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Aequationes Mathematicae

On the functional equation $x + f(y + f(x)) = y + f(x + f(y))$

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Abstract. For an abelian group $(G, +, 0)$ we consider the functional equation

$$f : G \to G, \ \ x + f(y + f(x)) = y + f(x + f(y)) \quad (\forall x, y \in G), \tag{1}$$

most times together with the condition

$$f(0) = 0. \tag{0}$$

Our main question is whether a solution of (1) $\wedge$ (0) must be additive, i.e., an endomorphism of G. We shall answer this question in the negative (Example 3.14) Rätz (Aequationes Math 81:300, 2011).

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1. Introduction, notation, preliminaries, and some history

We denote by $S(G)$ the set of all solutions of (1) and put

$$S_0(G) := \{f \in S(G);\ f(0) = 0\}. \tag{2}$$

The symbol $:=$ means that the right-hand side defines the left-hand side. $=_{(\ldots)}=$ is a short form of quotation of $(\ldots)$, and $\Box$ marks the end of a proof. i_A denotes the identity mapping of the set A and $\underline{a}$ the constant mapping with value a. $\mathbb{P}, \mathbb{N}, \mathbb{N}^0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ stand for the sets of prime numbers, positive integers, nonnegative integers, integers, rational and real numbers, respectively. For every $n \in \mathbb{N}, f^n$ means the n-th iterate of $f : G \to G$. Throughout the paper, $(G, +, 0)$ or $(G, +)$ or G denotes an abelian group.

For every $n \in \mathbb{Z}$, $\omega_n : G \to G$ defined by $\omega_n(x) := nx$ $(\forall x \in G)$ is in $\mathrm{End}(G)$, i.e. ω_n is an endomorphism of G. For every $z \in G, \langle z\rangle$ denotes the subgroup of G generated by z, and $t_z : G \to G$ is the translation $x \mapsto x + z$ $(\forall x \in G)$ of G by z. We use 0 for the identity element of G as well as for

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the integer 0; it will always be clear from the context what is meant. We freely use the fact that for abelian groups A and B we have $A \times B \cong A \oplus B$, e.g., by denoting the elements of $A \oplus B$ as ordered pairs $(a, b) \in A \times B$ whenever we find it convenient. For every $n \in \mathbb{N}$, we let $\mathbb{Z}_n$ stand for the cyclic group with n elements, most times written as $\{0, \ldots, n-1\}$.

The following remarks are easily verified.

Remark 1.1. If G and H are abelian groups and $\varphi : G \to H$ is an isomorphism, if $f : G \to G, g : H \to H, g = \varphi \circ f \circ \varphi^{-1}$, then

(a) $f \in S(G) \Longrightarrow g \in S(H)$; $f \in S_0(G) \Longrightarrow g \in S_0(H)$.
(b) $f \in \mathrm{End}(G) \Longrightarrow g \in \mathrm{End}(H)$.
(c) $S_0(G) \subset \mathrm{End}(G) \Longrightarrow S_0(H) \subset \mathrm{End}(H)$.

Remark 1.2. The abelian group G is a unitary $\mathbb{Z}$-module in a natural way. We shall tacitly use the corresponding computation rules many times.

In particular, for $f : G \to G$: $f(0) = 0 \Longleftrightarrow f(n \cdot 0) = n\, f(0)$ $(\forall n \in \mathbb{Z})$.

In [2], Brillouët-Belluot asked what the continuous functions in $S(\mathbb{R})$ are. By a connectedness argument, Jarczyk and Jarczyk [3] showed that there are none.

Balcerowski [1] found many interesting and fundamental properties of solutions of (1); we list here some of them:

(B1) $f \in S(G) \Longrightarrow f$ is injective.
(B2) $f \in S(G) \Longrightarrow 0 \in f(G)$.
(B3) $f \in S_0(G) \Longrightarrow$
$$f^2(x) + x = f(x) \;\; (\forall x \in G). \tag{3}$$
(B4) $f \in S_0(G) \Longrightarrow f^3 = -i_G$ and f is odd.
(B5) $f \in \mathrm{End}(G) \Longrightarrow [f \in S_0(G) \Longleftrightarrow (3)]$.
(B6) $f \in S(G), z \in G \Longrightarrow f \circ t_z \in S(G)$.
(B7) $f \in S_0(G) \Longrightarrow 2f \in \mathrm{End}(G)$.
(B8) $f \in S_0(G), \omega_2$ injective $\Longrightarrow f \in \mathrm{End}(G)$.
(B9) $f \in S(\mathbb{R}) \Longrightarrow f$ is nowhere continuous.
(Cf. [1, Lemma 1, Corollary 1, Remark 1, Corollary 2, Lemma 2, Remark 2, Theorem 1, Corollary 3, Corollary 4]). (B9) strengthens and confirms the main result of [3].
(B10) **Open question** [1, Remark 3]): Is $S_0(G) \subset \mathrm{End}(G)$ true in general? I.e., can the injectivity of ω_2 in (B8) be deleted?

(B6) and (B1) above can slightly be sharpened:

Remark 1.3. (B6′) $S(G) = \{f \circ t_z;\ f \in S_0(G),\ z \in G\}$,
(B1′) $f \in S(G) \Longrightarrow f$ is bijective.

Proof. (B6′): $B := \{f \circ t_z;\ f \in S_0(G),\ z \in G\}$. Now $B \subset S(G)$ follows from $S_0(G) \subset S(G)$ and (B6). Conversely, let $h \in S(G)$. By (B2) there exists

$z \in G$ with $h(z) = 0$. Define $f := h \circ t_z$, and by (B6) $f \in S(G)$, moreover $f(0) = h(z) = 0$, so that we even have $f \in S_0(G)$ and $h = f \circ t_{-z}$, so $h \in B$, finally $S(G) \subset B$, in the total $S(G) = B$. – (B1′): If $f \in S(G)$, then by (B6′) there are $f' \in S_0(G)$ and $z \in G$ such that $f = f' \circ t_z$. By (B4) $(f')^3 = -i_G$, so f' must be bijective, and together with the bijectivity of t_z we get that of f. □

(B6′) says that it is sufficient to consider $S_0(G)$ rather than $S(G)$ and ensures

$$S(G) \neq \emptyset \iff S_0(G) \neq \emptyset. \tag{4}$$

2. New results

Lemma 2.1. (a) $\omega_1 = i_G \in S_0(G) \iff G = \{0\}$.
(b) $\omega_{-1} = -i_G \in S_0(G) \iff 3G = \{0\}$.

Proof. (a) $\Longrightarrow$: Let $x, y \in G$ be arbitrary. Then $x+y+x =_{(1)} = y+x+y$, i.e., $x = y$, so $\operatorname{card} G = 1, G = \{0\}$. – $\Longleftarrow$ is trivial.
(b) ω_{-1} is an involution and in $\operatorname{End}(G)$. So $\omega_{-1} \in S_0(G) \Leftarrow_{(B5)}\Rightarrow \omega_{-1}^2 + \omega_1 = \omega_{-1} \iff 3\omega_1 = 0 \iff 3G = \{0\}$. □

Corollary 2.2. *There are no continuous functions in* $S(\mathbb{R})$.

After the proofs given in [3] and [1] (cf. (B9)), we proceed completely differently.

Proof. Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f \in S(\mathbb{R})$. By (B2) there exists $z \in \mathbb{R}$ with $f(z) = 0$. For $h := f \circ t_z$ we get $h \in S(\mathbb{R})$ by (B6) and $h(0) = f(z) = 0$, so $h \in S_0(\mathbb{R})$. (B4) implies $h^3 = -i_{\mathbb{R}}$, so $h^6 = i_{\mathbb{R}}$, moreover h is continuous. By a theorem of McShane [4], h must be an involution in all possible cases, i.e., $h^2 = i_{\mathbb{R}}$. Together with $h^3 = -i_{\mathbb{R}}$ we obtain $h = -i_{\mathbb{R}}$, so by Lemma 2.1(b) $h \notin S_0(\mathbb{R})$, a contradiction. □

Lemma 2.3. *Let* $(G_i)_{i \in I}$ *be a family of abelian groups,* $G := \prod_{i \in I} G_i$ *their (cartesian) product,* $G' := \bigoplus_{i \in I} G_i$ *their direct sum,* $f_i \in S_0(G_i) (\forall i \in I)$*, and* $f : G \to G, f : (x_i)_{i \in I} \mapsto (f_i(x_i))_{i \in I}$ *for all* $(x_i)_{i \in I} \in G$*. Then:*
(a) $f \in S_0(G)$.
(b) $f(G') \subset G'$.
(c) *If* $f' : G' \to G'$ *is the restriction of* f*, then* $f' \in S_0(G')$.

Proof. (a) is established by a straightforward computation. (b) If $x := (x_i)_{i \in I} \in G'$, then the support of x is finite. Since $f_i(0) =_{(0)}= 0\,(\forall i \in I)$, the support of $(f_i(x_i))_{i \in I}$ is finite as well, so $f(x) \in G'$. – (c) By (b), the restriction $f' : G' \to G'$ of f exists. G' is a subgroup of G, and by (a), f' clearly satisfies (1) on G' and (0), so $f' \in S_0(G')$. □

Lemma 2.4. *Every $f \in S(G)$ has exactly one fixed element.*

Proof. Let $f \in S(G)$. 1) Let $z, w \in G, f(z) = z, f(w) = w$. Put $x = z, y = w$ in (1): $z + f(w + z) = w + f(z + w)$, so $z = w$, i.e., f has at most one fixed element. – 2) Assume that f has no fixed element. By (B2) there exists $z \in G$ with $f(z) = 0$. Put $x = z$ in (1):

$$z + f(y + 0) = y + f(z + f(y)) \quad (\forall y \in G). \tag{5}$$

By (B1′) f is bijective. Define $g : G \to G, g(y) := f(z + f(y))(\forall y \in G)$, i.e., $g = f \circ t_z \circ f$, so

$$g \text{ is bijective.} \tag{6}$$

By assumption, $f(y) \neq y(\forall y \in G)$, so by (5) $z \neq f(z + f(y))(\forall y \in G)$, i.e. $z \neq g(y)(\forall y \in G)$, in contradiction to (6). Therefore, f has to have at least one fixed element. □

Lemma 2.4 confirms and explains Lemma 2.1(a).

Theorem 2.5. *If $f \in S_0(G)$, then*

$$f(ny) = nf(y) \quad (\forall y \in G, \ \forall n \in \mathbb{Z}), \tag{7}$$

i.e., f is $\mathbb{Z}$-homogeneous.

Proof. Let $f \in S_0(G)$. By (B1′) and (B4), f is bijective and odd. (7) trivially holds for all $y \in G$ and $n = 1$ as well as for $n = 0$, the latter by (2) or (0). Induction hypothesis: For some $n \in \mathbb{N}$

$$f(ky) = kf(y) \quad (\forall y \in G, \ 0 \leq k \leq n) \tag{H}$$

is assumed to hold.

(1) We first prove three auxiliary assertions:

$$f(ky) = kf(y) \quad (\forall y \in G, \ -n \leq k \leq n), \tag{*}$$

$$f^{-1}(kz) = kf^{-1}(z) \quad (\forall z \in G, \ -n \leq k \leq n), \tag{**}$$

$$f(y + (n-1)f^{-1}(y)) = f(y) + (n-1)y \quad (\forall y \in G). \tag{***}$$

Proof of (*). Let $y \in G$ be arbitrary. If $0 \leq k \leq n$, then the assertion holds by (H). Let $-n \leq k < 0$, so $0 < -k \leq n$. By (H) we have $f((-k)y) = (-k)f(y)$. Since $(-k)z = -kz$ (remember Remark 1.2) and f is odd, we get $-f(ky) = -kf(y)$, i.e., $f(ky) = kf(y)$. As $y \in G$ was arbitrary, (*) holds.

Proof of (**). Let $z \in G$ be arbitrary, $-n \leq k \leq n$, and $y := f^{-1}(z)$. Then $f(ky) =_{(*)}= kf(y)$, so $ky = f^{-1}f(ky) = f^{-1}(kf(y))$, i.e., $kf^{-1}(z) = f^{-1}(kf(y)) = f^{-1}(kz)$. Since $z \in G$ was arbitrary, (**) is established.

Proof of (***). Let $y \in G$ be arbitrary and $x := -f((n-1)y)$. Then (1) becomes $-f((n-1)y) + f(y + f(-f((n-1)y))) = y + f(-f((n-1)y) + f(y))$. $n \in \mathbb{N}$

implies $0 \le n-1 < n$, so by (H) $-(n-1)f(y) + f(y + f(-(n-1)f(y))) = y + f(-(n-1)f(y) + f(y))$ and again by (H) $-(n-1)f(y) + f(y + (n-1)f(-f(y))) = y + f((-n+2)f(y))$. Now $-n < -n+2 \le -1+2 = 1 \le n$, so (*) yields

$$-(n-1)f(y) + f(y + (n-1)f(-f(y))) = y + (-n+2)f^2(y).$$

By (B4), f is odd, so

$$-(n-1)f(y) + f(y - (n-1)f^2(y)) = y + (-n+2)f^2(y).$$

Furthermore $-f^2(y) = -f^3f^{-1}(y) =_{(B4)}= f^{-1}(y)$, therefore

$$-(n-1)f(y) + f(y + (n-1)f^{-1}(y)) = y + (n-2)f^{-1}(y), \text{ i.e.,}$$
$$f(y + (n-1)f^{-1}(y)) = y + (n-1)f(y) + (n-2)f^{-1}(y). \tag{8}$$

The right-hand side of (8) is $y + f(y) + (n-2)(f(y) + f^{-1}(y)) = y + f(y) + (n-2)(f^2f^{-1}(y) + f^{-1}(y)) =_{(3)}= y + f(y) + (n-2)ff^{-1}(y) = y + f(y) + (n-2)y = f(y) + (n-1)y$, so by (8) $f(y + (n-1)f^{-1}(y)) = f(y) + (n-1)y$. Since $y \in G$ was arbitrary, (***) holds.

(2) For arbitrary $y \in G$ and $x := f^{-1}(ny)$, (1) becomes $f^{-1}(ny) + f(y + ff^{-1}(ny)) = y + f(f^{-1}(ny) + f(y))$, i.e., $f^{-1}(ny) + f((n+1)y) = y + f(f^{-1}(ny) + f(y))$, so by (**)

$$nf^{-1}(y) + f((n+1)y) = y + f(nf^{-1}(y) + f(y)). \tag{9}$$

The right-hand side of (9) is $y + f((n-1)f^{-1}(y) + f^{-1}(y) + f(y)) =_{(3)}= y + f((n-1)f^{-1}(y) + y) =_{(***)}= y + f(y) + (n-1)y = f(y) + ny$, so (9) is $nf^{-1}(y) + f((n+1)y) = f(y) + ny$, i.e.,

$$f((n+1)y) = f(y) + ny - nf^{-1}(y). \tag{10}$$

The right-hand side of (10) is $f(y) + n(y - f^{-1}(y)) = f(y) + n(ff^{-1}(y) - f^{-1}(y)) =_{(3)}= f(y) + nf^2f^{-1}(y) = f(y) + nf(y) = (n+1)f(y)$, so (10) is $f((n+1)y) = (n+1)f(y)$. As $y \in G$ was arbitrary, we have $f((n+1)y) = (n+1)f(y)$ $(\forall y \in G)$. By the so-called second principle of induction, we have so far

$$f(ny) = nf(y) \quad (\forall y \in G,\ \forall n \in \mathbb{N}^0). \tag{11}$$

(3) Let $y \in G$ be arbitrary, $n \in \mathbb{Z}, n < 0$, hence $(-n) \in \mathbb{N}^0$. Then $f(ny) = f((-n)(-y)) =_{(11)}= (-n)f(-y) =_{(B4)}= (-n)(-f(y)) = nf(y)$. Together with (11), we have reached (7). □

Corollary 2.6. *If G is cyclic and $f \in S_0(G)$, then $f \in \mathrm{End}(G)$, i.e., f is additive.*

Proof. Say $G = \langle a \rangle$ for some $a \in G$. Let $x, y \in G$ be arbitrary. Then $x = ka, y = \ell a$ for suitable $k, \ell \in \mathbb{Z}$. So $f(x+y) = f(ka + \ell a) = f((k+\ell)a) =_{(7)}= (k+\ell)f(a) = kf(a) + \ell f(a) =_{(7)}= f(ka) + f(\ell a) = f(x) + f(y)$. Since $x, y \in G$ were arbitrary, $f \in \mathrm{End}(G)$. □

Example 2.7. $S(\mathbb{Z}) = \emptyset$.

In fact: assume $f \in S_0(\mathbb{Z})$. By Corollary 2.6 $f \in \mathrm{End}(\mathbb{Z})$, so by (B1′) $f \in \mathrm{Aut}(\mathbb{Z}) = \{i_{\mathbb{Z}}, -i_{\mathbb{Z}}\}$. By Lemma 2.1 $f \notin S_0(\mathbb{Z})$, contradicting the assumption. So $S_0(\mathbb{Z}) = \emptyset$, and by (4) $S(\mathbb{Z}) = \emptyset$. □

Remark 2.8. The integers

$$m_k := k^2 - k + 1 \quad (\forall k \in \mathbb{Z}) \tag{12}$$

are positive and odd and satisfy

$$m_{-k} = m_{k+1} \quad (\forall k \in \mathbb{Z}). \tag{13}$$

Lemma 2.9. $k \in \mathbb{Z} \Longrightarrow [\omega_k \in S_0(G) \Longleftrightarrow m_k G = \{0\}]$.

Proof. Let $k \in \mathbb{Z}$ be arbitrary. Since $\omega_k \in \mathrm{End}(G)$, we get from (B5) $\omega_k \in S_0(G) \Longleftrightarrow \omega_k^2 + \omega_1 = \omega_k \Longleftrightarrow \omega_{k^2-k+1} = \underline{0} \Longleftrightarrow m_k G = \{0\}$. □

Remark 2.10. For every $n \in \mathbb{N}$ we have $\mathrm{End}(\mathbb{Z}_n) = \{\omega_k : \mathbb{Z}_n \to \mathbb{Z}_n;\ k \in \mathbb{Z}\}$. But here $\omega_{n+k} = \omega_k (\forall k \in \mathbb{Z})$, so $\mathrm{End}(\mathbb{Z}_n) = \{\omega_0, \ldots, \omega_{n-1}\}$. Since $m_k \mathbb{Z}_n = \{0\} \Longleftrightarrow n | m_k (\forall k \in \mathbb{Z},\ \forall n \in \mathbb{N})$, we obtain from Lemma 2.9

$$S_0(\mathbb{Z}_n) = \{\omega_k;\ k \in \{0, \ldots, n-1\},\ n | m_k\} \quad (\forall n \in \mathbb{N}). \tag{14}$$

Corollary 2.11. *If* $n \in \mathbb{N}$ *is even, then* $S(\mathbb{Z}_n) = \emptyset$.

Proof. Since n is even and m_k odd $(\forall k \in \mathbb{Z})$, (14) implies $S_0(\mathbb{Z}_n) = \emptyset$, so by (4) $S(\mathbb{Z}_n) = \emptyset$. □

Example 2.12. It follows from (14) that $S_0(\mathbb{Z}_1) = \{\omega_0\}$ [cf. Lemma 2.1(a)], $S_0(\mathbb{Z}_3) = \{\omega_2\}$ [cf. Lemma 2.1(b)], $S_0(\mathbb{Z}_5) = \emptyset, S_0(\mathbb{Z}_7) = \{\omega_3, \omega_5\}, S_0(\mathbb{Z}_9) = \emptyset, S_0(\mathbb{Z}_{11}) = \emptyset$.

Example 2.13. $S(\mathbb{Q}) = \emptyset$.

In fact: Let $f \in S_0(\mathbb{Q})$. By (B8), $f \in \mathrm{End}(\mathbb{Q}, +)$, so by (B1′) $f \in \mathrm{Aut}(\mathbb{Q}, +)$. So there exists $c \in \mathbb{Q} \setminus \{0\}$ with $f(x) = cx (\forall x \in \mathbb{Q})$. From (B3) we get $c^2 x + x = cx (\forall x \in \mathbb{Q})$, therefore (put $x = 1$) $c^2 + 1 = c$. But $c^2 - c + 1 = (c - \frac{1}{2})^2 - \frac{1}{4} + 1 > 0 (\forall c \in \mathbb{Q})$, so c cannot exist. This means $S_0(\mathbb{Q}) = \emptyset$, and by (4) $S(\mathbb{Q}) = \emptyset$. □

Remark 2.14. For $p \in \mathbb{P}$ and $m \in \mathbb{N}, \mathbb{Z}_p^m$ is the additive group of the Galois field $\mathrm{GF}(p^m)$; if moreover p is odd, then $\omega_2 : \mathbb{Z}_p^m \to \mathbb{Z}_p^m$ is bijective, so by (B8) $S_0(\mathbb{Z}_p^m) \subset \mathrm{End}(\mathbb{Z}_p^m)$. Furthermore by (B1′) and (B5)

$$S_0(\mathbb{Z}_p^m) = \{f \in \mathrm{Aut}(\mathbb{Z}_p^m);\ f \text{ satisfies (3)}\} \quad (p \in \mathbb{P} \text{ odd},\ m \in \mathbb{N}). \tag{15}$$

We shall see in the next section that the situation for $p = 2$ is quite different.

3. The case $2G = \{0\}$

The condition $2G = \{0\}$ means that G is an elementary abelian 2-group, so (isomorphic to) a $\mathbb{Z}_2$-vector space [cf., e.g., [6], p.82, (9.14)]. In this section, where $2G = \{0\}, \dim G$ will always stand for $\dim_{\mathbb{Z}_2} G$.

Remark 3.1. For $2G = \{0\}$ and every $f : G \to G$ with $f(0) = 0$, we have

(a) f is $\mathbb{Z}$-homogeneous.
(b) f is even and odd.
(c) $2f = \underline{0} \in \mathrm{End}(G)$.

In fact: (a), (b), (c) follow from

$$ny = \begin{cases} 0 & n \in 2\mathbb{Z} \\ y & n \in 2\mathbb{Z}+1 \end{cases} \quad (\forall y \in G). \tag{16}$$

Therefore Theorem 2.5, the second half of (B4) as well as (B7) lose their power in the process of finding $S_0(G)$ in the case $2G = \{0\}$.

Lemma 3.2. *If $2G = \{0\}$ and $f \in S_0(G)$, then G is the disjoint union of $C_0 := \{0\}$ and, for $G \neq \{0\}$, of 3-cycles $C_x := \{x, f(x), x+f(x)\} (\forall x \in G\backslash\{0\})$ of f.*

Proof. Let $f \in S_0(G)$ be arbitrary. By (B1′), f is bijective. Define

$$x, y \in G;\ x \sim_f y \ :\Longleftrightarrow\ \exists k \in \mathbb{Z} \text{ with } y = f^k(x).$$

Then $\sim_f$ is an equivalence relation on G; let C_x denote the $\sim_f$-class of $x (\forall x \in G)$. By (B4) $f^3 = -i_G =_{(16)}= i_G$. So $C_x = \{x, f(x), f^2(x)\} =_{(B3)}= \{x, f(x), -x + f(x)\} =_{(16)}= \{x, f(x), x + f(x)\} (\forall x \in G)$, and this automatically becomes $\{0\}$ for $x = 0$, while for $x \in G \setminus \{0\}$, (B1′), (0), and Lemma 2.4 ensure $\mathrm{card}\, C_x = 3$. □

Lemma 3.3. *If $2G = \{0\}, f \in S_0(G)$, and*

$$H_x := C_x \cup \{0\} \quad (\forall x \in G), \tag{17}$$

then $H_0 = \{0\}, H_x \cong \mathbb{Z}_2^2$ $(\forall x \in G \setminus \{0\})$, and

$$f(H_x) = H_x \quad (\forall x \in G). \tag{18}$$

So all $H_x (x \in G)$ are f-invariant subgroups (subspaces) of G, and if $x \neq 0, H_x$ is isomorphic to the Klein four group.

Proof. (B3) and (16) imply $c = a + b$ for all pairwise distinct $a, b, c \in C_x$ for $x \neq 0$. Moreover $2a = 2b = 2c = 0$, so $H_x \cong \mathbb{Z}_2^2$ $(\forall x \in G\backslash\{0\})$ is clear, and $H_0 = \{0\}$ is trivial. As cycles of f, the $C_x (x \in G)$ satisfy $f(C_x) = C_x$, and (0) and (17) imply (18). □

Example 3.4. Let $2G = \{0\}$. Then

(a) $\dim G = 0 \Longrightarrow S_0(G) = \{i_G\}$.

(b) $\dim G = 1 \Longrightarrow S_0(G) = \emptyset$.
(c) $\dim G = 2 \Longrightarrow S_0(G) = \{f_1, f_2\}$, where f_1, f_2 are the two permutations of G with 0 as their unique fixed element, and these are additive.

Proof. (a) follows from Lemma 2.1(a). (b) We have $G \cong \mathbb{Z}_2$. By Corollary 2.11, $S_0(\mathbb{Z}_2) = \emptyset$, so by Remark 1.1, $S_0(G) = \emptyset$. (c) Let $f \in S_0(G)$. By Lemma 3.2, f must have the 1-cycle $C_0 = \{0\}$ and a unique 3-cycle, say $\{a, b, c\}$, disjoint to $\{0\}$. So, in cycle notation, $f = (0)(abc) =: f_1$ or $f = (0)(acb) =: f_2$, so $S_0(G) \subset \{f_1, f_2\}$. Moreover,

$$f_1^2 = f_2 \quad \text{and} \quad f_2^2 = f_1. \tag{19}$$

Conversely, let $f \in \{f_1, f_2\}$ and $x, y \in G$ be arbitrary.

Case 1: $x = y$. Then $f_1(x + y) = f_1(x + x) = f_1(0) = 0 = f_1(x) + f_1(x) = f_1(x) + f_1(y)$.

Case 2: $x \neq y$.

Case 2a: $x = 0,\ y \neq 0$. Then $f_1(x + y) = f_1(y) = 0 + f_1(y) = f_1(0) + f_1(y) = f_1(x) + f_1(y)$.

Case 2b: $x \neq 0,\ y \neq 0$, say $x = a, y = b$.

Then $f_1(x + y) = f_1(a + b) =_{\text{Lemma 3.3}}= f_1(c) = a =_{\text{Lemma 3.3}}= b + c = f_1(a) + f_1(b) = f_1(x) + f_1(y)$.

Thus in all three cases $f_1(x + y) = f_1(x) + f_1(y)$. Since $x, y \in G$ were arbitrary, $f_1 \in \operatorname{End}(G)$, hence $f_2 =_{(19)}= f_1^2 \in \operatorname{End}(G)$. Furthermore,

$$f_1 + f_2 = \frac{0 \quad a \qquad b \qquad c}{0\ b+c\ c+a\ a+b} = \frac{0\ a\ b\ c}{0\ a\ b\ c} = i_G, \tag{20}$$

hence $f_1^2 + i_G =_{(19)}= f_2 + i_G =_{(20)}= f_2 + (f_1 + f_2) = f_1$ and analogously $f_2^2 + i_G = f_2$. Therefore, f_1 and f_2 satisfy (3), and by (B5) $f_1, f_2 \in S_0(G)$. In total, $S_0(G) = \{f_1, f_2\}$. □

Corollary 3.5. *$2G = \{0\}, f \in S_0(G), x \in G \Longrightarrow f$ is additive on H_x.*

Proof. By (18) $f(H_x) = H_x$ and by Lemma 3.3 $\dim H_x \in \{0, 2\}$. Now the assertion follows from Example 3.4(a) and (c). □

Remark 3.6. For $G \cong \mathbb{Z}_2^2$ we got $S_0(G) \subset \operatorname{End}(G)$ in Example 3.4(c), in the absence of injectivity of $\omega_2 : G \to G$. So this latter condition is sufficient for $S_0(G) \subset \operatorname{End}(G)$ by (B8), but by no means necessary, as already noted in [1, p. 300, Remark 3].

Lemma 3.7. *Let $2G = \{0\}$. Then*
(a) *If $n \in \mathbb{N}^0, \dim G = n$, then $[S_0(G) \neq \emptyset \Longleftrightarrow n$ is even$]$.*
(b) *If $\dim G \geq \aleph_0$, then $S_0(G) \neq \emptyset$.*

Proof. (a) Clearly $G \cong \mathbb{Z}_2^n$. 1) $\Longrightarrow$: Let $S_0(G) \neq \emptyset, f \in S_0(G)$. By Lemma 3.2

$$2^n = \operatorname{card} G \equiv_3 1. \tag{21}$$

Assume that $n = 2k+1$ ($\exists k \in \mathbb{N}^0$). Then $2^n = 2^{2k+1} = 4^k \cdot 2 \equiv_3 1 \cdot 2 = 2$, a contradiction to (21). So n must be even. 2) $\Longleftarrow$: For $n = 0, S_0(G) \neq \emptyset$ by Example 3.4(a). Let $n \in \mathbb{N}$ *be* even. Then $G = \mathbb{Z}_2^2 \oplus \ldots \oplus \mathbb{Z}_2^2$ ($n/2$ direct summands) by the associativity of $\oplus$. By Example 3.4(c) $S_0(\mathbb{Z}_2^2) \neq \emptyset$, so by Lemma 2.3, $S_0(G) \neq \emptyset$.

(b) There exists an infinite set J with $G \cong (\mathbb{Z}_2)^{(J)}$ (direct sum of $\operatorname{card} J$ copies of $\mathbb{Z}_2$). If we put $J_0 := J \times \{0\}, J_1 := J \times \{1\}$, then $\operatorname{card}(J_0 \cup J_1) = \operatorname{card} J$, i.e. $G \cong (\mathbb{Z}_2)^{(J_0 \cup J_1)}$. Now the sets $I_j := \{(j,0),(j,1)\} (j \in J)$ form a partition of $J_0 \cup J_1$, so $\bigoplus_{j \in J} (\mathbb{Z}_2)^{I_j} \cong \mathbb{Z}_2^{(J_0 \cup J_1)} \cong G$. Since $\operatorname{card} I_j = 2 (\forall j \in J)$, we have $(\mathbb{Z}_2)^{I_j} \cong \mathbb{Z}_2^2 (\forall j \in J)$, so $G \cong (\mathbb{Z}_2^2)^{(J)}$. By Example 3.4(c) $S_0(\mathbb{Z}_2^2) \neq \emptyset$, so by Lemma 2.3(c) and Remark 1.1(a) $S_0(G) \neq \emptyset$.

□

Remark 3.8. Lemma 3.7 implies that $S_0(\mathbb{Z}_2^n) = \emptyset$ for odd $n \in \mathbb{N}$ and $S_0(\mathbb{Z}_2^4) \neq \emptyset$. For $n \in \mathbb{N}^0, n \leq 3$, we have seen so far (cf. Example 3.4) that $S_0(\mathbb{Z}_2^n) \subset \operatorname{End}(\mathbb{Z}_2^n)$. Does this also hold for $n = 4$?

Theorem 3.9. $S_0(\mathbb{Z}_2^4) \subset \operatorname{End}(\mathbb{Z}_2^4)$.

Proof. Let $f \in S_0(\mathbb{Z}_2^4)$ as well as $x, y \in \mathbb{Z}_2^4$ be arbitrary but fixed in the following.

Case 1. $H_x \subset H_y$ and/or $H_y \subset H_x$. Then $H_x \cup H_y$ is the larger one of H_x, H_y. By Corollary 3.5, f is additive on $H_x \cup H_y$. Since by (17) $x, y \in H_x \cup H_y$, we have

$$f(x+y) = f(x) + f(y). \tag{22}$$

Case 2. $H_x \not\subset H_y$ and $H_y \not\subset H_x$. Then $x \neq 0, y \neq 0$. By (17), $C_x = C_y$ would imply $H_x = H_y$, which is excluded in Case 2. So $C_x \neq C_y$, hence, as $\sim_f$-classes, $C_x \cap C_y = \emptyset$, and by (17) $H_x \cap H_y = \{0\}$. By Lemma 3.3, $\dim H_x = \dim H_y = 2$, hence $H_x \oplus H_y = \mathbb{Z}_2^4$. It is clear from Lemma 3.2 and $\operatorname{card} \mathbb{Z}_2^4 = 16$, that $\mathbb{Z}_2^4$ is the disjoint union of $C_0 = \{0\}$ and five 3-cycles of f.

We first prove three auxiliary statements (3.9.1), (3.9.2), (3.9.3).

(3.9.1). $x + y \neq 0, x + f(y) \neq 0, y + f(x) \neq 0$.

In fact, $x+y = 0$ would imply $x = y$, in contradiction to $C_x \cap C_y = \emptyset$. Furthermore, $x + f(y) = 0$ would lead to $f(y) = x$, i.e., to $x \sim_f y$, also contradicting $C_x \cap C_y = \emptyset$, and the third formula is obtained analogously.

(3.9.2). $C_x, C_y, C_{x+y}, C_{x+f(y)}, C_{y+f(x)}$ are pairwise disjoint.

In fact, $C_x \cap C_y = \emptyset$ was already an immediate consequence of Case 2.

Assume $C_x \cap C_{x+y} \neq \emptyset$. As $\sim_f$-classes, $C_x = C_{x+y}$, we have $x+y \in C_x \subset H_x$. But also $x \in H_x$, so $y = x+y \ + \ x \in H_x$. Since $y \neq 0$, we have by (17) $y \in C_x$, so $y \in C_x \cap C_y$, a contradiction. Therefore $C_x \cap C_{x+y} = \emptyset$ and, analogously, $C_y \cap C_{x+y} = \emptyset$.

Assume $x+f(y) \sim_f x$. Then $x+f(y) \in C_x \subset H_x$ and $f(y) = x \ + \ x+f(y) \in H_x$. Now $y \neq 0$, (0) and (B1′) require $f(y) \neq 0$, so by (17) $f(y) \in C_x$. But also $f(y) \in C_y$, a contradiction. So $C_{x+f(y)} \cap C_x = \emptyset$ and analogously $C_{y+f(x)} \cap C_y = \emptyset$.

Suppose $x \sim_f y+f(x)$. Then $y+f(x) \in C_x \subset H_x$, so $y = y+f(x) \ + \ f(x) \in H_x$, so, since $y \neq 0$, we have $y \in C_x$, i.e., $y \in C_x \cap C_y$, a contradiction. Therefore $C_x \cap C_{y+f(x)} = \emptyset$ and analogously $C_y \cap C_{x+f(y)} = \emptyset$.

Assume $x+y \sim_f y+f(x)$. Then $y+f(x) \in C_{x+y} \subset H_{x+y}$, so $x+f(x) = x+y+y+f(x) \in H_{x+y}$. Because of (0), $x \neq 0$ and Lemma 2.4 $f(x) \neq x$, so $x+f(x) \neq 0$, i.e., $x+f(x) \in C_{x+y}$, so $x+f(x) \in C_x \cap C_{x+y}$, which was already recognized above to be impossible. So $C_{x+y} \cap C_{y+f(x)} = \emptyset$ and analogously $C_{x+y} \cap C_{x+f(y)} = \emptyset$.

Finally suppose $x+f(y) \sim_f y+f(x)$. Since $x+f(y) \sim_f f(x+f(y))$, we have $f(x+f(y)) \in C_{y+f(x)} \subset H_{y+f(x)}$ and furthermore $f(x+f(y))+f(y+f(x)) \in H_{y+f(x)}$, so by (1) (remember $f \in S_0(\mathbb{Z}_2^4)$) $x+y \in H_{y+f(x)}$. By (3.9.1) $x+y \neq 0$, so $x+y \in C_{y+f(x)}$, in contradiction to $C_{x+y} \cap C_{y+f(x)} = \emptyset$, which is already established. Hence $C_{x+f(y)} \cap C_{y+f(x)} = \emptyset$.

(3.9.3). $f(x)+f(y) \in C_{x+y}$.

In fact, by (3.9.1) and (3.9.2), $C_x, C_y, C_{x+y}, C_{x+f(y)}, C_{y+f(x)}$ are the five 3-cycles of f, hence

$$\mathbb{Z}_2^4 = C_0 \dot\cup C_x \dot\cup C_y \dot\cup C_{x+y} \dot\cup C_{x+f(y)} \dot\cup C_{y+f(x)}. \tag{23}$$

$f(x)+f(y) \in C_0 = \{0\}$ would imply $f(x) = f(y)$, so by (B1) $x = y$, in contradiction to Case 2. So $f(x)+f(y) \notin C_0$.

Assume $f(x)+f(y) \in C_x$. Then $f(x)+f(y) \in H_x, f(y) = f(x)+f(x)+f(y) \in H_x$, so $f(y) \in H_x \cap H_y =_{\text{Case 2}}= \{0\}$, i.e., $f(y) = 0$, so by (B1) and (0) $y = 0$, a contradiction. Therefore $f(x)+f(y) \notin C_x$, and analogously $f(x)+f(y) \notin C_y$.

Suppose $f(x)+f(y) \in C_{x+f(y)} \subset H_{x+f(y)}$. Then $x+f(x) = x+f(y)+f(x)+f(y) \in H_{x+f(y)}$. But $x+f(x) \in_{\text{Lemma 3.2}}\in C_x$, so $x+f(x) \neq 0$, i.e., $x+f(x) \in C_x \cap C_{x+f(y)}$, a contradiction to (3.9.2). So $f(x)+f(y) \notin C_{x+f(y)}$ and analogously $f(x)+f(y) \notin C_{y+f(x)}$.

Now it follows from (23) that $f(x)+f(y) \in C_{x+y}$, and (3.9.3) is proved. (3.9.3) says by Lemma 3.2 that

$$f(x)+f(y) \in \{x+y, f(x+y), x+y+f(x+y)\}. \tag{24}$$

Case 2a. $f(x) + f(y) = x + y$. So $y + f(x) = x + f(y)$, i.e. $f(y + f(x)) = f(x + f(y))$, so by (1) $x = y$, which is excluded in Case 2. Therefore, Case 2a is impossible.

Case 2b. $f(x) + f(y) = x + y + f(x + y)$. We replace y by $y + f(x)$ in (1) and obtain $x + f(y + f(x) + f(x)) = y + f(x) + f(x + f(y + f(x)))$, i.e. $x + f(y) = y + f(x) + f(x + f(y + f(x)))$, i.e., $x + y + f(x) + f(y) = f(x + f(y + f(x)))$, so by the assumption of Case 2b $f(x + y) = f(x + f(y + f(x)))$, and by (B1) $x + y = x + f(y + f(x))$, i.e., $y = f(y + f(x))$. But $f(y + f(x)) \in C_{y+f(x)}$, so $y \in C_y \cap C_{y+f(x)}$, in contradiction to (3.9.2). Therefore, Case 2b is impossible too.

From (24) we get (22) $f(x) + f(y) = f(x + y)$. Therefore, (22) holds in the two complementary cases 1 and 2. Since $x, y \in \mathbb{Z}_2^4$ were arbitrary, we have $f \in \operatorname{End}(\mathbb{Z}_2^4)$, and Theorem 3.9 is proved. □

Remark 3.10. In (3.9.2), among other things, $x+y \not\sim_f x+f(y)$ was established. This shows that $+$ and $\sim_f$ are not compatible in the sense that $z, w, z', w' \in G$; $z \sim_f z'$, $w \sim_f w' \not\Rightarrow z + w \sim_f z' + w'$. So $\sim_f$ is not a congruence relation on the group $(G, +)$.

Corollary 3.11. *If $2G = \{0\}, f \in S_0(G)$ and M is a 4-dimensional subspace of G with $f(M) = M$, then f is additive on M.*

Proof. We have $M \cong \mathbb{Z}_2^4$, and the restriction $\tilde{f} : M \to M$ of f is available and in $S_0(M)$. By Theorem 3.9 and Remark 1.1(c) $S_0(M) \subset \operatorname{End}(M)$, so $\tilde{f} \in \operatorname{End}(M)$, i.e., f is additive on M. □

Corollary 3.12. *If $2G = \{0\}, f \in S_0(G)$, and $f(M) = M$ for all 4-dimensional subspaces of G, then $f \in \operatorname{End}(G)$.*

Proof. Let $x, y \in G$ be arbitrary. If $H_x \subset H_y$ and/or $H_y \subset H_x$, we proceed as in Case 1 of the proof of Theorem 3.9 to get (22) $f(x + y) = f(x) + f(y)$. In the opposite case, $\dim M = 4$ for $M := H_x \oplus H_y$, and by hypothesis we have $f(H_x \oplus H_y) = H_x \oplus H_y$. By Corollary 3.11, we arrive again at (22). Since $x, y \in G$ were arbitrary, $f \in \operatorname{End}(G)$. □

Remark 3.13. Corollaries 3.11 and 3.12 show the way how to possibly obtain a non-additive $f \in S_0(\mathbb{Z}_2^6)$; remember $S_0(\mathbb{Z}_2^5) = \emptyset$ by Lemma 3.7(a). The experience gained in the proof of Theorem 3.9 is that $\mathbb{Z}_2^4$ is just narrow enough for enforcing an $f \in S_0(\mathbb{Z}_2^4)$ to be additive.

Now we come to a serious contrast to Remark 3.8 and Theorem 3.9.

Example 3.14. $S_0(\mathbb{Z}_2^6) \not\subset \operatorname{End}(\mathbb{Z}_2^6)$, so the open question (B10) in Sect. 1 is answered in the negative.

Proof. 1) First we do some heuristics for finding a function in $S_0(\mathbb{Z}_2^6)\backslash\mathrm{End}(\mathbb{Z}_2^6)$. We assume that f is such a function. Non-additivity of f is manifested by the existence of two elements of $\mathbb{Z}_2^6$, say e_1 and e_3, with

$$f(e_1 + e_3) \neq f(e_1) + f(e_3). \tag{25}$$

By Lemmas 3.2 and 3.3, C_x and $H_x (x \in \mathbb{Z}_2^6)$ are available for f. It follows from (25) and Corollary 3.5 that $H_{e_1} \not\subset H_{e_3}$ and $H_{e_3} \not\subset H_{e_1}$, therefore $e_1 \neq 0, e_3 \neq 0, C_{e_1} \cap C_{e_3} = \emptyset, H_{e_1} \cap H_{e_3} = \{0\}, \dim H_{e_1} = \dim H_{e_3} = 2$ and $\dim(H_{e_1} \oplus H_{e_3}) = 4$; cf. the beginning of Case 2 in the proof of Theorem 3.9. We now put

$$e_2 := f(e_1) \in H_{e_1}, \quad e_4 := f(e_3) \in H_{e_3} \tag{26}$$

and see that $\{e_1, e_2, e_3, e_4\}$ is a basis of the subspace $H_{e_1} \oplus H_{e_3}$. Assume for a moment that $f(H_{e_1} \oplus H_{e_3}) \subset H_{e_1} \oplus H_{e_3}$. Then, since f is injective by (B1) and $H_{e_1} \oplus H_{e_3}$ is finite, we would have $f(H_{e_1} \oplus H_{e_3}) = H_{e_1} \oplus H_{e_3}$, so by Corollary 3.11, f would be additive on $H_{e_1} \oplus H_{e_3}$, in contradiction to (25). Therefore

$$f(H_{e_1} \oplus H_{e_3}) \not\subset H_{e_1} \oplus H_{e_3}. \tag{27}$$

More explicitely, we find

$$f(e_1 + e_3) \notin H_{e_1} \oplus H_{e_3}. \tag{28}$$

Namely, (0), (B1), Lemma 2.4, (25), and the pairwise disjointness of $C_{e_1}, C_{e_3}, C_{e_1+e_3}, C_{e_1+f(e_3)}, C_{e_3+f(e_1)}$ (true in analogy to (3.9.2)) prevent the equality of $f(e_1 + e_3)$ to any one of the elements of $H_{e_1} \oplus H_{e_3}$. Because $H_{e_2} =_{(26)}= H_{f(e_1)} = H_{e_1}$ and $H_{e_4} = H_{e_3}$, we get similarly

$$f(e_2 + e_4) \notin H_{e_1} \oplus H_{e_3}, \tag{29}$$

and we define

$$e_5 := f(e_1 + e_3), \quad e_6 := f(e_2 + e_4). \tag{30}$$

Furthermore, (B4), the linear independence of $\{e_1, \ldots, e_4\}$, (28), (29) lead to $e_1 + e_3 \not\sim_f e_2 + e_4$, so to $H_{e_1+e_3} \cap H_{e_2+e_4} = \{0\}$. So by (30), $\{e_5, e_6\}$ is linearly independent and $e_5, e_6 \notin H_{e_1} \oplus H_{e_3}$. Therefore, $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is a basis of $\mathbb{Z}_2^6$. By Lemma 3.2 and because of $\mathrm{card}\, \mathbb{Z}_2^6 = 64, \mathbb{Z}_2^6$ is the disjoint union of $C_0 = \{0\}$ and 21 3-cycles of f. So far some heuristic thoughts on what an $f \in S_0(\mathbb{Z}_2^6) \setminus \mathrm{End}(\mathbb{Z}_2^6)$ has to look like.

2) On the basis of part 1) of this proof and keeping in mind (3) and special cases of (1) as necessary conditions for a function to belong to $S_0(\mathbb{Z}_2^6)$, we now construct the bijective function $f_0 : \mathbb{Z}_2^6 \to \mathbb{Z}_2^6$ as follows:

2.1) Let $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ be a basis of $\mathbb{Z}_2^6$ and put

$$f_0(e_1) := e_2,\ f_0(e_3) := e_4,\ f_0(e_1 + e_3) := e_5,\ f_0(e_2 + e_4) := e_6 \tag{31}$$

[cf. (26) and (30)]. Furthermore

$$f_0(0) := 0. \tag{32}$$

No matter how we extend (31) to a function f_0 with domain $\mathbb{Z}_2^6$, this extension will certainly not be additive since

$$f_0(e_1 + e_3) = e_5 \neq e_2 + e_4 = f_0(e_1) + f_0(e_3). \tag{33}$$

We write in the following $z \mapsto w$ for $f_0(z) = w(z, w \in \mathbb{Z}_2^6)$.

(31) with the assistance of (3) generates the following four 3-cycles of f_0

$$e_1 \mapsto e_2 \mapsto e_1 + e_2 \mapsto e_1, \tag{34}$$

$$e_3 \mapsto e_4 \mapsto e_3 + e_4 \mapsto e_3, \tag{35}$$

$$e_1 + e_3 \mapsto e_5 \mapsto e_1 + e_3 + e_5 \mapsto e_1 + e_3, \tag{36}$$

$$e_2 + e_4 \mapsto e_6 \mapsto e_2 + e_4 + e_6 \mapsto e_2 + e_4. \tag{37}$$

2.2) For possibly finding the remaining seventeen 3-cycles of f_0, our procedure is to put appropriate elements of $\mathbb{Z}_2^6$ into (1) in the places of x and y with the aim to determine a new pair $(w, f_0(w))$, of course by the aid of 3-cycles already computed; we then complete the 3-cycle by means of (3). This program can in fact be realized, e.g., in the order in which the 3-cycles are listed in the following table, where the last element of every line is sent by f_0 to the first one.

x	y	resulting 3-cycle of f_0	
e_1+e_2	e_3	$e_1+e_2+e_4 \mapsto e_1+e_2+e_3+e_5 \mapsto e_3+e_4+e_5$	(38)
e_3+e_4	e_1	$e_2+e_3+e_4 \mapsto e_1+e_3+e_4+e_5 \mapsto e_1+e_2+e_5$	(39)
e_3	e_2	$e_1+e_2+e_3 \mapsto e_2+e_3+e_6 \mapsto e_1+e_6$	(40)
e_1	e_4	$e_1+e_3+e_4 \mapsto e_1+e_4+e_6 \mapsto e_3+e_6$	(41)
e_1+e_3	e_3	$e_3+e_5 \mapsto e_4+e_6 \mapsto e_3+e_4+e_5+e_6$	(42)
e_1+e_3	e_1	$e_1+e_5 \mapsto e_2+e_6 \mapsto e_1+e_2+e_5+e_6$	(43)
e_5	e_1	$e_2+e_5 \mapsto e_1+e_4+e_5+e_6 \mapsto e_1+e_2+e_4+e_6$	(44)
e_5	e_3	$e_4+e_5 \mapsto e_2+e_3+e_5+e_6 \mapsto e_2+e_3+e_4+e_6$	(45)
e_2	e_6	$e_1+e_2+e_6 \mapsto e_2+e_3+e_4+e_5 \mapsto e_1+e_3+e_4+e_5+e_6$	(46)
e_2	$e_2+e_3+e_6$	$e_1+e_3+e_6 \mapsto e_2+e_4+e_5+e_6 \mapsto e_1+e_2+e_3+e_4+e_5$	(47)
e_1+e_6	e_1	$e_2+e_3 \mapsto e_2+e_3+e_4+e_5+e_6 \mapsto e_4+e_5+e_6$	(48)
e_2+e_6	e_1+e_2	$e_5+e_6 \mapsto e_1+e_2+e_3+e_4+e_5+e_6 \mapsto e_1+e_2+e_3+e_4$	(49)
e_1	e_3	$e_1+e_4 \mapsto e_1+e_2+e_4+e_5+e_6 \mapsto e_2+e_5+e_6$	(50)
e_1+e_3	e_2+e_4	$e_2+e_4+e_5 \mapsto e_1+e_3+e_5+e_6 \mapsto e_1+e_2+e_3+e_4+e_6$	(51)
e_1+e_2	e_4+e_5	$e_1+e_4+e_5 \mapsto e_3+e_5+e_6 \mapsto e_1+e_3+e_4+e_6$	(52)
$e_2+e_5+e_6$	e_5	$e_1+e_2+e_3+e_6 \mapsto e_2+e_3+e_5 \mapsto e_1+e_5+e_6$	(53)
$e_2+e_3+e_4+e_6$	e_1	$e_3+e_4+e_6 \mapsto e_1+e_2+e_4+e_5 \mapsto e_1+e_2+e_3+e_5+e_6$	(54)

Now $f_0 : \mathbb{Z}_2^6 \to \mathbb{Z}_2^6$ is explicitly defined by (32), (34),. . ., (54), is indeed bijective since the cycles were computed form a partition of $\mathbb{Z}_2^6$ and is not additive by (33). It remains to show that $f_0 \in S_0(\mathbb{Z}_2^6)$ by inspecting the validity of (1) for f_0 and all pairs (x, y) of $\mathbb{Z}_2^6 \times \mathbb{Z}_2^6$. It causes no principal problem to do this by hand, but it was done electronically. I cordially thank Hanspeter

Bieri and Heinz Bruggesser, University of Bern, for their valuable assistance by writing and performing the corresponding computer program.

In total, $f_0 \in S_0(\mathbb{Z}_2^6) \setminus \mathrm{End}(\mathbb{Z}_2^6)$, and Example 3.14 is established. □

Corollary 3.15. *If* $2G = \{0\}$ *and* $\dim G \geq \aleph_0$ *or* $\dim G \in 2\mathbb{N}$ *and* ≥ 6, *then* $S_0(G) \not\subset \mathrm{End}(G)$.

Proof. By Lemma 3.7, $S_0(G) \neq \emptyset$ in both cases. There are $\mathbb{Z}_2$-linear subspaces M, N of G with $G = M \oplus N$ and $\dim M = 6$; therefore $M \cong \mathbb{Z}_2^6$. By Example 3.14 and Remark 1.1(c) $S_0(M) \not\subset \mathrm{End}(M)$; let $g_0 \in S_0(M) \setminus \mathrm{End}(M)$. Since $\dim N \geq \aleph_0$ or $\dim N \in 2\mathbb{N}^0$, Lemma 3.7 guarantees $S_0(N) \neq \emptyset$; choose $g_1 \in S_0(N)$. By Lemma 2.3(a), $g : G \to G$ defined by $g(x_1, x_2) := (g_0(x_1), g_1(x_2))(\forall (x_1, x_2) \in M \times N = M \oplus N = G)$ belongs to $S_0(G)$. But $g_0 \notin \mathrm{End}(M)$ implies $g \notin \mathrm{End}(G)$, so $S_0(G) \not\subset \mathrm{End}(G)$. □

References

[1] Balcerowski, M.: On the functional equation $x + f(y + f(x)) = y + f(x + f(y))$. Aequationes Math. **75**, 297–303 (2008)

[2] Brillouët-Belluot. N., Problem 15. Report of Meeting. The Thirty-eighth International Symposium on Functional Equations (Noszvaj, 2000). Aequationes Math. **61**, 304 (2001)

[3] Jarczyk, J., Jarczyk, W.: On a problem of N Brillouët-Belluot. Aequationes Math. **72**, 198–200 (2006)

[4] McShane, N.: On the periodicity of homeomorphisms of the real line. Am. Math. Mon. **68**, 562–563 (1961)

[5] Rätz, J.: On the functional equation $x + f(y + f(x)) = y + f(x + f(y))$. Report of Meeting. The Forty-eighth International Symposium on Functional Equations (Batz-sur-Mer, 2010). Aequationes Math **81**, 300 (2011)

[6] Suzuki, M.: Group theory I Grundlehren der mathematischen Wissenschaften 247. Springer, Berlin (1982)

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