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Asymptotics and Bounds for Multivariate Gaussian Tails

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Let $\{X_n, n \geqslant 1\}$ be a sequence of centered Gaussian random vectors in $\mathbb{R}^d, d \geqslant 2$. In this paper we obtain asymptotic expansions $(n \to \infty)$ of the tail probability $P\{X_n > t_n\}$ with $t_n \in \mathbb{R}^d$ a threshold with at least one component tending to infinity. Upper and lower bounds for this tail probability and asymptotics of discrete boundary crossings of Brownian Bridge are further discussed.

KEY WORDS: Tail asymptotics; Gaussian random sequences; Discrete boundary crossings; Brownian bridge; Quadratic programming.

1. INTRODUCTION

Let X be a mean zero Gaussian random vector in $\mathbb{R}^d, d \geqslant 2$ with non-singular covariance matrix Σ and $t \in \mathbb{R}^d$ a fixed threshold. With the main impetus from results of Dai and Mukherjea[5] and Hashorva and Hüsler[10] we deal in this paper with the asymptotic behaviour of the tail probability

$$P\{X > t\} \tag{1.1}$$

if at least one of the components of t tends to infinity.

Indeed, there are several application of tail asymptotics of Gaussian random vectors, see e.g. Dai and Mukherjea[5], Elnaggar and Mukherjea[6], Raab[13], Mukherjea and Stephens[12] among many others. As mentioned in the first paper, solving tail asymptotic problems under a multivariate setup is not an easy task. Nevertheless, various results are known under specific

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restrictions; upper and lower bounds are given by Savage[15] who proved that

$$[1-\langle 1/(\Sigma^{-1}\boldsymbol{t}),\, \Sigma^{-1}(1/(\Sigma^{-1}\boldsymbol{t}))\rangle] \leqslant \frac{\boldsymbol{P}\{\boldsymbol{X}\geqslant \boldsymbol{t}\}}{\varphi_{\boldsymbol{X}}(\boldsymbol{t})\prod_{i=1}^{d}\langle \boldsymbol{e}_i,\, \Sigma^{-1}\boldsymbol{t}\rangle} \leqslant 1, \qquad (1.2)$$

holds with $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ the scalar product in $\mathbb{R}^d$, $\varphi_{\boldsymbol{X}}$ the density function of $\boldsymbol{X}$ and $\boldsymbol{e}_i, i=1,\dots,d$ the ith unit vector in $\mathbb{R}^d$ if the Savage condition

$$\Sigma^{-1}\boldsymbol{t} > (0,\dots,0)^{\top} \in \mathbb{R}^d \qquad (1.3)$$

holds. So if one component of $\boldsymbol{t}$ goes to ∞, then the asymptotics of the tail probability of interest follows easily. The situation becomes more difficult if the Savage condition (1.3) does not hold. Condition (1.3) is relaxed for instance in the context of multivariate Mills Ratio in Gjacjauskas[7] and Steck[16], Satish[14]. See Mukherjea and Stephens[12] and Tong[17] for related results.

Also in Dai and Mukherjea[5] condition (1.3) is not assumed. The asymptotic expansion is obtained therein for special non-singular covariance matrix Σ and threshold $\boldsymbol{t}$ with equal components. Hashorva and Hüsler[10] and Hashorva and Hüsler[8] consider general covariance matrix Σ. From the former paper we know that the asymptotic behaviour of the tail probability of interest is closely related to a quadratic programming problem. This fact is actually retrieved if one refers to the Large Deviation Principles, see e.g. Wlodzimierz[18].

With a mixture of new and old ideas, we extend several previous results for the case that the Savage condition does not hold. Their proofs are derived using a simple probabilistic approach. Important examples as well as asymptotics for discrete boundary crossings of Brownian Bridge are further discussed.

Outline of the rest of the paper: In the next section we introduce some basic notation and give a preliminary result needed for the proof of the main results. In the third section we discuss the asymptotic behaviour of (1.1). We treat first simple thresholds; a general result is then obtain letting both $\boldsymbol{X}$ and the threshold depend on n. Special parametric thresholds are treated in the last section where we derive an asymptotic result for a discrete boundary crossing probability of a Brownian Bridge.

2. NOTATION AND A PRELIMINARY RESULT

Let in the sequel $I \subset \{1,\dots,d\}, d \geqslant 2$ denote a non-empty index with $|I|$ elements, and put $J := \{1,\dots,d\}\backslash I$. Random Gaussian vectors in $\mathbb{R}^d$

are denoted by capital letters say $\boldsymbol{X}$, their density and distribution function is denoted by $\varphi_{\boldsymbol{X}}$ and $\Phi_{\boldsymbol{X}}$, respectively.

For a given vector $\boldsymbol{x} \in \mathbb{R}^d$ we write $\boldsymbol{x}_I$ the vector obtained by deleting the components of $\boldsymbol{x}$ in J. If $|I| = d$ we drop the subscript. Similar notation $A_{II}, A_{IJ}, A_{JI}, A_{JJ}$ are used for submatrices of a given matrix $A \in \mathbb{R}^{d\times d}$. For simplicity we write A_I, A_J instead of A_{II}, A_{JJ}. Further the following notation for vectors in $\mathbb{R}^d$ is used

$$\begin{aligned} \boldsymbol{x} &> \boldsymbol{y}, \quad \text{if } x_i > y_i, \quad \forall i = 1, \dots, d,\\ \boldsymbol{x} &\geqslant \boldsymbol{y}, \quad \text{if } x_i \geqslant y_i, \quad \forall i = 1, \dots, d, \end{aligned}$$

$$\langle \boldsymbol{x}_I, \boldsymbol{y}_I \rangle := \sum_{i \in I} x_i y_i, \quad \|\boldsymbol{x}_I\|^2 := \langle \boldsymbol{x}_I, \boldsymbol{x}_I \rangle, \quad I \subset \{1, \dots, d\},$$

$$a\boldsymbol{x} := (ax_1, \dots, ax_d)^\top, \quad a \in \mathbb{R}, \quad \boldsymbol{c}\boldsymbol{x} := \operatorname{diag}(\boldsymbol{c})\boldsymbol{x} = (c_1 x_1, \dots, c_d x_d)^\top, \quad \boldsymbol{c} \in \mathbb{R}^d,$$

with $\operatorname{diag}(\boldsymbol{c})$ the diagonal matrix corresponding to the vector $\boldsymbol{c}$ and

$$\boldsymbol{0} := (0, \dots, 0)^\top \in \mathbb{R}^d, \quad \boldsymbol{1} := (1, \dots, 1)^\top \in \mathbb{R}^d, \quad \boldsymbol{\infty} = (\infty, \dots, \infty)^\top \in \mathbb{R}^d.$$

To this end we solve a quadratic programming problem needed for the proof of the main results.

Proposition 2.1. Let $\Sigma \in \mathbb{R}^{d\times d}$ be a positive definite correlation matrix and let $\boldsymbol{b} \notin (-\infty, 0]^d$ be a fixed vector. Then the quadratic programming problem

$$\mathscr{P}(\Sigma^{-1}, \boldsymbol{b}): \text{ minimise } \langle \boldsymbol{x}, \Sigma^{-1}\boldsymbol{x} \rangle \text{ under the linear constraint } \boldsymbol{x} \geqslant \boldsymbol{b}$$

has a unique solution $\boldsymbol{b}^*$ and there exists a unique non-empty index set $I \subset \{1, \dots, d\}$ so that

$$\boldsymbol{b}_I^* = \boldsymbol{b}_I > \boldsymbol{0}_I \text{ and if } |J| \geqslant 0 \text{ then } \boldsymbol{b}_J^* = \Sigma_{JI}(\Sigma_I)^{-1} \geqslant \boldsymbol{b}_J, \tag{2.1}$$

$$\langle \boldsymbol{e}_i, (\Sigma_I)^{-1}\boldsymbol{b}_I \rangle > 0, \quad \forall i \in I, \tag{2.2}$$

$$\alpha := \min_{x \geqslant \boldsymbol{b}} \langle \boldsymbol{x}, \Sigma^{-1}\boldsymbol{x} \rangle = \langle \boldsymbol{b}^*, \Sigma^{-1}\boldsymbol{b}^* \rangle = \langle \boldsymbol{b}_I, (\Sigma_I)^{-1}\boldsymbol{b}_I \rangle > 0, \tag{2.3}$$

with $\boldsymbol{e}_i$ the ith unit vector in $\mathbb{R}^{|I|}$. Further for any vector $\boldsymbol{c} \in \mathbb{R}^d$ we have

$$\langle \boldsymbol{c}, \Sigma^{-1}\boldsymbol{b}^* \rangle = \langle \boldsymbol{c}_I, (\Sigma_I)^{-1}\boldsymbol{b}_I^* \rangle = \langle \boldsymbol{c}_I, (\Sigma_I)^{-1}\boldsymbol{b}_I \rangle \tag{2.4}$$

and for the special case $\boldsymbol{b} = b_0\boldsymbol{1}, b_0 > 0$ we have $2 \leqslant |I| \leqslant d$.

Proof. The proof follows immediately from Proposition 2.1 of Hashorva and Hüsler[8] recalling further that if $|J| \geqslant 1$, then $(\Sigma^{-1})_{JI} = -(\Sigma^{-1})_J \Sigma_{JI} (\Sigma_I)^{-1}$ holds. Note in passing that both Σ^{-1}, $(\Sigma_I)^{-1}$ exist since Σ, Σ_I are both positive definite matrices, and further (2.4) follows also by Lemma 4.1 of Bischoff *et al.*[2] □

3. MAIN RESULTS

Let $\boldsymbol{X}$ be a mean zero Gaussian random vector in $\mathbb{R}^d, d \geqslant 2$ with positive definite covariance matrix Σ and let $\boldsymbol{t}$ be a given threshold. In this section we investigate the rate of convergence to 0 for the tail probability in (1.1) as $\|\boldsymbol{t}\| \to \infty$. It is worth treating first the case of simple thresholds $\boldsymbol{t} = t\boldsymbol{b} + \boldsymbol{c}$ with $\boldsymbol{b}$, $\boldsymbol{c}$ two fixed vectors and $t > 0$. Since the threshold depends only on t we reduce the dimensionality problem for the threshold. It follows that the Savage condition can be stated only in terms of $\boldsymbol{b}$ and Σ for t sufficiently large. In the next theorem we deal with a particular case, where a Gaussian random vector $\boldsymbol{X}$ behaves asymptotically like $\boldsymbol{X}_I$, for $t \to \infty$ with I a non-empty index set. In this way we reduce further the dimensionality problem for the Gaussian random vector itself, treating instead a certain subvector.

Theorem 3.1. Let $\boldsymbol{X}$ be a centered Gaussian random vector in $\mathbb{R}^d$, $d \geqslant 2$ with positive definite covariance matrix Σ and let $\boldsymbol{t} = t\boldsymbol{b} + \boldsymbol{c}$ be a given threshold with $t > 0$, $\boldsymbol{b} \notin (-\infty, 0]^d$ and $\boldsymbol{c} \in \mathbb{R}^d$. Assume that there exist two non-empty disjoint index sets I, J such that $I \cup J = \{1, \ldots, d\}$ and a vector $\boldsymbol{b}^* \in \mathbb{R}^d$ satisfying

$$\boldsymbol{b}^*_J = \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{b}_I \geqslant \boldsymbol{b}_J. \tag{3.1}$$

Then we have for any $\varepsilon > 0$ and for all t sufficiently large

$$\left| \frac{\boldsymbol{P}\{\boldsymbol{X} > \boldsymbol{t}\}}{\boldsymbol{P}\{\boldsymbol{X}_I > \boldsymbol{t}_I\}} - \boldsymbol{P}\{\boldsymbol{Y}_J \geqslant \boldsymbol{b}_{J,\infty} - \boldsymbol{c}^*_J\} \right| < \varepsilon, \tag{3.2}$$

with

$$\boldsymbol{b}_{J,\infty} := \lim_{t \to \infty} t(\boldsymbol{b}_J - \boldsymbol{b}^*_J) \leqslant \boldsymbol{0}_J, \quad \boldsymbol{c}^*_J := \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{c}_I$$

and $\boldsymbol{Y}_J$ a Gaussian random vector in $\mathbb{R}^{|J|}$ with mean $-\boldsymbol{c}_J$ and positive definite covariance matrix $\Sigma_J - \Sigma_{JI}(\Sigma_I)^{-1}\Sigma_{IJ}$.

Proof. By the assumptions we obtain conditioning on $\boldsymbol{X}_I$

$$\begin{aligned}
\boldsymbol{P}\{\boldsymbol{X} \geqslant t\} &= \int_{\boldsymbol{x}_I \geqslant \boldsymbol{t}_I} \boldsymbol{P}\{\boldsymbol{X}_J \geqslant \boldsymbol{t}_J | \boldsymbol{X}_I = \boldsymbol{x}_I\} d\Phi_{\boldsymbol{X}_I}(\boldsymbol{x}_I) \\
&= \int_{\boldsymbol{y}_I \geqslant \boldsymbol{0}_I} \boldsymbol{P}\{\boldsymbol{X}_J \geqslant \boldsymbol{t}_J | \boldsymbol{X}_I = (\boldsymbol{t} + t^{-1}\boldsymbol{y})_I\} d\Phi_{t(\mathbf{X}_I - \boldsymbol{t}_I)}(\boldsymbol{y}_I) \\
&= \int_{\boldsymbol{y}_I \geqslant \boldsymbol{0}_I} \boldsymbol{P}\{\boldsymbol{Y}_J \geqslant t\boldsymbol{b}_J - \Sigma_{JI}(\Sigma_I)^{-1}(\boldsymbol{t} + t^{-1}\boldsymbol{y})_I\} d\Phi_{t(\mathbf{X}_I - \boldsymbol{t}_I)}(\boldsymbol{y}_I)
\end{aligned}$$

with $\boldsymbol{Y}_J$ a Gaussian random vector with mean $-\boldsymbol{c}_J$ and covariance matrix $\Sigma_J^* := \Sigma_J - \Sigma_{JI}(\Sigma_I)^{-1}\Sigma_{IJ}$. Clearly, since Σ is positive definite, $(\Sigma_I)^{-1}$ exists and Σ_J^* is positive definite. Next, (3.1) implies

$$\boldsymbol{P}\{\boldsymbol{X} \geqslant t\} = \int_{\boldsymbol{y}_I \geqslant \boldsymbol{0}_I} \boldsymbol{P}\{\boldsymbol{Y}_J \geqslant t(\boldsymbol{b}_J - \boldsymbol{b}_J^*) - \Sigma_{JI}(\Sigma_I)^{-1}(\boldsymbol{c} + t^{-1}\boldsymbol{y})_I\} d\Phi_{t(\mathbf{X}_I - \boldsymbol{t}_I)}(\boldsymbol{y}_I),$$

hence by the fact that

$$\boldsymbol{P}\{\boldsymbol{Y}_J \geqslant t(\boldsymbol{b}_J - \boldsymbol{b}_J^*) - \Sigma_{JI}(\Sigma_I)^{-1}(\boldsymbol{c} + t^{-1}\boldsymbol{y})_I\} \to \boldsymbol{P}\{\boldsymbol{Y}_J \geqslant \boldsymbol{b}_{J,\infty} - \boldsymbol{c}_J^*\}, \quad t \to \infty$$

holds uniformly (function is non-increasing in $\boldsymbol{y}_I$) the claim follows. □

Remark 3.2. (i) Clearly the restriction $\boldsymbol{b} \notin (-\infty, 0]^d$ above implies that $\lim_{t\to\infty} \boldsymbol{P}\{\boldsymbol{X} > t\boldsymbol{b} + \boldsymbol{c}\} = 0$. (ii) Probabilities of the form $\boldsymbol{P}\{\boldsymbol{X}_I \geqslant \boldsymbol{t}_I, \boldsymbol{X}_J \leqslant \boldsymbol{t}_J\}$ treated in Dai and Mukherjea[5] and Mukherjea and Stephens[12] can be written as the tail probability in (1.1).

Corollary 3.3. With the notation of Theorem 3.1, if the Savage condition (1.3) holds for the covariance matrix of the Gaussian random vector $\boldsymbol{X}_I$ and threshold $\boldsymbol{b}_I$, then we have as $t \to \infty$

$$\begin{aligned}
&\boldsymbol{P}\{\boldsymbol{X} > t\boldsymbol{b} + \boldsymbol{c}\} \\
&\quad = \frac{(1 + o(1))\exp(-\alpha t^2/2 - t\langle \boldsymbol{c}, \Sigma^{-1}\boldsymbol{b}^*\rangle - \langle \boldsymbol{c}_I, (\Sigma_I)^{-1}\boldsymbol{c}_I\rangle/2)\boldsymbol{P}\{\boldsymbol{Y}_J \geqslant \boldsymbol{b}_{J,\infty} - \boldsymbol{c}_J^*\}}{(2\pi)^{|I|/2}|\Sigma_I|^{1/2} t^{|I|} \Pi_{i\in I}\langle \boldsymbol{e}_i, (\Sigma_I)^{-1}\boldsymbol{b}_I\rangle}
\end{aligned} \tag{3.3}$$

where $\alpha = \min_{\boldsymbol{x} \geqslant \boldsymbol{b}} \langle \boldsymbol{x}, \Sigma^{-1}\boldsymbol{x}\rangle = \langle \boldsymbol{b}^*, \Sigma^{-1}\boldsymbol{b}^*\rangle > 0$, $\boldsymbol{b}_I^* = \boldsymbol{b}_I$, $\boldsymbol{b}_J^*$ as in (3.1) and $\boldsymbol{b}_{J,\infty}$ has components 0 or $-\infty$ depending on the fact whether the respective components of $\boldsymbol{b}_J^* - \boldsymbol{b}_J$ are 0 or positive.

Proof. By the above theorem and (1.2) we obtain

$$\begin{aligned}
&\boldsymbol{P}\{\boldsymbol{X} > t\boldsymbol{b}+\boldsymbol{c}\}\\
&= \frac{(1+o(1))\exp(-t^2\langle \boldsymbol{b}_I, (\Sigma_I)^{-1}\boldsymbol{b}_I\rangle/2 - t\langle \boldsymbol{c}_I, (\Sigma_I)^{-1}\boldsymbol{b}_I\rangle - \langle \boldsymbol{c}_I, (\Sigma_I)^{-1}\boldsymbol{c}_I\rangle/2)}{(2\pi)^{|I|/2}|\Sigma_I|^{1/2} t^{|I|}\prod_{i\in I}\langle \boldsymbol{e}_i, (\Sigma_I)^{-1}\boldsymbol{b}_I\rangle}\\
&\quad \times \boldsymbol{P}\{\boldsymbol{Y}_J \geqslant \boldsymbol{b}_{J,\infty} - \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{c}_I\}, \quad t\to\infty.
\end{aligned}$$

In light of Large Deviation Principle (see e.g. Wlodzimierz[18]) it follows that $\langle \boldsymbol{b}_I, (\Sigma_I)^{-1}\boldsymbol{b}_I\rangle$ is the attained minimum of the quadratic programming problem: minimise $\langle \boldsymbol{x}, \Sigma^{-1}\boldsymbol{x}\rangle$ for $\boldsymbol{x} \geqslant \boldsymbol{b}$, thus the claim follows using further (2.3) and (2.4). $\square$

Indeed, in view of the above derivations and Proposition 2.1, an asymptotic expansion of the tail probability of interest for simple thresholds can be easily obtained. We formulate this result in the next theorem.

Theorem 3.4. Let $\boldsymbol{X}$, $\boldsymbol{Y}_J$, $\boldsymbol{b}_{J,\infty}$, $\overset{*}{\boldsymbol{c}}_J$ and $\boldsymbol{t} = t\boldsymbol{b}+\boldsymbol{c}$ be as in Theorem 3.1. Then (3.3) holds with I the unique non-empty minimal index set of $\{1,\dots,d\}$ such that $\min_{\boldsymbol{x}\geqslant\boldsymbol{b}}\langle \boldsymbol{x}, \Sigma^{-1}\boldsymbol{x}\rangle = \langle \boldsymbol{b}_I, (\Sigma_I)^{-1}\boldsymbol{b}_I\rangle > 0$. Further, if $|I| = d$ then put $\boldsymbol{P}\{\boldsymbol{Y}_J \geqslant \boldsymbol{b}_{J,\infty} - \overset{*}{\boldsymbol{c}}_J\}$ equal 1.

Remark 3.5. (i) It is possible to obtain another mathematical description of the index set I, namely using the result of Theorem 2.1 of Dai and Mukherjea[5] it follows that I is the maximal index set such that (2.2) holds.
(ii) For $\boldsymbol{c} = \boldsymbol{0}$ the asymptotic expansion above is shown in Corollary 4.2 of Hashorva and Hüsler[10]. Similar results are shown in Dai and Mukherjea[5] too.
(iii) Theorem 3.4 follows also from Theorem 4.1 of Hashorva and Hüsler[10]. The latter is proved using a direct approach that differs from the one used above.
(iv) Clearly, the index set I in Theorem 3.4 is unique. Further $|I| = d$ is satisfied iff the Savage condition (1.3) holds. If $\boldsymbol{t} = t\boldsymbol{1}+\boldsymbol{c}$ then $2 \leqslant |I| \leqslant d$.

We present next four examples.

Example 1. Let $\boldsymbol{X}$ be a centered Gaussian random vector with covariance matrix

$$\Sigma = (1-\rho)\mathscr{I} + \rho\boldsymbol{1}\boldsymbol{1}^\top, \tag{3.4}$$

where $\rho \in (-1/(d-1), 1)$ and $\mathscr{I}$ the identity matrix in $\mathbb{R}^{d\times d}$, $d \geqslant 2$.

Applying Lemma 2.1 of Dai and Mukherjea[5] we get for any non-empty index set $I \subset \{1, \dots, d\}$

$$(\Sigma_I)^{-1} = r_1 \mathscr{I}_I - r_2 \mathbf{1}_I \mathbf{1}_I^\top,$$

with

$$r_1 := \frac{1}{1-\rho}, \quad r_2 := \frac{\rho}{(1-\rho)(1+(|I|-1)\rho)}.$$

For this example the Savage condition can be easily checked. It holds in the special case $\boldsymbol{b} = b_0 \mathbf{1}$ with $b_0 > 0$. For thresholds such that the Savage condition does not hold, it is easy to check condition (3.1). By Proposition 2.1 there exists $I \subset \{1, \dots, d\}$ such that $(\Sigma_I)^{-1} \boldsymbol{b}_I > \mathbf{0}_I$. Further we get

$$\alpha = r_1 \langle \boldsymbol{b}_I, \mathscr{I}_I \boldsymbol{b}_I \rangle - r_2 \langle \boldsymbol{b}_I, (\mathbf{1}\mathbf{1}^\top)_I \boldsymbol{b}_I \rangle = r_1 \|\boldsymbol{b}_I\|^2 - r_2 \langle \mathbf{1}_I, \boldsymbol{b}_I \rangle^2 > 0,$$

$$\langle \boldsymbol{b}^*, \Sigma^{-1} \boldsymbol{c} \rangle = r_1 \langle \boldsymbol{c}_I, \boldsymbol{b}_I \rangle - r_2 \langle \mathbf{1}_I, \boldsymbol{b}_I \rangle \langle \mathbf{1}_I, \boldsymbol{c}_I \rangle, \quad \langle \boldsymbol{c}_I, (\Sigma_I)^{-1} \boldsymbol{c}_I \rangle = r_1 \|\boldsymbol{c}_I\|^2 - r_2 \langle \mathbf{1}_I, \boldsymbol{c}_I \rangle^2$$

and

$$\langle \boldsymbol{e}_i, (\Sigma_I)^{-1} \boldsymbol{b}_I \rangle = r_1 b_i - r_2 \langle \mathbf{1}_I, \boldsymbol{b}_I \rangle > 0, \forall i \in I, \quad |\Sigma_I| = (1-\rho)^{|I|-1}(1+(|I|-1)\rho) > 0,$$

hence under the assumptions of Corollary 3.3 we obtain

$$\begin{aligned} &\boldsymbol{P}\{\boldsymbol{X} > t\boldsymbol{b} + \boldsymbol{c}\} \\ &= \frac{(1+o(1)) \exp(-t^2[r_1 \|\boldsymbol{b}_I\|^2 - r_2 \langle \mathbf{1}_I, \boldsymbol{b}_I \rangle^2]/2 - t[r_1 \langle \boldsymbol{c}_I, \boldsymbol{b}_I \rangle - r_2 \langle \mathbf{1}_I, \boldsymbol{b}_I \rangle \langle \mathbf{1}_I, \boldsymbol{c}_I \rangle])}{(2\pi)^{|I|/2}[(1-\rho)^{|I|-1}(1+(|I|-1)\rho)]^{1/2} t^{|I|} \Pi_{i \in I}[r_1 b_i - r_2 \langle 1_I, \boldsymbol{b}_I \rangle]} \\ &\quad \times \exp(-[r_1 \|\boldsymbol{c}_I\|^2 - r_2 \langle \mathbf{1}_I, \boldsymbol{c}_I \rangle^2]/2) \boldsymbol{P}\{\boldsymbol{Y}_J \geqslant \boldsymbol{b}_{J,\infty} - \boldsymbol{c}_J^*\}, \quad t \to \infty, \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{b}_{J,\infty} &= \lim_{t \to \infty} t[\boldsymbol{b}_J - ((1-\rho)\mathscr{I} + \rho \mathbf{1}\mathbf{1}^\top)_{JI} (r_1 \mathscr{I} - r_2 \mathbf{1}\mathbf{1}^\top)_I \boldsymbol{b}_I], \\ \boldsymbol{c}_J^* &= ((1-\rho)\mathscr{I} + \rho \mathbf{1}\mathbf{1}^\top)_{JI} (r_1 \mathscr{I} - r_2 \mathbf{1}\mathbf{1}^\top)_I \boldsymbol{c}_I \end{aligned}$$

and $\boldsymbol{Y}_J$ with mean $-\boldsymbol{c}_J$ and positive definite covariance matrix $(r_1 \mathscr{I}_J - r_2 (\mathbf{1}\mathbf{1}^\top)_J)^{-1}$.

Remark 3.6. The asymptotic for this case is discussed in Dai and Mukherjea[5] and in Hashorva and Hüsler[10]. Other results in connection with records are obtained in Hashorva and Hüsler[8] for special threshold.

Example 2. We consider next the case $\boldsymbol{b}=\mathbf{1}\in\mathbb{R}^3$, $\boldsymbol{c}\in\mathbb{R}^3$. Let further

$$\Sigma=\begin{pmatrix} 1 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 1 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 1 \end{pmatrix}.$$

Suppose for simplicity that $\sigma_{12}<\min(\sigma_{13},\sigma_{23})$. By Lemma 2.5 of Hashorva and Hüsler[9] we have

$$\min_{\boldsymbol{x}\geqslant(1,1,1)}\langle \boldsymbol{x},\,\Sigma^{-1}\boldsymbol{x}\rangle=\begin{cases}\tilde{\sigma}, & \text{if } 1+\sigma_{12}-\sigma_{13}-\sigma_{23}>0,\\ 2/(1+\sigma_{12}), & \text{otherwise,}\end{cases}$$

where

$$\tilde{\sigma}:=\frac{3-2(\sigma_{12}+\sigma_{13}+\sigma_{23})-\sigma_{12}^2-\sigma_{13}^2-\sigma_{23}^2+2(\sigma_{12}\sigma_{13}+\sigma_{12}\sigma_{23}+\sigma_{13}\sigma_{23})}{1+2\sigma_{12}\sigma_{13}\sigma_{23}-\sigma_{12}^2-\sigma_{13}^2-\sigma_{23}^2}$$

and

$$J=\begin{cases}\emptyset, & \text{if } 1+\sigma_{12}-\sigma_{13}-\sigma_{23}>0,\\ \{3\}, & \text{otherwise.}\end{cases}$$

If $|J|=1$ we get

$$\begin{aligned}&\boldsymbol{P}\{\boldsymbol{X}>t(1,1,1)+\boldsymbol{c}\}\\ &=(1+o(1))A(1+\sigma_{12})^2\left(2\pi t^2\sqrt{1-\sigma_{12}^2}\right)^{-1}\\ &\quad\times\exp\big(-t[t+(c_1+c_2)/2]/(1+\sigma_{12})\\ &\quad-(c_1^2-2c_1c_2\sigma_{12}+c_2^2)/(2(1-\sigma_{12}^2))\big),\quad t\to\infty,\end{aligned}$$

with $A:=\boldsymbol{P}\{X_3>c_3-c_3^*\}$ if $1+\sigma_{12}-\sigma_{13}-\sigma_{23}=0$ and $A:=1$ if $1+\sigma_{12}-\sigma_{13}-\sigma_{23}<0$.

Clearly, when $|J|=0$ we can use the Savage bounds since (1.3) holds. Note in passing that asymptotic results for trivariate case are also obtained in Mukherjea and Stephens[12].

Example 3. Next we investigate the special case $d=2$ and derive tail asymptotics for a simple threshold by straightforward calculations. Let therefore X,Y be two standard Gaussian random variables with correlation $\rho\in(-1,1)$ and let $a\in(0,1]$ be a fixed constant. For all $t>0$ we may write

$$
\begin{aligned}
\boldsymbol{P}\{X > at, Y > t\} &= \int_{y>t} \boldsymbol{P}\{X > at | Y = y\} \varphi_Y(y) dy \\
&= \frac{1}{t} \int_{s>0} \boldsymbol{P}\left\{ X > [t(a-\rho) - \rho s/t] / \sqrt{1-\rho^2} \right\} \varphi_Y(t + s/t) ds.
\end{aligned}
$$

The asymptotic for $a > \rho$ (which implies the Savage condition) is clear. If $a \leqslant \rho$ we get by straightforward calculations

$$
\boldsymbol{P}\{X > at, Y > t\} = \frac{(1 + o(1)) \exp(-t^2/2)}{t\sqrt{2\pi}(1(a=\rho)+1)}, \quad t \to \infty,
$$

with $1(\cdot)$ the indicator fuction.

Remark 3.7. The bivariate case is dealt within Lemma 3.1 of Mukherjea and Stephens[(12)], Lemma 2.1 of Elnaggar and Mukherjea[(16)] and Example 3 of Hashorva and Hüsler[(10)].

Example 4. Consider the random vector $\boldsymbol{X} = (B_M(z_1), \ldots, B_M(z_d))^\top$, $d \geqslant 2$ with $B_M(s), s \in [0, \infty)$ a Brownian Motion. Let $0 < z_1 < \cdots < z_d < \infty$ and $\boldsymbol{t} = t\mathbf{1} + \boldsymbol{c}, t > 0, \boldsymbol{c} \in \mathbb{R}^d$. The inverse of the covariance matrix of $\boldsymbol{X}$ is

$$
\Sigma^{-1} = \begin{pmatrix}
\left(\frac{1}{z_1} + \frac{1}{z_2 - z_1}\right) & -\frac{1}{z_2 - z_1} & 0 & \cdots & 0 \\
-\frac{1}{z_2 - z_1} & \left(\frac{1}{z_2 - z_1} + \frac{1}{z_3 - z_2}\right) & -\frac{1}{z_3 - z_2} & & \vdots \\
0 & -\frac{1}{z_3 - z_2} & \left(\frac{1}{z_3 - z_2} + \frac{1}{z_4 - z_3}\right) & \ddots & 0 \\
\vdots & & \ddots & \ddots & -\frac{1}{z_d - z_{d-1}} \\
0 & \cdots & 0 & -\frac{1}{z_d - z_{d-1}} & \frac{1}{z_d - z_{d-1}}
\end{pmatrix}. \tag{3.5}
$$

The solution of quadratic programming problem $\mathscr{P}(\Sigma^{-1}, \mathbf{1})$ is the unit vector $\mathbf{1} \in \mathbb{R}^d$. Further we have

$$
I = \{1\}, \quad J = \{2, \ldots, d\}, \quad \alpha = \langle \mathbf{1}, \Sigma^{-1}\mathbf{1} \rangle = z_1^{-1}, \ \boldsymbol{b}_{J,\infty} = \mathbf{0}_J, \quad \Sigma_{JI}(\Sigma_I)^{-1}\mathbf{1}_I = \mathbf{1}_J^\top
$$

and

$$
\Sigma_J - \Sigma_{JI}(\Sigma_I)^{-1}\Sigma_{IJ} = \begin{pmatrix}
z_2 - z_1 & z_2 - z_1 & z_2 - z_1 & \cdots & z_2 - z_1 \\
z_2 - z_1 & z_3 - z_1 & z_3 - z_1 & & \vdots \\
z_2 - z_1 & z_3 - z_1 & z_4 - z_1 & \ddots & z_4 - z_1 \\
\vdots & & \ddots & \ddots & z_{d-1} - z_1 \\
z_2 - z_1 & \cdots & \cdots & z_{d-1} - z_1 & z_d - z_1
\end{pmatrix},
$$

hence using Corollary 3.3 we get

$$\begin{aligned}
&\boldsymbol{P}\{B_M(z_1) > t + c_1, \dots, B_M(z_d) > t + c_d\}\\
&\quad = (1 + o(1))(2\pi t^2/z_1)^{-1/2} \exp(-(t + c_1)^2/(2z_1))\\
&\qquad \times \boldsymbol{P}\{B_M(z_2) - B_M(z_1) > c_2 - c_1, \dots, B_M(z_d) - B_M(z_1) > c_d - c_1\}.
\end{aligned}$$

Indeed, the asymptotics of (1.1) depends in general on both threshold and the covariance matrix Σ. Instead of dealing with simple thresholds, we consider next special covariance matrices. In the following theorem we derive two simple upper and lower bounds by imposing a restriction on the covariance matrix.

Theorem 3.8. Let $\boldsymbol{X}$ be a standard Gaussian random vector in $\mathbb{R}^d$ with positive definite covariance matrix Σ. Assume that there exist two non-empty disjoint index sets $I, J, I \cup J = \{1, \dots, d\}$ such that the matrix $\Sigma_{JI}(\Sigma_I)^{-1}$ has non-negative elements. Then we have for all $\boldsymbol{t}_I \geqslant \boldsymbol{0}_I$

$$\frac{\boldsymbol{P}\{\boldsymbol{X} > \boldsymbol{t}\}}{\boldsymbol{P}\{\boldsymbol{X}_I > \boldsymbol{t}_I\}} \geqslant \boldsymbol{P}\{\boldsymbol{Y}_J \geqslant \boldsymbol{t}_J - \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{t}_I\}, \tag{3.6}$$

whereas if $\Sigma_{JI}(\Sigma_I)^{-1}$ has non-positive elements then

$$\frac{\boldsymbol{P}\{\boldsymbol{X} > \boldsymbol{t}\}}{\boldsymbol{P}\{\boldsymbol{X}_I > \boldsymbol{t}_I\}} \leqslant \boldsymbol{P}\{\boldsymbol{Y}_J \geqslant \boldsymbol{t}_J - \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{t}_I\} \tag{3.7}$$

holds with $\boldsymbol{Y}_J$ a mean zero Gaussian random vector with positive definite covariance matrix $\Sigma_J - \Sigma_{JI}(\Sigma_I)^{-1}\Sigma_{IJ}$.

Proof. Along the lines of the proof of the first theorem we get

$$\boldsymbol{P}\{\boldsymbol{X} \geqslant \boldsymbol{t}\} = \int_{\boldsymbol{y}_I \geqslant \boldsymbol{0}_I} \boldsymbol{P}\{\boldsymbol{Y}_J \geqslant \boldsymbol{t}_J - \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{t}_I - \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{y}_I\} d\Phi_{\boldsymbol{X}_I - \boldsymbol{t}_I}(\boldsymbol{y}_I),$$

with $\boldsymbol{Y}_J$ a centered Gaussian random vector with covariance matrix $\Sigma_J - \Sigma_{JI}(\Sigma_I)^{-1}\Sigma_{IJ}$. Thus the proof follows easily by the assumptions. □

Example 5. Consider the same Gaussian random vector $\boldsymbol{X}$ as in the first example. We choose here $I = \{2, \dots, d\}$, so $J = \{1\}$. Simple calculations yield

$$\Sigma_{JI}(\Sigma_I)^{-1} = \left(\frac{\rho}{1 + (d-2)\rho}\right)\boldsymbol{1}_I,$$

hence in view of (3.6) we obtain for $\rho \geqslant 0$ and threshold $\boldsymbol{t}$ with positive components

$$\boldsymbol{P}\{\boldsymbol{X} > \boldsymbol{t}\} \geqslant \boldsymbol{P}\{X_2 > t_2, \dots, X_d > t_d\}\boldsymbol{P}\left\{Y_1 \geqslant t_1 - \frac{\rho \sum_{j=2}^d t_j}{1+(d-2)\rho}\right\},$$

where Y_1 is a centered Gaussian random variable with variance $(1+(d-2)\rho)[(1-\rho)(1+(d-1)\rho)]^{-1}$. Iterating we get

$$\boldsymbol{P}\{\boldsymbol{X} > \boldsymbol{t}\} \geqslant \prod_{i=1}^d \boldsymbol{P}\left\{Y_i \geqslant t_i - \frac{\rho \sum_{j=i+1}^d t_j}{1+(d-i+1)\rho}\right\}, \tag{3.8}$$

where $Y_i, i=1, \dots, d$ are centered Gaussian random variable with variance $(1+(d-i-1)\rho)[(1-\rho)(1+(d-i)\rho)]^{-1}$ and $\sum_{d+1}^d t_i =: 0$. Further, by (3.7) the reversed inequalities hold.

Example 6. We retake Example 4. Put $I=\{1\}$ and $J=\{2, \dots, d\}$. Since the matrix $\Sigma_{JI}(\Sigma_I)^{-1}$ has all entries equal 1, if $t_1 > 0$ we get using (3.6)

$$\boldsymbol{P}\{(B_M(z_1), \dots, B_M(z_d)) > \boldsymbol{t}\} \geqslant \boldsymbol{P}\{B_M(z_1) > t_1\}\boldsymbol{P}\{\boldsymbol{Y}_J > (t_2 - t_1, \dots, t_d - t_1)\},$$

with $\boldsymbol{Y}_J$ a centered Gaussian rancom vector in $\mathbb{R}^{d-1}$ with inverse covariance matrix $(\Sigma^{-1})_J$ and Σ^{-1} as in (3.5). On the other hand we have for any $\boldsymbol{t} \in \mathbb{R}^d$

$$\boldsymbol{P}\{(B_M(z_1), \dots, B_M(z_d)) > \boldsymbol{t}\} \leqslant \boldsymbol{P}\{B_M(z_1) > t_1\},$$

hence the lower bound above captures the speed of convergence to 0 if t_1 goes to ∞ and $t_i - t_1, i=2, \dots, d$ remains bounded.

Example 7. We discuss next asymptotics for $\boldsymbol{X} = (B_0(z_1), \dots, B_0(z_d))^\top$, $d \geqslant 4$ with $B_0(s), s \in [0,1]$ a Brownian Bridge and $0 < z_1 < \dots < z_d < 1$. The inverse covariance matrix of $\boldsymbol{X}$ is

$$\Sigma^{-1} = \begin{pmatrix} \left(\frac{1}{z_1} + \frac{1}{z_2 - z_1}\right) & -\frac{1}{z_2-z_1} & 0 & \dots & 0 \\ -\frac{1}{z_2-z_1} & \left(\frac{1}{z_2-z_1} + \frac{1}{z_3-z_2}\right) & -\frac{1}{z_3-z_2} & & \vdots \\ 0 & -\frac{1}{z_3-z_2} & \left(\frac{1}{z_3-z_2} + \frac{1}{z_4-z_3}\right) & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\frac{1}{z_d-z_{d-1}} \\ 0 & \dots & 0 & -\frac{1}{z_d-z_{d-1}} & \left(\frac{1}{z_d-z_{d-1}} + \frac{1}{1-z_d}\right) \end{pmatrix}. \tag{3.9}$$

Taking $I=\{1,d\}$, $J=\{3,\dots,d-1\}$ implies that $\Sigma_{IJ}(\Sigma_I)^{-1}$ has positive elements. Hence by the above result we have for $\boldsymbol{t}\in(0,\infty)^d$

$$\begin{aligned}
&\boldsymbol{P}\{(B_0(z_1),\dots,B_0(z_d))>\boldsymbol{t}\}\\
&\quad\geqslant \boldsymbol{P}\{B_0(z_1)>t_1,\,B_0(z_d)>t_d\}\\
&\qquad\times\boldsymbol{P}\{\boldsymbol{Y}_J>\boldsymbol{t}_J+((\Sigma^{-1})_J)^{-1}(t_1/(z_1-z_2),0,\dots,0,t_d/(z_{d-1}-z_d))^\top\},
\end{aligned}\tag{3.10}$$

with $\boldsymbol{Y}_J$ centered with inverse covariance matrix $(\Sigma^{-1})_J$. Since further

$$\boldsymbol{P}\{(B_0(z_1),\dots,B_0(z_d))>\boldsymbol{t}\}\leqslant\boldsymbol{P}\{B_0(z_1)>t_1,\,B_0(z_d)>t_d\}$$

we see that the lower bound in (3.10) captures the speed of convergence to zero if $\max(t_1,t_2)\to\infty$ and $\boldsymbol{t}_J-\Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{t}_I$ remains bounded.

A more general discussion is to consider thresholds with an involved dependence between components. If $\boldsymbol{t}=t\boldsymbol{b}$ then for all $t>0$

$$\min_{\boldsymbol{x}\geqslant\boldsymbol{t}}\langle\boldsymbol{x},\Sigma^{-1}\boldsymbol{x}\rangle=t\min_{\boldsymbol{x}\geqslant\boldsymbol{b}}\langle\boldsymbol{x},\Sigma^{-1}\boldsymbol{x}\rangle\tag{3.11}$$

which simplifies the arguments considerably. Clearly, this scaling property does not hold for general thresholds.

In the next theorem we discuss general thresholds, letting both threshold and the covariance matrix of the underlying Gaussian random vector depend on n. More precisely, we are interested in the asymptotic behaviour of the tail probability $\boldsymbol{P}\{\boldsymbol{X}_n\geqslant\boldsymbol{t}_n\}$, with $\|\boldsymbol{t}_n\|\to\infty$ as $n\to\infty$ and $\boldsymbol{X}_n, n\geqslant 1$ Gaussian random vectors in $\mathbb{R}^d, d\geqslant 2$.

In the following we use a simplified notation; for example we write $\boldsymbol{t}_{n,I}$ instead of $(\boldsymbol{t}_n)_I$ and similarly for matrices.

Theorem 3.9. Let $\{\boldsymbol{X}_n, n\geqslant 1\}$ be a sequence of Gaussian random vectors in $\mathbb{R}^d, d\geqslant 2$ with positive definite covariance matrix Σ_n and $\boldsymbol{t}_n$ thresholds so that $\lim_{n\to\infty}\|\boldsymbol{t}_n\|=\infty$. Let $\{\boldsymbol{\lambda}_n, n\geqslant 1\}$ be a sequence of positive vectors in $\mathbb{R}^d$. Assume that for large $n\in\mathbb{N}$ there exist two non-empty disjoint index sets $I, J, I\cup J=\{1,\dots,d\}$ such that

$$\lim_{n\to\infty}\operatorname{diag}(\boldsymbol{\lambda}_{n,J})\left(\boldsymbol{t}_{n,J}-\Sigma_{n,JI}(\Sigma_{n,I})^{-1}\boldsymbol{t}_{n,I}\right)=\boldsymbol{t}_{J,\infty},\tag{3.12}$$

$$\lim_{n\to\infty}\operatorname{diag}(\boldsymbol{\lambda}_{n,J})\left(\Sigma_{n,J}-\Sigma_{n,JI}(\Sigma_{n,I})^{-1}\Sigma_{n,IJ}\right)\operatorname{diag}(\boldsymbol{\lambda}_{n,J})=\Sigma_J^*,\tag{3.13}$$

hold with $\boldsymbol{t}_{J,\infty} < \infty_J$, $\Sigma_J^* \in \mathbb{R}^{|J|\times|J|}$. Then for any $\varepsilon > 0$ we have as $n \to \infty$

$$\left| \frac{\boldsymbol{P}\left\{\boldsymbol{X}_n > \boldsymbol{t}_n\right\}}{\boldsymbol{P}\left\{\boldsymbol{X}_{n,I} > \boldsymbol{t}_{n,I}\right\}} - \boldsymbol{P}\left\{\boldsymbol{Z}_J \geqslant \boldsymbol{t}_{J,\infty}\right\} \right| < \varepsilon, \tag{3.14}$$

with $\boldsymbol{Z}_J$ a mean zero Gaussian random vector with covariance matrix Σ_J^*.

If additionally for all $i \in I$

$$\lim_{n\to\infty} \langle \boldsymbol{e}_i, (\Sigma_{n,I})^{-1} \boldsymbol{t}_{n,I} \rangle = \infty \tag{3.15}$$

then

$$\boldsymbol{P}\left\{\boldsymbol{X}_n > \boldsymbol{t}_n\right\} = \frac{(1+o(1)) \exp\left(-\left\langle \boldsymbol{t}_I^{(n)}, (\Sigma_{n,I})^{-1} \boldsymbol{t}_I^{(n)} \right\rangle \big/ 2\right) \boldsymbol{P}\left\{\boldsymbol{Z}_J \geqslant \boldsymbol{t}_{J,\infty}\right\}}{(2\pi)^{|I|/2} |\Sigma_{n,I}|^{1/2} \Pi_{i\in I} \left\langle \boldsymbol{e}_i, (\Sigma_{n,I})^{-1} \boldsymbol{t}_{n,I} \right\rangle}, \quad n \to \infty. \tag{3.16}$$

Proof. Let $\boldsymbol{Y}_{n,J}$ be a mean zero Gaussian random vector in $\mathbb{R}^{|J|}$ with positive definite covariance matrix $\Sigma_{n,JJ} - \Sigma_{n,JI}(\Sigma_{n,I})^{-1}\Sigma_{n,IJ}$ and put $\boldsymbol{Z}_{n,J} := \boldsymbol{\lambda}_n \boldsymbol{Y}_{n,J}$. Let further $\{z_n, n \geqslant 1\}$ be a positive sequence converging to ∞. As in the proof of the first theorem we obtain for n large

$$\begin{aligned}
&\boldsymbol{P}\left\{\boldsymbol{X}_n \geqslant \boldsymbol{t}_n\right\} \\
&= \int_{\boldsymbol{y}_I \geqslant \boldsymbol{0}_I} \boldsymbol{P}\{\boldsymbol{X}_{n,J} \geqslant \boldsymbol{t}_{n,J} | \boldsymbol{X}_{n,I} = \boldsymbol{t}_{n,I} + z_n^{-1} \boldsymbol{y}_I\} d\Phi_{z_n(\boldsymbol{X}_{n,I} - \boldsymbol{t}_{n,I})}(\boldsymbol{y}_I) \\
&= \int_{\boldsymbol{y}_I \geqslant \boldsymbol{0}_I} \boldsymbol{P}\left\{\boldsymbol{Y}_{n,J} \geqslant \boldsymbol{t}_{n,J} - \Sigma_{n,JI}(\Sigma_{n,I})^{-1}\left(\boldsymbol{t}_n + z_n^{-1}\boldsymbol{y}\right)_I\right\} d\Phi_{z_n(\boldsymbol{X}_{n,I} - \boldsymbol{t}_{n,I})}(\boldsymbol{y}_I) \\
&= \int_{y_I \geqslant 0_I} \boldsymbol{P}\Big\{\boldsymbol{Z}_{n,J} \geqslant \boldsymbol{\lambda}_n \left(\boldsymbol{t}_{n,J} - \Sigma_{n,JI}(\Sigma_{n,I})^{-1}\boldsymbol{t}_{n,I}\right) \\
&\qquad\qquad - z_n^{-1}\boldsymbol{\lambda}_n \left(\Sigma_{n,JI}(\Sigma_{n,I})^{-1}\boldsymbol{y}_I\right)\Big\} d\Phi_{z_n(\boldsymbol{X}_{n,I} - \boldsymbol{t}_{n,I})}(\boldsymbol{y}_I).
\end{aligned}$$

Choosing now z_n so that

$$\lim_{n\to\infty} z_n^{-1}\boldsymbol{\lambda}_{n,J}\left(\Sigma_{n,JI}(\Sigma_{n,I})^{-1}\boldsymbol{y}_I\right) = \boldsymbol{0}_J$$

holds for all $\boldsymbol{y}_I > \boldsymbol{0}_I$ and applying (3.13) yields the uniform convergence of the integrand on $[0,\infty)^{|J|}$ to $\boldsymbol{P}\{\boldsymbol{Z}_J \geqslant \boldsymbol{t}_{J,\infty}\}$, with $\boldsymbol{Z}_J$ a Gaussian random vector with mean zero and covariance matrix Σ_J^*. Next, by (3.15) and (2) we obtain as $n \to \infty$

$$\boldsymbol{P}\left\{\boldsymbol{X}_{n,I} > \boldsymbol{t}_{n,I}\right\} = (1+o(1))\left((2\pi)^I |\Sigma_{n,I}|\right)^{1/2} \exp\left(-\left\langle \boldsymbol{t}_I^{(n)}, (\Sigma_{n,I})^{-1}\boldsymbol{t}_I^{(n)}\right\rangle \Big/ 2\right) \times \prod_{i\in I}\left\langle \boldsymbol{e}_i, (\Sigma_{n,I})^{-1}\boldsymbol{t}_{n,I}\right\rangle^{-1},$$

thus the proof follows using further Proposition 2.1. □

Remark 3.10. (i) A good candidate for $\boldsymbol{\lambda}_n$ is the vector such that $1/(\boldsymbol{\lambda}_n\boldsymbol{\lambda}_n)$ is the main diagonal of the matrix $((\Sigma_n)^{-1})_J$. If $\Sigma_n \to \Sigma, n\to\infty$ with Σ a positive definite matrix, then we can take also $\boldsymbol{\lambda}_n = \mathbf{1}$. For the latter case Theorem 3.9 implies Theorem 4.1 of Hashorva and Hüsler[10].
(ii) The square matrix Σ_J^* in (3.13) can be semi-positive definite.
(iii) If for all large $n \in \mathbb{N}$, we have $\min_{\boldsymbol{x}\geqslant \boldsymbol{t}_n} \langle \boldsymbol{x}, \Sigma_n^{-1}\boldsymbol{x}\rangle = \left\langle \boldsymbol{t}_I^{(n)}, (\Sigma_{n,I})^{-1}\boldsymbol{t}_I^{(n)}\right\rangle$, then in the above theorem we have $\boldsymbol{t}_{J,\infty} \leqslant \mathbf{0}_J$.

Example 8. Let $0 < z_{n1} < z_{n2} < \cdots < z_{nd} < \infty$ and B_M a standard Brownian Motion in $[0,\infty)$. Put $\boldsymbol{X}_n = (B_M(z_{n1}), \ldots, B_M(z_{nd}))^\top$ and $\boldsymbol{t}_n = t_n\mathbf{1}, t_n > 0$. We investigate $\boldsymbol{P}\{\min_{i=1,\ldots,n} B_M(z_{n1}) > t_n\}$, with $t_n \to \infty$ as $n \to \infty$. The unique solution of the quadratic programming problem $\mathscr{P}(\Sigma_n^{-1}, t_n\mathbf{1})$, with Σ_n^{-1} the inverse covariance matrix of $\boldsymbol{X}_n$ is the vector $t_n\mathbf{1}$. Moreover

$$I := I_n = \{1\}, \quad J := J_n = \{2,\ldots,d\}, \quad \alpha_n = t_n^2/z_{n1}, \quad \Sigma_{n,JI}(\Sigma_{n,I})^{-1} = \mathbf{1}_J^\top,$$

so we have $\boldsymbol{t}_{J,\infty} = \mathbf{0}_J$ for any positive sequence $\boldsymbol{\lambda}_n, n \geqslant 1$. Consider next $\boldsymbol{\lambda}_n$ as in (i) of Remark 3.10 and suppose that $z_{n(i+1)} = iq_n z_{n1}, i = 1,\ldots,d-1$ with $q_n > 1$. Then we get

$$(\Sigma_J^*)_{kl} = \lim_{n\to\infty}\sqrt{\frac{z_{n(k+1)} - z_{n1}}{z_{n(l+1)} - z_{n1}}} = \sqrt{\frac{k}{l}}, \quad 1 \leqslant k \leqslant l \leqslant d-1,$$

hence (3.12) holds. If further $\lim_{n\to\infty} t_n^2/z_{n1} = \infty$, then (3.15) is satisfied, thus we get

$$\boldsymbol{P}\left\{\min_{i=1,\ldots,d} B_M(z_{ni}) > t_n\right\} = (1+o(1))\left(2\pi t_n^2/z_{n1}\right)^{-1/2} \exp(-t_n^2/(2z_{n1}))\boldsymbol{P}\{\boldsymbol{Z}_J \geqslant \mathbf{0}_J\}, \quad n\to\infty,$$

with $\boldsymbol{Z}_J$ a centered Gaussian random vector with covariance matrix Σ_J^*.

Example 9. Let $(\xi_1, \eta_1), \dots, (\xi_n, \eta_n)$ be iid random vectors with standard Gaussian components so that $\mathrm{Corr}\{\xi_i, \eta_i\} = \rho_i \in (-1, 1)$ and put $\boldsymbol{t}_n = (t_n, a_n t_n)$ with $\lim_{n\to\infty} t_n = \infty$ and $a_n > 0, n \in \mathbb{N}$. Let $\boldsymbol{\lambda}_n$ be as in (i) of Remark 3.10 and take $I = I_n = \{1\}, J = J_n = \{2\}, n \in \mathbb{N}$, hence $\Sigma_j^* = 1$ and $\langle \boldsymbol{e}_1, (\Sigma_{n,I})^{-1} t_n \rangle = t_n$, thus condition (3.15) is fulfilled. Now if we assume further that a_n, ρ_n are such that

$$\lim_{n\to\infty} \frac{t_n(a_n - \rho_n)}{\sqrt{1-\rho_n^2}} = t^*$$

holds, then by the above theorem

$$\boldsymbol{P}\{\xi_n > t_n, \eta_n > a_n t_n\} = (1 + o(1)(2\pi t_n^2)^{-1/2} \boldsymbol{P}\{\xi_1 > t^*\} \exp(-t_n^2/2), \quad n \to \infty.$$

Note that for $a_n \leqslant \rho_n$ we have $\boldsymbol{P}\{\xi_1 > t^*\} \in [1/2, 1]$. If $t^* = \infty$ the asymptotic above is not exact.

The next corollary is important if we consider a single random vector $\boldsymbol{X}$ and a sequence of thresholds depending on n.

Corollary 3.11. Let $\boldsymbol{X}$ be a standard Gaussian random vector in $\mathbb{R}^d$ with positive definite covariance matrix Σ and let $\{\boldsymbol{t}_n, n \geqslant 1\}$ be a sequence of thresholds such that $\lim_{n\to\infty} \|\boldsymbol{t}_n\| = \infty$. Suppose that there exists a minimal index set $I \subset \{1, \dots, d\}$ so that $|I| < d$ and

$$\lim_{n\to\infty} \left(\boldsymbol{t}_{n,J} - \Sigma_{JI}(\Sigma_I)^{-1}\boldsymbol{t}_{n,I}\right) = \boldsymbol{t}_{J,\infty} < \boldsymbol{\infty}_J, \tag{3.17}$$

$$\lim_{n\to\infty} \left\langle \boldsymbol{e}_i - (\Sigma_I)^{-1}\boldsymbol{t}_{n,I}\right\rangle = \infty, \quad \forall i \in I \tag{3.18}$$

hold. Then we have

$$\boldsymbol{P}\left\{\boldsymbol{X} > \boldsymbol{t}_n\right\} = \frac{(1+o(1))\exp\left(-\left\langle \boldsymbol{t}_I^{(n)}, (\Sigma_I)^{-1}\boldsymbol{t}_I^{(n)}\right\rangle \big/ 2\right) \boldsymbol{P}\left\{\boldsymbol{Z}_J \geqslant \boldsymbol{t}_{J,\infty}\right\}}{(2\pi)^{|I|/2}|\Sigma_I|^{1/2}\Pi_{i\in I}\left\langle \boldsymbol{e}_i, (\Sigma_I)^{-1}\boldsymbol{t}_{n,I}\right\rangle}, \quad n \to \infty \tag{3.19}$$

with $\boldsymbol{Z}_J$ a mean zero Gaussian random vector with positive definite covariance matrix $\Sigma_J - \Sigma_{JI}(\Sigma_I)^{-1}\Sigma_{IJ}$.

4. ASYMPTOTIC RESULTS FOR PARAMETRIC THRESHOLDS

In this section we consider a tractable special case where the threshold is parametrised. Such problems arise when considering discrete and continuous boundary crossing probabilities. More precisely, let $\{X(s)+h(s), s \in [0,1]\}$ be a separable Gaussian random process with trend function h and let $u:[0,1]\to\mathbb{R}$ be a function. Further denote by $G_n, n\geqslant 1$ the grid of points $0=z_0<z_{n1}<\cdots<z_{nd}<1, d\geqslant 2$.

There are several statistical problems where the asymptotics of the boundary crossing probability

$$\boldsymbol{P}\{X(s)+t_n h(s)>u(s),\ \forall s\in[0,1]\},\qquad t_n\to\infty$$

is of particular interest, see e.g. Bischoff *et al.*[1,3,4]. A good approximation of the above probability is the discrete boundary crossing probability

$$\boldsymbol{P}\{X(z_{ni})>u(z_{ni})-t_n h(z_{ni}),\ \forall i\in\{1,\ldots,d\}\}.$$

If $\boldsymbol{X}$ is a Gaussian random process, then we can find the asymptotic behaviour ($t_n\to\infty$) of the above probability using Theorem 3.9. Since for any $\boldsymbol{t}_n>\boldsymbol{0}$

$$\boldsymbol{P}\{X(s)+t_n h(s)>u(s),\forall s\in[0,1]\}\leqslant \inf_{(z_{n1},\ldots,z_{nd})\in[0,1]^d,d\geqslant 1}\boldsymbol{P}\{X(z_{ni})>u(z_{ni})-t_n h(z_{ni}),\ \forall i\in\{1,\ldots,d\}\},$$

we get further an upper asymptotic bound for the continuous boundary crossing probability above.

In the following we deal with the special case $X=B_0$ a Brownian Bridge in [0,1]. Let $\boldsymbol{t}_n:=t_n\boldsymbol{h}_n-\boldsymbol{u}_n$, where $\boldsymbol{u}_n:=(u(z_{n1}),\ldots,u(z_{nd}))^\top, \boldsymbol{h}_n:=(h(z_{n1}),\ldots,h(z_{nd}))^\top$. Thus the threshold $\boldsymbol{t}_n$ is parametrised by t_n,u,h and the grid $z_{n1},\ldots,z_{nd}$.

If $\lim_{n\to\infty} z_{ni}=z_i\in(0,1), i=1,\ldots,d$, we denote the new grid of points $0=z_0<z_1<\cdots<z_d<z_{d+1}=1$ by G. Further for any function $g:[0,1]\to\mathbb{R}$ the polygon lines with nodes in $(0,g(0),(z_{n1},g(z_{n1})),\ldots,(1,g(1)))$ and $(0,g(0),(z_1,g(z_1)),\ldots,(1,g(1)))$ are denoted by $\underline{g}_n$ and $\underline{g}$, respectively. We formulate now the main result of this section.

Theorem 4.1. Let $h:[0,1]\to\mathbb{R}, u:[0,1]\to\mathbb{R}$ be two continuous functions such that $h(0)=h(1)=0, u(0)>0$ and $h(t_0)>0$ for some $t_0\in(0,1)$ and let $G_n, n\geqslant 2$ and G be as above. Denote by $\tilde{\underline{h}}_n, \tilde{\underline{h}}$ the smallest upper concave polygons of $\underline{h}_n$ and $\underline{h}$, respectively. Assume that $\tilde{\underline{h}}_n, \tilde{\underline{h}}$ are such that no three nodes of these polygons are in a line. If $\lim_{n\to\infty} t_n=\infty$, then we have

$$\begin{aligned}&\boldsymbol{P}\{(B_0(z_{n1}),\ldots,B_0(z_{nd}))>t_n\boldsymbol{h}_n-\boldsymbol{u}_n\}\\&\quad=\frac{(1+o(1))\exp\left(-t_n^2\|\underline{\tilde{h}}_n'\|^2/2+t_n\int_0^1 u_n(s)d(-\underline{\tilde{h}}_n'(s))-\langle\boldsymbol{u}_{n,I},(\Sigma_{n,I})^{-1}\boldsymbol{u}_{n,I}\rangle/2\right)}{(2\pi)^{|I|/2}|\Sigma_{n,I}|^{1/2}t_n^{|I|}\Pi_{i\in I}\langle\boldsymbol{e}_i,(\Sigma_{n,I})^{-1}\boldsymbol{h}_{n,I}\rangle},\\&\qquad\qquad\qquad n\to\infty,\end{aligned}\tag{4.1}$$

with Σ_n the covariance matrix of $(B_0(z_{n1}),\ldots,B_0(z_{nd}))^\top$, $I\subseteq\{1,\ldots,d\}$ the minimal index set such that the polygons through $(z_i,h(z_i)), i=0,\ldots,d+1$ and through $(z_i,\tilde{h}(z_i)), i\in I\cup\{0,d+1\}$ are equal. Further $\|\cdot\|$ is the norm of the reproducing kernel Hilbert space corresponding to the Brownian Bridge $B_0(t), t\in[0,1]$ and $\langle\boldsymbol{e}_i,(\Sigma_{n,I})^{-1}\boldsymbol{h}_{n,I}\rangle>0, \forall i\in I$.

Proof. The inverse covariance matrix of the centered Gaussian vector $(B_0(t_{n1}),\ldots,B_0(t_{nd}))$ denoted by Σ_n^{-1} is

$$\begin{pmatrix}\left(\frac{1}{z_{n_1}}+\frac{1}{z_{n_2}-z_{n_1}}\right) & -\frac{1}{z_{n_2}-z_{n_1}} & 0 & \cdots & 0\\ -\frac{1}{z_{n_2}-z_{n_1}} & \left(\frac{1}{z_{n_2}-z_{n_1}}+\frac{1}{z_{n_3}-z_{n_2}}\right) & -\frac{1}{z_{n_3}-z_{n_2}} & & \vdots\\ 0 & -\frac{1}{z_{n_3}-z_{n_2}} & \left(\frac{1}{z_{n_3}-z_{n_2}}+\frac{1}{z_{n_4}-z_{n_3}}\right) & \ddots & 0\\ \vdots & & \ddots & \ddots & -\frac{1}{z_{nd}-z_{n,d-1}}\\ 0 & \cdots & 0 & -\frac{1}{z_{nd}-z_{n,d-1}} & \left(\frac{1}{z_{n,d}-z_{n,d-1}}+\frac{1}{1-z_{nd}}\right)\end{pmatrix}.$$

We determine for $n\in\mathbb{N}$ index sets I_n, J_n by solving the quadratic programming problem $\mathscr{P}(\Sigma_n^{-1},t_n\boldsymbol{h}_n-\boldsymbol{u}_n)$. Since $\lim_{n\to\infty}\boldsymbol{u}_n/t_n=0$ and the scaling property mentioned in (3.11), we need to solve instead $\mathscr{P}(\Sigma_n^{-1},\boldsymbol{h}_n)$. Next, by Proposition 2.1 there exists the minimal non-empty index set $I_n\subset\{1,\ldots,d\}$ and $\tilde{\boldsymbol{h}}_n\in\mathbb{R}^d$ so that

$$\begin{aligned}\min_{\boldsymbol{x}\geqslant\boldsymbol{h}_n}\left\langle\boldsymbol{x},\Sigma_n^{-1}\boldsymbol{x}\right\rangle&=\left\langle\tilde{\boldsymbol{h}}_n,\Sigma_n^{-1}\tilde{\boldsymbol{h}}_n\right\rangle\\&=\left\langle\tilde{\boldsymbol{h}}_{n,I_n},(\Sigma_{n,I_n})^{-1}\tilde{\boldsymbol{h}}_{n,I_n}\right\rangle=\left\langle\boldsymbol{h}_{n,I_n},(\Sigma_{n,I_n})^{-1}\boldsymbol{h}_{n,I_n}\right\rangle>0.\end{aligned}$$

Lemma 4.2 of Bischoff *et al.*[2] implies that $\tilde{\boldsymbol{h}}_n:=(\tilde{h}(z_{n1}),\ldots,\tilde{h}(z_{nd}))$ with $\tilde{h}\geqslant h$ the smallest concave majorant of h. Furthermore the index set I_n is such that the polygon lines through $\{(0,\tilde{h}(0)),(z_{ni},\tilde{h}(z_{ni})),(1,\tilde{h}(1))|i\in I_n\}$ and through $\{(0,\tilde{h}(0)),(z_{n1},\tilde{h}(z_{n1})),\ldots,(1,\tilde{h}(1))\}$ are equal. By the continuity of h,u and the fact that $z_{ni}\to z_i, i\leqslant d$ as $n\to\infty$ we get

$$\lim_{n\to\infty}\boldsymbol{h}_n=\boldsymbol{h}:=(h(z_1),\ldots,h(z_d))^\top,\quad \lim_{n\to\infty}\tilde{\boldsymbol{h}}_n=\tilde{\boldsymbol{h}}:=(\tilde{h}(z_1),\ldots,\tilde{h}(z_d))^\top$$

and

$$\lim_{n\to\infty} \boldsymbol{u}_n = \boldsymbol{u} := (u(z_1), \ldots, u(z_d))^\top.$$

Moreover for all large n we have $I_n = I$ with I the minimal index set corresponding to the quadratic programming problem $\mathscr{P}(\Sigma^{-1}, \boldsymbol{h})$, where Σ^{-1} is the inverse covariance matrix of $(B_0(z_1), \ldots, B_0(z_d))^\top$. Now by the fact that no three points of the polygon $\underline{\tilde{h}}_n, \underline{\tilde{h}}$ are in a line, we have that $\tilde{\boldsymbol{h}}_{n,J} > \boldsymbol{h}_{n,J}, \tilde{\boldsymbol{h}}_J > \boldsymbol{h}_J$ if $|J| > 0$. Assume for simplicity that J is not empty. Putting $\boldsymbol{\lambda}_n = 1$ we get (3.12) holds with $\boldsymbol{t}_{J,\infty} = -\infty_J$. Note in passing that $|J| = 0$ iff the polygons are concave. Using Lemma 4.2 of Bischoff *et al.* (2) we obtain

$$\min_{\boldsymbol{x} \geqslant \boldsymbol{h}_n} \left\langle \boldsymbol{x}, \Sigma_n^{-1} \boldsymbol{x} \right\rangle = \int_0^1 \left(\underline{\tilde{h}}_n'(s) \right)^2 ds =: \left\| \underline{\tilde{h}}_n' \right\|^2$$

and

$$\left\langle \boldsymbol{u}_n, \Sigma_n^{-1} \boldsymbol{h}_n \right\rangle = \left\langle \boldsymbol{u}_{n,I}, (\Sigma_{n,I})^{-1} \boldsymbol{h}_{n,I} \right\rangle = \int_0^1 u_n(s) d\left(-\underline{\tilde{h}}_n'(s) \right)$$

with $\underline{\tilde{h}}_n'$ the right continuous derivative of $\tilde{h}_n$. Since $\lim_{n\to\infty} z_{ni} = z_i$ the proof follows easily by Theorem 3.9. □

Remark 4.2. (i) The right continuous derivatives of $\underline{\tilde{h}}_n, \underline{\tilde{h}}$ have bounded variation, thus $\int_0^1 u_n(s) d\underline{\tilde{h}}_n'(s)$ and $\int_0^1 u(s) d\underline{\tilde{h}}'(s)$ are well defined. (ii) If the points of the grid G_n become dense in [0,1] with $d = d_n \to \infty$ as $n \to \infty$ then

$$\lim_{n\to\infty} \left\langle (\boldsymbol{u}_n)_I, (\Sigma_{n,I})^{-1} (\boldsymbol{h}_n)_I \right\rangle = \int_0^1 u(s) d\left(-\underline{\tilde{h}}_n'(s) \right)$$

and

$$\lim_{n\to\infty} \min_{\boldsymbol{x} \geqslant \boldsymbol{h}_n} \left\langle \boldsymbol{x}, \Sigma_n^{-1} \boldsymbol{x} \right\rangle = \int_0^1 \left(\underline{\tilde{h}}'(s) \right)^2 ds = \left\| \underline{\tilde{h}}' \right\|^2.$$

(iii) Results for Brownian Motion can be obtained along the same lines.

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