

On a functional equation related to projections of abelian groups

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Dedicated in friendship to Professor János Aczél on his eightieth birthday

Summary. For an abelian group $(G, +, 0)$ we consider the functional equation

$$f : G \longrightarrow G, f(x + y + f(y)) = f(x) + 2f(y) \quad (\forall x, y \in G), \quad (1)$$

most times together with the condition $f(0) = 0$. A solution of (1) is always idempotent. Our main question is as to whether it must be additive, i.e., a projection of the abelian group G .

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1. Introduction, notation and preliminaries

The subject of this paper is the composite nonlinear functional equation (1) above. We consider it here in the case of an abelian group $(G, +, 0)$ (cf. [12], [13]). The situation for a not necessarily abelian group will be considered in a forthcoming paper. The similar looking equation

$$f : G \longrightarrow G, f(x + f(y)) = f(x) + f(y) \quad (\forall x, y \in G) \quad (1')$$

has been thoroughly investigated ([2], pp. 326–327; [5], p. 6.16, (14)).

There are, however, serious differences between the theories of (1) and (1'), respectively, e.g., in (1'), the image $f(G)$ must be a subgroup of G while this is not so in (1) (cf. Example 2.7 below). Furthermore, neither (1) nor (1') implies the other.

Our main interest is in the question as to whether (1) implies

$$f(x + y) = f(x) + f(y), \quad f[f(x)] = f(x) \quad (\forall x, y \in G), \quad (2)$$

expressing that f is an idempotent endomorphism, i.e., a projection of G . It is easily seen that (2) implies (1), which means that additivity and idempotency of f can be compressed into the single functional equation (1). It was proposed in

[3] to prove (1) \iff (2) for $G = \mathbb{R}$, which amounts to saying that (1) constitutes a characterization of the projections of the \mathbb{Q} -vector space $\mathbb{R}_{[\mathbb{Q}]}$ onto its \mathbb{Q} -linear subspaces. For abelian groups other than \mathbb{R} , the situation may be quite different: A solution of (1) need not be additive (cf. Example 2.7 or Lemma 4.7 a), d)).

For solutions f of (1) many features of the projections of G will be observed (Section 2), but here they are, to an essential extent, obtained in the possible absence of additivity of f . Section 3 deals with the cases of divisible, locally cyclic, torsion-free, and indecomposable abelian groups while section 4 treats that of torsion groups.

Notation. The symbol $:=$ means that the right-hand side defines the left-hand side, $=(\dots)$ is a short form of quotation of (\dots) , and \square marks the end of a proof. i_A denotes the identity mapping of the set A and \underline{a} the constant mapping with value a . $\mathbb{P}, \mathbb{N}, \mathbb{N}^0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ stand for the sets of prime numbers, positive integers, nonnegative integers, integers, rational and real numbers, respectively.

Throughout the paper, $(G, +, 0)$ or $(G, +)$ or G denotes an abelian group.

$\omega_2 : G \rightarrow G$, $\omega_2(x) := 2x$ ($\forall x \in G$), is an endomorphism of G , i.e., is in $\text{End}(G)$. For $x \in G$, we use $\text{ord } x$ for the order of x , and for $M \subset G$, $\langle M \rangle$ is the subgroup of G generated by M ; $\langle y \rangle := \langle \{y\} \rangle$ ($\forall y \in G$). $H \hookrightarrow G$ expresses that the group H is embeddable into G , i.e., that H is isomorphic to a subgroup of G . We use 0 for the identity element of G as well for the integer zero; it will always be clear from the context what is meant. We freely use the fact that for abelian groups A and B we have $A \times B \cong A \oplus B$, e.g., by denoting the elements of $A \oplus B$ as ordered pairs $(\alpha, \beta) \in A \times B$ whenever we find it convenient (cf. also Remark 1.1 below). For every $n \in \mathbb{N}$, we let \mathbb{Z}_n stand for the cyclic group with n elements, most times written as $\{0, \dots, n-1\}$, and $\mathbb{Z}(p^\infty)$ is the Prüfer quasicyclic group ($\forall p \in \mathbb{P}$).

Finally we put

$$\begin{aligned} (1_0) &:= (1) \wedge f(0) = 0, \\ S(G) &:= \{f : G \rightarrow G; f \text{ is a solution of (1)}\}, \\ S_0(G) &:= \{f : G \rightarrow G; f \text{ is a solution of (1)}_0\}. \end{aligned}$$

The following remarks are easily verified.

Remark 1.1. If G and H are abelian groups and $\varphi : G \rightarrow H$ is an isomorphism, $f : G \rightarrow G$, $g : H \rightarrow H$, $g = \varphi \circ f \circ \varphi^{-1}$, then

- a) $f \in S(G) \implies g \in S(H)$; $f \in S_0(G) \implies g \in S_0(H)$.
- b) $S_0(G) \subset \text{End}(G) \implies S_0(H) \subset \text{End}(H)$.

Remark 1.2. If M is a set and $f : M \rightarrow M$ is an arbitrary mapping, then $f(M) = \{y \in M; f(y) = y\}$ if and only if $f \circ f = f$.

Remark 1.3. For an abelian group $(G, +, 0)$, the following statements are equiv-

alent:

- i) $\omega_2 : G \longrightarrow G$ is injective.
- ii) G has no elements of order 2.
- iii) $x \in G \implies \text{ord } x \in \mathbb{N} \text{ odd, or } \text{ord } x = \infty$.

Remark 1.4. Two of the most significant features of abelian groups $(G, +, 0)$ are that all $\omega_n : G \longrightarrow G$, $\omega_n(x) := nx$ ($\forall x \in G, \forall n \in \mathbb{N}$) are endomorphism of G and that G is a unitary \mathbb{Z} -module in a natural way. We tacitly shall use this many times.

2. Results for arbitrary abelian groups

Almost throughout the paper we shall suppose $f \in S_0(G)$, but not in Lemmas 2.1 and 2.3 where the results also hold in the absence of $f(0) = 0$.

Lemma 2.1. *If $f \in S(G)$, then:*

- a) $f[-f(y)] = -f(y)$ ($\forall y \in G$).
- b) $z \in f(G) \implies -z \in f(G)$.
- c) $f[f(y)] = f(-y) + 2f(y)$ ($\forall y \in G$).
- d) $y \in G \implies [f[f(y)] = f(y) \iff f(-y) = -f(y)]$.
- e) $f(G) = \{y \in G; f(y) = y\}$.
- f) $f \circ f = f$, i.e., f is idempotent.
- g) f is odd.

Proof. a) For $y \in G$ arbitrary and $x := -f(y)$ we get $f(y) = f(x + y + f(y)) \stackrel{(1)}{=} f(x) + 2f(y) = f[-f(y)] + 2f(y)$, i.e., a) holds.

b) If $z \in f(G)$, there exists $y \in G$ with $z = f(y)$, so $-z = -f(y) \stackrel{(a)}{=} f[-f(y)] \in f(G)$.

c) For $y \in G$ and $x := -y$ we obtain $f[f(y)] \stackrel{(1)}{=} f(-y) + 2f(y)$.

d) \implies : $f(y) = f[f(y)] \stackrel{(c)}{=} f(-y) + 2f(y)$, so $f(-y) = -f(y)$. \iff : $f[f(y)] \stackrel{(c)}{=} f(-y) + 2f(y) = f(y)$.

e) Let $z \in f(G)$ be arbitrary. By a) $f(-z) = -z$, so $f[f(-z)] = f(-z)$, and for $y := -z$ by d) $f(-(-z)) = -f(-z)$, i.e., $f(-z) = -f(z)$ and finally $-z = f(-z) = -f(z)$, so $f(z) = z$, which establishes $f(G) \subset \{y \in G; f(y) = y\}$. The reverse inclusion is obvious, so the assertion holds.

f) follows from e) and Remark 1.2.

g) For every $y \in G$ we have by f) $f[f(y)] = f(y)$, so by d) $f(-y) = -f(y)$. \square

Lemma 2.2. *$S(G) = S_0(G)$ if and only if $\omega_2 : G \longrightarrow G$ is injective.*

Proof. If ω_2 is not injective, then by Remark 1.3 G has an element a of order 2, and then $\underline{a} \in S(G) \setminus S_0(G)$. If ω_2 is injective and $f \in S(G)$, then by Lemma 2.1 g) $f(0) = -f(0)$, $2f(0) = 0$, $f(0) = 0$, so $f \in S_0(G)$, and $S(G) = S_0(G)$ holds. \square

Lemma 2.3. $f \in S(G)$, $x, y \in G \implies [f(x) = f(y) \iff f(x - y) = f(0)]$.

Proof. \implies : Let $f(x) = f(y)$. Then $f(x - y) + 2f(y) \stackrel{(1)}{=} f((x - y) + y + f(y)) = f(x + f(y)) = f(x + f(x)) = f(0 + x + f(x)) \stackrel{(1)}{=} f(0) + 2f(x) = f(0) + 2f(y)$, so $f(x - y) = f(0)$. \impliedby : Let be $f(x - y) = f(0)$. Then $f(x) + 2f(0) \stackrel{(1)}{=} f(x + 0 + f(0)) = f(x + 0 + f(x - y)) = f(y + (x - y) + f(x - y)) \stackrel{(1)}{=} f(y) + 2f(x - y) = f(y) + 2f(0)$, therefore $f(x) = f(y)$. \square

In the Introduction we formulated our main question in the provisional form (1) $\stackrel{?}{\implies}$ (2). With respect to Lemma 2.1 f) the idempotency part of (2) is guaranteed by (1), so a more concise form of the main question reads: Does (1₀) imply additivity of f ? or, equivalently, does $S_0(G) \subset \text{End}(G)$ hold?

Lemma 2.4. *If $f \in S_0(G)$ and $K_f := f^{-1}(\{0\})$, we have:*

- a) K_f is a subgroup of G , and f is constant on every coset $x + K_f$ ($x \in G$) of K_f . (So K_f is the group of periods of f .)
- b) $z \in K_f \setminus \{0\}, w \in f(G) \implies z + w \notin f(G)$.
- c) The values of f on distinct cosets of K_f are distinct.
- d) If M is a subsemigroup of G with $K_f \subset M$ and if N is a subset of G , then $f(M \cap N) = f(M) \cap f(N)$.

Proof. a) $f \in S_0(G)$ ensures $0 \in K_f$. The constancy assertion follows from

$$x \in G, y \in K_f \implies f(x + y) = f(x + y + f(y)) \stackrel{(1)}{=} f(x) + 2f(y) = f(x). \quad (3)$$

If $y, z \in K_f$, then $f(y + z) \stackrel{(3)}{=} f(y) \stackrel{(3)}{=} f(0) = 0$, hence $y + z \in K_f$. Finally, if $y \in K_f$, then $f(-y) \stackrel{(3)}{=} f(-y + y) = f(0) = 0$, so $-y \in K_f$, and K_f is a subgroup of G .

b) By Lemma 2.1 e) $f(w) = w$. Assume that $z + w \in f(G)$, then again by Lemma 2.1 e) $f(z + w) = z + w$, so $z + w = f(z + w) \stackrel{(a)}{=} f(w) = w$, a contradiction to $z \neq 0$.

c) Let be $x, y \in G, f(x) = f(y)$. By Lemma 2.3 $f(x - y) = f(0) = 0$, i.e., $x - y \in K_f$, and so x and y belong to the same coset of K_f in G . The contrapositive version of this is our assertion.

d) $f(M \cap N) \subset f(M) \cap f(N)$ is clear. If $v \in f(M) \cap f(N)$, then there exist $x \in M, y \in N$ with $v = f(x) = f(y)$, so, as in the proof of c), $y - x \in K_f$, so $y - x \in M$, therefore $y = (y - x) + x \in M + M \subset M$. Since also $y \in N$, i.e., $y \in M \cap N$, and $v = f(y) \in f(M \cap N)$, which proves the reverse inclusion $f(M) \cap f(N) \subset f(M \cap N)$. \square

Remark 2.5. a) K_f is called the *kernel* of f although f need not be additive (Example 2.7). Despite this fact, parts a), c), d) of Lemma 2.4 perfectly imitate the behavior of a group homomorphism while part b) need not hold there.

b) For $f \in S_0(G)$, Lemma 2.4 a) opens the possibility of defining $\bar{f}: G/K_f \rightarrow G$, $\bar{f}(x + K_f) := f(x)$ ($\forall x \in G$) and considering \bar{f} instead of f in the sequel. For

the situation of the functional equation (1₀) this does not seem to be profitable for several reasons: The properties of f heavily depend on those of the image set $f(G)$, and clearly $\bar{f}(G/K_f) = f(G)$; G/K_f depends on f , which would “individualize” the framework of the equation; for returning to f , a lifting of \bar{f} would be unavoidable. So we prefer not to pursue this way. For a related situation cf. Remark 2.10 below.

Lemma 2.6. *For $f \in S_0(G)$ and $K_f := f^{-1}(\{0\})$, we have:*

- a) $f(x - f(x)) = 0 \ (\forall x \in G)$.
- b) $K_f \cap f(G) = \{0\}$, and every $x \in G$ has a unique representation $x = x_1 + x_2$ with $x_1 \in K_f, x_2 \in f(G)$. We shall abbreviate this fact from now on by $G = K_f \oplus f(G)$, even in the case that $f(G)$ is not a subgroup of G .
- c) $f(x) \in x + K_f = K_f + x \ (\forall x \in G)$.

Proof. a) For every $x \in G$, Lemma 2.1 f) ensures $f(x) = f[f(x)]$, which implies $f(x - f(x)) = f(0) = 0$ by Lemma 2.3.

b) Let $x \in G$ be arbitrary. Then by a) $x = (x - f(x)) + f(x) \in K_f + f(G)$, so $G = K_f + f(G)$. Let be $y \in K_f \cap f(G)$. $y \in K_f$ means $f(y) = 0$; $y \in f(G)$ means by Lemma 2.1 e) $f(y) = y$, so in the total $y = 0$, i.e. $K_f \cap f(G) = \{0\}$. For the uniqueness of $x = x_1 + x_2, x_1 \in K_f, x_2 \in f(G)$, the usual argument in the case of two subgroups K_f and $f(G)$ cannot be applied here because $f(G)$ need not be a subgroup of G . Assume that $x \in G, x = x_1 + x_2 = y_1 + y_2$ with $x_1, y_1 \in K_f, x_2, y_2 \in f(G)$. Then $f(x_2) =_{(3)} f(x_1 + x_2) = f(y_1 + y_2) =_{(3)} f(y_2)$, so by Lemma 2.1 e) $x_2 = y_2$ and finally also $x_1 = y_1$.

c) If $x \in G, x = x_1 + x_2, x_1 \in K_f, x_2 \in f(G)$, then $f(x) = f(x_1 + x_2) =_{(3)} f(x_2) =_{(L. 2.1 e)} x_2 \in K_f + x_2 = K_f + x_1 + x_2 = K_f + x$. □

Example 2.7. Let be $G := \mathbb{Z}_2^3 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ the elementary abelian 2-group of dimension 3, $e_1 := (1, 0, 0), e_2 := (0, 1, 0), e_3 := (0, 0, 1)$.

Let $f_0 : G \rightarrow G$ be defined by means of the following table:

x	0	e_1	e_2	e_3	$e_1 + e_2$	$e_2 + e_3$	$e_1 + e_3$	$e_1 + e_2 + e_3$
$f_0(x)$	0	0	e_2	e_3	e_2	$e_1 + e_2 + e_3$	e_3	$e_1 + e_2 + e_3$

It can be immediately checked that

$$y + f_0(y) \in \{0, e_1\} = K_{f_0} \ (\forall y \in G), \tag{4}$$

$$f_0(x + e_1) = f_0(x) \quad (\forall x \in G). \tag{5}$$

For arbitrary $x, y \in G$ we get $x + y + f_0(y) \in_{(4)} \{x, x + e_1\}$, so $f_0(x + y + f_0(y)) \in \{f_0(x), f_0(x + e_1)\} =_{(5)} \{f_0(x)\}$, i.e., $f_0(x + y + f_0(y)) = f_0(x) = f_0(x) + 2f_0(y)$, so $f_0 \in S_0(G)$, but f_0 is not additive: $f_0(e_2 + e_3) = e_1 + e_2 + e_3 \neq e_2 + e_3 = f_0(e_2) + f_0(e_3)$. Moreover, $K_{f_0} = \{0, e_1\}$ is a subgroup of G , but $f_0(G) = \{0, e_2, e_3, e_1 + e_2 + e_3\}$ is not. However, $G = K_{f_0} \oplus f_0(G)$, so, together

with (5), Lemmas 2.4 a) and 2.6 b) are confirmed. (4) and $y - f_0(y) = y + f_0(y)$ ($\forall y \in G$) ensure Lemma 2.6 a).

This example shows moreover that if G is an abelian group, A a subgroup of G , B a subset of G with $G = A \oplus B$, then B need not be a subgroup of G .

The great importance of $K_f := f^{-1}(\{0\})$ und $f(G)$ for $f \in S_0(G)$ is underlined by

Corollary 2.8. $f_1, f_2 \in S_0(G) \implies [f_1 = f_2 \iff K_{f_1} = K_{f_2} \wedge f_1(G) = f_2(G)]$.

Proof. \implies is trivial. \iff : $K := K_{f_1} = K_{f_2}$. Let be $x \in G$ arbitrary. By Lemma 2.6 b) there exist $x_1 \in K, x_2 \in f_1(G) = f_2(G)$ with $x = x_1 + x_2$, so $f_1(x) = f_1(x_1 + x_2) \stackrel{(3)}{=} f_1(x_2) \stackrel{(\text{L. 2.1 e})}{=} x_2 = f_2(x_2) = f_2(x_1 + x_2) = f_2(x)$, i.e., $f_1 = f_2$. \square

Next, a connection to the functional equation

$$\begin{aligned} f : F &\longrightarrow F; a, b, c, p, q, r \in F, abpq \neq 0, \\ f(ax + by + c) &= pf(x) + qf(y) + r \quad (\forall x, y \in F) \end{aligned} \quad (\text{DL})$$

becomes visible. The beautiful theory of (DL) has been developed by Z. Daróczy [4] and L. Losonczy [9] in their Ph.D. works supervised by J. Aczél for a field F of characteristic 0. A very modest side case of (DL) occurs in Lemma 2.9 below ($a = p = 1, b = q = 2, c = r = 0$), but $(G, +)$ need not be the additive group of a field nor will it have a priori any divisibility properties such as $(F, +)$ does in the most perfect sense: it is divisible and torsion-free.

Lemma 2.9. *If $f \in S_0(G)$, then*

$$f(x + 2y) = f(x) + 2f(y) \quad (\forall x, y \in G). \quad (6)$$

The converse is false.

Proof. 1) Let be $x, y \in G$ arbitrary. By Lemma 2.6 a) $y - f(y) \in K_f$, so $f(x + 2y) = f(x + y + f(y) + y - f(y)) \stackrel{(3)}{=} f(x + y + f(y)) \stackrel{(1)}{=} f(x) + 2f(y)$. - 2) For $G = \mathbb{Q}, f := 2i_{\mathbb{Q}}$ is a solution of (6), but not of (1_0) : it is not idempotent. This expresses the contrast between linearity of (6) and nonlinearity of (1_0) . \square

Remark 2.10. For abelian groups A, B and the linear functional equation

$$h : A \longrightarrow B, h(x + 2y) = h(x) + 2h(y) \quad (\forall x, y \in A); h(0) = 0 \quad (6'_0)$$

we state without proof some important properties of solutions h of $(6'_0)$ the analogues of which for $f \in S_0(G)$ are indicated in brackets.

- a) h is \mathbb{Z} -homogeneous and a fortiori odd, but not additive in general (Lemma 2.13 b), Example 2.7).

- b) $2h$ is additive (Lemma 2.13 i)).
 c) If $\omega_2 : A \rightarrow A$ is surjective, then h is additive (Theorem 3.2; the proof of [1], p. 66, Theorem 1, is easily adapted to the present situation).
 d) If $\omega_2 : B \rightarrow B$ is injective, then h is additive (Theorem 3.9).

Lemma 2.11. For $f \in S(G)$ we have:

- a) $i_G - f$ idempotent $\iff f(0) = 0$.
 b) $(i_G - f) \in S(G) \iff f$ is additive.

Proof. a) For every $x \in G$ we obtain $(i_G - f) \circ (i_G - f)(x) = (i_G - f)(x - f(x)) = x - f(x) - f(x - f(x)) \stackrel{(\text{L. 2.1 f}), (\text{L. 2.3})}{=} (i_G - f)(x) - f(0)$, and the assertion follows at once from this identity.

b) *Case 1:* $f(0) \neq 0$. Then f is not additive, and by a) $(i_G - f)$ is not idempotent, so by Lemma 2.1 f) $(i_G - f) \notin S(G)$, and assertion b) holds here.

Case 2: $f(0) = 0$. We put $g := i_G - f$ and get for all $x, y \in G$: $g(x+y+g(y)) = x + y + g(y) - f(x + y + g(y)) = x + y + y - f(y) - f(x + y + y - f(y)) = x + 2y - f(y) - f((x + y) + (y - f(y))) \stackrel{(\text{Case 2; L. 2.6 a})}{=} x + 2y - f(y) - f((x + y) + (y - f(y))) + f(y - f(y)) \stackrel{(\text{1})}{=} x + 2y - f(y) - f(x + y) - 2f(y - f(y)) = x + 2y - f(y) - f(x + y) = (x - f(x)) + 2(y - f(y)) - f(x + y) + f(x) + f(y)$, i.e.

$$g(x + y + g(y)) = g(x) + 2g(y) + f(x) + f(y) - f(x + y) \quad (\forall x, y \in G). \quad (7)$$

So by (7) $(i_G - f) = g \in S(G)$ if and only if f is additive. \square

Lemma 2.12. For $f \in S_0(G)$, the following statements are equivalent:

- (i) $u, v \in f(G) \implies u + v \in f(G)$.
 (ii) $f(G)$ is a subgroup of G .
 (iii) f is additive.

By Lemma 2.1 f), f then is the projection of G onto $f(G)$ along K_f , and K_f and $f(G)$ are direct summands of G .

Proof. (i) \implies (iii): Let $x, y \in G$ be arbitrary. By Lemma 2.6 b) there exist $x_1, y_1 \in K_f$ and $x_2, y_2 \in f(G)$ with $x = x_1 + x_2, y = y_1 + y_2$, so $f(x + y) = f(x_1 + y_1 + x_2 + y_2) \stackrel{(\text{3})}{=} f(x_2 + y_2)$. By (i) $x_2 + y_2 \in f(G)$, so by Lemma 2.1 e) $f(x_2 + y_2) = x_2 + y_2 = f(x_2) + f(y_2) \stackrel{(\text{3})}{=} f(x_1 + x_2) + f(y_1 + y_2) = f(x) + f(y)$, i.e., (iii) holds.

(iii) \implies (ii) \implies (i) is trivial. \square

Lemma 2.13. If $f \in S_0(G)$, then:

- a) $v \in f(G), n \in \mathbb{Z} \implies nv \in f(G)$ ($f(G)$ is \mathbb{Z} -homogeneous).
 b) $x \in G, n \in \mathbb{Z} \implies f(nx) = nf(x)$ (f is \mathbb{Z} -homogeneous; cf. Remark 2.10 a)).
 c) $u, v \in f(G); r, s \in \mathbb{Z} \implies ru + 2sv \in f(G)$.
 d) $(i_G - f)(G) = K_f$.
 e) $f(G) = (i_G - f)^{-1}(\{0\})$.

- f) $(i_G - f)(nx) = n(i_G - f)(x) \ (\forall x \in G, \forall n \in \mathbb{Z})$.
- g) $(i_G - f)(G) = \{y \in G; (i_G - f)(y) = y\}$.
- h) $x \in G, \text{ord } x = n \in \mathbb{N} \implies \text{ord } f(x) \mid n$.
- i) $k \in \mathbb{Z} \implies 2kf \in \text{End}(G)$.

Proof. a) For $S := \{n \in \mathbb{Z}; v \in f(G) \implies nv \in f(G)\}$ we show $S = \mathbb{Z}$. $0 \in S$ since $0v = 0 = f(0) \in f(G)$ for all $v \in f(G)$, and $1 \in S$ is trivial. Assume $n \in S, u \in f(G)$ arbitrary. By definition of $S, nu \in f(G)$, say $f(x) = nu$ for a suitable $x \in G$. Then $f(x + u + f(u)) \stackrel{(1)}{=} f(x) + 2f(u) \stackrel{\text{L. 2.1 e)}}{=} f(x) + 2u = nu + 2u = (n + 2)u$, i.e., $(n + 2)u \in f(G)$. As $u \in f(G)$ was arbitrary, $n + 2 \in S$. $f(x - u + f(-u)) \stackrel{(1)}{=} f(x) + 2f(-u) \stackrel{\text{L. 2.1 g)}}{=} f(x) - 2f(u) = nu - 2u = (n - 2)u$, i.e., $(n - 2)u \in f(G)$. Since $u \in f(G)$ was arbitrary, $n - 2 \in S$. Together with $0, 1 \in S$ we obtain $2\mathbb{Z} \subset S, 2\mathbb{Z} + 1 \subset S$, so $S = \mathbb{Z}$.

b) For $x \in G, n \in \mathbb{Z}, x = x_1 + x_2$ with $x_1 \in K_f, x_2 \in f(G)$ (Lemma 2.6 b) we get $f(nx) = f(n(x_1 + x_2)) = f(nx_1 + nx_2)$. Now $nx_1 \in K_f$ by Lemma 2.4 a), and $nx_2 \in f(G)$ by a), so $f(nx_1 + nx_2) \stackrel{(3)}{=} f(nx_2) \stackrel{\text{L. 2.1 e)}}{=} nx_2 = nf(x_2) \stackrel{(3)}{=} nf(x_1 + x_2) = nf(x)$.

c) There exist $x, y \in G$ with $f(x) = u, f(y) = v$. Then $f(rx + sy + f(sy)) \stackrel{(1)}{=} f(rx) + 2f(sy) \stackrel{\text{b)}}{=} rf(x) + 2sf(y) = ru + 2sv$, so $ru + 2sv \in f(G)$.

d) If $z \in (i_G - f)(G)$, then $z = (i_G - f)(x) = x - f(x)$ for some $x \in G$, and by Lemma 2.6 a) $z \in K_f$. Conversely, if $z \in K_f$, then $z - f(z) = z, (i_G - f)(z) = z, z \in (i_G - f)(G)$.

e) For every $y \in G$ we have: $y \in f(G) \stackrel{\text{L. 2.1 e)}}{\iff} f(y) = y \iff (i_G - f)(y) = 0 \iff y \in (i_G - f)^{-1}(\{0\})$.

f) $x \in G$ and $n \in \mathbb{Z}$ imply $(i_G - f)(nx) = nx - f(nx) \stackrel{\text{b)}}{=} nx - nf(x) = n(i_G - f)(x)$.

g) follows from Lemma 2.11 a) and Remark 1.2.

h) $nx = 0$ implies $0 = f(nx) \stackrel{\text{b)}}{=} nf(x)$, so $\text{ord } f(x) \in \mathbb{N}, \text{ord } f(x) \mid n$.

i) Let be $x, y \in G$ arbitrary. Then $2kf(x + y) \stackrel{\text{b)}}{=} kf(2x + 2y) \stackrel{\text{L. 2.9)}}{=} k(f(2x) + 2f(y)) \stackrel{\text{b)}}{=} k(2f(x) + 2f(y)) = 2kf(x) + 2kf(y)$. \square

Remark 2.14. The property of $f(G)$ in Lemma 2.13 c) does not imply that $f(G)$ is a subgroup of G ; it cannot because of Example 2.7 where $f(G) = \{0, e_2, e_3, e_1 + e_2 + e_3\}$.

The notion of purity in abelian group theory was introduced by H. Prüfer [11] for subgroups ("Servanzuntergruppe") and turned out to be extremely important and handy. We find it useful for our purposes to extend it to subsets rather than using it only for subgroups of our abelian group G . Purity says that the subset inherits a divisibility property from G :

Definition 2.15. A subset M of G is called *pure in G* or a *pure subset of G* if for all $y \in M, n \in \mathbb{N}, n|y$ in G implies $n|y$ in M , i.e., if the existence of $x \in G$ with $nx = y$ implies that of $x' \in M$ with $nx' = y$, i.e., if $M \cap nG \subset nM$.

Lemma 2.16. For $f \in S_0(G)$ we have:

- a) $f(G)$ is a pure subset of G .
- b) K_f is a pure subgroup of G .

Proof. a) Let be $y \in f(G)$, $n \in \mathbb{N}$, $n|y$ in G . There exists $x \in G$ with $y = nx$, so $y =_{(L. 2.1 e)} f(y) = f(nx) =_{(L. 2.13 b)} nf(x)$ with $f(x) \in f(G)$, so $f(G)$ is pure in G .

b) By Lemma 2.4 a), K_f is a subgroup of G . Let be $y \in K_f$, $n \in \mathbb{N}$, $n|y$ in G . There exists $x \in G$ with $y = nx$, and by Lemma 2.13 d), $y \in (i_G - f)(G)$. So $y =_{(L. 2.13 g)} (i_G - f)(y) = (i_G - f)(nx) =_{(L. 2.13 f)} n(i_G - f)(x)$ where $(i_G - f)(x) \in (i_G - f)(G) =_{(L. 2.13 d)} K_f$. Therefore $n|y$ in K_f , and K_f is pure in G . \square

Remark 2.17.

- a) For every abelian group G , the subgroups $\{0\}$ and G are pure in G , which is immediate from Definition 2.15.
- b) *Definition:* If $\{0\}$ and G are the only pure subgroups of G , then G is called *pure-simple*.
- c) Purity is a relative property (exactly like openness and closedness for subsets of a topological space): Every subgroup H of G is pure in itself (cf. part a)) but not necessarily pure in G . This fact is emphasized by the formulation “*pure in...*”.
- d) An abelian group G is pure-simple if and only if it is isomorphic to a subgroup of $(\mathbb{Q}, +)$ or $\mathbb{Z}(p^\infty)$ for some $p \in \mathbb{P}$ ([6], p. 119, Ex.7). So \mathbb{Z} is not pure in \mathbb{Q} , which is obvious.
- e) By Lemma 2.16, K_f and $f(G)$ are pure in G for every $f \in S_0(G)$. This is not true for an arbitrary group endomorphism: Take $G := \mathbb{Z}_4 = \{0, 1, 2, 3\}$, $f := \omega_2 : \mathbb{Z}_4 \rightarrow \mathbb{Z}_4$. Then $f \in \text{End}(G)$, $K_f = \{0, 2\} = f(G)$. From $\mathbb{Z}_4 \hookrightarrow \mathbb{Z}(2^\infty)$ it follows by d) that \mathbb{Z}_4 is pure-simple, so $\{0, 2\}$ is not pure in \mathbb{Z}_4 . This example also shows that purity in G is not preserved under endomorphisms of G .
- f) In the problem of computing $S_0(G)$, Lemma 2.16 often constitutes a helpful and efficient instrument, e.g. in the following Theorem 2.18.
- g) It is clear that always $\{0, i_G\} \subset S_0(G)$.

Theorem 2.18. If G is pure-simple, then $S_0(G) = \{0, i_G\}$. The converse is not true (cf. Remark 3.15 a)).

Proof. By Lemma 2.16, K_f and $f(G)$ are pure in G for all $f \in S_0(G)$. Therefore the only possible cases are, by Lemma 2.6 b) and Def. 2.17 b), $K_f = \{0\}$, $f(G) = G$ or $K_f = G$, $f(G) = \{0\}$. In the first case $f = i_G$ (by Lemma 2.1 e)), and in the second case $f = 0$. Together with Remark 2.17 g) the assertion follows. \square

Theorem 2.18 already deals with a special class of abelian groups (pure-simple). This kind of results is the subject of the next section. We finish this section with a simple and useful tool for the case that, for $f \in S_0(G)$, the subgroup K_f (Lemma 2.4 a)) has finite index in G .

Lemma 2.19. *If $f \in S_0(G)$ and K_f has finite index k in G , then:*

- a) $v \in f(G) \implies \text{ord } v \leq k$.
- b) *If there exists $v \in f(G)$ with $\text{ord } v = k$, then f is additive.*
- c) $k = 2$ implies additivity of f .

Proof. By definition, $k = \text{card}(G/K_f)$, by Lemma 2.4 a) $\text{card } f(G) \leq k$, and by Lemma 2.4 c) $k \leq \text{card } f(G)$, hence $\text{card } f(G) = k$.

- a) By Lemma 2.13 a) $\langle v \rangle \subset f(G)$, so $\text{ord } v = \text{card } \langle v \rangle \leq \text{card } f(G) = k$.
- b) $\text{ord } v = k$ means $\langle v \rangle = f(G)$, and by Lemma 2.12, f is additive.
- c) By a) $\text{ord } v \leq k = 2$ for every $v \in f(G)$. Since $\text{card } f(G) = k = 2$, there must exist $v \in f(G)$ with $\text{ord } v = 2$, and additivity of f follows from b). \square

Remark 2.20.

- a) In Example 2.7, $k = \text{card } f_0(\mathbb{Z}_2^3) = 4$ but $\text{ord } v \leq 2 (\forall v \in \mathbb{Z}_2^3)$. Hence Lemma 2.19 a) is verified while the hypothesis of Lemma 2.19 b) is not satisfied, and accordingly f_0 is not additive.
- b) Lemma 2.19 c) can be read as follows: If $f \in S_0(G)$ exists and if $\text{card } f(G) = k = 2$, then f must be additive. But $k = 2$ alone implies by no means the existence of an $f \in S_0(G)$ with $\text{card } f(G) = 2$: $G := \mathbb{Z}_4$ is pure-simple (cf. Remark 2.17 e)), so by Theorem 2.18, $S_0(G)$ cannot contain an element f with $\text{card } f(G) = 2$.

3. Results for special classes of abelian groups

Definition 3.1.

- a) If $p \in \mathbb{P}$, G is called *p-divisible* iff $G = pG$. (Of course every divisible G is *p-divisible*.)
- b) G is called *uniquely p-divisible* iff $\omega_p : G \rightarrow G$ is bijective.
- c) A subset M of G is called *p-divisible* iff $\omega_p(M) \subset M$ and the restriction $\tilde{\omega}_p : M \rightarrow M$ of ω_p is surjective.

Theorem 3.2. *If G is 2-divisible, then $S_0(G) \subset \text{End}(G)$ (cf. Remark 2.10 c)).*

Proof. Let be $f \in S_0(G)$ and $x, y \in G$ arbitrary. Since $G = 2G$, there exist $v, w \in G$ with $2v = x$, $2w = y$. It follows that $f(x + y) = f(2v + 2w) = f(2(v+w)) \stackrel{(\text{L. 2.13 b))}}{=} 2f(v+w) \stackrel{(\text{L. 2.13 i))}}{=} 2f(v) + 2f(w) = f(2v) + f(2w) = f(x) + f(y)$. Since x, y were arbitrary, we have $f \in \text{End}(G)$. \square

In favor of a possibly higher flexibility in the applications, we state an “individualized” version of Theorem 3.2:

Theorem 3.3. *If $f \in S_0(G)$ and $f(G)$ is 2-divisible, then f is additive.*

Proof. We first note that by Lemma 2.13 a), $\omega_2[f(G)] \subset f(G)$ automatically holds. Let $u, v \in f(G)$ be arbitrary. By Definition 3.1c) there exist $u', v' \in f(G)$ with $2u' = u, 2v' = v$, so there exist $x, y \in G$ with $f(x) = u', f(y) = v'$. Therefore $u+v = 2u'+2v' = 2f(x)+2f(y) \stackrel{(\text{L. 2.13 i})}{=} 2f(x+y) \stackrel{(\text{L. 2.13 b})}{=} f(2(x+y)) \in f(G)$, so $u+v \in f(G)$. As $u, v \in f(G)$ were arbitrary, additivity of f follows from Lemma 2.12. \square

Next we look at locally cyclic groups G .

Definition 3.4. G is called *locally cyclic* iff every finitely generated subgroup of G is cyclic. (Clearly such a group is automatically abelian, and every cyclic group is locally cyclic.)

Remark 3.5. G is locally cyclic if and only if one of the following pairwise equivalent conditions holds:

- (i) G is a direct limit of cyclic groups ([6], p. 58, Exercise 1 b)).
- (ii) G is isomorphic to a subgroup of $(\mathbb{Q}, +)$ or $(\mathbb{Q}/\mathbb{Z}, +)$ ([14], p. 194, Theorem 1).
- (iii) The lattice of subgroups of G is distributive (Theorem of Ore, [10], p. 267, Theorem 4).

By the way, (ii), $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)$, and Remark 2.17 d) imply that every pure-simple G is locally cyclic, so that we may expect some extension of Theorem 2.18.

Theorem 3.6. *If G is locally cyclic, then $S_0(G) \subset \text{End}(G)$, but in general $S_0(G) \neq \{\underline{0}, i_G\}$.*

Proof. 1) Let be $f \in S_0(G)$ and $x, y \in G$ arbitrary. By assumption, there exists $d \in G$ with $\langle \{x, y\} \rangle = \langle d \rangle$, so $x = rd, y = sd$ for some $r, s \in \mathbb{Z}$, and $f(x+y) = f(rd+sd) = f((r+s)d) \stackrel{(\text{L. 2.13 b})}{=} (r+s)f(d) = rf(d)+sf(d) = f(rd)+f(sd) = f(x) + f(y)$. Since x, y were arbitrary, $f \in \text{End}(G)$, i.e., $S_0(G) \subset \text{End}(G)$.

2) $G := \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ is locally cyclic, and $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$. The subgroups of \mathbb{Z}_6 , namely $\{0\}, \{0, 3\}, \{0, 2, 4\}, \mathbb{Z}_6$, form a distributive lattice (cf. Remark 3.5 (iii)). All of them are pure in \mathbb{Z}_6 , as direct summands of \mathbb{Z}_6 ([7], p. 14 (h)). By part 1) of this proof, $S_0(\mathbb{Z}_6) \subset \text{End}(\mathbb{Z}_6)$. It is well known that $\text{End}(\mathbb{Z}_6) = \{\omega_k; k \in \{0, 1, 2, 3, 4, 5\}\}$, and precisely $\omega_0 = \underline{0}$, $\omega_1 = i_{\mathbb{Z}_6}$, ω_3 and ω_4 are idempotent, hence by Lemma 2.1 f) and by (2) \implies (1) we have $S_0(\mathbb{Z}_6) = \{\omega_0, \omega_1, \omega_3, \omega_4\}$,

i.e., $S_0(\mathbb{Z}_6)$ consists of the projections of \mathbb{Z}_6 onto its four subgroups along the respective complementary subgroup, and $S_0(G) = \{\underline{0}, i_G\}$ is violated. \square

The next aspect we are going to consider is the non-existence of elements of order 2 in G , i.e., the case where $\omega_2 : G \rightarrow G$ is injective (cf. Remark 1.3).

Definition 3.7. G is called *2-torsion-free* (sometimes *2-cancellable*) if $\omega_2 : G \rightarrow G$ is injective. (Notice that then $S(G) = S_0(G)$ by Lemma 2.2.) Evidently every torsion-free G and a fortiori every free abelian G is 2-torsion-free.

Lemma 3.8. *If G is arbitrary, if $f \in S_0(G)$ and if*

$$w \in G, 2w \in f(G) \implies w \in f(G), \quad (8)$$

then f is additive.

Proof. Let be $u, v \in f(G)$ arbitrary. Then by Lemma 2.13 c) with $r = 2, s = 1$, we get $2(u + v) = 2u + 2v \in f(G)$, so by (8) $u + v \in f(G)$, and by Lemma 2.12 f is additive. \square

Theorem 3.9. *G 2-torsion-free implies $S_0(G) \subset \text{End}(G)$ (cf. Remark 2.10 d)).*

Proof. Assume $f \in S_0(G)$ and $w \in G, 2w \in f(G)$. By Lemma 2.1 e) $2w = f(2w)$, so by Lemma 2.13 b) $2w = 2f(w)$. Injectivity of ω_2 yields $w = f(w) \in f(G)$. This means that (8) holds, and by Lemma 3.8 f is additive, i.e. $S_0(G) \subset \text{End}(G)$. \square

As with Theorems 3.2 and 3.3, we present an individualized and sharpened version of Theorem 3.9:

Theorem 3.10. *If $f \in S_0(G)$ and if K_f contains no element of order 2, then f is additive.*

Proof. Let be $w \in G, 2w \in f(G)$. Then $2w =_{(\text{L.2.1 e})} f(2w) =_{(\text{L. 2.13 b})} 2f(w)$, hence $2(w - f(w)) = 0$. By Lemma 2.6 a) $w - f(w) \in K_f$, and the assumption on K_f leads to $w - f(w) = 0, w = f(w) \in f(G)$, i.e., (8) holds, and the assertion follows from Lemma 3.8. \square

Remark 3.11. (8) need not hold when $f(G)$, instead of K_f , has no element of order 2: $G := \mathbb{Z}_2 = \{0, 1\}, f = \underline{0}, w := 1$. This shows that (8) is sufficient but not necessary for f to be additive.

We terminate this section by considering indecomposability of G .

Definition 3.12. G is said to be (*directly*) *indecomposable* if for any subgroups A, B of $G, G = A \oplus B$ implies $A = \{0\}$ or $B = \{0\}$.

Remark 3.13. If G is indecomposable, then G cannot be a mixed group, i.e., it must be a torsion group or torsion-free ([7], p. 22, Theorem 10). We establish now a connection to our functional equation (1_0) :

Theorem 3.14. *For the conditions*

- (i) G is pure-simple,
- (ii) G is indecomposable,
- (iii) $S_0(G) = \{\underline{0}, i_G\}$,

we have (i) $\not\Leftarrow$ (ii) \Leftarrow (iii).

Proof. (i) \implies (ii): If there were subgroups $A, B \neq \{0\}$ with $G = A \oplus B$, then A, B would be pure in G ([7], p. 14 (h)), a contradiction to (i).

(ii) $\not\Leftarrow$ (i): Let for some $p \in \mathbb{P}$ the group G be the additive group $(J_p, +)$ of p -adic integers. By a theorem of R. Baer, J_p is indecomposable ([7], p. 46, Theorem 18). If J_p were pure-simple, it would be embeddable into a countable group by Remark 2.17 d), contradicting the fact that J_p is uncountable. Therefore (i) is violated.

(ii) \implies (iii): *Case 1:* G is not torsion-free. By Remark 3.13, G is a torsion group, and by [7], p. 22, Theorem 10, it is embeddable into $\mathbb{Z}(p^\infty)$ for some $p \in \mathbb{P}$, so pure-simple by Remark 2.17 d). Now (iii) follows from Theorem 2.18.

Case 2: G is torsion-free. By Theorem 3.9, $S_0(G) \subset \text{End}(G)$. Let be $f \in S_0(G)$ arbitrary. Then $f(G)$ is a subgroup of G , and by Lemma 2.6 b) $G = K_f \oplus f(G)$, which is here a decomposition into subgroups. By (ii) $f(G) = \{0\}$ or $f(G) = G$, i.e., $f = \underline{0}$ or, by Lemma 2.1 e), $f = i_G$, and (iii) holds again.

(iii) \implies (ii): If there were subgroups $A, B \neq \{0\}$ of G with $G = A \oplus B$, then the projection f_1 of G onto A along B would be additive and idempotent, hence in $S_0(G)$, a contradiction to (iii). \square

Remark 3.15.

- a) Theorem 3.14 (iii) $\not\Leftarrow$ (i) shows that the converse of Theorem 2.18 is not true.
- b) The group J_p of p -adic integers is not pure-simple. Therefore it must possess a pure subgroup H with $\{0\} \neq H \neq J_p$. Since $S_0(J_p) = \{\underline{0}, i_{J_p}\}$ by Theorem 3.14, H cannot be K_f or $f(G)$ for an f in $S_0(J_p)$. This shows that Lemma 2.16 a), b) have no converses.

4. Results for torsion groups

So far the main question as to whether $S_0(G) \subset \text{End}(G)$ has been answered for several classes of abelian groups. The answer is affirmative for G torsion-free (Theorem 3.9) and a fortiori for the case $G = \mathbb{R}$ mentioned in the Introduction.

We shall now focus on the case of a torsion group G and begin by some auxiliary results.

Definition 4.1. If G is an abelian group, H a subgroup of G , $x \in H$, $p \in \mathbb{P}$, then $h_{p,H}(x) := \sup\{k \in \mathbb{N}^0; p^k | x \text{ in } H\}$ (∞ allowed) is called the p -height of x relative to the subgroup H .

It is clear from Definitions 2.15 and 4.1 that for a pure subgroup H of G and $x \in H$ we have $h_{p,H}(x) = h_{p,G}(x)$, in which case we merely write $h_p(x)$.

Lemma 4.2. ([7], p. 21, Lemma 8.) *If $p \in \mathbb{P}$ and if G is a p -group such that $x \in G$ and $\text{ord } x = p$ imply $h_p(x) = \infty$, then G is divisible. (For arbitrary abelian groups G this is not true: consider \mathbb{Z} or $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}$.)*

The next lemma is one of the splitting theorems of Kulikov.

Lemma 4.3. ([8], p. 180, Lemma.) *If $p \in \mathbb{P}$, if G is a p -group and if $x \in G$, $\text{ord } x = p$, $h_p(x) = k \in \mathbb{N}^0$, then there exists $y \in G$ with $\text{ord } y = p^{k+1}$, $x \in \langle y \rangle \cong \mathbb{Z}_{p^{k+1}}$ and $G = \langle y \rangle \oplus C$ for an appropriate subgroup C of G .*

Theorem 4.4. *For $k \in \mathbb{N}$, H a 2-divisible subgroup of G , $G = \mathbb{Z}_{2^k} \oplus H$ we have $S_0(G) \subset \text{End}(G)$.*

Proof. We use $G \cong \mathbb{Z}_{2^k} \times H$ and write the elements of G as ordered pairs (cf. Remark 1.1 and the Notation).

- 1) By hypothesis $2^k H = H$, so $2^k(\{0\} \times H) = \{0\} \times H$ for the given k . Let be $f \in S_0(G)$ arbitrary. By Lemma 2.13 b), f is \mathbb{Z} -homogeneous, therefore

$$f(\{0\} \times H) = f(2^k(\{0\} \times H)) = 2^k f(\{0\} \times H). \quad (9)$$

Let be $(\eta_1, \eta_2) \in f(\{0\} \times H)$ arbitrary. Then by (9) $(\eta_1, \eta_2) \in 2^k f(\{0\} \times H)$, so there exists $(\zeta_1, \zeta_2) \in f(\{0\} \times H)$ with $(\eta_1, \eta_2) = 2^k(\zeta_1, \zeta_2) = (2^k \zeta_1, 2^k \zeta_2) = (0, 2^k \zeta_2) \in \{0\} \times H$, so $f(\{0\} \times H) \subset \{0\} \times H$. This guarantees that the restriction $f_2 : \{0\} \times H \rightarrow \{0\} \times H$ of f exists, and clearly $f_2 \in S_0(\{0\} \times H)$, so by Theorem 3.2 $f_2 \in \text{End}(\{0\} \times H)$, i.e.,

$$f \in S_0(G) \implies \text{The restriction } f_2 : \{0\} \times H \rightarrow \{0\} \times H \text{ of } f \text{ is additive.} \quad (10)$$

- 2) If ξ_0 is a generator of \mathbb{Z}_{2^k} , the general element of G has the form $(l\xi_0, \xi)$ ($l \in \mathbb{Z}$, $\xi \in H$). $2H = H$ ensures the existence of $\xi' \in H$ with $2\xi' = \xi$. Let now again be $f \in S_0(G)$, $l \in \mathbb{Z}$, $\xi \in H$ arbitrary. $f(l\xi_0, \xi) = f((l\xi_0, 0) + (0, \xi)) = f((l\xi_0, 0) + 2(0, \xi')) \stackrel{(6)}{=} f(l\xi_0, 0) + 2f(0, \xi') \stackrel{(L. 2.13 b)}{=} lf(\xi_0, 0) + f(0, \xi)$, i.e.,

$$f(l\xi_0, \xi) = lf(\xi_0, 0) + f_2(0, \xi) \quad (\forall f \in S_0(G), \forall l \in \mathbb{Z}, \forall \xi \in H). \quad (11)$$

3) Finally let be $f \in S_0(G)$; $l, m \in \mathbb{Z}$; $\xi, \eta \in H$ arbitrary. Then $f((l\xi_0, \xi) + (m\xi_0, \eta)) = f((l+m)\xi_0, \xi + \eta) \stackrel{(11)}{=} (l+m)f(\xi_0, 0) + f_2(0, \xi + \eta) \stackrel{(10)}{=} (l+m)f(\xi_0, 0) + f_2(0, \xi) + f_2(0, \eta) \stackrel{(11)}{=} f(l\xi_0, \xi) + f(m\xi_0, \eta)$, so $f \in \text{End}(G)$. \square

Remark 4.5. Caution: For $f \in S_0(\mathbb{Z}_{2^k} \oplus H)$, the inclusion $f(\mathbb{Z}_{2^k} \times \{0\}) \subset \mathbb{Z}_{2^k} \times \{0\}$ need not hold: Put $H = \mathbb{Z}(2^\infty)$, $\eta_0 \in \mathbb{Z}(2^\infty)$, $\text{ord } \eta_0 = 2^k$, $f(l\xi_0, \xi) := (0, l\eta_0 + \xi)$ ($\forall l \in \mathbb{Z}, \forall \xi \in \mathbb{Z}(2^\infty)$). This contrast to the situation in the foregoing proof is mainly due to the fact that H may contain an isomorphic copy of \mathbb{Z}_{2^k} (as $\mathbb{Z}(2^\infty)$ does). It can be shown however, that if $\omega_2 : H \rightarrow H$ is bijective, then $f(\mathbb{Z}_{2^k} \times \{0\}) \subset \mathbb{Z}_{2^k} \times \{0\}$ does hold.

Theorem 4.6. For $k, l \in \mathbb{N}$ $G := \mathbb{Z}_{2^l} \oplus \mathbb{Z}_{2^k}$ we have $S_0(G) \subset \text{End}(G)$.

Proof. Again we use $G = \mathbb{Z}_{2^l} \times \mathbb{Z}_{2^k}$ in the sense of our notational convention. Without loss of generality we assume $l \leq k$.

1) We prove first

$$\begin{aligned} M \subset G \quad \mathbb{Z}\text{-homogeneous and pure in } G, \\ \emptyset \neq M \neq \{(0, 0)\} \implies \exists v \in M \text{ with } \text{ord } v = 2^l. \end{aligned} \tag{12}$$

In fact: Let be $(\xi', \eta') \in M \setminus \{(0, 0)\}$, say $\text{ord } (\xi', \eta') = 2^m$ where $1 \leq m \leq k$; notice that $\max\{\text{ord } x; x \in G\} = 2^k$. $(\xi, \eta) := 2^{m-1}(\xi', \eta') \in M$ since M is \mathbb{Z} -homogeneous, and $\text{ord } (\xi, \eta) = 2$. For a moment, we find the “ \mathbb{Q}/\mathbb{Z} notation” optimal: $\mathbb{Z}_{2^l} = \{0, 1/2^l, \dots, (2^l - 1)/2^l\}$, $\mathbb{Z}_{2^k} = \{0, 1/2^k, \dots, (2^k - 1)/2^k\}$. Now (ξ, η) must be $(1/2, 0)$ or $(0, 1/2)$ or $(1/2, 1/2)$, and we define $(\gamma, \delta) := (1/2^l, 0)$, $(0, 1/2^l)$ or $(1/2^l, 1/2^l)$ in the preceding three cases for (ξ, η) , respectively. Then in either case $(\xi, \eta) = 2^{l-1}(\gamma, \delta)$, i.e., $2^{l-1} | (\xi, \eta)$ in G . Since M is pure in G , we get $2^{l-1} | (\xi, \eta)$ in M , i.e., there exists $(\alpha, \beta) \in M$ with $2^{l-1}(\alpha, \beta) = (\xi, \eta) \neq (0, 0)$. But $2^l(\alpha, \beta) = 2(\xi, \eta) = (0, 0)$, so $\text{ord } (\alpha, \beta) = 2^l$, and (12) holds with $v := (\alpha, \beta)$.

2) Let be $f \in S_0(G)$ arbitrary. If $f = (0, 0)$ or $f = i_G$, then $f \in \text{End}(G)$. So let be $f \notin \{(0, 0), i_G\}$. Then K_f and $f(G)$ are nonempty \mathbb{Z} -homogeneous pure subsets of G (Lemmas 2.4 a), 2.13 a), 2.16) different from $\{(0, 0)\}$. By (12)

$$\text{both } K_f \text{ and } f(G) \text{ contain an element of order } 2^l. \tag{13}$$

$2^{l+k} = \text{card } G \stackrel{(\text{L. 2.6 b))}}{=} \text{card } (K_f \oplus f(G)) = \text{card } K_f \cdot \text{card } f(G)$, therefore $\text{card } K_f | 2^{k+l}$, $\text{card } f(G) | 2^{k+l}$. If neither K_f nor $f(G)$ contained an element of order 2^k , then $\text{ord } u | 2^{k-1}$ ($\forall u \in K_f$) and $\text{ord } v | 2^{k-1}$ ($\forall v \in f(G)$), so $\text{ord } (u + v) | \text{lcm}(\text{ord } u, \text{ord } v) | 2^{k-1}$, i.e., $\text{ord } y | 2^{k-1}$ ($\forall y \in G$), in contradiction to $\max\{\text{ord } x; x \in G\} = 2^k$. So

$$\text{at least one of } K_f, f(G) \text{ contains an element of order } 2^k. \tag{14}$$

For facilitating the formulations, we put $\{M_1, M_2\} := \{K_f, f(G)\}$. Without loss of generality, let be $v_1 \in M_1$ with $\text{ord } v_1 = 2^k$ and $v_2 \in M_2$ with $\text{ord } v_2 = 2^l$

(see(14) and (13)). \mathbb{Z} -homogeneity of M_1, M_2 implies $\langle v_1 \rangle \subset M_1, \langle v_2 \rangle \subset M_2$, so

$$\text{card } M_1 \geq 2^k \quad \text{and} \quad \text{card } M_2 \geq 2^l. \quad (15)$$

Now $\text{card } M_1 \cdot \text{card } M_2 = \text{card } G = 2^{l+k}$, which makes strict inequality in (15) impossible, so $M_1 = \langle v_1 \rangle, M_2 = \langle v_2 \rangle$, and $f(G)$ must be a subgroup of G . By Lemma 2.12, $f \in \text{End}(G)$. \square

Next we extend Example 2.7.

Lemma 4.7. *If A, B, C are abelian groups, $G := A \oplus B \oplus C, D := A \oplus 2B \oplus 2C, \alpha_0 \in A, \text{ord } \alpha_0 = 2$, and if we define*

$$f : G \longrightarrow G, \quad f(\alpha, \beta, \gamma) := \begin{cases} (0, \beta, \gamma) & \text{if } x = (\alpha, \beta, \gamma) \in D \\ (\alpha_0, \beta, \gamma) & \text{if } x = (\alpha, \beta, \gamma) \in G \setminus D \end{cases} \quad (16)$$

then we have

- a) $f \in S_0(G)$.
- b) $K_f = A \oplus \{0\} \oplus \{0\}$.
- c) If $2B = B$ and $2C = C$, then $f \in \text{End}(G)$.
- d) If $2B \neq B$ and $2C \neq C$, then $f \notin \text{End}(G)$.

Proof. a) It follows at once from (16) that

$$y + f(y) \in D \quad \forall y \in G. \quad (17)$$

Let be $x, y \in G$. If $x \in D, y \in G$, then by (17) $x+y+f(y) \in D$. If $x+y+f(y) \in D$, then $x = (x+y+f(y) - (y+f(y))) \in D$ by (17), so

$$x, y \in G \implies [x \in D \iff x+y+f(y) \in D]. \quad (18)$$

Inspecting the four cases for $x, y \in G$ to belong or not to belong to D yields, with the help of (16) and (18), $f \in S_0(G)$.

b) follows immediately from (16).

c) We have $D = G$, and by (16) f is the projection of G onto $\{0\} \oplus B \oplus C$ along K_f , so $f \in \text{End}(G)$.

d) Let be $\beta_0 \in B \setminus 2B, \gamma_0 \in C \setminus 2C$. Then $(\alpha_0, \beta_0, 0), (\alpha_0, 0, \gamma_0), (0, \beta_0, \gamma_0) \in G \setminus D$, and $f(\alpha_0, \beta_0, 0) + f(\alpha_0, 0, \gamma_0) \stackrel{(16)}{=} (\alpha_0, \beta_0, 0) + (\alpha_0, 0, \gamma_0) = (\alpha_0, \beta_0, \gamma_0)$ while $f((\alpha_0, \beta_0, 0) + (\alpha_0, 0, \gamma_0)) = f(\alpha_0, \beta_0, \gamma_0) \stackrel{(16)}{=} (\alpha_0, \beta_0, \gamma_0)$, so $f \notin \text{End}(G)$. \square

The main results on $S_0(G)$ for abelian torsion groups G is based upon the corresponding situation over their 2-components. So our next result concerns 2-groups where of course divisibility and 2-divisibility are equivalent.

Theorem 4.8. *For an abelian 2-group G_2 , the following statements are equivalent:*

- (i) $S_0(G_2) \subset \text{End}(G_2)$.
- (ii) G_2 has the form (ii₁) or (ii₂) or (ii₃) where

- (ii₁) G_2 is 2-divisible,
(ii₂) $G_2 \cong \mathbb{Z}_{2^k} \oplus C_1, k \in \mathbb{N}, C_1$ a 2-divisible abelian group,
(ii₃) $G_2 \cong \mathbb{Z}_{2^l} \oplus \mathbb{Z}_{2^k}$ for some $k, l \in \mathbb{N}$.

Proof. (ii) \implies (i): (ii₁) \implies (i) by Theorem 3.2, (ii₂) \implies (i) by Theorem 4.4, and (ii₃) \implies (i) by Theorem 4.6.

(i) \implies (ii). We prove the contrapositive $\neg(\text{ii}) \implies \neg(i)$, and $\neg(\text{ii})$ means $\neg(\text{ii}_1) \wedge \neg(\text{ii}_2) \wedge \neg(\text{ii}_3)$. $\neg(\text{ii}_1)$ says that G_2 is not (2-)divisible. By Lemma 4.2 there exists $x \in G_2$ with $\text{ord } x = 2$ and $h_2(x) =: k_1 \in \mathbb{N}^0$. By Lemma 4.3

$$G_2 \cong \mathbb{Z}_{2^{k_1+1}} \oplus C_1 \quad (19)$$

for a suitable subgroup C_1 of G_2 , uniquely determined up to isomorphism. Because of $\neg(\text{ii}_2)$, C_1 cannot be (2-)divisible. By Lemma 4.2 there exists $y \in C_1$ with $\text{ord } y = 2$ and $h_{2,C_1}(y) =: k_2 \in \mathbb{N}^0$. C_1 is a direct summand of G_2 , so purity of C_1 in G_2 is guaranteed, and 2-height of y in C_1 is the same as the one in G_2 . By Lemma 4.3 $C_1 \cong \mathbb{Z}_{2^{k_2+1}} \oplus C_2$ for a suitable subgroup C_2 of C_1 , therefore also of G_2 . Together with (19) and the associativity of \oplus we obtain

$$G_2 \cong \mathbb{Z}_{2^{k_1+1}} \oplus \mathbb{Z}_{2^{k_2+1}} \oplus C_2. \quad (20)$$

$\neg(\text{ii}_3)$ now requires that $C_2 \neq \{0\}$. As a subgroup of G_2 , C_2 is a 2-group and furthermore contains an element z of order 2. The first two summands in (20) are not 2-divisible, and from (20) and Lemma 4.7 a), d) $S_0(G_2) \setminus \text{End}(G_2) \neq \emptyset$, i.e., $\neg(i)$ holds. \square

Our main result on $S_0(G)$ for torsion groups G now reads as follows:

Theorem 4.9. *For an abelian torsion group G and its 2-component G_2 , the following statements are equivalent:*

- (a) $S_0(G) \subset \text{End}(G)$,
(b) $S_0(G_2) \subset \text{End}(G_2)$,
(c) G_2 has the form (ii₁) or (ii₂) or (ii₃) of Theorem 4.8 (ii).

Proof. (b) \iff (c) is Theorem 4.8.

(a) \implies (b): Let be $f_2 \in S_0(G_2)$ arbitrary. For the p -components G_p of G ($p \in \mathbb{P}, p \neq 2$) we put $f_p : G_p \longrightarrow G_p, f_p = \underline{0}$. Since $G = \bigoplus_{p \in \mathbb{P}} G_p$, every $x \in G$ has the form $x = \sum_{p \in \mathbb{P}} x_p$ with uniquely determined summands $x_p \in G_p$ ($p \in \mathbb{P}$), all but finitely many being zero. We define

$$f : G \longrightarrow G, f(x) := \sum_{p \in \mathbb{P}} f_p(x_p) = f_2(x_2) \quad (\forall x = \sum_{p \in \mathbb{P}} x_p \in G). \quad (21)$$

Clearly $f(0) = 0$, and f is well-defined by (21). Let be $x, y \in G$ arbitrary. Then $f(x+y+f(y)) = f(\sum x_p + \sum y_p + f(\sum y_p)) \stackrel{(21)}{=} f(\sum x_p + \sum y_p + \sum f_p(y_p)) = f(\sum(x_p + y_p + f_p(y_p))) \stackrel{(21)}{=} f_2(x_2 + y_2 + f_2(y_2)) \stackrel{(1)}{=} f_2(x_2) + 2f_2(y_2) \stackrel{(21)}{=} f(x) + 2f(y)$

$f(x) + 2f(y)$, i.e., $f \in S_0(G)$, and by (a) $f \in \text{End}(G)$. (21) also expresses that $f_2 : G_2 \rightarrow G_2$ is the restriction of f , hence $f_2 \in \text{End}(G_2)$, and (b) holds.

(b) \implies (a): Let be $f \in S_0(G)$ arbitrary. By Lemma 2.13 h) we have $f(G_2) \subset G_2$ as well as $f(H) \subset H$ for $H := \bigoplus_{p \in \mathbb{P} \setminus \{2\}} G_p$. The restrictions $f_2 : G_2 \rightarrow G_2$ and $f_H : H \rightarrow H$ of f are in $S_0(G), S_0(H)$, respectively. By (b) $f_2 \in \text{End}(G_2)$ and, since H is 2-divisible, $f_H \in \text{End}(H)$ by Theorem 3.2. Let be $x \in G = G_2 \oplus H$ arbitrary, say $x = x_2 + x_H = x_2 + 2x'_H \in G_2 \oplus 2H = G_2 \oplus H = G$. Then $f(x) = f(x_2 + 2x'_H) \stackrel{\text{L. 2.9}}{=} f(x_2) + 2f(x'_H) \stackrel{\text{L. 2.13 b)}}{=} f(x_2) + f(2x'_H) = f(x_2) + f(x_H) = f_2(x_2) + f_H(x_H)$, and $f_2 \in \text{End}(G_2), f_H \in \text{End}(H)$ ensure $f \in \text{End}(G)$, i.e., (a) holds. \square

5. Concluding remarks

Remark 5.1. The techniques used in parts (a) \implies (b) and (b) \implies (a) of the proof of Theorem 4.9 are merely special cases of those for a systematic treatment of the functional equation (1) over general direct sums of abelian groups.

Remark 5.2. A brief sketch for a prominent example of a direct sum is given: $G := \mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)$. \mathbb{Q}/\mathbb{Z} is divisible and locally cyclic (cf. Remark 3.5), so that Theorems 3.2 or 3.6 ensure $S_0(\mathbb{Q}/\mathbb{Z}) \subset \text{End}(\mathbb{Q}/\mathbb{Z})$. For every $p \in \mathbb{P}$ the p -component of \mathbb{Q}/\mathbb{Z} is $\mathbb{Z}(p^\infty)$, and for every $f \in S_0(\mathbb{Q}/\mathbb{Z})$, Lemma 2.13 h) implies

$$f(\mathbb{Z}(p^\infty)) \subset \mathbb{Z}(p^\infty) \quad (\forall p \in \mathbb{P}), \tag{22}$$

hence the restrictions $f_p : \mathbb{Z}(p^\infty) \rightarrow \mathbb{Z}(p^\infty)$ ($p \in \mathbb{P}$) of f are available, and $f_p \in S_0(\mathbb{Z}(p^\infty))$ ($\forall p \in \mathbb{P}$). Additivity of f leads to

$$f(x) = \sum_{p \in \mathbb{P}} f_p(x_p) \quad \left(\forall x = \sum x_p \in \mathbb{Q}/\mathbb{Z} \right). \tag{23}$$

But now all the $\mathbb{Z}(p^\infty)$ ($p \in \mathbb{P}$) are pure-simple (cf. Remark 2.17 d)), and from Theorem 2.18 we get $f_p \in \{0, i_{\mathbb{Z}(p^\infty)}\}$, so that the general solution $f \in S_0(\mathbb{Q}/\mathbb{Z})$ appears in the lucid form $\mathbb{P}_0(f) := \{p \in \mathbb{P}; f_p = 0\}, \mathbb{P}_1(f) := \{p \in \mathbb{P}; f_p = i_{\mathbb{Z}(p^\infty)}\}$, $f(x) = \sum_{p \in \mathbb{P}_1(f)} x_p$ ($\forall x = \sum x_p \in \mathbb{Q}/\mathbb{Z}$). So f is the projection of \mathbb{Q}/\mathbb{Z} onto $f(\mathbb{Q}/\mathbb{Z}) = \bigoplus_{p \in \mathbb{P}_1(f)} \mathbb{Z}(p^\infty)$ along $K_f = \bigoplus_{p \in \mathbb{P}_0(f)} \mathbb{Z}(p^\infty)$.

Remark 5.3. For torsion-free abelian groups G , Theorem 3.9 guarantees $S_0(G) \subset \text{End}(G)$, while for abelian torsion groups, Theorem 4.9 provides necessary and sufficient conditions for $S_0(G) \subset \text{End}(G)$. For mixed abelian groups G we have to expect much more complicated circumstances. Thus, e.g., in general no analogue of (22) and the subsequent decomposition (23) of $f \in S_0(G)$ will occur (cf. also Remark 4.5). We finish by an example which lies outside the scopes of Theorems 3.2, 3.6, 3.9, and 3.14.

Example 5.4. $G := \mathbb{Z}_2 \oplus \mathbb{Z}$ is a splitting mixed abelian group; we write $\mathbb{Z}_2 = \{0, 1\}$. Then $(1, 0)$ is the only element of G of order 2, and for any $f \in S_0(G)$, Lemma 2.13 h) implies $f(1, 0) \in \{(0, 0), (1, 0)\}$. If $f(1, 0) = (0, 0)$, then $f(1, \beta) \stackrel{(\text{L. 2.4 a))}}{=} f(0, \beta) \stackrel{(\text{L. 2.13 b))}}{=} \beta f(1, 0) (\forall \beta \in \mathbb{Z})$, i.e., f is additive. If $f(1, 0) = (1, 0)$, then $(1, 0) \notin K_f$, and K_f does not contain an element of order 2, so by Theorem 3.10 f is again additive. In the total $S_0(\mathbb{Z}_2 \oplus \mathbb{Z}) \subset \text{End}(\mathbb{Z}_2 \oplus \mathbb{Z})$. The six elements of $S_0(\mathbb{Z}_2 \oplus \mathbb{Z})$ are: $\underline{0}$, $i_{\mathbb{Z}_2 \oplus \mathbb{Z}}$, the projections $(\varepsilon, \beta) \mapsto (0, \beta)$ and $(\varepsilon, \beta) \mapsto (\varepsilon, 0)$, and finally the mappings $(\varepsilon, \beta) \mapsto (\beta \cdot 1, \beta)$, $(\varepsilon, \beta) \mapsto (\varepsilon + \beta \cdot 1, 0)$, where $(\varepsilon, \beta) \in \mathbb{Z}_2 \oplus \mathbb{Z}$. The last two solutions show that $f(\{0\} \times \mathbb{Z}) \subset \{0\} \times \mathbb{Z}$ is not valid in general.

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