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COMPACT SURFACES AS CONFIGURATION SPACES OF MECHANICAL LINKAGES

BY

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ABSTRACT

There exists a homeomorphism between any compact orientable closed surface and the configuration space of an appropriate mechanical linkage defined by a weighted graph embedded in the Euclidean plane.

1. Introduction

A **mechanical linkage** $\mathcal{G}$ is a mechanism in the Euclidean plane $\mathbb{R}^2$ that is built up exclusively from rigid bars joined along flexible links. Some links of the linkage may be pinned down with respect to a fixed frame of reference. The **configuration space** $[\mathcal{G}]$ of a mechanical linkage $\mathcal{G}$ is the totality of all its admissible positions in the Euclidean plane. In section 4 we present the constructive proof of the main result:

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THEOREM 1.1: *Let Σ_g be any compact orientable closed surface of genus $g \in \mathbb{N} \cup \{0\}$. Then there exists a mechanical linkage $\mathcal{S}_g$, such that its configuration space $[\mathcal{S}_g]$ is homeomorphic to Σ_g.*

Recently the authors proved a universality theorem for configuration spaces of mechanical linkages, [7]. It is a matter of an explicit construction in the sense that starting from a given compact real algebraic variety, addition and multiplication of the defining polynomials are realized with configurations. So the theorem of Nash-Tognoli, cf. [1], implies that only some components of the resulting configuration space are homeomorphic to a given compact differentiable manifold. Notice that using similar proofs different universality theorems for configuration spaces have already been established and are summarized in a work of M. Kapovich and J. Millson, [9].

To prove Theorem 1.1 we construct for all $g \in \mathbb{N}$ a mechanical linkage with only $3 + 2g$ bars and $3 + 2g$ links, such that its configuration space is homeomorphic to the compact orientable closed surface of genus g. It is known, cf. [4] or [8], that the set of non-singular configuration spaces of a planar 5-polygon contains Σ_g with $g \leq 4$. Starting with a simple 5-polygon for which the configuration space is a torus, we add two edges connected by a link to increase the genus of the surface by one. To control the induction we only make use of topological and geometrical arguments, i.e. the configuration space is computed with a fibration over the admitted locations of an appropriate vertex of the linkage. The final explicitly constructed mechanical linkage $\mathcal{S}_g$ with $[\mathcal{S}_g] \approx \Sigma_g$ is easy to control, as Figure 1 shows.

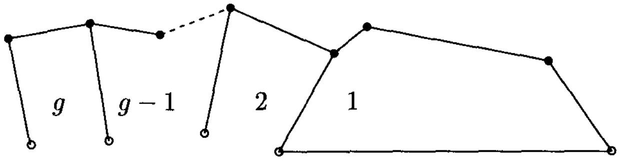

Figure 1. The mechanical linkage $\mathcal{S}_g$.

ACKNOWLEDGEMENT: We are grateful to J.-C. Hausmann who introduced us to the work of his former graduate student A. Wenger, cf. [11], and to P. Mani-Levitska for numerous conversations.

2. Preliminaries

Let us give an exact mathematical definition of a mechanical linkage:

Definition 2.1: The triple $\mathcal{G} = (V, E, d)$ consisting of

(1) a set of **vertices** $V = V_{fix} \cup V_{free}$, with $V_{fix} = \{V_1, \ldots, V_m\}$ and $V_{free} = \{V_{m+1}, \ldots, V_n\}$,

(2) a set of **edges** $E = \{\{V_{i_1}, V_{j_1}\}, \{V_{i_2}, V_{j_2}\}, \ldots, \{V_{i_k}, V_{j_k}\}\}$ such that $i_l, j_l \in \{1, \ldots, n\}$, $i_l \neq j_l$, where any two vertices in V are connected by a sequence of elements of E, and

(3) a **weight function** d: $E \to \mathbb{R}_+$, that attaches to every edge $\{V_{i_l}, V_{j_l}\}$ in E a length (weight) $d(V_{i_l}, V_{j_l}) \in \mathbb{R}_+$,

is called a **connected weighted graph.**

Definition 2.2: Let $\mathcal{G} = (V, E, d)$ be a connected weighted graph.

(1) The graph $\mathcal{G}$ is called a **mechanical linkage**, if $\mathcal{G}$ is **realizable** in $\mathbb{R}^2$, i.e. if a mapping ξ: $V \to \mathbb{R}^2$ exists, such that $|\xi(V_i) - \xi(V_j)| = d(V_i, V_j)$ for all $\{V_i, V_j\} \in E$.

(2) A **realization** of $\mathcal{G} = (V, E, d)$ is the evaluation $\xi(V) = (\xi(V_1), \ldots, \xi(V_n))$ in $\mathbb{R}^{2n}$ with $|\xi(V_i) - \xi(V_j)| = d(V_i, V_j)$ for all $\{V_i, V_j\} \in E$.

Denote by $\mathcal{G} \subset \mathcal{G}' = (V', E', d')$ a mechanical linkage $\mathcal{G} = (V, E, d)$, such that $V \subset V'$, $E \subset E'$ and $d = d'|_E$. Notice that we often abuse notation by identifying the mapping ξ to its evaluation $\xi(V)$. The configuration space of a mechanical linkage is defined as a subset of $\mathbb{R}^{2n}$ with the natural topology:

Definition 2.3: Let $\mathcal{G} = (V, E, d)$ be a mechanical linkage and $\{p_1, \ldots, p_m\}$ fixed points in $\mathbb{R}^2$ with $m \geq 2$, such that $|p_i - p_j| = d(V_i, V_j)$ for all $\{V_i, V_j\} \in E$ with $V_i, V_j \in \{V_1, \ldots, V_m\} = V_{fix}$. Then the **configuration space** of $\mathcal{G}$ is defined by

$$\begin{aligned}[\mathcal{G}] &= \{\xi \text{ realization of } \mathcal{G};\ \xi(V_{fix}) = (p_1, \ldots, p_m)\} \\ &= \{(x_1, \ldots, x_n) \in (\mathbb{R}^2)^n;\ x_j = p_j \ \ \forall j \in \{1, \ldots, m\} \text{ and} \\ &\qquad\qquad\qquad |x_i - x_j| = d(V_i, V_j) \ \ \forall \{V_i, V_j\} \in E\}\end{aligned}$$

with the topology induced by the Euclidean metric of $\mathbb{R}^{2n}$.

Definition 2.4: Let $\mathcal{G} = (V, E, d)$ be a mechanical linkage and $V_j \in V$. Then $\mathbb{W}_{\mathcal{G}}(V_j) = \{\xi(V_j);\ \xi \in [\mathcal{G}]\} \subset \mathbb{R}^2$ is called the **work space** of the vertex V_j.

Next we introduce a fibration π: $F \to X$ which drops the requirement of local triviality:

Definition 2.5: A **splitted fibration** is a surjective map π: $F \to X$, where F and X are topological spaces and $\pi^{-1}(x)$ is the **fiber** over $x \in X$.

Two splitted fibrations $\pi\colon F \to X$ and $\pi'\colon F' \to X'$ are called **equivalent**, if homeomorphisms $f\colon F \to F'$ and $g : X \to X'$ exist, such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{g} & X' \end{array}$$

commutes.

This notion is related to the definition of the configuration space and the work space of a mechanical linkage: let $\mathcal{G} = (V, E, d)$ be a mechanical linkage and $\mathbb{W}_{\mathcal{G}}(V_j)$ the work space of the vertex $V_j \in V$. Then we obtain a splitted fibration

$$\begin{aligned} \pi_{\mathcal{G}}\colon [\mathcal{G}] &\longrightarrow \mathbb{W}_{\mathcal{G}}(V_j) \\ \xi &\longmapsto \xi(V_j) \end{aligned}$$

and the fiber above $p \in \mathbb{W}_{\mathcal{G}}(V_j)$ consists of all realizations $\xi \in [\mathcal{G}]$ with $\xi(V_j) = p$, in particular $\pi_{\mathcal{G}}^{-1}(\mathbb{W}_{\mathcal{G}}(V_j)) = [\mathcal{G}]$. Notice that $\mathbb{W}_{\mathcal{G}}(V_j)$ is a compact subset of $\mathbb{R}^2$ since $\mathcal{G}$ is supposed to be connected.

Finally, for a mechanical linkage $\mathcal{G}$ we consider the polynomials $L_{i_1 j_1}, \ldots, L_{i_k j_k}$ defined by $L_{i_l j_l} := |\xi(V_{i_l}) - \xi(V_{j_l})|^2$ in $\mathbb{R}[X_1, \ldots, X_{2n}]$ with $\{V_{i_l}, V_{j_l}\} \in E$, such that V_{i_l}, V_{j_l} are not both elements of V_{fix}. We say that the k-tuple $p := (d^2(V_{i_1}, V_{j_1}), \ldots, d^2(V_{i_k}, V_{j_k}))$ of $\mathcal{G}$ is **regular** if $p \in \mathbb{R}^k$ is a regular value of the map $L := (L_{i_1 j_1}, \ldots, L_{i_k j_k})\colon \mathbb{R}^{2n} \to \mathbb{R}^k$, otherwise p is **critical**. A first result about the topological behaviour of $[\mathcal{G}]$ allowing small perturbations of p is given:

PROPOSITION 2.6: *Let p be as above for a mechanical linkage $\mathcal{G}$ and suppose that an open neighborhood U of $p \in \mathbb{R}^k$ exists, such that for all mechanical linkages $\mathcal{G}'$ with $p' \in U$ we have $[\mathcal{G}'] \approx [\mathcal{G}]$. Then $[\mathcal{G}]$ is an orientable smooth manifold of dimension $2(n-m)-k$.*

Proof: If p is regular then $L^{-1}(p) = [\mathcal{G}]$ is an orientable smooth manifold by the Implicit Function Theorem. Conversely, if p is critical, then a regular $p' \in U$ exists, since by the Lemma of Sard the critical values of $L\colon \mathbb{R}^{2n} \to \mathbb{R}^k$ have measure zero in $\mathbb{R}^k$. This completes the proof since $[\mathcal{G}] \approx [\mathcal{G}'] = L^{-1}(p')$. ∎

3. n-Polygons

A n-polygon is a special mechanical linkage built up by a cyclic arrangement of its edges:

Definition 3.1: An n-**polygon** $\mathcal{P}_l$ is a mechanical linkage, such that

(1) $V = \{V_1, \ldots, V_n\}$ with $V_{fix} = \{V_1, V_2\}$,
(2) $E = \{\{V_1, V_2\}, \{V_2, V_3\}, \ldots, \{V_n, V_1\}\}$ and
(3) $l = (l_1, \ldots, l_n)$, such that $l_j = d(V_j, V_{j+1}) \in \mathbb{R}_+$ with indices modulo n.

In the proof of Theorem 1.1 we assume some knowledge about the configuration space of a special 5-polygon whose computation in Example 3.2 is prepared by some considerations on 4-polygons: the configuration space of a 4-polygon $\mathcal{P}_l$ is discussed in [3] using Morse Theory, but no classification depending on the length tuple $l = (l_1, l_2, l_3, l_4)$ is given. We state a complete classification, using either the cited article or elementary geometric considerations, i.e. the work space of the vertex V_3 and its splitted fibration:

Since we require that $\mathcal{P}_l$ is realizable, there is $l_j \leq \sum_{i \neq j} l_i$ for all $j \in \{1, \ldots, 4\}$. If we have equality in the above condition for one $j \in \{1, \ldots, 4\}$ then $[\mathcal{P}_l] = pt$, (point), otherwise we consider the pair of relations $\theta = (\theta_1, \theta_2)$, where

$$(l_1 + l_2) \;\; \theta_1 \;\; (l_3 + l_4) \quad \text{and} \quad |l_1 - l_2| \;\; \theta_2 \;\; |l_3 - l_4|$$

for $\theta_1, \theta_2 \in \{>, =, <\}$. For the classification we distinguish the following two cases:

(i) If $l_1 \neq l_2$ or $l_3 \neq l_4$ and
$\theta \in \{(<,<), (>,>)\}$ then $[\mathcal{P}_l] \approx S^1$ (manifold);
$\theta \in \{(<,>), (>,<)\}$ then $[\mathcal{P}_l] \approx S^1 \amalg S^1$ (disjoint union of two S^1, manifold);
$\theta \in \{(=,<), (=,>), (<,=), (>,=)\}$ then $[\mathcal{P}_l] \approx S^1 \vee S^1$ (one point union of two S^1);
$\theta = (=,=)$ then $[\mathcal{P}_l] \approx ((S^1, s_0) \cup (S^1, s_0'))/\{s_0 \sim s_0', -s_0 \sim -s_0'\}$.

(ii) If $l_1 = l_2$ and $l_3 = l_4$ and
$\theta \in \{(<,=), (>,=)\}$ then $[\mathcal{P}_l] \approx ((S^1, s_0) \cup (S^1, s_0'))/\{s_0 \sim s_0', -s_0 \sim -s_0'\}$;
$\theta = (=,=)$ then $[\mathcal{P}_l] \approx ((S^1, s_0) \cup (S^1, s_0') \cup (S^1, s_0''))/\{s_0 \sim s_0'', -s_0'' \sim s_0', -s_0' \sim -s_0\}$.

Thus the configuration space of a 4-polygon is given up to homeomorphism.

Example 3.2: Consider the 5-polygon $[\mathcal{P}_l]$ with $l = (9, 3, 5, 1, 3)$ as shown in Figure 2; then $[\mathcal{P}_l] \approx \Sigma_1$. In fact, take orthogonal coordinates of $\mathbb{C}$, such that $\xi(V_1) = 0$ and $\xi(V_2) = l_1 e^{i0}$. Then it is easy to see that the angle φ defined by $\arg(\xi(V_5))$ satisfies $\varphi \in [-\varphi_0, \varphi_0]$ with $\varphi_0 := \arccos(\frac{1}{6})$ for any $\xi \in [\mathcal{P}_l]$, in terms of the work space $\mathbb{W}_{\mathcal{P}_l}(V_5) = \{l_5 e^{i\varphi};\, \varphi \in [-\varphi_0, \varphi_0]\}$. Using the above classification for 4-polygons we deduce $[\mathcal{P}_{l(\varphi)}]$ as a function of $l(\varphi) := (d(\varphi), 3, 5, 1)$, where $d(\varphi) := |\xi(V_2) - \xi(V_5)| \in [6, 9]$.

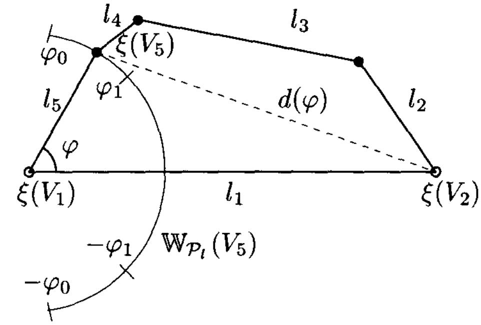

Figure 2. 5-polygon $\mathcal{P}_l$ and work space $\mathbb{W}_{\mathcal{P}_l}(V_5)$.

If $\varphi = \varphi_0$ we have $d(\pm\varphi_0) = 9$, thus $\left[\mathcal{P}_{l(\pm\varphi_0)}\right] = pt$. For $\varphi = \varphi_1 := \arccos(\frac{41}{54})$ we have $d(\pm\varphi_1) = 7$ which implies $\theta = (>, =)$, thus $\left[\mathcal{P}_{l(\pm\varphi_1)}\right] \approx S^1 \vee S^1$. Let be $\varphi \in \,]-\varphi_0, -\varphi_1[\, \cup \,]\varphi_1, \varphi_0[\, =: I_- \cup I_+$, so $d(\varphi) \in \,]7,9[$ which implies $\theta = (>, >)$ and thus $\left[\mathcal{P}_{l(\varphi)}\right] \approx S^1$. If $\varphi \in \,]-\varphi_1, \varphi_1[\, =: M$ we have $d(\varphi) \in \,]6,7[$ which implies $\theta = (>, <)$, so $\left[\mathcal{P}_{l(\varphi)}\right] \approx S^1 \amalg S^1$. Since the map from $[\mathcal{P}_l]$ to $[-\varphi_0, \varphi_0]$ which assigns to any realization the angle φ defines a Morse function, we obtain $\pi_{\mathcal{P}_l} : [\mathcal{P}_l] \to \mathbb{W}_{\mathcal{P}_l}(V_5)$ with fibers $\pi_{\mathcal{P}_l}^{-1}(l_5 e^{i\varphi}) = \left[\mathcal{P}_{l(\varphi)}\right]$: for all points $(l_2', l_3', l_4', l_5') \in U_{1/10}(3,5,1,3) \subset \mathbb{R}^4$ the splitted fibrations $\pi_{\mathcal{P}_l} : [\mathcal{P}_l] \to \mathbb{W}_{\mathcal{P}_l}(V_5)$ and $\pi_{\mathcal{P}_l'} : [\mathcal{P}_{l'}] \to \mathbb{W}_{\mathcal{P}_{l'}}(V_5)$ with $l' = (l_1, l_2', l_3', l_4', l_5')$ are equivalent, thus by Proposition 2.6 the configuration space $[\mathcal{P}_l]$ is an orientable compact two-dimensional manifold.

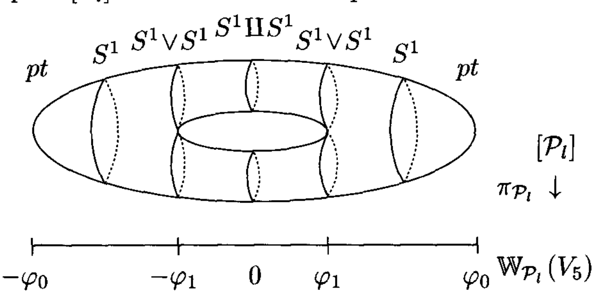

Figure 3. Splitted fibration over the work space $\mathbb{W}_{\mathcal{P}_l}(V_5) \approx [-\varphi_0, \varphi_0]$.

The Euler characteristic χ of the fiber space

$$\pi_{\mathcal{P}_l}^{-1}\{l_5 e^{i\varphi};\, \varphi \in U\} \approx \begin{cases} pt & \text{for } \; U = \{-\varphi_0\}, \{\varphi_0\} \\ S^1 \vee S^1 & \text{for } \; U = \{-\varphi_1\}, \{\varphi_1\} \\ I_\pm \times S^1 & \text{for } \; U = I_-, I_+ \\ M \times (S^1 \amalg S^1) & \text{for } \; U = M \end{cases}$$

illustrated in Figure 3 finally determines the genus of $[\mathcal{P}_l]$: use $\chi(A \times B) = \chi(A) \cdot \chi(B)$ to get

$$\chi\left(\pi_{\mathcal{P}_l}^{-1}\{l_5 e^{i\varphi};\ \varphi \in U\}\right) = \begin{cases} \chi(pt) & = \ 1 \quad \text{for} \quad U = \{-\varphi_0\}, \{\varphi_0\} \\ \chi(S^1 \vee S^1) & = -1 \quad \text{for} \quad U = \{-\varphi_1\}, \{\varphi_1\} \\ \chi(I_\pm) \cdot \chi(S^1) & = \ 0 \quad \text{for} \quad U = I_-, I_+ \\ \chi(M) \cdot \chi(S^1 \amalg S^1) & = \ 0 \quad \text{for} \quad U = M \end{cases}$$

and therefore $\chi([\mathcal{P}_l]) = 0$. We obtain $[\mathcal{P}_l] \approx \Sigma_1$.

Remark 3.3: The reader is invited to look for a 5-polygon $\mathcal{P}_l$ such that $[\mathcal{P}_l] \approx S^2$. Denote this mechanical linkage by $\mathcal{S}_0$. Notice that by the computation method as proposed above one can get a full classification of the configuration spaces for 5-polygons.

4. Proof of Theorem 1.1

Before stating an inductive construction of the mechanical linkage $\mathcal{S}_g$ with $[\mathcal{S}_g] \approx \Sigma_g$ we need two lemmas presented in the first and second parts of the proof. The inductive step consists of cutting the configuration space of an assumed mechanical linkage, such that the borders are either homeomorphic to $S^1 \amalg S^1$ (g even) or to S^1 (g odd), throwing away one of the connected components, duplicating the other component and then pasting the canonically isometric borders. We obtain that the genus of the configuration space is increased by one.

All this can be done mechanically by adding two edges to a given linkage, whose sizes are determined by Lemma 4.1; see Figures 1 and 5. This gives enough freedom to execute the cut at the right position in each inductive step. Lemma 4.2 states the the resulting gluing procedure during this step as illustrated in Figures 6 and 7.

1. Assume a mechanical linkage $\mathcal{G} = (V, E, d)$ with $\{O, A\} \in E$, $d(O, A) = r$ and $O \in V_{fix}$ with $\xi(O) = p \in \mathbb{R}^2$, such that $\mathbb{W}_{\mathcal{G}}(A) = \{q(\varphi) = p + re^{i\varphi};\ \varphi \in [\alpha, \beta]\}$ for $\beta - \alpha < \pi$, where the angles are measured according to any direction in p.

We define a mechanical linkage $\mathcal{G}' = (V', E', d') \supset \mathcal{G}$, such that $V'_{free} = V_{free} \cup \{A'\}$, $V'_{fix} = V_{fix} \cup \{O'\}$, $E' = E \cup \{\{O', A'\}, \{A, A'\}\}$, $d(O', A') = r'$, $d(A, A') = s$ and $d'|_E = d$. In the next lemma we lay down the lengths r', s and $\xi(O') = p' \in \mathbb{R}^2$; see also Figure 4.

LEMMA 4.1: *Let $\mathcal{G}$ be a mechanical linkage with $\mathbb{W}_{\mathcal{G}}(A)$ and $\delta \in]\alpha, \beta[$. Then $\mathcal{G}'$ exists, such that*

(i) $\mathbb{W}_{\mathcal{G}'}(A) = \{q(\varphi);\ \varphi \in [\alpha, \delta]\} \subset \mathbb{W}_{\mathcal{G}}(A)$,

(ii) $\mathbb{W}_{\mathcal{G}'}(A') = \{q'(\varphi');\ \varphi' \in [\alpha', \beta']\}$, *where* $q'(\varphi') = p' + r'e^{i\varphi'}$ *and* $\beta' - \alpha' < \pi$,

(iii) *if* $\delta' = \frac{1}{2}(\beta' - \alpha')$ *then the maps* $F_{[\alpha',\delta']} : \mathbb{W}_{\mathcal{G}'}(A) \to \{q'(\varphi');\ \varphi' \in [\alpha', \delta']\}$ *and* $F_{[\delta',\beta']}$: $\mathbb{W}_{\mathcal{G}'}(A) \to \{q'(\varphi');\ \varphi' \in [\delta', \beta']\}$ *with* $F_{[\alpha',\delta']}, F_{[\delta',\beta']}(\xi(A)) := \xi(A')$ *are homeomorphisms.*

The angle φ *is measured in* p *and* φ' *in* p', *both with respect to the direction given by* $p' - p$.

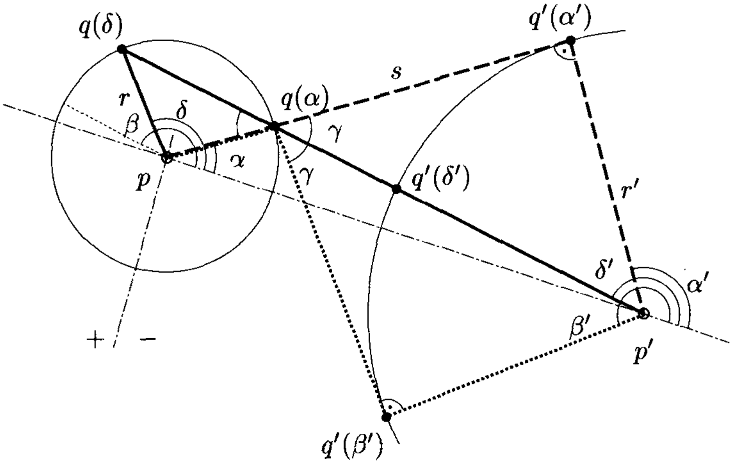

Figure 4. Part of the mechanical linkage $\mathcal{G}'$: a realization with $R = 0$ (dashed), a realization with $R = -d$ (lined) and a realization with $R \in]-d, 0[$ (dotted).

Proof: First let us describe the exact shape of the mechanical linkage $\mathcal{G}'$ to get (i) and (ii). Put $p' \in \mathbb{R}^2$ on the ray from $q(\delta)$ through $q(\alpha)$, and $q'(\alpha') \in \mathbb{R}^2$ on the ray from p through $q(\alpha)$, such that the rectangular triangle $q(\alpha), p', q'(\alpha')$ with right angle at $q'(\alpha')$ has sidelength $|p' - q(\alpha)| = s + r' - 2r\cos\gamma$, $|p' - q'(\alpha')| = r'$ and $|q'(\alpha') - q(\alpha)| = s$, where $\gamma := \pi/2 - (\delta - \alpha)/2$. We obtain two defining equations $s = (s + r' - 2r\cos\gamma)\cos\gamma$ and $(s + r' - 2r\cos\gamma)^2 = r'^2 + s^2$, thus $s = 2r(\cos\gamma)^2(\sin\gamma + \cos\gamma - 1)^{-1}$ and $r' = r(\sin\gamma + \cos\gamma + 1)$, which shows the existence of $\mathcal{G}'$.

The oriented input angle $\varphi := \measuredangle_p(p', \xi(A)) \in [\alpha, \delta]$ and the oriented output angle $\varphi' := \pi - \measuredangle_{p'}(p, \xi(A')) \in [\alpha', \beta']$, are related by *Freudenstein's Equation*

$$2rd\cos\varphi + 2r'd\cos(\varphi' - \pi) - (r^2 + r'^2 + d^2 - s^2) = 2rr'\cos(\varphi - \varphi' + \pi)$$

where $d := |p - p'|$; cf. Section 5.2 in [2], Formulas (5.6) and (5.7).

It suffices to show that $f(\varphi) := \varphi'$ is strictly increasing if f: $[\alpha, \delta] \to [\alpha', \delta']$ and strictly decreasing if f: $[\alpha, \delta] \to [\delta', \beta']$ respectively. The Formula (3.11)

$$\frac{df}{d\varphi}(\varphi) = \frac{R}{R + d}$$

for the first derivative of the input-output relation $f \in C^1[\alpha, \delta]$ supplied for the *Carter-Hall Construction* in Subsection 3.4.2 of [2] is useful. Here $R := \pm|p - I|$ where $I =$ *line* l_1 *through* p *and* $p' \cap$ *line* l_2 *through* $\xi(A)$ *and* $\xi(A')$. Remember I is empty or equal to l_1 if and only if $l_1 \parallel l_2$, then $R := \pm\infty$. The sign convention is *plus* if p lies between I and p' else *minus*. Determine R as a function of the input angle φ:

- Let $\varphi = \alpha$. Then $\varphi' \in \{\alpha', \beta'\}$. If $\varphi' = \alpha'$ then $p, p', \xi(A')$ is a triangle with $\measuredangle_{\xi(A')}(p, p') = \pi/2$, hence $R = 0$ (dashed realization in Figure 4). If $\varphi' = \beta'$ then $\measuredangle_{\xi(A)}(p, \xi(A')) = \pi - 2\gamma$, thus $R \in]-d, 0[$ (dotted realization in Figure 4).
- Let $\varphi \in]\alpha, \delta[$. Then $|\xi(A) - p'| < s + r'$, thus exactly two positions for $\xi(A')$ on $\mathbb{W}_{\mathcal{G}'}(A')$ can occur. If $\varphi' \in]\alpha', \delta'[$ then the triangle $\xi(A), p', \xi(A')$ is positively oriented, and $\measuredangle_{\xi(A)}(p, \xi(A')) \in]\gamma, \pi[$. Then either l_1 and l_2 cut where $R > 0$, or $l_l \parallel l_2$ then $R = \pm\infty$, or l_1 and l_2 cut where $R < -d$. If $\varphi' \in]\delta', \beta'[$ then the triangle $\xi(A), p', \xi(A')$ is negatively oriented, and $\measuredangle_{\xi(A)}(p, \xi(A')) \in]\gamma, \pi - 2\gamma[$. Then l_1 and l_2 cut where $R \in]-d, 0[$.
- Let $\varphi \to \delta$. Then $\varphi' \to \delta'$ and the quadrilateral $p, p', \xi(A'), \xi(A)$ degenerates to the triangle $p, p', \xi(A')$ (lined realization in Figure 4). Therefore if $\varphi' \to \delta'$ from above/below then $R \to -d$ from above/below.

We get $\frac{df}{d\varphi}(\varphi) > 0$ where $f(\varphi) \in]\alpha', \delta']$ for all $\varphi \in]\alpha, \delta]$, and $\frac{df}{d\varphi}(\varphi) < 0$ where $f(\varphi) \in [\delta', \beta'[$ for all $\varphi \in]\alpha, \delta]$, thus (iii). ∎

2. The inductive construction of a mechanical linkage $\mathcal{S}_g$ with $[\mathcal{S}_g] \approx \Sigma_g$ builds on the knowledge about the following gluing procedure: consider the mechanical linkages $\mathcal{G}$, $\mathcal{G}'$ as presented above and set $\mathbb{W}_{\mathcal{G}'}(A')_{[\alpha',\delta']} = \{q'(\varphi');\ \varphi' \in [\alpha', \delta']\}$, $\mathbb{W}_{\mathcal{G}'}(A')_{[\delta',\beta']} = \{q'(\varphi');\ \varphi' \in [\delta', \beta']\}$, cf. Figure 5,

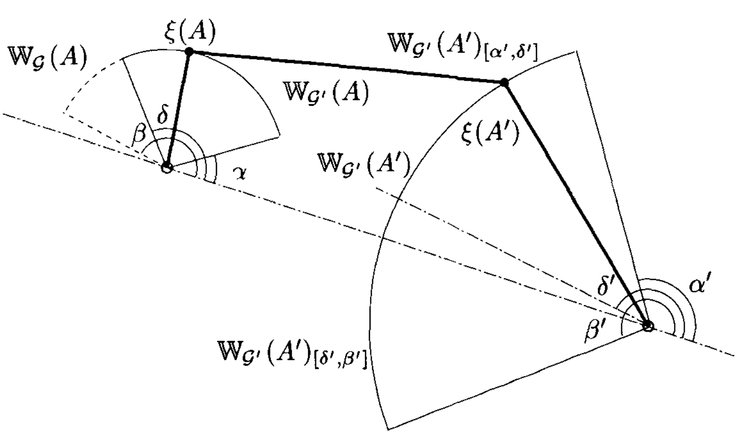

Figure 5. Part of $\mathcal{G}'$ with a realization in general position.

then by

$$\begin{tikzcd}
\{\xi\in\pi_{\mathcal{G}'}^{-1}(\mathbb{W}_{\mathcal{G}'}(A));\xi(A')\in\mathbb{W}_{\mathcal{G}'}(A')_{[\alpha',\delta']}\} \arrow[r, "\approx"] & \pi_{\mathcal{G}'}^{-1}\circ F^{-1}_{[\alpha',\delta']}(\mathbb{W}_{\mathcal{G}'}(A')_{[\alpha',\delta']}) \arrow[d, "\approx"] \\
{[\mathcal{G}]_{[\alpha,\delta]} := \pi_{\mathcal{G}}^{-1}(\mathbb{W}_{\mathcal{G}'}(A))} \arrow[u, "h_{[\alpha',\delta']}"] & \pi_{\mathcal{G}'}^{-1}(\mathbb{W}_{\mathcal{G}'}(A')_{[\alpha',\delta']}) =: [\mathcal{G}']_{[\alpha',\delta']}
\end{tikzcd}$$

there is $[\mathcal{G}]_{[\alpha,\delta]}\approx[\mathcal{G}']_{[\alpha',\delta']}$, where $h_{[\alpha',\delta']}\colon \xi(V)\mapsto(\xi(V),F_{[\alpha',\delta']}\circ\xi(A),p')$ defines a homeomorphism, and $[\mathcal{G}]_{[\alpha,\delta]}\approx\pi_{\mathcal{G}'}^{-1}(\mathbb{W}_{\mathcal{G}'}(A')_{[\delta',\beta']}) =: [\mathcal{G}']_{[\delta',\beta']}$ respectively. The space $[\mathcal{G}]_{[\alpha,\delta]}$ is illustrated in Figure 6.

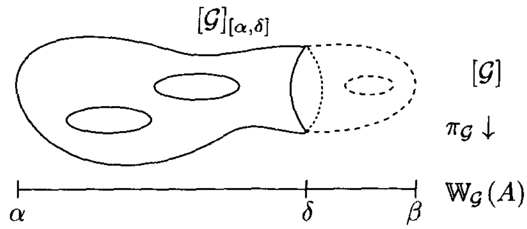

Figure 6. Splitted fibration over $\mathbb{W}_{\mathcal{G}}(A)\approx[\alpha,\beta]$ of $\mathcal{G}$.

Notice that $\pi_{\mathcal{G}}^{-1}(\xi(A))\approx\pi_{\mathcal{G}'}^{-1}(\xi(A'))$ gives a one to one correspondence between the fibers over $\mathbb{W}_{\mathcal{G}'}(A)$ in $[\mathcal{G}]$ and $\mathbb{W}_{\mathcal{G}'}(A')$ in $[\mathcal{G}']$ for all $\xi\in[\mathcal{G}']$. In particular the subspaces $[\mathcal{G}']_{[\alpha',\delta']}$ and $[\mathcal{G}']_{[\delta',\beta']}$ in $[\mathcal{G}']$ are glued by the identity at the common border $\pi_{\mathcal{G}'}^{-1}(q'(\delta'))$ as shown in Figure 7.

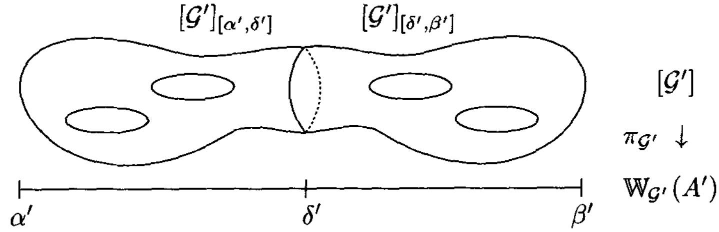

Figure 7. Splitted fibration over $\mathbb{W}_{\mathcal{G}'}(A')\approx[\alpha',\beta']$ of $\mathcal{G}'$.

Summarizing we conclude:

LEMMA 4.2: *The splitted fibrations $\pi_{\mathcal{G}'}\colon[\mathcal{G}']_{[\alpha',\delta']}\to\mathbb{W}_{\mathcal{G}'}(A')_{[\alpha',\delta']}$, $\pi_{\mathcal{G}'}\colon[\mathcal{G}']_{[\delta',\beta']}\to\mathbb{W}_{\mathcal{G}'}(A')_{[\delta',\beta']}$, $\pi_{\mathcal{G}}\colon[\mathcal{G}]_{[\alpha,\delta]}\to\mathbb{W}_{\mathcal{G}'}(A)$ are equivalent, and there is*

$$[\mathcal{G}']\approx[\mathcal{G}']_{[\alpha',\delta']}\cup_{\iota'}[\mathcal{G}']_{[\delta',\beta']}\approx[\mathcal{G}]_{[\alpha,\delta]}\cup_{\iota}[\mathcal{G}]_{[\alpha,\delta]},$$

where $\iota'\colon \pi_{\mathcal{G}'}^{-1}(q'(\delta')) \to [\mathcal{G}']_{[\delta',\beta']}$ *and* $\iota\colon \pi_{\mathcal{G}}^{-1}(q(\delta)) \to [\mathcal{G}]_{[\alpha,\delta]}$ *are the inclusions.*

3. Departing from a mechanical linkage $\mathcal{S}_g$, $g \in \mathbb{N}$ with $[\mathcal{S}_g] \approx \Sigma_g$ we give $\mathcal{S}_{g+1}$ corresponding to the construction of $\mathcal{G}' \supset \mathcal{G}$ in the preceding parts of the proof. First we need a sequence of special subsets of a closed interval $I \subset \mathbb{R}$: $U_1 < \cdots < U_n$ is called an **increasing decomposition of** I, if

(1) $U_1 \cup \cdots \cup U_n = I$,

(2) $U_i \cap U_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$ and

(3) $U_i < U_j :\Leftrightarrow \sup(U_i) \leq \inf(U_j)$ for all $i, j \in \{1, \ldots, n\}$, $i < j$.

Let $g = 1$ and define $\mathcal{S}_1 = \mathcal{P}_l$ as in Example 3.2 by changing the denotations so that V_1 of $\mathcal{P}_l$ corresponds to O_1 of $\mathcal{S}_1$, V_2 to O_0, V_5 to A_1 and l_5 to r_1 respectively, i.e. $O_0 := V_2, O_1 := V_1 \in V_{fix}$. Then

$$\mathbb{W}_{\mathcal{S}_1}(A_1) = \left\{r_1 e^{i\varphi_1};\ \varphi_1 \in [\alpha_1, \beta_1]\right\}$$

where $\alpha_1 = -\arccos(\frac{1}{6})$ and $\beta_1 = \arccos(\frac{1}{6})$, therefore $\beta_1 - \alpha_1 < \pi$. In particular there is $[\mathcal{S}_1] \approx \Sigma_1$ and $\pi_{\mathcal{S}_1} : [\mathcal{S}_1] \to \mathbb{W}_{\mathcal{S}_1}(A_1)$ with

$$(*_1) \qquad \pi_{\mathcal{S}_1}^{-1}\left\{r_1 e^{i\varphi_1};\ \varphi_1 \in U\right\} \approx \begin{cases} pt & \text{for } U = \{\alpha_1\}, \{\beta_1\} \\ S^1 \vee S^1 & \text{for } U = \{\gamma_1^-\}, \{\gamma_1^+\} \\ U \times S^1 & \text{for } U = I_1, I_2 \\ U \times (S^1 \amalg S^1) & \text{for } U = M_1 \end{cases}$$

where I_1, I_2, M_1 are open intervals and $\{\alpha_1\} < I_1 < \{\gamma_1^-\} < M_1 < \{\gamma_1^+\} < I_2 < \{\beta_1\}$ is an increasing decomposition of $[\alpha_1, \beta_1]$; see Figure 3.

We assume a mechanical linkage $\mathcal{S}_g$ with $[\mathcal{S}_g] \approx \Sigma_g$ for $g \in \mathbb{N}$ and

$$\mathbb{W}_{\mathcal{S}_g}(A_g) = \left\{q_g(\varphi_g) = p_g + r_g e^{i\varphi_g};\ \varphi_g \in [\alpha_g, \beta_g]\right\}$$

with $\beta_g - \alpha_g < \pi$ for the vertex A_g of $\mathcal{S}_g$. In addition we assume $\pi_{\mathcal{S}_g}\colon [\mathcal{S}_g] \to \mathbb{W}_{\mathcal{S}_g}(A_g)$, such that

$(*_g)$

$$\pi_{\mathcal{S}_g}^{-1}\left\{q_g(\varphi_g);\ \varphi_g \in U\right\} \approx \begin{cases} pt & \text{for } U = \{\alpha_g\}, \{\beta_g\} \\ S^1 \vee S^1 & \text{for } U = \{\gamma_1^-\}, \{\gamma_1^+\}, \ldots, \{\gamma_g^-\}, \{\gamma_g^+\} \\ U \times S^1 & \text{for } U = I_1, \ldots, I_{g+1} \\ U \times (S^1 \amalg S^1) & \text{for } U = M_1, \ldots, M_g \end{cases}$$

where I_j, M_j are open intervals and $\{\alpha_g\} < I_1 < \{\gamma_1^-\} < M_1 < \{\gamma_1^+\} < I_2 < \cdots < I_g < \{\gamma_g^-\} < M_g < \{\gamma_g^+\} < I_{g+1} < \{\beta_g\}$ is an increasing decomposition of $[\alpha_g, \beta_g]$.

By Lemma 4.1 there exists a mechanical linkage $\mathcal{S}_{g+1} \supset \mathcal{S}_g$, such that

$$\mathbb{W}_{\mathcal{S}_{g+1}}(A_g) = \{q_g(\varphi_g);\ \varphi_g \in [\alpha_g, \delta_g]\},$$
$$\mathbb{W}_{\mathcal{S}_{g+1}}(A_{g+1}) = \left\{p_{g+1} + r_{g+1} e^{i\delta_{g+1}};\ \varphi_{g+1} \in [\alpha_{g+1}, \beta_{g+1}]\right\}$$

and $\beta_{g+1} - \alpha_{g+1} < \pi$ for any $\delta_g \in]\alpha_g, \beta_g[$. Let us distinguish two cases: if g is even, then take $\delta_g \in M_{g/2+1}$ and conversely, if g is odd, then $\delta_g \in I_{(g+1)/2+1}$. By Lemma 4.2 we obtain $[\mathcal{S}_{g+1}] \approx [\mathcal{S}_g]_{[\alpha_g,\delta_g]} \cup_\iota [\mathcal{S}_g]_{[\alpha_g,\delta_g]}$ where $[\mathcal{S}_g]_{[\alpha_g,\delta_g]} := \pi_{\mathcal{S}_g}^{-1}(\mathrm{W}_{\mathcal{S}_{g+1}}(A_g))$ and $\iota : \pi_{\mathcal{S}_g}^{-1}(q_g(\delta_g)) \to [\mathcal{S}_g]_{[\alpha_g,\delta_g]}$ is the inclusion. The above description of $\pi_{\mathcal{S}_g} : [\mathcal{S}_g] \to \mathrm{W}_{\mathcal{S}_g}(A_g)$ implies that $[\mathcal{S}_{g+1}]$ is an orientable compact two-dimensional manifold since $\pi_{\mathcal{S}_g}^{-1}(q_g(\delta_g)) = \mathrm{bd}[\mathcal{S}_g]_{[\alpha_g,\delta_g]}$ is homeomorphic to $S^1 \amalg S^1$ (g even) or to S^1 (g odd). This allows us to compute $\chi([\mathcal{S}_{g+1}]) = 2(1-(g+1))$, thus $[\mathcal{S}_{g+1}] \approx \Sigma_{g+1}$. For $\pi_{\mathcal{S}_{g+1}}$: $[\mathcal{S}_{g+1}] \to \mathrm{W}_{\mathcal{S}_{g+1}}(A_{g+1})$ the relation $(*_{g+1})$ holds where I_j, M_j are open intervals and $\{\alpha_{g+1}\} < I_1 < \{\gamma_1^-\} < M_1 < \{\gamma_1^+\} < I_2 < \cdots < I_{g+1} < \{\gamma_{g+1}^-\} < M_{g+1} < \{\gamma_{g+1}^+\} < I_{g+2} < \{\beta_{g+1}\}$ is an increasing decomposition of $[\alpha_{g+1}, \beta_{g+1}]$, so we obtain identical properties for $\mathcal{S}_{g+1}$ as assumed in $\mathcal{S}_g$. This completes the proof. ∎

Remark 4.3: To make the inductive construction of $\mathcal{S}_g$ more easily we used Lemma 4.2 to add handles $g-1$ times. This defines a mechanical linkage with $3+2g$ edges. However, adding two edges to the construction may double the number of the genus. By suitable binary encoding of g one can achieve a configuration space homeomorphic to Σ_g with at most $5+2\log_2(g)$ edges.

Remark 4.4: For $\mathcal{S}_g$ we have $V_{fix} = \{O_0, O_1, \ldots, O_g\}$ pinned down in the plane with dim aff$\{\xi(O_0), \xi(O_1), \ldots, \xi(O_g)\} = 2$ whenever $g \geq 2$. Adding all edges of $\mathrm{Pot}_2(V_{fix}) := \{\{O_i, O_j\}; O_i, O_j \in V_{fix}, i \neq j\}$ to the set E and extending the weight function d on $\mathrm{Pot}_2(V_{fix})$ defines a mechanical linkage $\tilde{\mathcal{S}}_g$. If the configuration space is introduced as *all realizations of $\tilde{\mathcal{S}}_g$ in the plane modulo proper Euclidean motions*, then we obtain $[\mathcal{S}_g] \amalg [\mathcal{S}_g] \approx \Sigma_g \amalg \Sigma_g$ as the configuration space of $\tilde{\mathcal{S}}_g$.

Remark 4.5: The work [7] makes available a constructive method to produce a huge mechanical linkage with one component of its configuration space being a projective plane, a Klein Bottle or even a non-orientable compact closed surface of any genus. But in contrast to the orientable case it seems to be much harder to find such an easy analyzable mechanical linkage, whose configuration space is exactly a non-orientable compact closed surface. Because of Proposition 2.6 this may be impossible.

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