,}$$

in contradiction to $y_{j_0}^{h+1} > y_{j_0}^h$. \square

To any integer solution x_j^i, y_j^i of this linear system we can associate a function $\delta' \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$ by setting $\delta'(c_j^i) = x_j^i$ for $1 \leq i \leq h - 1$, $\delta'(c_j^i) = y_j^i$ for $h \leq i \leq h + 1$ and $\delta'(I) = 0$ for every injective module I . Under the same conditions we will speak of a function $\delta' \in (\bigoplus_{\bar{Y}; Y \text{ indec}} \mathbb{Z}\bar{Y})^*$ as a solution of the linear system (14)–(16).

LEMMA 3.4. *The unique solution of the linear system (14)–(16) is given by $x_j^i = a_j^i$ and $y_j^i = b_j^i$. In particular $a_j^h = 2b_j^h$ is an even integer for all j .*

Proof. With respect to uniqueness, it is obvious that the values of x_j^i are determined by the equations in (14) and (15). The values y_j^i are given by $y_j^i = x_j^h/2$ by (14) and Lemma 3.3.

The equations in (14) are obviously satisfied by a_j^i and b_j^i .

Because of (13) and $a_1^s + a_0^s - a_0^{s+1} - a_1^{s-1} = 1$ it remains for (15) to show that

$$a_j^i + a_{j-1}^i - a_{j-1}^{i+1} - a_j^{i-1} = 0$$

for $c_u^{s-t} \neq c_j^i \neq c_1^s$. By (12) we have

$$0 = 2a_i - a_{i+1} - a_{i-1} = \sum_j (a_j^i + a_{j-1}^i - a_{j-1}^{i+1} - a_j^{i-1})$$

for $s \neq i \neq s-t$ and

$$0 = 2a_{s-t} - a_{s-t+1} - a_{s-t-1} + 1 = \sum_{j \neq u} (a_j^{s-t} + a_{j-1}^{s-t} - a_{j-1}^{s-t+1} - a_j^{s-t-1})$$

for $i = s-t$ and

$$0 = 2a_s - a_{s+1} - a_{s-1} - 1 = \sum_{j \neq 1} (a_j^s + a_{j-1}^s - a_{j-1}^{s+1} - a_j^{s-1})$$

for $i = s$. However because of (6) each of the summands on the right-hand side is less than or equal to zero. Thus each summand on the right-hand side is zero.

Concerning (16) we remark that for $h \leq i \leq h+1$ we have

$$0 = 2b_i - a_{h-1} = \sum_j (b_j^i + b_{j-2}^i - a_{j-1}^{h-1})$$

by Lemma 3.2. Since each summand on the right-hand side is less than or equal to zero by (7) it is zero.

Finally we have $a_j^h = 2b_j^h$ by (14) and Lemma 3.3. \square

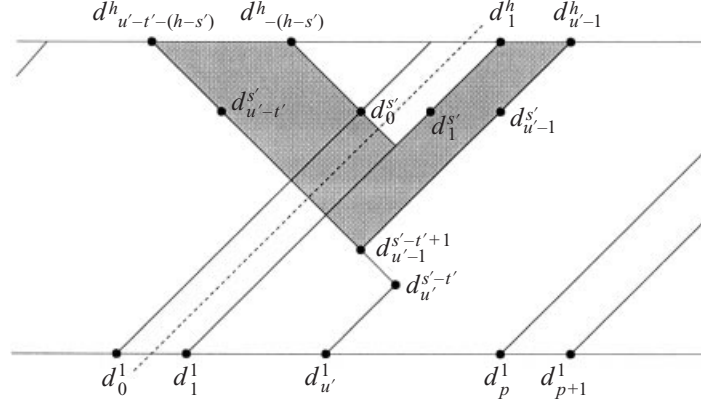
This means that every $\delta \in \gamma(\mathcal{S}'_N)$ is the unique solution of a linear system which itself depends on some vertex c_{u-1}^{s-t+1} . In the next section we will show that c_{u-1}^{s-t+1} is uniquely determined by c_1^s .

3.2. Computation of c_{u-1}^{s-t+1}

Note that the vertices c_1^s and c_{u-1}^{s-t+1} have to be positioned in $\mathbb{Z}D_{3m}/\tau^{2m-1}$ in such a way that the unique solution of the linear system (14)–(16) takes values in \mathbb{N} . We will show that this condition implies that $m+1 \leq s \leq 2m$ and determines c_{u-1}^{s-t+1} as c_{s-m}^{2m-s+1} .

It is possible to solve this problem directly by examining the linear system (14)–(16), but this procedure is rather complicated. Therefore we use a different method. We will ignore the equations of (16) and only use that a_h^j is even. Furthermore we use a covering technique to simplify the computations.

To this purpose we consider the stable translation-quiver $\mathbb{Z}A_h$ with vertices d_j^i where $1 \leq i \leq h$ and $j \in \mathbb{Z}$. There are arrows from d_j^i to d_j^{i+1} and from d_j^{i+1} to d_{j+1}^i for $1 \leq i \leq h-1$. The translation τ is given by $\tau(d_j^i) = d_{j-1}^i$. In particular we are interested in two quotients of $\mathbb{Z}A_h$, namely $Q_1 := \mathbb{Z}A_h/\tau^p$ and $Q_2 := \mathbb{Z}A_h/\tau^{2p}$.

FIGURE 3. Q_2 .

Now let $s', t', u' \in \mathbb{N}$ with $2 \leq t' \leq s' \leq h-1$ and $1 \leq u' \leq p$. Let $\delta_r : \{\text{vertices of } Q_r\} \rightarrow \mathbb{Z}$ for $r = 1, 2$ be a function satisfying

$$\delta_r(d_j^1) = \begin{cases} 1 & \text{if } d_j^1 = d_{u'-1}^{s'-t'+1} \\ 0 & \text{otherwise,} \end{cases} \quad (17)$$

$$\delta_r(d_j^1) + \delta_r(d_{j-1}^1) - \delta_r(d_{j-1}^2) = \begin{cases} -1 & \text{if } d_j^1 = d_{u'}^{s'-t'} \\ 0 & \text{otherwise,} \end{cases} \quad (18)$$

$$\delta_r(d_j^i) + \delta_r(d_{j-1}^i) - \delta_r(d_{j-1}^{i+1}) - \delta_r(d_j^{i-1}) = \begin{cases} 1 & \text{if } d_j^i = d_1^{s'} \\ -1 & \text{if } d_j^i = d_{u'}^{s'-t'} \\ 0 & \text{otherwise,} \end{cases} \quad (19)$$

for $2 \leq i \leq h-1$.

The values of $\delta_r(d_j^1)$ are determined by condition (17). The values of $\delta_r(d_j^i)$ with $i \geq 2$ are determined by the values of $\delta_r(d_j^{i'})$ with $i' < i$ because of conditions (18) and (19). Thus there exists exactly one such function δ_r .

The function δ_2 is easy to calculate.

LEMMA 3.5. *If $u' - t' \leq 1$ then the function δ_2 is given by $\delta_2(d_j^i) = 1$ if the vertex d_j^i lies in the shaded area (including the boundary) of Figure 3 and $\delta_2(d_j^i) = 0$ otherwise.*

Proof. It is straightforward to check that Equations (17), (18) and (19) hold. \square

We define the function $\delta'_1 : \{\text{vertices of } Q_1\} \rightarrow \mathbb{Z}$ by $\delta'_1(d_j^i) = \delta_2(d_j^i) + \delta_2(d_{j+p}^i)$. Of course δ'_1 satisfies Equations (17), (18) and (19), from which we see that $\delta'_1 = \delta_1$ and consequently

$$\delta_1(d_j^i) = \delta_2(d_j^i) + \delta_2(d_{j+p}^i). \quad (20)$$

We note the following.

LEMMA 3.6. *If $u' - t' \leq 1$ then $\delta_1(d_j^i)$ is an even integer if and only if $\delta_2(d_j^i) = \delta_2(d_{j+p}^i)$.*

Proof. Since δ_2 takes only values in $\{0, 1\}$ by Lemma 3.5 the claim follows from (20). \square

Let us now consider again our function $\delta \in \gamma(\mathcal{S}'_N)$. This function induces a function $\bar{\delta} : \{\text{vertices of } Q_1\} \rightarrow \mathbb{Z}$ by $\bar{\delta}(d_j^i) = a_j^i$. We set $s' = s$, $t' = t$ and $u' = \tilde{u}$.

Then the function $\bar{\delta}$ satisfies Equations (17), (18) and (19) by Lemma 3.4 and consequently $\bar{\delta} = \delta_1$.

LEMMA 3.7. *If $\delta \in \gamma(\mathcal{S}'_N)$ is the unique solution of the linear system (14)–(16) then we have $\tilde{u} - t \leq 1$.*

Proof. Suppose the lemma is false. Then $1 < \tilde{u} - t \leq \tilde{u} \leq p$ and in consequence none of the integers $2p, 1, p, p+1$ is congruent modulo $2p$ to an integer in $\{\tilde{u} - t, \tilde{u} - t + 1, \dots, \tilde{u} - 1\}$. By (17)–(19) and since $2 \leq t < p$ we have

$$\delta_2(d_j^s) = \begin{cases} 1 & \text{if } d_j^s = d_{j'}^s \text{ with } j' \in \{\tilde{u} - t, \tilde{u} - t + 1, \dots, \tilde{u} - 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Hence by (20) and Remark 3.1

$$\begin{aligned} 0 &= \delta_2(d_1^s) + \delta_2(d_{1+p}^s) + \delta_2(d_{2p}^s) + \delta_2(d_p^s) \\ &= \bar{\delta}(d_1^s) + \bar{\delta}(d_p^s) \\ &= \delta(N) + \delta(\tau N) \\ &\geq 1 \end{aligned}$$

which is obviously a contradiction. \square

We are able now to determine \tilde{u} and t by means of s .

LEMMA 3.8. *If $\delta \in \gamma(\mathcal{S}'_N)$ solves the linear system (14)–(16) then $c_{u-1}^{s-t+1} = c_{s-m}^{2m-s+1}$ and $m+1 \leq s \leq 2m$ holds. In particular $t = 2(s-m)$ and $\tilde{u} = t/2 + 1$.*

Proof. By Lemma 3.7 we have $\tilde{u} - t \leq 1$. Therefore we can apply Lemma 3.5 to describe δ_2 . On the other hand, we know from Lemma 3.4 that $\delta_1(d_j^h) = \bar{\delta}(d_j^h) = a_j^h$ is always an even integer. In view of Lemma 3.6 this means that $\delta_2(d_j^h) = \delta_2(d_{j+p}^h)$ for all j . Thus we have (see Figure 3)

$$\begin{aligned} \tilde{u} - 1 - 1 &\equiv -(h-s) - (\tilde{u} - t - (h-s)) \pmod{2p}, \\ -(h-s) + p &\equiv \tilde{u} - 1 \pmod{2p}. \end{aligned}$$

Substituting p by $2m-1$ and h by $3m-1$ we calculate that $\tilde{u} = t/2 + 1$ and $t = 2(s-m)$. Since $2 \leq t \leq s$ by Lemma 3.2 we get $2 \leq 2(s-m) \leq s$ which is equivalent to $m+1 \leq s \leq 2m$. \square

Hence the existence of a proper degeneration to the indecomposable N corresponding to the vertex c_1^s implies that $m+1 \leq s \leq 2m$. Furthermore, if $M \leq_{\text{deg}} N$ is a proper degeneration to N , then $\gamma((\bar{M}, \bar{N})) \in (\bigoplus_{\bar{Y}: Y \text{ indec}} \mathbb{Z}\bar{Y})^*$ is the unique solution of the linear system (14)–(16) with $c_{u-1}^{s-t+1} = c_{s-m}^{2m-s+1}$, by Lemma 3.4 and Lemma 3.8. Hence there exists up to isomorphism at most one module M degenerating to N .

On the other hand, let $m + 1 \leq s \leq 2m$ and N correspond to the vertex c_1^s . We set $s' = s$, $t' = 2(s - m)$ and $u' = t'/2 + 1$ and define δ_2 according to (17), (18) and (19). Since $u' - t' \leq 1$ the function δ_2 is described by Lemma 3.5. In consequence we have $\delta_2(d_j^h) = \delta_2(d_{j+p}^h)$. We define δ by

$$\delta(c_j^i) = \begin{cases} \delta_2(d_j^i) + \delta_2(d_{j+p}^i) & \text{if } 1 \leq i \leq h - 1 \\ 1/2(\delta_2(d_j^h) + \delta_2(d_{j+p}^h)) & \text{if } h \leq i \leq h + 1. \end{cases}$$

Then δ takes value in \mathbb{N} and solves the linear system (14)–(16). Indeed (14) is a consequence of (17), (15) of (18) and (19), and (16) can be checked easily using Lemma 3.5. Thus $\delta \in \gamma(\mathcal{S}'_N)$ which means in view of Lemma 2.3 that there exists a proper degeneration to N . This completes the proof of Theorem 1.1.

Note that the function δ constructed above for the indecomposable module N corresponding to the vertex c_1^s describes the module M degenerating to N in the following way. Let P_1, \dots, P_m be representatives of the isomorphism classes of projective indecomposable Λ -modules. By Lemma 3.4 and (5)

$$M \cong M_1 \oplus \bigoplus_{i=1}^m P_i^{\delta(\text{rad } P_i)}$$

where M_1 corresponds to the vertex c_{s-m+1}^{2m-s} if $s < 2m$ and $M_1 = 0$ if $s = 2m$.

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