

The variety of quasi-Stone algebras does not have the amalgamation property

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ABSTRACT. We give an example showing that the variety of quasi-Stone algebras does not have the amalgamation property.

The question whether the variety **QSA** of quasi-Stone algebras has the amalgamation property is posed as an open problem in [2]. In this note, we show that the answer is negative by providing a counterexample. In particular, this also provides a counterexample to the claim made in [1] that the class of all finite quasi-Stone algebras has the amalgamation property.

An algebra $(L; \wedge, \vee, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ is a *quasi-Stone algebra* (in the following: a QSA) if $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the unary operation $'$ satisfies the following conditions for all $a, b \in L$:

- (QS1) $0' = 1$ and $1' = 0$,
- (QS2) $(a \vee b)' = a' \wedge b'$ (the \vee -DeMorgan law),
- (QS3) $(a \wedge b)' = a' \vee b''$ (the weak \wedge -DeMorgan law),
- (QS4) $a \wedge a'' = a$,
- (QS5) $a' \vee a'' = 1$ (the Stone identity).

We write QSA's as pairs $(L, ')$ where L stands for the underlying bounded distributive lattice. QSA-homomorphisms are defined in the obvious way.

In [1], it is shown that the category of all QS-spaces together with QS-maps is dually equivalent with the category **QSA**. Here, a *QS-space* is a pair (X, \mathcal{E}) consisting of a Priestley space X and an equivalence relation \mathcal{E} on X satisfying certain conditions.

For a given QSA $(L, ')$, its QS-space is constructed as follows: Let $X = D(L)$ be the (standard) Priestley space of all prime filters of L and set

$$\mathcal{E} = \{(P, Q) \in D(L) \times D(L) \mid P \cap B(L) = Q \cap B(L)\},$$

where $B(L) = \{a' \mid a \in L\}$ is the skeleton of L . Then (X, \mathcal{E}) is the dual QS-space of $(L, ')$. We write $[x]_{\mathcal{E}}$ for the \mathcal{E} -class of $x \in X$, and $\mathcal{E}(U) = \bigcup_{x \in U} [x]_{\mathcal{E}}$ for any subset $U \subseteq X$.

QS-maps are defined as follows: Let (X, \mathcal{E}) and (Y, \mathcal{F}) be QS-spaces. Then a continuous, order preserving map $\varphi: X \rightarrow Y$ is a *QS-map* if

$$\mathcal{E}(\varphi^{-1}(U)) = \varphi^{-1}(\mathcal{F}(U))$$

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for each clopen increasing set $U \subseteq Y$. For a given QSA-homomorphism $f: L \rightarrow K$, its dual QS-map is given by $D(f): D(K) \rightarrow D(L)$ with $D(f)(P) = f^{-1}(P)$ for $P \in D(K)$. Crucially for our purposes, QSA-embeddings correspond bijectively to onto QS-maps.

Now we can present our example of a tuple (L, M, N, i, j) , where L, M, N are QSA's and $i: L \rightarrow M, j: L \rightarrow N$ are QSA-embeddings, which can not be amalgamated within **QSA**. The failure of amalgamation will be shown by means of the duality described above.

Let L be the three-element Stone algebra ($0' = 1$ and $a' = 1' = 0$), M the four-element Boolean lattice, and N a six-element Stone algebra as defined in Figure 1:

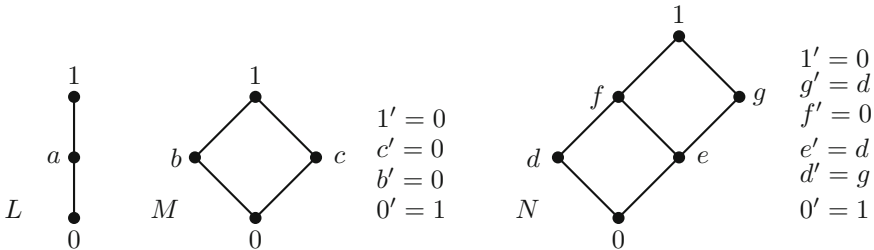


FIGURE 1

We choose QSA-embeddings $i: L \hookrightarrow M$ such that $i(0) = 0, i(a) = b, i(1) = 1$, and $j: L \hookrightarrow N$ such that $j(0) = 0, j(a) = f, j(1) = 1$. The corresponding QS-spaces are given in Figure 2 (putting $X = D(L), Y = D(M)$, and $Z = D(N)$):

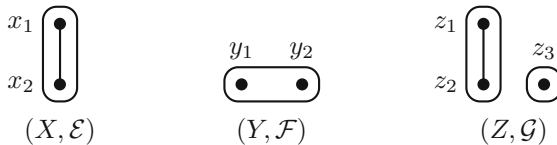
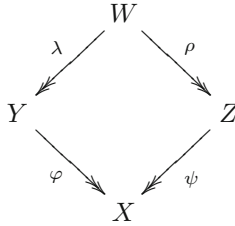


FIGURE 2

With it we have the onto QS-maps $\varphi = D(i): Y \twoheadrightarrow X$ with $\varphi(y_1) = x_1, \varphi(y_2) = x_2$, and $\psi = D(j): Z \twoheadrightarrow X$ with $\psi(z_1) = \psi(z_3) = x_1, \psi(z_2) = x_2$.

Assume that there is a QSA K and embeddings $h: M \hookrightarrow K, k: N \hookrightarrow K$ amalgamating (L, M, N, i, j) , i.e., such that $h \circ i = k \circ j$. Let (W, \mathcal{H}) be the dual space of K with $W = D(K)$, and let $\lambda = D(h), \rho = D(k)$ be the duals of h and k , respectively. Then, by duality, the following diagram commutes:



Since ρ is onto, there is some $w_1 \in W$ such that $\rho(w_1) = z_3$. But then $\rho(w) = z_3$ for all $w \in [w_1]_{\mathcal{H}}$ because $\{z_3\}$ is a clopen increasing set, and therefore $[w_1]_{\mathcal{H}} \subseteq \mathcal{H}(\rho^{-1}(\{z_3\})) = \rho^{-1}(\mathcal{G}(\{z_3\})) = \rho^{-1}(\{z_3\})$. Thus, for all $w \in [w_1]_{\mathcal{H}}$, we have $\psi \circ \rho(w) = x_1$, and by the commutativity of the diagram, it follows that also $\varphi \circ \lambda(w) = x_1$ for all $w \in [w_1]_{\mathcal{H}}$. This implies that $[w_1]_{\mathcal{H}} \subseteq \lambda^{-1}(\{y_1\}) \subseteq \lambda^{-1}(\mathcal{F}(\{y_2\}))$ and that $[w_1]_{\mathcal{H}} \cap \mathcal{H}(\lambda^{-1}(\{y_2\})) = \emptyset$. Hence, $\lambda^{-1}(\mathcal{F}(\{y_2\})) \neq \mathcal{H}(\lambda^{-1}(\{y_2\}))$ which is a contradiction, since $\{y_2\}$ is a clopen increasing set.

A different counterexample has been obtained independently by S. Solovjov (private communication).

REFERENCES

- [1] Gaitán, H.: Priestley duality for quasi-Stone algebras. *Studia Logica* **64**, 83–92 (2000)
- [2] Sankappanavar, N. H., Sankappanavar, H. P.: Quasi-Stone algebras. *Math. Logic Quart.* **39**, 255–268 (1993)

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