J Geom Anal (2012) 22:633–665
DOI 10.1007/s12220-010-9205-5

Uniformly Quasiregular Maps on the Compactified Heisenberg Group

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Received: 18 June 2009 / Published online: 4 December 2010

Abstract We show the existence of a non-injective uniformly quasiregular mapping acting on the one-point compactification $\bar{\mathbb{H}}^1 = \mathbb{H}^1 \cup \{\infty\}$ of the Heisenberg group $\mathbb{H}^1$ equipped with a sub-Riemannian metric. The corresponding statement for arbitrary quasiregular mappings acting on sphere $\mathbb{S}^n$ was proven by Martin (Conform. Geom. Dyn. 1:24–27, 1997). Moreover, we construct uniformly quasiregular mappings on $\bar{\mathbb{H}}^1$ with large-dimensional branch sets. We prove that for any uniformly quasiregular map g on $\bar{\mathbb{H}}^1$ there exists a measurable CR structure μ which is equivariant under the semigroup Γ generated by g. This is equivalent to the existence of an equivariant horizontal conformal structure.

Keywords Uniformly quasiregular mappings · Heisenberg group

Mathematics Subject Classification (2000) Primary 30D05 · Secondary 58F23

Communicated by Fulvio Ricci.

The authors were supported by Swiss National Science Foundation, European Research Council Project GALA and European Science Foundation Project HCAA.

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1 Introduction

Quasiconformal and quasiregular maps play a crucial role in geometric function theory and new developments target generalizations of these notions to the abstract metric-measure setting as in the work of Heinonen and Koskela [12, 14]. An important class of spaces where such general results work is the setting of Carnot groups, in particular the setting of Heisenberg groups which are the simplest examples of non-commutative stratified groups. In this setting the theory of quasiconformal and quasiregular maps has been considered by various authors. Korányi and Reimann focused on quasiconformal mappings ([21, 23] and [22]), while Heinonen, Holopainen, and Rickman [13, 17] were the first ones to consider quasiregular maps in the Heisenberg/Carnot setting.

As seen from the papers of Markina [24] or Dairbekov [7], many analytic regularity properties of quasiregular maps in the Carnot setting are almost as good as the corresponding statements in Euclidean spaces. It is therefore of general interest to find examples of quasiconformal and quasiregular maps on Heisenberg or Carnot groups with given non-trivial properties. Let us recall that—as presented in Rickman's monograph [31]—in the Euclidean setting there is a great collection of classical examples of quasiregular maps illustrating the richness of the theory. In the setting of Carnot groups it is much harder to construct examples due to the highly complicated structure of the underlying sub-Riemannian geometry.

In this paper we take a step in the direction of constructing interesting examples of quasiregular maps. We shall work in the setting of the compactified first Heisenberg group which can be identified with the unit sphere in $\mathbb{C}^2$. In this setting we shall construct uniformly quasiregular maps—even with almost full-dimensional branch set. Related to a semigroup Γ generated by uniformly quasiregular maps we prove the existence of an equivariant CR structure. This is interesting also from the point of view of several complex variables because by a result which goes back to Poincaré [30] there are no non-injective CR maps acting on the standard unit sphere in $\mathbb{C}^2$. The only semigroup of CR maps (with respect to the standard CR structure on $S^3 \subseteq \mathbb{C}^2$) must be the restriction to S^3 of a subgroup of the conformal automorphisms of the unit ball in $\mathbb{C}^2$.

The paper is organized as follows: in Sect. 2 we fix the notation and recall the sub-Riemannian geometric setting of the Heisenberg groups. Here we also formulate some basic definitions and recall previous results on quasiconformal and quasiregular mappings in this setting. In Sect. 3 we construct a uqr map on the compactified Heisenberg group $\bar{\mathbb{H}}^1$ starting from the winding map and using the flow method. In Sect. 4 we construct uqr maps on $\bar{\mathbb{H}}^1$ with branch set of Hausdorff dimension close to 4. We use in the construction similar ideas as in [3]. Section 5 is devoted to the proof of the existence of a CR structure (or equivalently, a horizontal conformal structure) which is equivariant under a given countable, Abelian semigroup of uqr maps acting on $\bar{\mathbb{H}}^1$. The corresponding statement for the Riemann sphere is known as the Sullivan–Tukia theorem [1]. Section 6 is for final remarks and open questions.

2 Notation and Preliminaries

A quasiregular map f with a uniform control of the distortion of all its iterates is called uniformly quasiregular (uqr). In the Riemannian case such maps are studied in [19] and they are always conformal with respect to some measurable Riemannian structure. The first examples of such mappings acting on the sphere were found in [18] and further in more general Riemannian manifolds in [29]. One of the main goals of the present paper is to construct uqr maps in the setting of sub-Riemannian geometry of the compactified Heisenberg groups.

In our model for the Heisenberg group $\mathbb{H}^n$ we take $\mathbb{R}^{2n+1}$ as the underlying space and provide it with the group multiplication

$$p \cdot p' = (x + x', y + y', t + t' - 2x \cdot y' + 2y \cdot x')$$
$$\text{for } p = (x, y, t),\ p' = (x', y', t') \in \mathbb{H}^n,$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $x \cdot y$ denotes the standard scalar product on $\mathbb{R}^n$.

It is sometimes more appropriate to write points in the Heisenberg group in complex notation as follows $p = (x, y, t) =: (z, t) \in \mathbb{C}^n \times \mathbb{R}$. The above group law in this notation becomes

$$(z, t)(z', t') = (z + z', t + t' + 2\,\mathrm{Im}\, z \cdot \bar{z}'),$$

where $z \cdot z'$ is the standard complex scalar product in $\mathbb{C}^n$.

The left-invariant vector fields at points $p \in \mathbb{H}^n$ are given by

$$X_j(p) = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n$$
$$Y_j(p) = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n$$
$$T(p) = \frac{\partial}{\partial t}$$

and they form a basis of the Lie algebra of the Heisenberg group. Denote by HT the horizontal tangent bundle of $\mathbb{H}^n$, that is the subbundle of the tangent bundle $T\mathbb{H}^n$ with fibers

$$HT_p = \mathrm{span}\{X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p)\}, \quad p \in \mathbb{H}^n.$$

The Heisenberg group is equipped with the norm

$$|(x, y, t)| = \left((|x|^2 + |y|^2)^2 + t^2\right)^{1/4}$$

and the Heisenberg distance

$$d_H(p, q) = |p^{-1}q|, \quad p, q \in \mathbb{H}^n, \tag{2.1}$$

which is equivalent to the Carnot–Carathéodory metric based on the curve length of horizontal curves (see [10]).

The vector fields

$$Z_j = \frac{1}{2}\left(X_j - \mathrm{i}Y_j\right) = \frac{\partial}{\partial z_j} + \mathrm{i}\bar{z}_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n$$

span a left-invariant CR structure $T^{1,0}$ on $\mathbb{H}^n$. More precisely, the complex n-dimensional subbundle $T^{1,0}$ of the complexified $2n+1$-dimensional tangent bundle $T\mathbb{H}^n \otimes \mathbb{C}$ makes the Heisenberg group a CR manifold of hypersurface type. The total space $T\mathbb{H}^n \otimes \mathbb{C}$ is spanned by $T^{1,0}$, its complex conjugate bundle $T^{0,1} = \bar{T}^{1,0}$ spanned by vector fields

$$\bar{Z}_j = \frac{1}{2}\left(X_j + \mathrm{i}Y_j\right) = \frac{\partial}{\partial \bar{z}_j} - \mathrm{i}z_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n$$

and one additional direction given by T. For more details on CR manifolds see [8] and also Sect. 5.

Denote further by $Q = 2n + 2$ the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$.

Next, we define for an open subset $U \subset \mathbb{H}^n$ and $1 \le p < \infty$ the *horizontal Sobolev space* $HW^{1,p}(U)$ as the space of functions $u \in L^p(U)$ for which $X_1u, \dots, X_nu$, $Y_1u, \dots, Y_nu$ exist in the sense of distributions and belong to $L^p(U)$. A function $f = (f_1, \dots, f_{2n+1}) : U \to \mathbb{H}^n$, $U \subseteq \mathbb{H}^n$ is said to be of class $HW^{1,p}(U)$ if each component of f belongs to $HW^{1,p}(U)$. The local space $HW^{1,p}_{loc}(U)$ is defined analogously.

A (*generalized*) *contact map* $f : U \to \mathbb{H}^n$, $U \subset \mathbb{H}^n$ open, is a function in $HW^{1,1}_{loc}(U)$ for which the tangent vectors $X_jf(p)$, $Y_jf(p)$, $j = 1, \dots, n$, belong to $HT_{f(p)}$ for almost all $p \in U$, where

$$X_jf(p) = (X_jf_1(p), \dots, X_jf_{2n+1}(p)) \quad \text{and}$$
$$Y_jf(p) = (Y_jf_1(p), \dots, Y_jf_{2n+1}(p)).$$

For such a map the *formal horizontal differential* $Hf_*(p) : HT_p \to HT_{f(p)}$ can be defined a.e. by setting $Hf_*(p)X_j = X_jf(p)$, $Hf_*(p)Y_j = Y_jf(p)$, $j = 1, \dots, n$ and then extending linearly from the basis vectors to HT_p. If $f \in HW^{1,2}_{loc}(U)$, the resulting map $Hf_*(p)$ itself extends uniquely to a homomorphism $f_*(p)$ of the Lie algebra of $\mathbb{H}^n$ (the *formal Pansu differential*) for almost all $p \in U$. With respect to the basis $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ it is given by

$$f_*(p) = \begin{pmatrix} Hf_*(p) & 0 \\ 0 & \lambda(p) \end{pmatrix} = \begin{pmatrix} X_1f_1(p) & \cdots & Y_nf_1(p) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ X_1f_{2n}(p) & \cdots & Y_nf_{2n}(p) & 0 \\ 0 & \cdots & 0 & \lambda(p) \end{pmatrix},$$

where

$$\lambda(p) = \sum_{j=1}^{n} X_k f_j(p) Y_k f_{n+j}(p) - Y_k f_j(p) X_k f_{n+j}(p)$$

for an arbitrary $k = 1, \dots, n$.

If U is an open set in $\mathbb{H}^n$ we say that a continuous mapping $f : U \to \mathbb{H}^n$ is K-*quasiregular* if

- $f \in HW^{1,Q}_{loc}(U)$,
- f is a (generalized) contact map,
- there exists $K < \infty$ such that

$$|Hf_*(p)|^Q \le K \det f_*(p)$$

holds a.e. p in U, where we denoted $|Hf_*(p)| = \max_{\xi \in HT_p,\ |\xi|=1} |Hf_*(p)\xi|$.

This is the analytic definition of quasiregularity studied in [6]. In [13], quasiregular mappings on Carnot groups were first studied under more stringent smoothness assumptions. Yet it turned out that the properties of quasiregular mappings which have been established in [13] also hold for the definition given above.

A K-quasiregular homeomorphism $f : U \to V$ between open sets in $\mathbb{H}^n$ is *quasiconformal*. The basic theory of quasiconformal maps in the Heisenberg group has been developed by Korányi and Reimann in [21–23].

In [7] and [13] they further show that non-constant quasiregular mappings defined on Heisenberg groups are discrete open maps and almost everywhere differentiable in the sense of Pansu, with nonzero differential. For the composition of two quasiregular mappings $f, g : \mathbb{H}^n \to \mathbb{H}^n$ it follows that $K_{f \circ g} \le K_f \cdot K_g$, in particular $K_{f^m} \le (K_f)^m$ for $m \in \mathbb{N}$. This shows that under composition the constant of quasiregularity may (and in general will) grow exponentially with m. Our intention is to study those quasiregular mappings for which this growth is forbidden. In this paper we show that on the compactified Heisenberg group there is an abundance of such mappings.

Let us note first that $\bar{\mathbb{H}}^n$—the one-point compactification of the Heisenberg group—can be defined analogously to the one-point compactification of the complex plane by performing a CR generalization of stereographic projection. We describe this by following [20]. Define a Siegel domain

$$D = \{\tilde{\zeta} = (\zeta, \zeta_0) \mid \operatorname{Im} \zeta_0 - |\zeta|^2 > 0\} \subset \mathbb{C}^{n+1},$$

where the elements of $\mathbb{C}^{n+1}$ are written in the form $\tilde{\zeta} = (\zeta, \zeta_0)$ where $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ and $\zeta_0 \in \mathbb{C}$. The norm $|\zeta|^2 = \zeta \cdot \bar{\zeta}$ is the standard Euclidean norm in $\mathbb{C}^n$. The Heisenberg group $\mathbb{H}^n$ operates simply transitively (analogously as real numbers act on the upper one-dimensional complex half-plane via translations) on D by

$$\mathbb{H}^n \times D \to D$$

$$((z,t), (\zeta, \zeta_0)) \mapsto (\zeta + z, \zeta_0 + t + 2\mathrm{i}\zeta \cdot \bar{z} + \mathrm{i}|z|^2)$$

and this operation extends to the boundary ∂D as well. This gives a unique correspondence of an element $(z,t) \in \mathbb{H}^n$ with an element $(z,t)(0,0) = (z, t + \mathrm{i}|z|^2) \in \partial D$. Under this identification the CR structure of $\mathbb{H}^n$ defined above coincides with the CR structure induced by the standard complex structure in $\mathbb{C}^{n+1}$ since the holomorphic subspaces coincide at the origin and therefore everywhere via holomorphic action. The boundary of the Siegel domain is further identified with the unit sphere

$$\partial B = \{\tilde{w} = (w, w_0) \in \mathbb{C}^n \times \mathbb{C} \mid |w|^2 + |w_0|^2 = 1\}$$

via Cayley transform $C : B \to D$:

$$C(w, w_0) = \left(\frac{\mathrm{i}w}{1+w_0}, \mathrm{i}\frac{1-w_0}{1+w_0}\right)$$

which is a holomorphic bijection extending to a bijection between boundaries

$$\partial B \backslash \{(0,-1)\} \to \partial D.$$

The differential of the Cayley map maps the horizontal subbundle of ∂B onto the horizontal subbundle of ∂D.

The CR stereographic projection

$$\pi : \partial B \backslash \{(0,-1)\} \to \mathbb{R}^{2n+1}$$

is then defined as the composition of C followed by the projection $(\zeta, \zeta_0) \mapsto (\zeta, \operatorname{Re} \zeta_0)$. The mapping π can then be extended to a map from ∂B to the one-point compactification of $\mathbb{R}^{2n+1}$:

$$\pi(w, w_0) = \left(\frac{\mathrm{i}w}{1+w_0}, \frac{2\operatorname{Im} w_0}{|1+w_0|^2}\right)$$

and the inverse map is given by

$$\pi^{-1}(z,t) = \left(\frac{2z}{\mathrm{i}(1+|z|^2)+t}, \frac{\mathrm{i}(1-|z|^2)-t}{\mathrm{i}(1+|z|^2)+t}\right).$$

For the sake of simplicity, we will concentrate in the subsequent discussion on $n = 1$.

One can use the chart at infinity

$$\Phi_0 : (z,t) \mapsto \left(\frac{-z}{|z|^2 - \mathrm{i}t}, \frac{-t}{|z|^4 + t^2}\right)$$

to extend in the obvious manner the notion of quasiregularity to mappings $f : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$. More precisely, $f : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ is said to be K-quasiregular if each point $p \neq \infty$ has a neighborhood $U \subset \mathbb{H}^1$ such that either $f : U \to \mathbb{H}^1$ is K-quasiregular (if $f(p) \in \mathbb{H}^1$) or $\Phi_0 \circ f : U \to \mathbb{H}^1$ is K-quasiregular (if $f(p) = \infty$). Moreover, for $p = \infty$, there exists a neighborhood U of 0 in $\mathbb{H}^1$ such that $f \circ \Phi_0$ is K-quasiregular on U (if $f(\infty) \in \mathbb{H}^1$) or $\Phi_0 \circ f \circ \Phi_0$ is K-quasiregular on U (if $f(\infty) = \infty$).

Quasiregular mappings acting on $\bar{\mathbb{H}}^1$ can be identified with those mappings acting on $\partial B \subseteq \mathbb{C}^2$ which distort the standard CR structure in a controlled way. Quasiconformal mappings acting on strictly pseudoconvex hypersurfaces in $\mathbb{C}^n$ have been studied by Korányi and Reimann [22], Tang [33, 34], and also by Dragomir and Tomassini [8].

In our construction, we will define mappings $f : \mathbb{H}^1 \to \mathbb{H}^1$ and then extend them to $\bar{f} : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ by setting

$$\bar{f}(p) := \begin{cases} f(p) & p \in \mathbb{H}^n \\ \infty & p = \infty. \end{cases} \tag{2.2}$$

If $f : \mathbb{H}^1 \to \mathbb{H}^1$ is quasiregular (or quasiconformal) and satisfies the following two conditions:

$$\lim_{d(p,0)\to\infty} d(f(p), f(0)) = \infty \tag{2.3}$$

and

$$N(f, \Omega) := \sup_{q\in\mathbb{H}^1} N(q, f, \Omega) = \sup_{q\in\mathbb{H}^1} \mathrm{card}\{f^{-1}(q) \cap \Omega\} < \infty \quad \text{for all } \Omega \subset \mathbb{H}^1 \tag{2.4}$$

then $\bar{f} : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ will also be quasiregular (or quasiconformal).

First note that condition (2.3) guarantees the continuity of the extension $\bar{f}$. Obviously, we can choose for $p \neq \infty$ a neighborhood $U \subset \mathbb{H}^1$ such that $\bar{f}(U) \subset \mathbb{H}^1$. The map $\bar{f}|_U = f|_U : U \to \mathbb{H}^1$ is quasiregular by assumption.

Now consider the point $p = \infty$ and the map $g := \Phi_0 \circ \bar{f} \circ \Phi_0$. By continuity of g it is possible to choose a neighborhood U of 0 in $\mathbb{H}^1$ such that $g(U) \subseteq B(0, 1) \subset \mathbb{H}^1$. Note that

$$\Phi_0 \circ \bar{f} \circ \Phi_0|_{U\setminus\{0\}} = \Phi_0 \circ f \circ \Phi_0|_{U\setminus\{0\}} : U \setminus \{0\} \to \mathbb{H}^1$$

is contact as a composition of contact mappings (the map Φ_0 is 1-quasiconformal). Since the statement in the definition of a contact mapping needs to hold only almost everywhere, an additional point does not matter and it follows that g is contact on U. Moreover,

$$|H_* g(p)|^4 \leq K \det g_*(p) \quad \text{a.e. } p \in U$$

for f K-quasiregular (and Φ_0 1-quasiconformal). It remains to show that $g \in HW^{1,4}_{loc}(U)$. Obviously, we have $g \in L^4_{loc}(U)$, since g is continuous. Moreover, the horizontal derivative Xg_1 exists in the distributional sense and satisfies

$$\begin{aligned} \int_U (Xg_1(p))^4 \, \mathrm{d}\mathcal{L}^3(p) &\leq \int_U ((Xg_1(p))^2 + (Xg_2(p))^2 + (Yg_1(p))^2 \\ &\quad + (Yg_2(p))^2)^2 \, \mathrm{d}\mathcal{L}^3(p) \\ &\leq C_1 \int_U |H_* g(p)|^4 \, \mathrm{d}\mathcal{L}^3(p) \\ &\leq C_1 K \int_U \det g_*(p) \, \mathrm{d}\mathcal{L}^3(p) \end{aligned}$$

$$= C_1 K \int_{g(U)} N(q, g, U)\, \mathrm{d}\mathcal{L}^3(p)$$

$$\leq C_1 \cdot K \cdot N(f, \Phi_0(U)) \cdot \mathcal{L}^3(g(U)) < \infty,$$

where we have used in the second-to-last line a change of variable formula from [37]; see also [25]. It follows that $Xg_1 \in L^4_{loc}(U)$. An analogous reasoning holds for the other weak horizontal derivatives.

Altogether, this shows that the map $\bar{f}$, defined as in (2.2), is quasiregular (or quasiconformal) on $\bar{\mathbb{H}}^1$, provided that f has the same properties and conditions (2.3) and (2.4) hold.

We call a quasiregular mapping $f : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ *uniformly quasiregular* (uqr) if f and all the iterates $f \circ f$, $f \circ f \circ f, \dots$ are K-quasiregular in the above sense with the same $K < \infty$ independently of the iterate f^n, $n \in \mathbb{N}$.

The *branch set* of f is denoted by

$$B_f = \{p \in \bar{\mathbb{H}}^1 \mid f \text{ is not locally injective in } p\}.$$

The uqr map in our consideration will be obtained as a composition of a quasiregular map $\bar{f}$ of type (2.2) together with a conformal mapping on $\bar{\mathbb{H}}^1$.

It has been shown by Korányi and Reimann [21] that all smooth conformal (1-quasiconformal) maps are compositions of left translations

$$L_q : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1, \qquad p \mapsto q \cdot p \quad \text{for } q \in \bar{\mathbb{H}}^1,$$

dilations

$$\delta_r : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1, \qquad (z, t) \mapsto (rz, r^2 t) \quad \text{for } r > 0,$$

rotations around the t-axis

$$m_\phi : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1, \qquad (z, t) \mapsto (ze^{\mathrm{i}\phi}, t) \quad \text{for } \phi \in \mathbb{R}, \tag{2.5}$$

and the orientation preserving inversion on the unit sphere

$$\Phi_0 : (z, t) \mapsto \left(\frac{-z}{|z|^2 - \mathrm{i}t}, \frac{-t}{|z|^4 + t^2} \right).$$

This inversion mapping Φ_0 takes a simple form when transported via π to act on ∂B:

$$\pi^{-1} \circ \Phi_0 \circ \pi(w, w_0) = (-w, -w_0).$$

It has been proved by Capogna in [5] by removing the regularity assumption that indeed *all* 1-quasiconformal mappings defined on a domain in $\bar{\mathbb{H}}^1$ are necessarily group actions. The corresponding Liouville type theorem for 1-quasiregular maps is due to Dairbekov [6].

Denote a ball of radius r centered at p with respect to metric d_H briefly by $B(p, r)$. Then both L_p and m_ϕ are isometries for the Heisenberg distance and

$$B(p, r) = L_p \delta_r B(0, 1) \quad \text{and} \quad m_\phi(B(p, r)) = B(m_\phi(p), r).$$

Obviously, the conformal maps defined above are uqr as they come from holomorphic automorphisms of $B \subseteq \mathbb{C}^2$ restricted to ∂B. It is the purpose of the present paper to provide examples of *non-injective* uqr mappings on $\bar{\mathbb{H}}^1$. The main tool for constructing such maps is the flow method due to Korányi and Reimann [21, 23]. They first demonstrated the existence of nontrivial smooth quasiconformal maps in $\mathbb{H}^n$. In the first Heisenberg group $\mathbb{H}^1$ consider C^2-vector field

$$v = -\frac{1}{4}(Y\varrho)X + \frac{1}{4}(X\varrho)Y + \varrho T, \tag{2.6}$$

where ϱ is an arbitrary sufficiently smooth real-valued function in $\mathbb{H}^1$, which we call the potential of the vector field v. Then v generates a local one-parameter group (F_s) of contact transformations which are quasiconformal, provided the second horizontal derivatives of ϱ are bounded. The functions F_s come as solutions of the following differential equation

$$\frac{\partial F_s}{\partial s}(x, y, t) = v(F_s(x, y, t))$$

with initial condition $F_0(x, y, t) = (x, y, t)$.

3 Construction of a Non-injective uqr Map from the Winding Map

In this section we prove the existence of a non-injective uqr map $f : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$. The proof is an adaptation of the conformal trap method from the Euclidean case [18, 26].

Theorem 3.1 *Suppose $k \geq 2$ is an integer and consider the non-constant quasiregular winding map $f : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ given in cylindrical coordinates by $(r, \varphi, t) \mapsto (r, k\varphi, kt)$. There exists a uniformly quasiregular mapping $g : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ such that $B_f = B_g$ holds.*

3.1 Winding Maps and the Flow Method

We present some technical preliminaries which serve as preparations for the proof of the existence of non-injective uqr maps. In [13], the so-called *winding map* is studied as an example of a quasiregular map with non-empty branch set. We will later use this mapping to produce a uqr counterpart. In order to do so, we need to represent it as a time-t-map of a certain flow.

The winding map can be best described by using *cylindrical coordinates* (r, φ, t). We will use the notation $p = (x, y, t)$ and $p = (r, \varphi, t) \in [0, \infty) \times \mathbb{R} \times \mathbb{R}$ for points in $\mathbb{H}^1$ simultaneously. Note that infinitely many triples (r, φ, t) correspond to a given point p, however, the correspondence can be made one-to-one (except on the t-axis) by restricting the angle φ to a half-open interval of length 2π, e.g., $\varphi \in (-\pi, \pi]$. More precisely, the cylindrical coordinates are given in terms of Cartesian coordinates (x, y, t) as

$$r = \sqrt{x^2 + y^2} \in (0, \infty), \qquad \varphi := \tan^{-1}\left(\frac{y}{x}\right) \in (-\pi, \pi], \qquad t := t \in \mathbb{R},$$

where the inverse tangent is suitably defined.

The function $f : \mathbb{H}^1 \to \mathbb{H}^1$ given in cylindrical coordinates as $(r, \varphi, t) \mapsto (r/2, 2\varphi, t/2)$ is an example of a nontrivial quasiregular map.

More generally, for a map $f_{(\alpha,\beta,\gamma)} : (r, \varphi, t) \mapsto (\alpha r, \beta\varphi, \gamma t)$, where $\beta \in \mathbb{Z}$, the matrix of $f_{(\alpha,\beta,\gamma)*}$ in the basis X, Y, T in a point $p = (r, \varphi, t)$, $r > 0$, is given by

$$f_{(\alpha,\beta,\gamma)*}(p) = \begin{pmatrix} \alpha\cos\varphi\cos\beta\varphi + \alpha\beta\sin\varphi\sin\beta\varphi & \alpha\sin\varphi\cos\beta\varphi - \alpha\beta\cos\varphi\sin\beta\varphi & 0 \\ \alpha\cos\varphi\sin\beta\varphi - \alpha\beta\sin\varphi\cos\beta\varphi & \alpha\sin\varphi\sin\beta\varphi + \alpha\beta\cos\varphi\cos\beta\varphi & 0 \\ 2r(\gamma - \alpha^2\beta)\sin\varphi & 2r(\alpha^2\beta - \gamma)\cos\varphi & \gamma \end{pmatrix}.$$

Hence the contact property is satisfied if and only if $\gamma = \alpha^2\beta \neq 0$. By studying the eigenvalues of $f^T_{(\alpha,\beta,\gamma)*} f_{(\alpha,\beta,\gamma)*}$ we deduce that the Lipschitz map $f_{(\alpha,\beta,\gamma)}$ is $K = \beta^2$ quasiregular.

In what follows we choose $\alpha = 1$ to keep the cylinders $\{(r, \varphi, t) \mid r = \text{constant}\}$ invariant and study the globally defined winding mappings:

Definition 3.2 The *winding mapping* of degree k, $k \geq 2$, is given in cylindrical coordinates as

$$f_{(1,k,k)} : \mathbb{H}^1 \to \mathbb{H}^1, \qquad (r, \varphi, t) \mapsto (r, k\varphi, kt).$$

The branch set $B_{f_{(1,k,k)}}$ of $f_{(1,k,k)}$ is the t-axis.

In what follows we need to represent $f_{(1,k,k)}$ locally as a flow of a vector field. More precisely, we need to know an explicit formula for potential functions ϱ_k of vector fields v_k that generate flows F_s^k such that $F_1^k = f_{(1,k,k)}$, locally around the point $e_0 = (x, y, t) = (1, 0, 0)$. Note, however, that it would not be possible to obtain $f_{(1,k,k)}$ as a flow on the whole space $\mathbb{H}^1$ since flows always define diffeomorphisms. But since the function $f_{(1,k,k)}$ is locally injective in the point e_0, it may still be obtained as a flow in a neighborhood of e_0.

Let us consider the open sector $U \subseteq \mathbb{H}^1$, given in cylindrical coordinates as $U = \{(r, \varphi, t) : r > 0, \varphi \in (-\frac{\pi}{k}, \frac{\pi}{k}), t \in \mathbb{R}\}$.

Lemma 3.3 *For any integer $k \geq 2$ there is a potential function $\varrho = \varrho_k$ defined on the slit space $\mathbb{H}^1_s := \mathbb{H}^1 \setminus \{(x, y, t) : x \leq 0, y = 0\}$ which generates on U a one-parameter group of quasiconformal transformations $F_s = F_s^k$, $s \in [0, 1]$ with the property that $F_1^k = f_{(1,k,k)}|_U$.*

Proof Let $k \geq 2$ be a fixed integer.

By writing the vector fields X, Y, T in cylindrical coordinates we obtain

$$X(r, \varphi, t) = \cos\varphi\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi} + 2r\sin\varphi\frac{\partial}{\partial t},$$

$$Y(r, \varphi, t) = \sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi} - 2r\cos\varphi\frac{\partial}{\partial t},$$

$$T(r,\varphi,t)=\frac{\partial}{\partial t},$$

for $r>0$. Let ϱ be a potential function in cylindrical coordinates. Vector fields related to ϱ which generate the flow of contact transformations are of the following form (in cylindrical coordinates):

$$v=\left(-\frac{1}{4r}\frac{\partial\varrho}{\partial\varphi}+\frac{r}{2}\frac{\partial\varrho}{\partial t}\right)\frac{\partial}{\partial r}+\left(\frac{1}{4r}\frac{\partial\varrho}{\partial r}\right)\frac{\partial}{\partial\varphi}+\left(-\frac{r}{2}\frac{\partial\varrho}{\partial r}+\varrho\right)\frac{\partial}{\partial t}.$$

Then the flow $F_s(r,\varphi,t)=(R_s,\theta_s,T_s)$, is obtained as a solution of the following system of differential equations:

$$\begin{aligned}\frac{\partial R_s}{\partial s}&=-\frac{1}{4R_s}\frac{\partial\varrho}{\partial\varphi}(F_s)+\frac{R_s}{2}\frac{\partial\varrho}{\partial t}(F_s),\\ \frac{\partial\theta_s}{\partial s}&=\frac{1}{4R_s}\frac{\partial\varrho}{\partial r}(F_s),\\ \frac{\partial T_s}{\partial s}&=-\frac{R_s}{2}\frac{\partial\varrho}{\partial r}(F_s)+\varrho(F_s),\end{aligned}$$

with initial condition

$$R_0(r,\varphi,t)=r,\qquad \theta_0(r,\varphi,t)=\varphi,\qquad T_0=t.$$

By choosing $\varrho(r,\varphi,t)=2(\ln k)r^2\varphi+(\ln k)t$ for $(r,\varphi,t)\in\{(r,\varphi,t):\ r>0,-\pi<\varphi<\pi,t\in\mathbb{R}\}$, we obtain

$$v(r,\varphi,t)=(\ln k)\varphi\frac{\partial}{\partial\varphi}+(\ln k)t\frac{\partial}{\partial t}. \tag{3.1}$$

and

$$\begin{aligned}\frac{\partial R_s}{\partial s}&=0,\\ \frac{\partial\theta_s}{\partial s}&=(\ln k)\theta_s(r,\varphi,t),\\ \frac{\partial T_s}{\partial s}&=(\ln k)T_s(r,\varphi,t),\end{aligned}$$

with initial condition

$$R_0(r,\varphi,t)=r,\qquad \theta_0(r,\varphi,t)=\varphi,\qquad T_0=t,$$

provided that

$$(R_s(r,\varphi,t),\theta_s(r,\varphi,t),T_s(r,\varphi,t))\in\{(r,\varphi,t):\ r>0,-\pi<\varphi<\pi,t\in\mathbb{R}\}$$

for all $(r,\varphi,t)\in U$. Let us note that we really have $\theta_s(r,\varphi,t)=k^s\varphi\in(-\pi,\pi)$ for all $(r,\varphi,t)\in U$ and for all $s\in[0,1]$. Thus this induces the correct flow $F_s(r,\varphi,t)=(r,k^s\varphi,k^st)$, $s\in[0,1]$. At $s=1$ we have $F_1(r,\varphi,t)=(r,k\varphi,kt)=f_{(1,k,k)}(r,\varphi,t)$.

Notice that $(r,\varphi,t)\mapsto(r,k^s\varphi,k^st)$, $\varphi\in\mathbb{R}$ defines a function from $\mathbb{H}^1$ to $\mathbb{H}^1$ only if $k^s\in\mathbb{N}$. Yet, this problem can be resolved by restricting φ to an appropriate interval. In this way we obtain for all $s\in[0,1]$ an injective contact map $F_s:U\to\mathbb{H}^1$. Let us further mention that $|ZZ_\varrho|=\ln k<\infty$, where $Z:=\frac{1}{2}(X-iY)$. This guarantees by a result due to Korányi and Reimann [21] that F_s, and in particular F_1, is quasiconformal (see also the previous remark on the quasiregularity of $f_{(1,k,k)}$). □

Lemma 3.4 *Let e_0, U and F_s be as before in Lemma 3.3. For any $a>0$ with $\overline{B}(e_0,a)\subseteq U$, there exists $b_0>0$ with $0<b_0<4b_0\le a$ such that*

$$d_H(e_0,F_s(p))\ge 4b_0 \quad \textit{for all } p\in\partial B(e_0,a) \textit{ and all } s\in[0,1].$$

Proof First note that $F_s(p)\neq e_0$ for all $p\in\partial B(e_0,a)$ and $s\in[0,1]$. Indeed, $(re^{ik^s\varphi},k^st)=(1,0)$ would imply that $r=1$, $t=0$ and $k^s\varphi=2\pi l$ for some $l\in\mathbb{Z}$. But since $\overline{B}(e_0,a)$ is contained in the sector U, we obtain $-\frac{\pi}{k}<\varphi=\frac{2\pi l}{k^s}<\frac{\pi}{k}$. This can only be fulfilled for $l=0$, but then $p=(1,0)=e_0\notin\partial B(e_0,a)$.

It follows that

$$d_H(e_0,F_s(p))>0 \quad \text{for all } p\in\partial B(e_0,a) \text{ and all } s\in[0,1].$$

Let us further note that $\min\{d_H(e_0,F_s(p)):\ (p,s)\in\partial B(e_0,a)\times[0,1]\}$ exists, since $(p,s)\to d_H(e_0,F_s(p))$ is a continuous function on the compact set $\partial B(e_0,a)\times[0,1]$. That is, there exist $p_0\in\partial B(e_0,a)$ and $s_0\in[0,1]$ such that

$$\min\{d_H(e_0,F_s(p)):\ (p,s)\in\partial B(e_0,a)\times[0,1]\}=d_H(e_0,F_{s_0}(p_0))=:c>0.$$

On the other hand, $c\le a$, since

$$\begin{aligned} c&=\min\{d_H(e_0,F_s(p)):\ (p,s)\in\partial B(e_0,a)\times[0,1]\}\le d_H(e_0,F_0(p))\\ &=d_H(e_0,p)=a.\end{aligned}$$

Now set $b_0:=\frac{c}{4}>0$. Then

$$d_H(e_0,F_s(p))\ge c=4b_0 \quad \text{for all } p\in\partial B(e_0,a),\ s\in[0,1].$$ □

3.2 Proof of Theorem 3.1

Let $k\ge 2$ be an arbitrary, but fixed integer, and consider the map

$$f:\mathbb{H}^1\to\mathbb{H}^1,\qquad (r,\varphi,t)\mapsto(r,k\varphi,kt).$$

Notice first that the map f itself is not uniformly quasiregular. Its n-th iterate is given by

$$f^n(r,\varphi,t)=(r,k^n\varphi,k^nt),$$

which is a quasiregular map with distortion $K_n=k^{2n}$. This shows that $K_n\to\infty$ as $n\to\infty$ and, as a consequence, f cannot be uniformly quasiregular since the distortion gets worse in each step of the iteration.

The mapping $f : \mathbb{H}^1 \to \mathbb{H}^1$ has degree $k < \infty$. Note that it leaves the t-axis invariant and $B_f = f(B_f) = t$-axis.

Choose any point $p_0 \in \mathbb{H}^1$ not lying on the t-axis with the following properties:

(1) There is a small ball $U_0 = B(p_0, r)$ about p_0 such that $f^{-1}U_0$ has components $U_1, \dots, U_k$ pairwise disjoint and such that $f : U_i \to U_0$ is injective for $i = 1, \dots, k$.
(2) $f(U_0)$ is disjoint from $\bigcup_{i=0}^k U_i$.

We can choose, for example, $p_0 = (1, \frac{\pi}{k}, 0)$ in cylindrical coordinate representation. Then $f(p_0) = (1, \pi, 0)$ and

$$f^{-1}\{p_0\} = \{p_1, \dots, p_k\},$$

where the points p_i can be written in cylindrical coordinates as $p_i = (1, \varphi_i, 0)$, where $\varphi_i \in (-\pi, \pi)$ is given by

$$\varphi_i := \begin{cases} \frac{\pi(1+2k(i-1))}{k^2} & \text{for } i \in \{1, \dots, \lfloor \frac{k^2-1}{2k} + 1 \rfloor\}, \\ \frac{\pi(1+2k(i-1))}{k^2} - 2\pi & \text{for } i \in \{\lceil \frac{k^2-1}{2k} + 1 \rceil, \dots, k\}. \end{cases} \tag{3.2}$$

Let further a,$b_1 > 0$ so small that $2b_1 < a$ and $\bar{B}(e_0, a) \subseteq U$, where e_0 and U are as in Lemma 3.3, and such that

(1) $B(p_i, a) \subset U_i$, $i = 0, \dots, k$,
(2) $B(p_0, b_1) \subset \bigcap_{i=1}^k f\,(B(p_i, a))$,
(3) $B\,(f(p_0), b_1) \subset f\,(B(p_0, a))\,.$

Then set $b := \min\{b_0, b_1\}$, where b_0 is as in Lemma 3.4.

To apply the conformal trap method we shall glue in our mapping f suitable rotations. To do that we define a modification $\tilde{f}$ as follows:

$$\tilde{f} := \begin{cases} f & \text{on } \bar{\mathbb{H}}^1 \setminus \bigcup_{i=0}^k B(p_i, a) \\ m_{(\frac{\pi}{k} - \varphi_i)} & \text{on } B(p_i, b),\ i = 1, \dots, k \\ m_{(\pi - \frac{\pi}{k})} & \text{on } B(p_0, b) \\ \text{qc extension} & \text{on } B(p_i, a) \setminus B(p_i, b),\ i = 0, \dots, k, \end{cases} \tag{3.3}$$

where m_ϕ denotes the rotation by an angle ϕ as defined in (2.5). To realize the last line in the definition of $\tilde{f}$, i.e., to show that we can make a quasiconformal transition from the rotation to f is the main technical difficulty of our proof. In the Euclidean case, the quasiconformal extension is obtained by Sullivan's version of the annulus theorem for quasiconformal mappings (see [36]). In our situation, the quasiconformal map appearing in the last line of the definition of $\tilde{f}$ will be defined using the flow technique of Korányi and Reimann as described below.

For $i = 1, \dots, k$, let φ_i be the angle given in (3.2) and set $\varphi_0 := \frac{\pi}{k}$. By applying rotations $m_{-\varphi_i}$ to balls $B(p_i, a)$ they are mapped onto the ball $B(e_0, a)$ where we modify the potential $\varrho = \varrho_k$ functions given by Lemma 3.3 as follows. We define first a function

$$\eta(p) := \begin{cases} 0 & \text{for } p \in \overline{B}(e_0, \frac{3}{2}b) \\ 1 & \text{for } p \in \mathbb{H}^1 \setminus (B(e_0, 2b) \cup \{(x,y,t):\ x \le 0, y = 0\}) . \end{cases}$$

Then choose any smooth extension $\tilde{\eta}$ of η on the slitted Heisenberg group $\mathbb{H}^1_s := \mathbb{H}^1 \setminus \{(x,y,t) : x \le 0, y = 0\}$ and define the modified potential

$$\tilde{\varrho}(p) := \varrho \cdot \tilde{\eta}(p) = \begin{cases} 0 & \text{for } p \in \overline{B}(e_0, \frac{3}{2}b) \\ \varrho \cdot \eta(p) & \text{for } p \in B(e_0, 2b) \setminus \overline{B}(e_0, \frac{3}{2}b) \\ \varrho(p) & p \in \mathbb{H}^1_s \setminus B(e_0, 2b). \end{cases}$$

According to this potential function we define the modified vector field $\tilde{v}$ on $\mathbb{H}^1_s$ by setting

$$\tilde{v} = \left(-\frac{1}{4r}\frac{\partial\tilde{\varrho}}{\partial r} + \frac{r}{2}\frac{\partial\tilde{\varrho}}{\partial t}\right)\frac{\partial}{\partial r} + \left(\frac{1}{4r}\frac{\partial\tilde{\varrho}}{\partial r}\right)\frac{\partial}{\partial\varphi} + \left(-\frac{r}{2}\frac{\partial\tilde{\varrho}}{\partial r} + \tilde{\varrho}\right)\frac{\partial}{\partial t}. \tag{3.4}$$

For each $p \in \mathbb{H}^1_s$ let $I_p \subseteq \mathbb{R}$ denote the open interval around 0 where the maximal solution of the differential equation $\Psi' = \tilde{v}(\Psi)$ for the initial value $\Psi(0) = p$ is defined. Then consider the flow

$$\begin{aligned} G : \{(s,p) :\ p \in \mathbb{H}^1_s,\ s \in I_p\} &\to \mathbb{H}^1_s \\ (s,p) &\mapsto G(s,p) = G_s(p). \end{aligned}$$

The vector field $\tilde{v}$ has been constructed such that the flow map $G_1|_{\bar{B}(e_0,a)} : \bar{B}(e_0,a) \to \mathbb{H}^1_s$ will have the following boundary behavior

$$G_1|_{\partial B(e_0,b)} = \mathrm{id} \quad \text{and} \quad G_1|_{\partial B(e_0,a)} = f.$$

We need to verify the following three properties:

(1) $p \in \bar{B}(e_0,a) \Rightarrow I_p \supseteq [0,1]$
(2) $p \in \partial B(e_0,a) \Rightarrow G_1(p) = f(p)$
(3) $p \in \partial B(e_0,b) \Rightarrow G_1(p) = p$.

For a given point $p \in B(e_0,a)$, we have either $[0,1] \subseteq I_p$ as desired or then there should exist $\xi < 1$ such that the flow line $s \mapsto G_s(p)$ of p comes arbitrarily close to the boundary of $\mathbb{H}^1_s$ as s approaches ξ (see [38]).

Lemma 3.4 shows that for $p \in \partial B(e_0,a)$ we have $d_H(G_s(p), e_0) \ge 4b$ for $s \in [0,1]$. Clearly, $G_s(p) \in \mathbb{H}^1_s$ for $s \in [0,1]$ and $p \in \partial B(e_0,a) \subseteq U$. It follows that the trajectory $\{G_s(p) : s \in [0,1]\}$ stays in the region $\mathbb{H}^1_s \setminus B(e_0,2b)$ where the modified vector field $\tilde{v}$ coincides with the vector field v from (3.1). Therefore $G_1(p) = f(p)$ for $p \in \partial B(e_0,a)$.

Next, we need to show that the flow map G_1 is defined for all points in the ball $\bar{B}(e_0,a)$.

We consider the vector field $\tilde{v} : \mathbb{H}^1_s \to \mathbb{R}^3$ as it has been defined in (3.4). Assume, to get a contradiction, that for some point $p \in \bar{B}(e_0,a)$ the solution $s \mapsto G_s(p)$ is defined on $[0, \xi)$ with $\xi < 1$ and it cannot be extended to $[0, 1]$. This would imply that $G_s(p)$ comes arbitrarily close to the boundary of $\mathbb{H}^1_s$ as $s \nearrow \xi$. In particular, there has to exist $s \in [0, \xi)$ such that $q := G_{s_0}(p) \in \partial B(e_0, a)$ (the orbit of p cannot be contained entirely in $B(e_0, a)$). Yet, we have seen above that for points $q = \partial B(e_0, a)$ the solution $G_s(q)$ exists for all $s \in [0, 1]$. This can be applied to $q := G_{s_0}(p)$ and will guarantee the existence of $G_1 : \bar{B}(e_0, a) \to \mathbb{H}^1$. As $\tilde{v} \equiv 0$ on $\bar{B}(e_0, \frac{3}{2}b)$, we find $G_1|_{\partial B(e_0,b)} = \mathrm{id}$.

We need to ensure that the resulting map G_1 is quasiconformal. In order to do so, let us note that the set

$$C := \{G_s(p) : s \in [0, 1], p \in \bar{B}(e_0, a)\}$$

is compact and entirely contained in $\mathbb{H}^1_s$. Moreover, $\tilde{\varrho}$ is a smooth function on $\mathbb{H}^1_s$ and hence the second horizontal derivatives of $\tilde{\varrho}$ are clearly bounded on C. This shows in particular that G_1 is a quasiconformal mapping (see [21, 23]).

To come to a conclusion, we have proved the existence of a quasiconformal map $G_1 : \bar{B}(e_0, a) \to \mathbb{H}^1$ with the property that

$$G_1 = \begin{cases} \mathrm{id} & \text{on } \overline{B}(e_0, b) \\ \text{qc extension} & \text{on } B(e_0, a) \setminus \overline{B}(e_0, b) \\ f & \text{on } \partial B(e_0, a). \end{cases}$$

Finally, we define

$$\tilde{f}(p) = \left(m_{k\varphi_i} \circ G_1 \circ m_{(-\varphi_i)}\right)(p)$$

for all $p \in B(p_i, a)$, and $i = 0, 1, \dots, k$, where $\varphi_0 = \frac{\pi}{k}$ and φ_i for $i = 1, \dots, k$ as in (3.2).

We will verify that the continuous extension of this map to the boundary of the spherical annulus $B(p_i, a) \setminus B(p_i, b)$ has the right boundary values, that is, it coincides with the definition given in (3.3). To this end, let us note that

$$m_{k\phi} \circ f \circ m_{-\phi} = f$$

and

$$m_{k\phi} \circ \mathrm{id} \circ m_{-\phi} = m_{k\phi-\phi}.$$

Then the desired result follows by observing that $\varphi_0 = \frac{\pi}{k}$ and $k\varphi_i = \frac{\pi}{k} \pmod{2\pi}$ for $i = 1, \dots, k$.

Thus $\tilde{f}|_{\partial B(p_i,a)} = f$ and $\tilde{f}|_{\partial B(p_i,b)}$ is a rotation. Hence, $\tilde{f}$ maps the ring $B(p_i, a) \setminus \bar{B}(p_i, b)$ to the domain enclosed by $\partial B(p_0, b)$ and $\partial f B(p_i, a)$ for $i = 1, \dots, k$ and the ring $B(p_0, a) \setminus \bar{B}(p_0, b)$ to the domain enclosed by $\partial B(f(p_0), b)$ and $\partial f B(p_0, a)$.

As explained earlier, the map $\tilde{f} : \mathbb{H}^1 \to \mathbb{H}^1$ can now be extended to a map on $\bar{\mathbb{H}}^1$ by setting $\tilde{f}(\infty) = \infty$. The conditions above imply that the map $\tilde{f} : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ is well defined and quasiregular.

Denote further by

$$\Phi := L_{p_0} \circ \delta_b \circ \Phi_0 \circ \delta_{\frac{1}{b}} \circ L_{p_0^{-1}}$$

a conformal inversion on the sphere $\partial B(p_0, b)$ and set

$$g = \Phi \circ \tilde{f} : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1.$$

We will show that g and all its iterates are uniformly quasiregular. This is because we have built the set $B(p_0, b)$ to be a conformal trap, where all the points, whose neighborhood is distorted, land only after the next iterate under g. This especially happens to all the points in the branch set. First, if $p \in B := B(p_0, b)$ then $g|B$ is conformal and $g(B) = \Phi(B(f(p_0), b)) \subset B$. Clearly then $g^m|B$ is conformal for every $m \geq 1$. Next, if $p \in \bar{\mathbb{H}}^1 \setminus \bigcup_{i=1}^k B(p_i, b)$ then $\tilde{f}(p) \in \bar{\mathbb{H}}^1 \setminus B$ and $g(p) \in B$. Therefore $g^m|\bar{\mathbb{H}}^1 \setminus \bigcup_{i=1}^k B(p_i, b)$ is quasiregular with a uniform bound on the distortion for each m by our first observation. Finally, if $p \in B(p_i, b)$, $i = 1, \dots, k$ then g is a conformal rotation followed by the conformal mapping Φ. Thus the iterates of g stay conformal at p until it passes into the complement of $\bigcup_{i=1}^k B(p_i, b)$. Under the next iterate it picks up some distortion before passing into trap B and the iterates again stay conformal.

Note that $B_g = B_f$ also holds. □

Remark 3.5 The construction in Theorem 3.1 can also be made for all winding mappings $(r, \varphi, t) \mapsto (ar, k\varphi, a^2kt)$ that are globally defined in $\bar{\mathbb{H}}^1$ with obvious modifications. The same holds for the example in [13] producing a branched branch set.

For a *quasiregular semigroup* Γ (see Sect. 5) the *Fatou set* of Γ is defined as

$$F(\Gamma) := \{p \in \bar{\mathbb{H}}^1 : \text{ there is an open } U,\ p \in U,\ \Gamma|_U \text{ normal}\},$$

where *normal* means that every sequence of Γ contains a locally uniformly convergent subsequence. The *Julia set* is then defined as $J(\Gamma) = \bar{\mathbb{H}}^1 \setminus F(\Gamma)$. If the semigroup is generated by a single uniformly quasiregular map, i.e., $\Gamma = \{g^n\}_{n\in\mathbb{N}}$, we will write $J(g)$ instead of $J(\Gamma)$.

The Julia set of the mapping g constructed in Theorem 3.1 is the Cantor set

$$J(g) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} g^{-n}\left(\bar{\mathbb{H}}^1 \setminus B\right) \subset \bigcup_{i=1}^{k} B(p_i, b). \tag{3.5}$$

For the corresponding statement in $\bar{\mathbb{R}}^n$, see [19]. The proof of (3.5) is completely analogous.

4 Uniformly Quasiregular Mappings with Large Branch Sets

In this section we refine the result of the previous section by showing the existence of a uqr map on $\bar{\mathbb{H}}^1$ with arbitrarily large dimensional branch set. In order to set the notation right, recall that we consider the Heisenberg group $\mathbb{H}^1$ with the distance d_H from (2.1) as a metric space. (The standard compactification $\bar{\mathbb{H}}^1$ of $\mathbb{H}^1$ yields a metric $d_{\bar{H}}$ on $\bar{\mathbb{H}}^1$ which is locally bi-Lipschitz equivalent with d_H.) With respect to this metric we shall consider the notions of Hausdorff measure and dimension. Let us recall that the Hausdorff dimension of the space $\bar{\mathbb{H}}^1$ with respect to the Heisenberg metric is equal to 4. This in fact already illustrates some kind of fractal feature of the Heisenberg group where the topological and Hausdorff dimensions do not coincide. For $0 \le \alpha \le 4$ we denote by $\mathcal{H}^\alpha$ the α-dimensional Hausdorff measure in the metric space $(\bar{\mathbb{H}}^1, d_{\bar{H}})$. The Hausdorff dimension of subsets $A \subseteq (\bar{\mathbb{H}}^1, d_{\bar{H}})$ will be also considered in this context.

Using the positivity of the Jacobian, it has been shown by Heinonen and Holopainen in [13] that the branch set of a quasiregular mapping in a Carnot group of type H cannot be arbitrarily big (see also [6]). More precisely, for the first Heisenberg group the following statement holds:

Theorem 4.1 [13] *Let $f : \Omega \to \mathbb{H}^1$, $\Omega \subseteq \mathbb{H}^1$, be a quasiregular mapping, non-constant in each component of Ω. Then the branch set B_f of f has vanishing 4-dimensional Hausdorff measure*: $\mathcal{H}^4(B_f) = 0$.

As quasiregular mappings satisfy the so-called *Lusin property*, that is, they map sets of $\mathcal{H}^4$-measure zero to sets of $\mathcal{H}^4$-measure zero, we have the following:

Corollary 4.2 *Let $f : \Omega \to \mathbb{H}^1$, $\Omega \subseteq \mathbb{H}^1$, be a quasiregular mapping, non-constant in each component of Ω. Then for the image of the branch set $f(B_f)$ we have $\mathcal{H}^4(f(B_f)) = 0$.*

The same statements are true for quasiregular mappings on the compactified Heisenberg group.

In this section, we will construct an example which shows that this result is sharp in the sense that the dimension of the branch set and its image can come arbitrarily close to 4. To do that we shall use the technique of [3], where quasiconformal mappings of the Heisenberg group have been constructed which change the dimensions of Cantor sets in arbitrary fashion. The following statement refines this result by placing the Cantor sets on the vertical axis.

Proposition 4.3 *For every $\varepsilon > 0$ there exist Cantor subset $S_1 \subseteq \mathbb{H}^1$ with* $\dim_H S_1 = 4 - \varepsilon$, *a Cantor subset S_2 of the t-axis, and a quasiconformal mapping $H : \mathbb{H}^1 \to \mathbb{H}^1$ such that $H(S_1) = S_2$.*

Proof The mapping $H : \mathbb{H}^1 \to \mathbb{H}^1$ is constructed as a composition $H = H_2 \circ H_1$, where H_1 is a quasiconformal mapping which reduces the Hausdorff dimension of S_1 as in [3] and H_2 maps the Cantor set $\tilde{S}_2 := H_1(S_1)$ onto a Cantor subset S_2 of the t-axis. The Cantor sets S_1, $\tilde{S}_2$ and S_2 will be obtained as invariant sets of certain conformal dynamical systems.

The method presented in [3] allows one to reduce the Hausdorff dimension of the higher-dimensional Cantor set S_1 in an arbitrary fashion, however, the resulting

Cantor set $\tilde{S}_2$ will not typically lie on the t-axis. A modification of the proof in [3] is needed to eventually map $\tilde{S}_2$ onto a Cantor subset of the t-axis. This approach will be sketched below (for technical details see the similar proof of Theorem 1.1 in [3]).

As shown in [3], one can construct for an arbitrarily small $\varepsilon > 0$ a Cantor subset S_1 of the unit ball $B(0,1)$ such that $\dim_H S_1 = 4 - \varepsilon$. More precisely, we can choose $N \geq 2$ and $r_1 = r_1(N, \varepsilon) > 0$ such that there exist disjoint closed balls $\bar{B}(p_i, r_1) \subset B(0,1)$, $i = 1, \ldots, N$, for which the associated conformal dynamical system $\mathcal{F} = \{f_1, \ldots, f_N\}$ defined by

$$f_i := L_{p_i} \circ \delta_{r_1} : B(0,1) \to B(p_i, r_1)$$

has $(4 - \varepsilon)$-dimensional invariant set S_1, i.e.,

$$S_1 = \bigcup_{i=1}^{N} f_i(S_1) \subseteq \bigcup_{i=1}^{N} B(p_i, r_1).$$

To ensure the equality $\dim_H S_1 = 4 - \epsilon$ we choose N and r_1 such that $N r_1^{4-\epsilon} = 1$. Similarly, we consider for $r_2 < r_1$ the associated conformal dynamical system $\mathcal{G} = \{g_1, \ldots, g_N\}$ defined by

$$g_i := L_{p_i} \circ \delta_{r_2} : B(0,1) \to B(p_i, r_2).$$

This will yield a smaller-dimensional invariant set $\tilde{S}_2 \subset \bigcup_{i=1}^{N} B(p_i, r_2)$. Here we require that $\dim_H S_2 = d < 2$ by the appropriate choice of $r_2 < r_1$ such that $N r_2^d = 1$.

Then a quasiconformal map $H_1 : \mathbb{H}^1 \to \mathbb{H}^1$ with $H_1(S_1) = \tilde{S}_2$ can be defined as in [3] using the dynamical systems $\mathcal{F}$ and $\mathcal{G}$.

We needed to choose $r_2 > 0$ as above, small enough such that it is possible to construct a Cantor set S_2 of the same dimension as $\tilde{S}_2$ in the t-axis and a quasiconformal map $H_2 : \mathbb{H}^1 \to \mathbb{H}^1$ which maps $\tilde{S}_2$ to S_2.

We shall explain the method for constructing the quasiconformal mapping H_2 in the following. The main idea is to use an iterative construction which defines the mapping piecewise on successive multirings occurring in the above dynamical construction. In the first step of the construction this mapping will satisfy $H_2 |_{\mathbb{H}^1 \setminus B(0,1)} = \mathrm{id} |_{\mathbb{H}^1 \setminus B(0,1)}$. Inside the unit ball, we define H_2 using the dynamics

$$\mathcal{G} = \{g_1, \ldots, g_N\}, \quad g_i := L_{p_i} \circ \delta_{r_2} : B(0,1) \to B(p_i, r_2)$$

and

$$\mathcal{H} = \{h_1, \ldots, h_N\}, \quad h_i := L_{q_i} \circ \delta_{r_2} : B(0,1) \to B(q_i, r_2),$$

where $q_1, \ldots, q_N$ denote points on the t-axis (then the invariant set associated with $\mathcal{H}$ will lie entirely in the t-axis). Now the mapping H_2 is defined inside $B(0,1)$ piecewise by setting

$$H_2 |_{g_{i_n} \circ \cdots \circ g_{i_1} A_0^{\delta}} := h_{i_n} \circ \cdots \circ h_{i_1} \circ H_0 \circ g_{i_1}^{-1} \circ \cdots \circ g_{i_n}^{-1},$$

where

$$A_0^\delta := B(0, 1+\delta) \Big\backslash \bigcup_{i=1}^{N} g_i B(0, 1-\delta), \qquad \tilde{A}_0^\delta := B(0, 1+\delta) \Big\backslash \bigcup_{i=1}^{N} h_i B(0, 1-\delta)$$

with $\delta > 0$ small enough such that one can construct a quasiconformal map $H_0 : A_0^\delta \to \tilde{A}_0^\delta$ which satisfies

$$H_0 = \begin{cases} \mathrm{id} & \text{on } R_0^\delta := B(0, 1+\delta) \setminus B(0, 1-\delta), \\ h_i \circ g_i^{-1} = L_{q_i p_i^{-1}} & \text{on } R_i^\delta := g_i R_0^\delta, \end{cases} \tag{4.1}$$

i.e., we need $R_i^\delta \cap R_j^\delta = \emptyset$ for $0 \le i, j \le N$, $i \ne j$ and $R_0^\delta \cap h_i R_0^\delta = \emptyset$ for $0 \le i \le N$. The properties of H_0 ensure that there is no ambiguous definition of H_2 on the intersection of the domains $g_{i_n} \circ \cdots \circ g_{i_1} A_0^\delta$ for various $n \in \mathbb{N}$. As in [3] we can conclude that the thus defined mapping $H_2 : \mathbb{H}^1 \setminus \tilde{S}_2 \to \mathbb{H}^1 \setminus S_2$ is quasiconformal as a finite composition of translations, dilations, and the quasiconformal map H_0.

It remains to construct a quasiconformal map $H_0 : A_0^\delta \to \tilde{A}_0^\delta$ which satisfies the conditions given in (4.1). This can be done using the flow method due to Korányi and Reimann. The idea is to find appropriate potentials for which the corresponding vector fields will generate flows (Ψ_s) of quasiconformal maps with Ψ_1 being either the identity or an appropriate left translation $L_{q_i p_i^{-1}} = h_i \circ g_i^{-1}$. Then we need to glue these potentials together in order to obtain a globally defined potential function ϱ for which the corresponding vector field will produce a flow (Ψ_s) with $\Psi_1 = H_0$.

Let us discuss in more detail how such a potential can be defined. Obviously, the vanishing potential $\varrho = 0$ will yield the identity map. To generate translation $L_{q_i p_i^{-1}}$, where $p_i = (x_i, y_i, t_i)$ and $q_i = (0, 0, t_i + a_i)$, consider the potential

$$\varrho_i := a_i - 4y_i x + 4x_i y.$$

The corresponding vector field according to (2.6) is given by

$$v_i = -x_i \frac{\partial}{\partial x} - y_i \frac{\partial}{\partial y} + (a_i - 2y_i x + 2x_i y) \frac{\partial}{\partial t}.$$

Then, the system

$$\begin{cases} \frac{\partial \Psi_{is}}{\partial s}(x, y, t) = v_i(\Psi_{is}(x, y, t)) \\ \Psi_{i0}(x, y, t) = (x, y, t) \end{cases}$$

has the solution

$$\begin{aligned} \Psi_{is}(x, y, t) &= (-x_i s + x, -y_i s + y, s(a_i - 2y_i x + 2x_i y) + t) \\ &= L_{(-sx_i, -sy_i, sa_i)}(x, y, t), \end{aligned}$$

in particular we obtain for $s = 1$

$$\Psi_{i1} = L_{(-x_i, -y_i, a_i)} = L_{q_i p_i^{-1}} = h_i \circ g_i^{-1},$$

as desired. Also note that for all points p the flow curve $s \mapsto \Psi_{is}(p)$ is simply a straight line connecting p to $L_{q_i p_i^{-1}}(p) = q_i p_i^{-1} p$.

Now we have to glue together the potentials ϱ_i for $i = 1, \dots, N$ to a globally defined potential ϱ that coincides with ϱ_i in a small neighborhood D_i^δ of the trajectory $s \mapsto \Psi_{is}(p_i)$ for all $i = 1, \dots, N$. To do that we have to define ϱ in such a way that the flow lines of points in R_i^δ stay inside the region where $\varrho = \varrho_i$ such that $\Psi_1(p) = H_0(p)$. We denote

$$D_i^\delta := \{L_{(-sx_i, -sy_i, sa_i)}(p) : \ p \in R_i^\delta, \ s \in [0, 1]\}$$

and define

$$\tilde{\varrho}_i := \begin{cases} \varrho_i & \text{on } \bar{D}_i^\delta, \\ \text{smooth extension} & \text{on } D_i^{2\delta} \setminus \bar{D}_i^\delta, \\ 0 & \text{on } \mathbb{H}^1 \setminus D_i^{2\delta}. \end{cases}$$

Finally, we set $\varrho := \sum_{i=1}^N \tilde{\varrho}_i$. In order to ensure that this is a well-defined function, we need to have that the sets $D_i^{2\delta}$ are pairwise disjoint and do not touch the annular domain $R_0^{2\delta}$. This yields restrictions on the choice of $\delta > 0$, the radius $r_2 < 1$, and the points $q_i = (0, 0, t_i + a_i)$ on the t-axis. First, we want to make sure that the flow curves of the points p_i, i.e., the line segments $l_{p_i q_i}$ connecting p_i to q_i, do not meet. The only points for which there might be an intersection of the flow lines are points on the t-axis and points $p_i = (r_i, \varphi_i, t_i)$, $p_j = (r_j, \varphi_j, t_j)$ with $\varphi_i = \varphi_j$ (in cylindrical coordinates). Yet, given N distinct points $q_1, \dots, q_N$ on the t-axis, this situation can be prevented simply by perturbing the points p_i a little. Note that this can be done in such a way that even the larger balls $B(p_i, r_1)$ still remain disjoint.

So we have N points $p_1, \dots, p_N$ in the unit ball and N points $q_1, \dots, q_N$ lying on the t-axis within the unit ball such that the connecting line segments $l_{p_i q_i}$ do not meet. Then we can choose $r_2 \in (0, r_1)$ and $\delta > 0$ such that the sets $D_i^{2\delta}$ are pairwise disjoint and lie entirely within $R_0^{2\delta}$. This guarantees that the potential ϱ is well defined and the corresponding vector field v generates a flow (Ψ_s) such that $\Psi_1 = H_0$. Note that H_0 is obviously quasiconformal on $\bigcup_{i=0}^N R_i^\delta$. The quasiconformality on $A_0^\delta \setminus \bigcup_{i=0}^N R_0^\delta$ follows from the fact that the potential ϱ is smooth and compactly supported, hence it has bounded second horizontal derivatives.

Once we have constructed a quasiconformal map H_0 with the properties (4.1), a quasiconformal map $H_2 : \mathbb{H}^1 \setminus \tilde{S}_2 \to \mathbb{H}^1 \setminus S_2$ can be defined as sketched above. It can be extended to a homeomorphism $H_2 : \mathbb{H}^1 \to \mathbb{H}^1$ with $H_2(\tilde{S}_2) = S_2$. We conclude by Theorem 1.3 from [4] that this extension is again quasiconformal. The composition with the previously defined function H_1 yields the desired map. □

Theorem 4.4 *For any $\varepsilon > 0$ there exists a uqr mapping $f : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ such that*

$$\dim_H B_f \geq 4 - \varepsilon.$$

Proof We consider the quasiconformal mapping $H : \mathbb{H}^1 \to \mathbb{H}^1$ from Proposition 4.3 and extend it to the compactified Heisenberg group by defining $H(\infty) = \infty$ (note that this is clearly possible since H is the identity map outside the unit ball). This new mapping will again be denoted by H. Next, we define $f := H^{-1} \circ g \circ H$, where g is the uqr mapping from Theorem 3.1. Observe that $f^n = H^{-1} \circ g^n \circ H$ for $n \geq 1$. Therefore $K_{f^n} \leq K_{H^{-1}} \cdot K_{g^n} \cdot K_H$ and f is a uqr map. Observe furthermore that

$$B_f = H^{-1}(t - \text{axis}) \supseteq S_1$$

which implies that

$$\dim_H B_f \geq \dim_H S_1 = 4 - \varepsilon.$$

□

Theorem 4.5 *For any* $\varepsilon > 0$ *there exists a uqr mapping* $f : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ *such that*

$$\dim_H f(B_f) \geq 4 - \varepsilon.$$

Proof Similarly as before, we consider the extension $H : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ of the quasiconformal mapping from Proposition 4.3.

Next, we define $f := H^{-1} \circ \Phi^{-1} \circ g \circ \Phi \circ H$, where $g = \Phi \circ \tilde{f}$ is the uqr mapping from Theorem 3.1 and Φ denotes the conformal inversion on $\partial B(p_0, b)$.

Observe that f is indeed uqr and

$$\begin{aligned} f(B_f) &= H^{-1}(\Phi^{-1}(g(t - \text{axis}))) = H^{-1}(\Phi^{-1}(\Phi(\tilde{f}(t - \text{axis}))) \\ &= H^{-1}(t - \text{axis}) \supseteq H^{-1}(S_2) = S_1 \end{aligned}$$

and hence $\dim_H f(B_f) \geq \dim_H S_1 = 4 - \varepsilon$. □

5 Equivariant Measurable Structures

In this section we show that given a quasiregular semigroup Γ generated by a uniformly quasiregular mapping g, e.g., by the mapping constructed above,

$$\Gamma = \{g^n : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1 \mid g \text{ uniformly quasiregular },\ n \in \mathbb{N}\},$$

it is possible to construct a measurable CR structure (or equivalently a measurable horizontal conformal structure) which is equivariant with respect to the elements of Γ.

Before coming to the proof of this main result, we fix the necessary notation and list a few important properties of contact mappings. Recall the following:

Definition 5.1 A map $f : U \to \mathbb{H}^1$, $U \subset \mathbb{H}^1$ open, is a (*generalized*) *contact map* if $f \in HW^{1,1}_{loc}(U)$ and the tangent vectors

$$Xf(p) := (Xf_1(p), Xf_2(p), Xf_3(p)) \quad \text{and} \quad Yf(p) := (Yf_1(p), Yf_2(p), Yf_3(p))$$

belong to the horizontal tangent space $HT_{f(p)}$ for almost all $p \in U$.

A mapping of class $HW^{1,1}_{loc}(U)$ is contact if and only if

$$-2f_2(p)Xf_1(p)+2f_1(p)Xf_2(p)+Xf_3(p)=0 \tag{5.1}$$

$$-2f_2(p)Yf_1(p)+2f_1(p)Yf_2(p)+Yf_3(p)=0, \tag{5.2}$$

for almost every $p \in U$.

It will be convenient to write the mapping $f=(f_1, f_2, f_3)$ in complex notation as $f=(f_I, f_3)$ with $f_I = f_1 + \mathrm{i} f_2$. Moreover, let us denote by $H^{\mathbb{C}}\mathbb{H}^1$ the complexified horizontal bundle of $\mathbb{H}^1$ which is given by

$$H^{\mathbb{C}}_p\mathbb{H}^1 := \operatorname{span}\{Z_p, \bar{Z}_p\}, \ \ p \in \mathbb{H}^1,$$

where $Z_p = \frac{1}{2}(X_p - \mathrm{i}Y_p)$ and $\bar{Z}_p = \frac{1}{2}(X_p + \mathrm{i}Y_p)$.

For a contact map f the *complexified horizontal tangent map* $Hf^{\mathbb{C}}_*(p)$ can be defined in almost all points p as the complex linear map $Hf^{\mathbb{C}}_*(p) : H^{\mathbb{C}}_p\mathbb{H}^1 \to H^{\mathbb{C}}_{f(p)}\mathbb{H}^1$ whose matrix representation in the bases $\{Z_p, \bar{Z}_p\}$ and $\{Z_{f(p)}, \bar{Z}_{f(p)}\}$ is

$$Hf^{\mathbb{C}}_*(p) = \begin{pmatrix} Zf_I(p) & \bar{Z}f_I(p) \\ Z\bar{f}_I(p) & \bar{Z}\bar{f}_I(p) \end{pmatrix}.$$

In the subsequent discussion, we will only consider mappings with $\det Hf^{\mathbb{C}}_*(p) \geq 0$ for a.e. p.

Using this complex notation, the existence of a constant $K \geq 1$ such that $|Hf_*(p)|^4 \leq K \det f_*(p)$ in the definition of quasiregularity turns out to be equivalent to the existence of $K \geq 1$ such that

$$\frac{|Zf_I(p)| + |\bar{Z}f_I(p)|}{|Zf_I(p)| - |\bar{Z}f_I(p)|} \leq \sqrt{K} \quad \text{a.e. } p.$$

It follows that a continuous contact map $f \in HW^{1,4}_{loc}(U)$ for which

$$\|\mu_f\| = \operatorname{ess\,sup}_p \left| \frac{\bar{Z}f_I(p)}{Zf_I(p)} \right| < 1$$

is quasiregular.

We will need the following *chain rule* for the real horizontal tangent map

$$H(g \circ f)_*(p) = Hg_*(f(p))Hf_*(p) \quad \text{a.e. } p \tag{5.3}$$

of quasiregular mappings. For $\mathcal{C}^1$-maps this can be seen by a direct computation, using the identities (5.1). The general case follows from the chain rule for Pansu differentials and the fact that qr maps satisfy Lusin's conditions and are almost everywhere Pansu differentiable with Pansu differential equal to the formal Pansu differential [6].

5.1 Measurable CR Structures

In this section we show that given a quasiregular semigroup Γ it is possible to construct a measurable CR structure which is equivariant with respect to the elements of Γ analogously as in the Riemannian case [18]. This will be the main theorem of this section (Theorem 5.3). To explain the situation in the Heisenberg group $\mathbb{H}^1$ we use the notation as in [21] where it was shown that a smooth (orientation-preserving) H-quasiconformal mapping (in the sense of Mostow) $f = (f_1, f_2, f_3)$ satisfies a tangential version of the classical Beltrami equation

$$\bar{Z} f_I = \mu Z f_I$$

where μ is the complex dilation of f satisfying $|\mu| < 1$ and

$$\frac{1+|\mu|}{1-|\mu|} \le H.$$

The mutual and quantitative equivalence of different definitions of quasiconformality is discussed in [23] (for Heisenberg groups) and in [11] (for more general Carnot groups). The complex function μ is interpreted to take values in the standard hyperbolic unit disk, and it takes the role of a CR structure as presented in [8] and [22]. The standard CR structure on $\mathbb{H}^1$ is given by the splitting $H_p^{\mathbb{C}}\mathbb{H}^1 = T_p^{1,0} \oplus T_p^{0,1}$ of $H_p^{\mathbb{C}}\mathbb{H}^1$ into the subspace of "holomorphic and antiholomorphic" vectors: $T_p^{1,0} = \operatorname{span}\{Z_p\}$ and $T_p^{0,1} = \overline{T_p^{1,0}}$. A general measurable CR structure on $\mathbb{H}^1$ is given by a measurable function $\nu : \mathbb{H}^1 \to \mathbb{D}$ with the property that

$$\|\nu\| := \operatorname{ess\,sup}_{p\in\mathbb{H}^1} |\nu(p)| < 1,$$

where the new subspace $T_{\nu,p}^{1,0}$ is defined as

$$T_{\nu,p}^{1,0} := \{Z' - \overline{\nu(p)} \cdot \overline{Z'} : Z' \in T_p^{1,0}\},$$

and $T_{\nu,p}^{0,1} = \overline{T_{\nu,p}^{1,0}}$.

Let us consider a quasiregular map $g : \mathbb{H}^1 \to \mathbb{H}^1$ and a measurable CR structure ν defined on the target space. Using g we can pull back ν to a measurable CR structure $g_\sharp \nu$ on the domain space by requiring the condition

$$Hg_*^{\mathbb{C}}(p)(T_{g_\sharp\nu,\ p}^{1,0}) = T_{\nu,\ g(p)}^{1,0}, \tag{5.4}$$

where $Hg_*^{\mathbb{C}}(p)$ is the complexified horizontal tangent map of g, which exists for points p where g is contact, i.e., for almost every point in $\mathbb{H}^1$.

We shall calculate explicitly the value of $g_\sharp\nu(p)$ for a.e. $p \in \mathbb{H}^1$ as follows. By (5.4) we can write

$$\begin{aligned} Hg_*^{\mathbb{C}}(p)(Z_p - \overline{g_\sharp\nu(p)} \cdot \overline{Z_p}) &= (Zg_I(p) \cdot Z_{g(p)} + Z\bar{g}_I(p) \cdot \overline{Z}_{g(p)}) \\ &\quad - \overline{g_\sharp\nu(p)} \cdot (\overline{Z}g_I(p) \cdot Z_{g(p)} + \overline{Z}\bar{g}_I(p) \cdot \overline{Z}_{g(p)}) \end{aligned}$$

$$= \alpha(Z_{g(p)} - \bar{\nu}(g(p))\overline{Z}_{g(p)}),$$

for some $\alpha \in \mathbb{C}$.

The equality of the coefficients of $Z_{g(p)}$ and $\overline{Z}_{g(p)}$ in the second equation above yields:

$$\begin{cases} Zg_I(p) - \overline{g_\sharp \nu(p)} \cdot \overline{Z} g_I(p) = \alpha \\ Z\bar{g}_I(p) - \overline{g_\sharp \nu(p)} \cdot \overline{Z}\bar{g}_I(p) = -\bar{\nu}(g(p)) \cdot \alpha, \end{cases}$$

which implies that

$$g_\sharp \nu(p) = \frac{\overline{Z} g_I(p) + \nu(g(p)) \cdot \overline{Zg_I(p)}}{Zg_I(p) + \nu(g(p)) \cdot \overline{\overline{Z} g_I(p)}}. \tag{5.5}$$

An important special case is when $\nu = 0$, i.e., we pull back the standard CR structure by g. In this case (5.5) reads as:

$$g_\sharp \nu(p) = \frac{\overline{Z} g_I(p)}{Zg_I(p)} =: \mu_g(p).$$

The resulting CR structure $\mu_g : \mathbb{H}^1 \to \mathbb{D}$ is the so-called Beltrami differential of g. A CR structure $\mu : \mathbb{H}^1 \to \mathbb{D}$ is called *realizable* if $\mu = \mu_g$ for some quasiconformal mapping $g : \mathbb{H}^1 \to \mathbb{H}^1$. In contrast to the planar case, in the Heisenberg group there is no measurable Riemann mapping theorem, and so not every CR structure is realizable. In general it is a difficult problem to characterize the realizable CR structures on $\mathbb{H}^1$ (see [22] for results in this direction).

Coming back to (5.5) in its general form, we can denote by $a = \overline{Z} g_I(p)$ and $b = Zg_I(p)$ and observe that (5.5) can be written in the form

$$g_\sharp \nu(p) = T_g(p)(\nu(g(p))), \tag{5.6}$$

where $T_g(p) : \mathbb{D} \to \mathbb{D}$ is the Möbius transformation

$$T_g(p)(z) = \frac{a + \bar{b}z}{b + \bar{a}z}, \tag{5.7}$$

with $|a| < |b|$.

Definition 5.2 Let Γ be a semigroup of quasiregular mappings on $\bar{\mathbb{H}}^1$. We say that a CR structure $\mu : \bar{\mathbb{H}}^1 \to \mathbb{D}$ is Γ-equivariant if

$$g_\sharp \mu(p) = \mu(p) \quad \text{for a.e. } p \in \bar{\mathbb{H}}^1 \text{ and all } g \in \Gamma.$$

The main result of this section is the following theorem stating the existence of an equivariant CR structure for a semigroup of uqr mappings on $\bar{\mathbb{H}}^1$.

Theorem 5.3 *Let Γ be a countable, Abelian semigroup of uqr mappings acting on $\bar{\mathbb{H}}^1$. Then there exists a Γ-equivariant CR structure μ on $\bar{\mathbb{H}}^1$.*

Proof According to (5.6) and Definition 5.2 we have to find a measurable function $\mu : \bar{\mathbb{H}}^1 \to \mathbb{D}$ with $\|\mu\| < 1$ and such that

$$\mu(p) = T_g(p)(\mu(g(p))) \quad \text{for a.e. } p \in \bar{\mathbb{H}}^1 \text{ and all } g \in \Gamma. \tag{5.8}$$

The idea of finding such μ is based on the hyperbolic center method of Tukia [35] who proved a similar statement for groups of planar quasiconformal maps. The idea was later adapted to the case of semigroups of quasiregular maps by Iwaniec and Martin [18] acting on higher-dimensional Euclidean spheres. We shall begin our proof as in [18] by considering the so called "local groups" generated by Γ.

To do that, we use a version of Picard's theorem for quasiregular maps in H-type Carnot groups [13, 25], which guarantees that $g \in \Gamma$ omits at most finitely many points; together with the result saying that the branch set and its image are of null measure $\mathcal{H}^4(B_g) = \mathcal{H}^4(g(B_g)) = 0$ for all $g \in \Gamma$ [13, 37].

Because of the above facts and the assumption that Γ is countable we can construct a full measure set $U \subset \bar{\mathbb{H}}^1$ with the following properties:

(1) $g(U) = U = g^{-1}(U)$ for all $g \in \Gamma$,
(2) $Hg_*^{\mathbb{C}}(p)$ is defined with $\det Hg_*^{\mathbb{C}}(p) \neq 0$ for all $p \in U$ and $g \in \Gamma$,
(3) $|\mu_g(p)| < \alpha < 1$ for all $p \in U$ and $g \in \Gamma$.

For $p \in U$ we define the "local group" Γ_p of Γ at p as follows: a map $h \in \Gamma_p$ if there is some neighborhood V of p in which h can be written in the form: $h = h_1 \circ h_2 : V \to \bar{\mathbb{H}}^1$, where $h_2 \in \Gamma$ and h_1 is a branch of the inverse of some element of Γ restricted to $h_2(V)$. As in [18] one can check the following two essential properties of Γ_p:

(1) for $p \in U$ and $g \in \Gamma$ we have that

$$\Gamma_{g(p)} \circ g := \{h \circ g : h \in \Gamma_{g(p)}\} = \Gamma_p, \tag{5.9}$$

(2) if $h \in \Gamma_p$ then $h : V \to \bar{\mathbb{H}}^1$ is K-quasiconformal with K independent of p and h.

For $p \in U$ we associate the collection of CR structures generated by Beltrami differentials of mappings $h \in \Gamma_p$ as

$$CR_p := \{\mu_h(p) : h \in \Gamma_p\}.$$

Let us note at this point that as a consequence of the chain rule we have the following composition formula for the Beltrami differentials of quasiregular maps:

$$\mu_{h\circ g}(p) = \frac{\bar{Z}(h \circ g)_I(p)}{Z(h \circ g)_I(p)} = \frac{\overline{Z} g_I(p) + \mu_h(g(p)) \cdot \overline{Z g_I}(p)}{Z g_I(p) + \mu_h(g(p)) \cdot \bar{Z}\overline{g_I}(p)} = T_g(p)(\mu_h(g(p))), \tag{5.10}$$

where $T_g(p)$ is the Möbius transformation as in (5.7).

Combining relations (5.10) and (5.9) we obtain

$$CR_p = T_g(p)(CR_{g(p)}), \tag{5.11}$$

which is the set valued version of the desired relation (5.8).

Let us remark, however, that while (5.11) holds for all $g \in \Gamma$ we still need to find a single-valued solution of (5.8) i.e., a function $p \mapsto \mu(p)$ such that (5.8) holds for all $g \in \Gamma$. This is done by the hyperbolic center method. The idea is to view $CR_p \subset \mathbb{D}$ as a subset of the hyperbolic disc, and associate with it the unique closed hyperbolic disc which contains CR_p and has the smallest radius. The center of this disc is called the hyperbolic center of CR_p and is denoted by $\mu(p) \in \mathbb{D}$. For $g \in \Gamma$ we use the fact that $T_g(p) : \mathbb{D} \to \mathbb{D}$ is a Möbius transformation and hence an isometry of the hyperbolic disc. Relation (5.11) says that the set CR_p is an isometric image by $T_g(p)$ of the set $CR_{g(p)}$. Now, the hyperbolic center of any set is mapped by an isometry to the hyperbolic center of its image. Therefore we have:

$$\mu(p) = T_g(p)(\mu(g(p))), \quad \text{for all } p \in U \text{ and } g \in \Gamma,$$

which is exactly relation (5.8).

To finish the proof we remark that, because for $p \in U$, $h \in \Gamma_p$ is quasiconformal with a constant independent of p and h the set $CR_p \subset \mathbb{D}$ lies in a fixed compact set for all $p \in U$. The same is true for the hyperbolic center $\mu(p)$ showing that $\|\mu\| < 1$. □

We conclude this section with two remarks related to the statement of Theorem 5.3.

Remark 5.4 A variant of Theorem 5.3 need not be true for non-commutative semigroups with more than one generator. In [16] a planar counterexample to Theorem 5.3 for semigroups with two generators is constructed.

Recall that in the case of the Riemann sphere, the Sullivan–Tukia theorem [1] states that an Abelian uqr semigroup Γ is conjugate to a semigroup of rational maps defined as $\Gamma' = f \circ \Gamma \circ f^{-1}$, where f is the solution to the Beltrami equation $\bar{\partial} f = \mu \partial f$ and μ is the equivariant complex structure of Γ. On the other hand, it is known from several complex variables by a result which goes back to Poincaré [30] and Tanaka [32] that the only CR semigroup map acting on the unit sphere in $\mathbb{C}^2$ must be the restriction of a subgroup of automorphisms of the unit ball (see [9] for related results in more general setting).

Remark 5.5 It follows that in the present setting of non-injective maps in Γ the associated equivariant CR structure μ given in Theorem 5.3 will not be realizable, i.e., there exists no solution f to the Beltrami equation $\bar{Z} f = \mu Z f$.

5.2 Measurable Horizontal Conformal Structures

To define a horizontal conformal structure on $\bar{\mathbb{H}}^1$ we start with an inner product $\langle \cdot, \cdot \rangle_G$ on the horizontal bundle HT of $\mathbb{H}^1$ by setting

$$\langle u, v \rangle_{G(p)} = \langle G(p)u, v \rangle_p$$

in the fiber $HT_p = \operatorname{span}\{X_p, Y_p\}$, where $G : p \mapsto G(p)$ is a measurable map $\mathbb{H}^1 \to S(2)$ and $p \mapsto \langle \cdot, \cdot \rangle_p$ on the right-hand side is a Euclidean inner product with orthonormal X_p and Y_p. The space $S(2)$ of symmetric positive definite 2×2 matrices G

with real entries and with determinant 1 can be equipped with a metric that becomes isometric to the hyperbolic disk $\mathbb{D}$ via bijective correspondence

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \mapsto \mu = \frac{g_{11} - g_{22} + 2ig_{12}}{g_{11} + g_{22} + 2}. \tag{5.12}$$

This correspondence reflects the correspondence between measurable horizontal conformal structures and measurable CR structures as it is illustrated in Theorem 5.6.

We call G a *horizontal conformal structure* if it is essentially bounded with respect to the hyperbolic metric in $S(2)$ ([19]). A quasiregular mapping $g : \mathbb{H}^1 \to \mathbb{H}^1$ preserves the given structure if

$$\langle Hg_*(p)u, Hg_*(p)v\rangle_{G(g(p))} = \lambda(p)\langle u, v\rangle_{G(p)} \tag{5.13}$$

holds almost everywhere for some positive real-valued function $\lambda : \mathbb{H}^1 \to \mathbb{R}_+$. Here $Hg_*(p)$ is the real horizontal tangent map of g at p; $Hg_*(p) : HT_p \to HT_{g(p)}$ for which the matrix representation in the bases $\{X_p, Y_p\}$ and $\{X_{g(p)}, Y_{g(p)}\}$ is given by

$$Hg_*(p) = \begin{pmatrix} Xg_1(p) & Yg_1(p) \\ Xg_2(p) & Yg_2(p) \end{pmatrix}.$$

Condition (5.13) implies then that

$$Hg_*^T(p)G(g(p))Hg_*(p) = \lambda(p)G(p)$$

holds for almost every $p \in \mathbb{H}^1$. The condition $\det G = 1$ implies that

$$\lambda(p) = \det Hg_*(p) =: J_H(g_I, p)$$

agrees with the horizontal Jacobian determinant almost everywhere. Hence the horizontal Beltrami equation in the real form reads

$$Hg_*^T(p)G(g(p))Hg_*(p) = J_H(g_I, p)G(p). \tag{5.14}$$

Given a semigroup Γ of qr maps acting on $\bar{\mathbb{H}}^1$, we say that a horizontal conformal structure G is Γ-*equivariant* if (5.14) holds for a.e. $p \in \mathbb{H}^1$ and all $g \in \Gamma$. The existence of such an equivariant horizontal conformal structure turns out to be equivalent to the existence of an equivariant CR structure stated in Theorem 5.3. Yet another characterization of the existence of a Γ-equivariant structure is that all $g \in \Gamma$ are solutions to the following differential equation

$$\bar{Z}g_I(p) = \alpha(p, g(p))Zg_I(p) + \beta(p, g(p))\overline{Zg_I}(p), \tag{5.15}$$

with

$$\alpha(p, g(p)) := \frac{\eta(p)}{\sqrt{|\eta(p)|^2 + 1} + \sqrt{|\eta(g(p))|^2 + 1}} \quad \text{and}$$

and

$$\beta(p, g(p)) := \frac{-\eta(g(p))}{\sqrt{|\eta(g(p))|^2+1}+\sqrt{|\eta(p)|^2+1}}$$

for some function η. This is summarized in the following theorem:

Theorem 5.6 *Let* Γ *be a semigroup of quasiregular self-mappings of a domain* $U \subseteq \bar{\mathbb{H}}^1$. *The following statements are equivalent*:

(1) *there exists a measurable* Γ*-equivariant horizontal conformal structure* G *on* U,
(2) *there exists a measurable bounded function* $\eta : U \to \mathbb{C}$ *such that* (5.15) *holds almost everywhere in* U *for all* $g \in \Gamma$,
(3) *there exists a measurable* Γ*-equivariant CR structure* μ *on* U.

Proof The functions G, η and μ are related through the following identities:

$$\begin{aligned} G(p) &= \begin{pmatrix} g_{11}(p) & g_{12}(p) \\ g_{12}(p) & g_{22}(p) \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{Re}\eta(p)+\sqrt{|\eta(p)|^2+1} & \operatorname{Im}\eta(p) \\ \operatorname{Im}\eta(p) & -\operatorname{Re}\eta(p)+\sqrt{|\eta(p)|^2+1} \end{pmatrix} \\ &= \frac{1}{1-|\mu(p)|^2}\begin{pmatrix} 2\operatorname{Re}\mu(p)+1+|\mu(p)|^2 & 2\operatorname{Im}\mu(p) \\ 2\operatorname{Im}\mu(p) & -2\operatorname{Re}\mu(p)+1+|\mu(p)|^2 \end{pmatrix} \end{aligned}$$

as well as

$$\eta(p) = \frac{1}{2}(g_{11}(p)-g_{22}(p)+\mathrm{i}2g_{12}(p)) = \frac{2}{1-|\mu(p)|^2}\mu(p)$$

and

$$\mu(p) = \frac{g_{11}(p)-g_{22}(p)+\mathrm{i}2g_{12}(p)}{g_{11}(p)+g_{22}(p)+2} = \frac{\eta(p)}{\sqrt{|\eta(p)|^2+1}+1}.$$

The computations to prove the equivalence of the above real and complex horizontal Beltrami equations (1) and (2) follow the proof of the corresponding planar equations (10.29) and (10.32) in [1] verbatim. It can be seen by a direct computation that both the matrix given with respect to η and the one given with respect to μ are symmetric, positive definite, with determinant 1 and real entries.

Using the isometry (5.12), we find

$$\log|G| = d_{S(2)}(I, G) = d_{\mathbb{D}}(0, \mu) = \log\frac{1+|\mu|}{1-|\mu|},$$

hence G is essentially bounded with respect to the hyperbolic metric in $S(2)$ if and only if $\|\mu\| = \operatorname{ess\,sup}_{p\in U}|\mu(p)| < 1$. The last condition itself is equivalent to the boundedness of the function η. □

To further illustrate the connection between an equivariant horizontal conformal structure G and an equivariant CR structure μ it is illuminative to write down the quadratic form

$$q_p(u) := \langle G(p)u, u\rangle = \gamma(p)|z + \mu(p)\bar{z}|^2,$$

where we present the vector $u = (u_1, u_2) \in HT_p$ as a complex number $z = u_1 + \mathrm{i}u_2$, $\gamma(p) = \frac{1}{4}(g_{11}(p) + g_{22}(p)) + \frac{1}{2} \in \mathbb{R}_+$ and μ is related to G as described in Theorem 5.6.

As a counterpart for Theorem 10.3.4 in [1], we obtain the following result.

Theorem 5.7 *Let η be a bounded, measurable, complex-valued function defined on a domain U in $\bar{\mathbb{H}}^1$. Then the continuous contact mappings $g \in HW^{1,4}_{loc}(U, U)$ which solve the uniformly elliptic equation*

$$\bar{Z}g_I(p) = \frac{1}{\sqrt{|\eta(p)|^2 + 1} + \sqrt{|\eta(g(p))|^2 + 1}}\left(\eta(p)Zg_I(p) - \eta(g(p))\overline{Zg_I}(p)\right), \tag{5.16}$$

form a uqr semigroup closed under composition. The family of homeomorphic solutions forms a uniformly quasiconformal group.

Proof As remarked earlier, it will follow from $\|\mu_g\| < 1$ that g is quasiregular. Yet, this last condition is obviously satisfied for solutions of (5.16) since

$$|\mu_g(p)| = \left|\frac{\bar{Z}g_I(p)}{Zg_I(p)}\right| \leq \frac{|\eta(p)| + |\eta(g(p))|}{\sqrt{1 + |\eta(p)|^2} + \sqrt{1 + |\eta(g(p))|^2}}$$

and η is assumed to be a bounded function.

Then let us note that there is a horizontal conformal structure G such that a quasiregular map $g : U \to U$ is a solution of (5.16) if and only if

$$Hg_*^T(p)G(g(p))Hg_*(p) = J_H(g_I, p)G(p) \quad \text{for almost all } p \in U \tag{5.17}$$

(cf. the proof of Theorem 5.6). Let g and h be two solutions of (5.17) and let p be a point such that (5.17) is fulfilled for g in $h(p)$ and for h in p (almost every point is such a point). Using the chain rule (5.3), we conclude

$$\begin{aligned} &H(g \circ h)_*^T(p)G(g \circ h(p))H(g \circ h)_*(p) \\ &\quad = Hh_*^T(p)Hg_*^T(h(p))G(g(h(p)))Hg_*(h(p))Hh_*(p) \\ &\quad = J_H(g_I, h(p))Hh_*^T(p)G(h(p))Hh_*(p) \\ &\quad = J_H(g_I, h(p))J_H(h_I, p)G(p) \\ &\quad = J_H((g \circ h)_I, p)G(p). \end{aligned}$$

It follows that the appropriately regular solutions of (5.16) form a uqr semigroup under composition.

Similarly, one can prove that for a *homeomorphic* solution, i.e., a quasiconformal map g, the inverse function g^{-1} is again a solution of (5.17) and hence of (5.16). In order to see this, we use $Hg_*^{-1}(g(p)) = Hg_*(p)^{-1}$ and $J_H((g^{-1})_I, g(p)) = J_H(g_I, p)^{-1}$. □

One can also compare the set of solutions to (5.16) with the set of mappings preserving the standard CR structure. In the complex case, each solution of the corresponding equation is conformal (holomorphic) after an appropriate change of variables by a quasiconformal mapping. This does not hold in full generality in our situation since a given CR structure need not be realizable.

Remark 5.8 If we assume the CR structure μ is realizable for some quasiconformal map φ on $\bar{\mathbb{H}}^1$, that is $\mu = \mu_\varphi$ or in other words solving the equation

$$\bar{Z}\varphi_I = \mu Z\varphi_I,$$

then for any solution g of (5.16) the mapping $\varphi \circ g \circ \varphi^{-1}$ preserves the standard CR structure. Conversely, every function of form $\varphi^{-1} \circ h \circ \varphi$ where h preserves the standard CR structure satisfies the equation (5.16).

6 Final Comments and Open Questions

There is by now a quite elaborate theory of quasiregular mappings on Heisenberg and more general Carnot groups, however, many important problems are still open. It would be of great importance to have a toolkit of interesting examples for quasiregular maps akin to the case of Euclidean spaces [31].

The methods of construction of quasiregular maps in this paper are based on the flow-technique of Korányi and Reimann [21] and [23] and seem not be powerful enough to produce a Heisenberg analogue of quasiregular Zorich type maps omitting points from the target space. It has been shown in [17] that a non-constant quasiregular map $f : \mathbb{H}^n \to \mathbb{H}^n$ on the Heisenberg group equipped with a *Riemannian* structure cannot omit any value. Yet, it is still an open question whether such a result holds for Heisenberg groups with a *sub*-Riemannian structure.

In the Riemannian setting, so-called Lattès type mappings give many examples of uqr mappings. In the Riemann sphere these mappings are generated by semi-conjugating a dilation in the plane by the two periodic Weierstraß $\wp$-function. These mappings have been studied in the sphere case in [28] and on other compact manifolds in [2]. It is then natural to ask:

Question 6.1 Are there Lattès type mappings in the compactified Heisenberg group?

Among the Lattès type mappings there is a counterpart for planar power function acting on the n-sphere with codimension 1-sphere as a Julia set and two superattracting fixed points (origin and infinity). Hence this mapping has a uqr restriction acting on $\mathbb{R}^n$ as well as on $\mathbb{R}^n\setminus\{0\}$. The existence of a similar mapping on the compactified Heisenberg group is an open question. Moreover, we do not know the answer to the following:

Question 6.2 Are there non-injective uqr maps acting on the (non-compactified) Heisenberg group?

A further open question in this context is the following:

Question 6.3 Does every qr map $f : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ have a uqr counterpart?

More precisely, given a qr map $f : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ we would ask for the existence of a uqr map $g : \bar{\mathbb{H}}^1 \to \bar{\mathbb{H}}^1$ with the property that the two maps have the same branch set $B_g = B_f$. The corresponding Euclidean statement [26] follows by Sullivan's annulus quasiconformal extension theorem (see [36]). In the positive case a Stoïlow factorization for quasiregular mappings would follow also in the Heisenberg group case. In [27] we show that all quasiregular mappings f acting on the standard sphere have a factorization $f = g \circ h$, where g is uqr and h quasiconformal.

In [13] a quasiregular mapping is presented with branching branch set. With the techniques from this paper one can construct a uqr map whose branch set is branching along a Cantor set. Our results show that the branch set (and its image) for a uqr map of $\mathbb{H}^1$ can be arbitrarily large in dimension. On the other hand, we can recall a result due to Markina [24] which gives a lower bound on the dimension of the *image* of a branch set of a quasiregular map between Carnot groups, i.e., the image of a branch set cannot become arbitrarily small. For the first Heisenberg group, the precise statement is the following

Theorem 6.4 ([24]) *Let $f : \Omega \to \mathbb{H}^1$, $\Omega \subseteq \mathbb{H}^1$, be a quasiregular map with non-empty branch set $B_f \neq \emptyset$. Then*

$$\mathcal{H}^1(f(B_f)) > 0,$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure on $\mathbb{H}^1$ with respect to the Heisenberg distance.

Question 6.5 Is the above lower bound on the size of the branch set sharp?

The currently known smallest non-empty branch set for a Heisenberg qr map is the t-axis for the case of the winding map. The positive answer to the above question would follow from the existence of a quasiconformal map which maps the t-axis to a rectifiable curve in the Heisenberg group, which is exactly Question 25 in [15].

We think that the same methods can be used to produce uqr mappings also in higher-dimensional Heisenberg groups. The construction of an equivariant CR structure in the higher-dimensional case can be a more complicated task. Let us recall that in [18] Iwaniec and Martin prove the existence of a conformal structure which is equivariant under an Abelian uqr semigroup acting on Euclidean spheres.

Analogously to the case of $\bar{\mathbb{H}}^1$ one can introduce a horizontal conformal structure in higher-dimensional Heisenberg groups as well. Following the reasoning as in [18] one can prove for a countable Abelian uqr semigroup Γ acting on the higher-dimensional compactified Heisenberg groups $\mathbb{H}^n$ the existence of an invariant horizontal conformal structure. In the higher-dimensional case the connection between equivariant horizontal conformal structures and equivariant CR structures is not at all clear. It would be interesting to explore the analogous identity between quadratic forms given by $2n$-(real)dimensional horizontal conformal structure G and complex antilinear mapping $\mu : T^{1,0} \to T^{1,0}$. For the definition of higher-dimensional CR structures, we refer to [8, 22].

Acknowledgements We thank Lászlo Lempert and Jeremy Tyson for valuable communication on the subject of this paper. We wish we could have shared the joy of finding the uqr world in the Heisenberg group with Juha Heinonen. We will never forget his enthusiasm and encouragement towards new discoveries and better understanding in mathematics.

References

1. Astala, K., Iwaniec, T., Martin, G.: Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. Princeton Mathematical Series, vol. 48. Princeton University Press, Princeton (2009)
2. Astola, L., Kangaslampi, R., Peltonen, K.: Lattès type mappings on compact manifolds. Conform. Geom. Dyn. (2009, accepted for publication)
3. Balogh, Z.M.: Hausdorff dimension distribution of quasiconformal mappings on the Heisenberg group. J. Anal. Math. **83**, 289–312 (2001)
4. Balogh, Z.M., Koskela, P.: Quasiconformality, quasisymmetry and removability in Loewner spaces. Duke Math. J. **101**, 555–572 (2000)
5. Capogna, L.: Regularity of quasi-linear equations in the Heisenberg group. Commun. Pure Appl. Math. **50**(9), 867–889 (1997)
6. Dairbekov, N.S.: Mappings with bounded distortion on Heisenberg groups. Sib. Mat. Zh. **41**(3), 567–590 (2000)
7. Dairbekov, N.S.: Mappings with bounded distortion on two-step Carnot groups. In: Proceedings on Analysis and Geometry, Russian Novosibirsk Akademgorodok, 1999, pp. 122–155. Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk (2000)
8. Dragomir, S., Tomassini, G.: Differential Geometry and Analysis on CR Manifolds. Progress in Mathematics, vol. 246. Birkhäuser Boston, Boston (2006)
9. Forstnerič, F.: Extending proper holomorphic mappings of positive codimension. Invent. Math. **95**(1), 31–61 (1989)
10. Gromov, M.: Carnot-Carathéodory spaces seen from within. In: Sub-Riemannian Geometry. Progress in Mathematics, vol. 144, Birkhäuser, Basel (1996)
11. Heinonen, J.: Calculus on Carnot groups. In: Fall School in Analysis, Jyväskylä, 1994. Report, vol. 68, pp. 1–31. Univ. Jyväskylä, Jyväskylä (1995)
12. Heinonen, J.: Lectures on Analysis on Metric Spaces. Springer, New York (2001)
13. Heinonen, J., Holopainen, I.: Quasiregular maps on Carnot groups. J. Geom. Anal. **7**(1), 109–148 (1997)
14. Heinonen, J., Koskela, P.: Quasiconformal maps in metric spaces with controlled geometry. Acta Math. **181**, 1–61 (1998)
15. Heinonen, J., Semmes, S.: Thirty-three yes or no questions about mappings, measures, and metrics. Conform. Geom. Dyn. **1**, 1–12 (1997) (electronic)
16. Hinkkanen, A.: Uniformly quasiregular semigroups in two dimensions. Ann. Acad. Sci. Fenn. Math. **21**(1), 205–222 (1996)
17. Holopainen, I., Rickman, S.: Quasiregular mappings of the Heisenberg group. Math. Ann. **294**(4), 625–643 (1992)
18. Iwaniec, T., Martin, G.: Quasiregular semigroups. Ann. Acad. Sci. Fenn. Math. **21**(2), 241–254 (1996)
19. Iwaniec, T., Martin, G.: Geometric Function Theory and Non-Linear Analysis. Oxford Mathematical Monographs. Clarendon/Oxford University Press, New York (2001)
20. Korányi, A.: Geometric properties of Heisenberg-type groups. Adv. Math. **56**(1), 28–38 (1985)
21. Korányi, A., Reimann, H.M.: Quasiconformal mappings on the Heisenberg group. Invent. Math. **80**(2), 309–338 (1985)

22. Korányi, A., Reimann, H.M.: Quasiconformal mappings on CR manifolds. In: Complex Geometry and Analysis, Pisa, 1988. Lecture Notes in Math, vol. 1422, pp. 59–75. Springer, Berlin (1990)
23. Korányi, A., Reimann, H.M.: Foundations for the theory of quasiconformal mappings on the Heisenberg group. Adv. Math. **111**(1), 1–87 (1995)
24. Markina, I.: Hausdorff measure of the singular set of quasiregular maps on Carnot groups. Int. J. Math. Math. Sci. **35**, 2203–2220 (2003)
25. Markina, I., Vodopyanov, S.: On value distributions for quasimeromorphic mappings on $\mathbb{H}$-type Carnot groups. Bull. Sci. Math. **130**(6), 467–523 (2006)
26. Martin, G.J.: Branch sets of uniformly quasiregular maps. Conform. Geom. Dyn. **1**, 24–27 (1997) (electronic)
27. Martin, G., Peltonen, K.: Stoïlow factorization for quasiregular mappings in all dimensions. Proc. Am. Math. Soc. **138**, 147–151 (2010)
28. Mayer, V.: Uniformly quasiregular mappings of Lattès type. Conform. Geom. Dyn. **1**, 104–111 (1997) (electronic)
29. Peltonen, K.: Examples of uniformly quasiregular mappings. Conform. Geom. Dyn. **3**, 158–163 (1999) (electronic)
30. Poincaré, H.: Les fonctions analytiques de deux variables et la représentation conforme. Rend. Circ. Mat. Palermo **II**(23), 185–220 (1907)
31. Rickman, S.: Quasiregular Mappings. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 26. Springer, Berlin (1993)
32. Tanaka, N.: On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables. J. Math. Soc. Jpn. **14**, 397–429 (1962)
33. Tang, P.: Quasiconformal homeomorphisms on CR 3-manifolds with symmetries. Math. Z. **219**(1), 49–69 (1995)
34. Tang, P.: Regularity and extremality of quasiconformal homeomorphisms on CR 3-manifolds. Ann. Acad. Sci. Fenn. Math. **21**(2), 289–308 (1996)
35. Tukia, P.: On two-dimensional quasiconformal groups. Ann. Acad. Sci. Fenn., Ser. A 1 Math. **5**(1), 73–78 (1980)
36. Tukia, P., Väisälä, J.: Lipschitz and quasiconformal approximation and extension. Ann. Acad. Sci. Fenn., Ser. A 1 Math. **6**(2), 303–342 (1982)
37. Vodop'yanov, S.K.: Mappings with bounded distortion and with finite distortion on Carnot groups. Sib. Mat. Zh. **40**(4), 764–804 (1999)
38. Walter, W.: Ordinary Differential Equations. Graduate Texts in Mathematics, vol. 182. Springer, New York (1998). Translated from the sixth German (1996) edition by Russell Thompson, Readings in Mathematics