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A Saddlepoint Approximation to the Distribution of Inhomogeneous Discounted Compound Poisson Processes

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Abstract In this article we propose an accurate approximation to the distribution of the discounted total claim amount, where the individual claim amounts are independent and identically distributed and the number of claims over a specified period is governed by an inhomogeneous Poisson process. More precisely, we compute cumulant generating functions of such discounted total claim amounts under various intensity functions and individual claim amount distributions, and invert them by the saddlepoint approximation. We provide precise conditions under which the saddlepoint approximation holds. The resulting approximation is numerically accurate, computationally fast and hence more efficient than Monte Carlo simulation.

Keywords Cumulant generating function · Intensity function · Interest rate · Monte Carlo · Shot-noise process · Total claim amount

AMS 2000 Subject Classifications 60G55 · 41A60

1 Introduction

The Poisson process is among the simplest stochastic processes and it is popular in actuarial science. The compound Poisson process is commonly used to model the total claim amount generated by a portfolio of risks over a given period.

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Some important references of compound Poisson processes in the context of actuarial risk theory are, for example, Bühlmann (1970), Gerber (1979) and Mikosch (2004). For situations where the claim occurrence frequency is time dependent, the inhomogeneous Poisson process is more appropriate to model the number of claims than the homogeneous one. The distribution of the homogeneous compound Poisson process has been studied by several authors. Here we mention some of them. Esscher (1932, 1963) proposed an approximation based on a local Edgeworth expansion applied after an exponential tilt, or Esscher transform, of the underlying distribution, see Field and Ronchetti (1990) for example. Daniels (1954) derived a saddlepoint approximation to the density of the sample mean and showed that it corresponds to the tilted Edgeworth expansion. Embrechts et al. (1985) proposed a saddlepoint approximation for the total claim amount under the Pólya process (which is a mixed Poisson process with a gamma mixing distribution). Willmot (1989) provided an infinite series representation for the density of the total claim amount for which each summand is a convolution of power equal to its summation order. He also provided a renewal equation for this density. Jensen (1991) proposed saddlepoint approximations for homogeneous compound Poisson processes extended in the following ways: by adding a Wiener process, representing investment uncertainty, by taking the deterministic interest rate into account and by considering Markov modulation, meaning that there are various compound Poisson processes with different intensity parameters running in series and the switch to one to the next is determined by the transition of the state of an underlying Markov chain. For Markov-modulated compound Poisson processes in ruin theory, refer to Asmussen (2000).

In this article we consider both the inhomogeneity of the claim arrival process and the impact of interest rate on the determination of the total claim amount. We propose accurate analytical approximations of the distributions of total claim amounts for various choices of inhomogeneous Poisson processes and claim amount distributions, while taking the deterministic interest rate into account. These approximations are based on the saddlepoint approximation and we give precise conditions under which the saddlepoint approximation holds. We provide closed form or infinite series expressions of the cumulant generating functions of total claim amounts for the following pairs of intensity function and individual claim amount distribution: constant intensity and linear combination of exponentials claim amount distribution, constant intensity and gamma claim amount distribution, linear combination of exponentials intensity function and exponential claim amount distribution, gamma intensity function and exponential claim amount distribution, and polynomial intensity function and exponential claim amount distribution. Moreover, we give numerical comparisons with respect to Monte Carlo algorithms for simulating Poisson processes and comment on the drawbacks and the advantages of the two methods. The Monte Carlo methods used here are exact in the sense that any differences between the desired and the simulated distributions are due solely to the pseudo-random number generator and to the finiteness of the number of generations. On the other side, the saddlepoint approximation is only the leading term of an expansion, which can be found for example in Daniels (1987, Eq. 4.5). Nevertheless, there is no practical difference in terms of numerical accuracy between these two methods. Besides this, the saddlepoint approximation is conceptually simpler and computationally faster than Monte Carlo simulation. We also illustrate that, for some choices of the intensity function, Monte Carlo methods can fail while the saddlepoint maintains its high accuracy.

The remaining part of the article is organized as follows. In Section 2 we give a general result, under Lemma 2.1, from which we derive cumulant generating functions of various discounted total claim amounts, under Examples 2.1 to 2.5. In Section 3, under Result 3.1, we first adapt the Lugannani and Rice (1980) saddlepoint approximation to the case where the distribution to approximate has a positive probability mass at zero and then, under Lemma 3.1, we give precise conditions under which the saddlepoint approximation holds. The existence of the moment generating function of the individual claim amounts in a neighborhood of zero is neither a sufficient nor a necessary condition to have a saddlepoint approximation. In Section 4, we give practical details for the implementation of three chosen examples and we show numerical comparisons of the saddlepoint approximation with Monte Carlo methods for inhomogeneous Poisson processes. These Monte Carlo methods are briefly summarized in the Appendix. We end with some remarks in Section 5, which include some extensions to the case of time-varying interest rate.

2 Cumulant Generating Functions

We begin this section with a general result, under Lemma 2.1, from which we derive cumulant generating functions of various homogeneous and inhomogeneous discounted total claim amounts, under Examples 2.1 to 2.5. Suppose $X_1, X_2, \ldots > 0$ are independent and identically distributed (i.i.d.) individual claim amounts, $0 < T_1 < T_2 < \ldots$ are the corresponding times of claim occurrences, which are supposed independent of the individual claim amounts and which arise from an inhomogeneous Poisson process on R_+. Let us define the number of claims occurring during the fixed time interval $[0, t]$ by $N_t = \max\{k \geq 0 | T_k \leq t\}$, where $T_0 \stackrel{\text{def}}{=} 0$ for convenience. We assume that the Poisson process has an expectation function $\Lambda : \mathrm{R}_+ \to \mathrm{R}_+$ which is absolutely continuous with respect to the Lebesgue measure, i.e. for which there exists an intensity function $\lambda : \mathrm{R}_+ \to \mathrm{R}_+$ such that $\Lambda(s_2) - \Lambda(s_1) = \int_{s_1}^{s_2} \lambda(y)dy$, for all $0 \leq s_1 < s_2 < \infty$. We consider $Z(N_t) = \sum_{i=0}^{N_t} e^{r(t-T_i)} X_i$, with $X_0 \stackrel{\text{def}}{=} 0$, which has the two following practical interpretations. If $r > 0$, then $Z(N_t)$ is the total claim amount incurred during $[0, t]$ compounded to time t by the constant force of interest r of all past reimbursements made at the times $T_1, \ldots, T_{N_t}$ prior to t. If $r < 0$, then $Z(N_t)$ is the future total claim amount during $[0, t]$ discounted to time 0 by the interest rate $-r > 0$, of all future reimbursements at the future times $T_1, \ldots, T_{N_t}$ before t.

We generally denote by f_U the density function with respect to the Lebesgue measure of any random variable U. For $v \in \mathrm{R}$, we denote by $M_U(v) = \mathrm{E}[e^{vU}]$ and by $K_U(v) = \log\{\mathrm{E}[e^{vU}]\}$ the moment and the cumulant generating functions of U respectively. Whenever we have a sequence of identically distributed random variables $U_1, U_2, \ldots$, we denote unambiguously by f_U, M_U, and K_U the density, the moment and the cumulant generating functions of any random variable of this sequence.

Lemma 2.1 *Cumulant generating function of the discounted total claim amount.*

Assume that the moment generating function of X_1, $M_X(v) = \mathrm{E}[e^{vX_1}]$, exists for all $v \in (-\infty, c)$, where $0 \leq c \leq \infty$. Assume further that $\lambda(s)$ is bounded for all $s \in [0, t]$. Then, the cumulant generating function of $Z(N_t)$ is given by

$$K_{Z(N_t)}(v) = \int_0^t M_X(ve^{r(t-y)})\lambda(y)dy - \Lambda(t),$$

for all $t > 0$ and $v < \gamma_t \stackrel{\text{def}}{=} c\min\{1, e^{-rt}\}$.

Proof The order statistics property of the Poisson process states that for any $n \geq 1$ and $t > 0$, $(T_1, \ldots, T_{N_t})|(N_t = n) \sim (Y_{(1)}, \ldots, Y_{(n)})$, where $Y_1, Y_2, \ldots$ are independent random variables with common density

$$f_Y(y) \stackrel{\text{def}}{=} \frac{\lambda(y)}{\Lambda(t)}, \tag{1}$$

$y \in [0, t)$, $Y_{(1)} \leqslant \ldots \leqslant Y_{(n)}$ is the order statistics of $Y_1, \ldots, Y_n$, and where the symbol "$\sim$" denotes the equivalence in distribution. Let us assume $Y_1, Y_2, \ldots$ independent of all other random variables considered here. Consider a Borel-measurable function $\psi : \mathrm{R}^2 \to \mathrm{R}$. Then it follows from the order statistics property that

$$\sum_{i=0}^{N_t} \psi(T_i, X_i) \sim \sum_{i=0}^{N_t} \psi(Y_i, X_i), \tag{2}$$

where $\psi(0, 0) \stackrel{\text{def}}{=} 0$. For further details about this well-known result, interested readers may refer to, for example, Mikosch (2004). We consider the function $\psi(s, x) = e^{r(t-s)}x$, for $s \in [0, t]$, $r \in \mathrm{R}$ and $x > 0$. Since $\psi(s, x)$ is continuous on R^2, it is Borel-measurable on R^2. Let us define $W_{ti} = \psi(Y_i, X_i) = e^{r(t-Y_i)}X_i$, $i = 1, 2, \ldots$, and $W_{t0} \stackrel{\text{def}}{=} 0$. It follows from Eq. 2 that $Z(N_t) \sim \sum_{i=0}^{N_t} W_{ti}$. We have that $M_{N_t}(v) = \mathrm{E}[e^{vN_t}] = \exp\{\Lambda(t)[e^v - 1]\}$ is the moment generating function of N_t. With this we can compute the moment generating function of W_{t1} by

$$\begin{aligned} M_{W_t}(v) &= \mathrm{E}[e^{vW_{t1}}] = \mathrm{E}[\mathrm{E}[\exp\{ve^{r(t-Y_1)}X_1\}|Y_1]] \\ &= \int_0^t M_X(ve^{r(t-y)})f_Y(y)dy = \frac{1}{\Lambda(t)}\int_0^t M_X(ve^{r(t-y)})\lambda(y)dy. \end{aligned} \tag{3}$$

This expression can be used for computing the moment generating function of $Z(N_t)$ by

$$M_{Z(N_t)}(v) = \mathrm{E}[e^{v\sum_{i=0}^{N_t} W_{ti}}] = M_{N_t}(\log\{M_{W_t}(v)\}) = \exp\{\Lambda(t)[M_{W_t}(v) - 1]\}.$$

Hence the cumulant generating function of $Z(N_t)$ is $K_{Z(N_t)}(v) = \log\{M_{Z(N_t)}(v)\} = \Lambda(t)\ [M_{W_t}(v) - 1]$. We finally note that, because $\lambda(s)$ is assumed bounded for $s \in [0, t]$, $M_{W_t}(v)$ does exist if $M_X(ve^{r(t-y)})$ exists for almost every $y \in [0, t]$. Because the existence of $M_X(v)$ is assumed for all $v \in (-\infty, c)$, $M_{W_t}(v)$ exists for all $v < ce^{-rt}$, if $r > 0$, and for all $v < c$, if $r < 0$. □

We now show how the cumulant generating function of $Z(N_t)$ can be obtained for some important choices of the individual claim amount distribution and of the intensity function. We first give results for the homogeneous Poisson process and the following claim amount distributions: linear combination of exponentials, in

Example 2.1, and gamma, in Example 2.2. Then we give results with inhomogeneous Poisson processes with exponential claim amounts and with the following intensity functions: linear combination of exponentials, in Example 2.3, gamma, in Example 2.4, and polynomial, in Example 2.5.

Example 2.1 Constant intensity and linear combination of exponentials claim amount.

Let us consider a homogeneous Poisson process for the times of claims, with constant intensity function $\lambda(s) = \lambda \in (0, \infty)$, for all $s \in [0, t]$, and the linear combination of exponentials individual claim amount density $f_X(x) = \sum_{j=1}^{k} \alpha_j \nu_j e^{-\nu_j x}$, for all $x > 0$, where $\alpha_1, \ldots \alpha_k \in \mathrm{R}$ are chosen so that f_X is a density and $\nu_1, \ldots, \nu_k > 0$. Then

$$K_{Z(N_t)}(v) = \frac{\lambda}{r} \sum_{j=1}^{k} \alpha_j \log\left(\frac{\nu_j - v}{\nu_j - ve^{rt}}\right), \tag{4}$$

for all $v < \gamma_t = \min\{\nu_1, \ldots, \nu_k\} \min\{1, e^{-rt}\}$ and $r \neq 0$. Note that a necessary condition for f_X to be a density is $\sum_{j=1}^{k} \alpha_j = 1$, whereas $\alpha_j \geq 0$, $j = 1, \ldots, k$ is not one, as shown in Example 2.1 (continued) in Section 4.

The justification of Eq. 4 is the following. By the partial fraction decomposition we obtain

$$\begin{aligned} M_{W_t}(v) &= \frac{1}{\Lambda(t)} \lambda \sum_{j=1}^{k} \alpha_j \nu_j \int_0^t \frac{1}{\nu_j - ve^{r(t-y)}} dy = \frac{1}{\Lambda(t)} \frac{\lambda}{r} \sum_{j=1}^{k} \alpha_j \nu_j \int_1^{e^{-rt}} \left(\frac{1}{uve^{rt} - \nu_j}\right) \frac{du}{u} \\ &= \frac{1}{\Lambda(t)} \frac{\lambda}{r} \sum_{j=1}^{k} \alpha_j \left[\log\left(\frac{uve^{rt} - \nu_j}{u}\right)\right]_1^{e^{-rt}} = \frac{1}{\Lambda(t)} \frac{\lambda}{r} \sum_{j=1}^{k} \alpha_j \log\left(\frac{\nu_j - v}{\nu_j e^{-rt} - v}\right), \end{aligned} \tag{5}$$

when $v < \gamma_t$ and $r \neq 0$. Equation 4 follows after applying Lemma 2.1 to Eq. 5.

Example 2.2 Constant intensity and gamma claim amounts.

Let us consider a homogeneous Poisson process for the times of claim and a gamma claim amount density $f_X(x) = \nu^\alpha e^{-\nu x} x^{\alpha-1} / \Gamma(\alpha)$, for all $x, \nu > 0$ and $\alpha \in \mathrm{N} \backslash \{0\}$. Under this restriction on the values of α we obtain

$$K_{Z(N_t)}(v) = \frac{\lambda}{r} \log\left(\frac{\nu - v}{\nu - ve^{rt}}\right) + \frac{\lambda}{r} \sum_{i=1}^{\alpha-1} \binom{\alpha-1}{i} \frac{1}{i} \left[\left(\frac{ve^{rt}}{\nu - ve^{rt}}\right)^i - \left(\frac{v}{\nu - v}\right)^i\right], \tag{6}$$

where $\sum_{i=1}^{0} \stackrel{\text{def}}{=} 0$ and for all $v < \gamma_t = \nu \min\{1, e^{-rt}\}$ and $r \neq 0$.

For the proof of Eq. 6, we first compute

$$M_{W_t}(v) = \frac{1}{\Lambda(t)}\lambda\nu^{\alpha}\int_0^t \left(\frac{1}{\nu - ve^{r(t-y)}}\right)^{\alpha} dy = -\frac{1}{\Lambda(t)}\frac{\lambda\nu^{\alpha}}{r}\int_1^{e^{-rt}}\left(\frac{1}{\nu - uve^{rt}}\right)^{\alpha}\frac{du}{u}$$

$$= -\frac{1}{\Lambda(t)}\frac{\lambda\nu^{\alpha}}{r}\left(-\frac{1}{\nu^{\alpha}}\right)\left[\log\left(\frac{\nu - uve^{rt}}{u}\right) - \sum_{i=1}^{\alpha-1}\binom{\alpha-1}{i}\frac{(uve^{rt})^i}{i(\nu - uve^{rt})^i}\right]_1^{e^{-rt}}$$

$$= \frac{1}{\Lambda(t)}\frac{\lambda}{r}\left[\log\left(\frac{\nu - v}{\nu e^{-rt} - v}\right) + \sum_{i=1}^{\alpha-1}\binom{\alpha-1}{i}\frac{1}{i}\left(\frac{v}{\nu - v}\right)^i \right.$$

$$\left. \times \left\{1 - \left(\frac{\nu - v}{\nu e^{-rt} - v}\right)^{-i}\right\}\right], \qquad (7)$$

when $v < \gamma_t$ and $r \neq 0$. The case $\alpha = 1$ corresponds to Eq. 4 with $k = 1$. By applying Lemma 2.1 to Eq. 7 we obtain Eq. 6.

In Example 2.3 we consider a linear combination of exponential functions as intensity function. This choice is used in problems of radioactive decay, for example to model the emission of photons by k radioactive sources. It is also useful in actuarial problems, when one or more new technologies which prevent a particular risk are introduced, for example new anti-theft technologies.

Example 2.3 Linear combination of exponentials intensity and exponential claim amounts.

Consider the linear combination of exponentials intensity $\lambda(s) = \alpha_0 + \sum_{j=1}^{k}\alpha_j e^{-\theta_j s}$, for all $s \in [0, t]$, $\theta_1, \ldots, \theta_k > 0$, and the exponential claim amount density $f_X(x) = \nu e^{-\nu x}$, for all $x, \nu > 0$. Then we have

$$K_{Z(N_t)}(v) = \frac{\alpha_0}{r}\log\left(\frac{\nu - v}{\nu - ve^{rt}}\right) + \sum_{j=1}^{k}\alpha_j\left(\sum_{n=0}^{\infty}\frac{(\nu^{-1}ve^{rt})^n}{nr + \theta_j}(1 - e^{-(nr+\theta_j)t}) - \theta_j^{-1}(1 - e^{-\theta_j t})\right), \qquad (8)$$

for all v such that $|v| < \gamma_t = \nu\min\{1, e^{-rt}\}$ and $r \neq 0$.

For the proof of Eq. 8, we first compute

$$M_{W_t}(v) = \frac{1}{\Lambda(t)}\int_0^t \frac{1}{1 - \nu^{-1}ve^{r(t-y)}}\left(\alpha_0 + \sum_{j=1}^{k}\alpha_j e^{-\theta_j y}\right) dy$$

$$= \frac{1}{\Lambda(t)}\left(\frac{\alpha_0}{r}\log\left(\frac{\nu - v}{\nu e^{-rt} - v}\right) + \sum_{j=1}^{k}\alpha_j\sum_{n=0}^{\infty}\frac{(\nu^{-1}ve^{rt})^n}{nr + \theta_j}[1 - e^{-(nr+\theta_j)t}]\right), \qquad (9)$$

when $r \neq 0$ and $|v| < \gamma_t$. Equation 8 follows after applying Lemma 2.1 to Eq. 9 with the substitution $\Lambda(t) = \alpha_0 t + \sum_{j=1}^{k}\alpha_j\theta_j^{-1}(1 - e^{-\theta_j t})$. Note that these formulas hold also if any of $\theta_1, \ldots \theta_k$ are negative.

Example 2.4 Gamma intensity and exponential claim amounts.

Consider the gamma type intensity $\lambda(s) = \alpha_0 + \alpha_1 s^{a-1} e^{-bs}$, for all $s \in [0, t]$, a, $b > 0$, and the exponential claim amount density $f_X(x) = \nu e^{-\nu x}$, for all $x, \nu > 0$. Then we have

$$K_{Z(N_t)}(v) = \frac{\alpha_0}{r} \log\left(\frac{\nu - v}{\nu - v e^{rt}}\right) + \alpha_1 \sum_{n=1}^{\infty} \frac{(\nu^{-1} v e^{rt})^n}{(nr + b)^a} \Gamma(a, [nr + b]t), \tag{10}$$

for all v such that $|v| < \gamma_t = \nu e^{-rt}$ and $r > 0$ and where

$$\Gamma(a, x) = \int_0^x u^{a-1} e^{-u} du, \tag{11}$$

for all $a, x > 0$, is the incomplete gamma function.

The proof of Eq. 10 is the following. We cannot apply Lemma 2.1 directly because $\lambda(s)$ is unbounded for $s \in [0, t]$ when $a \in (0, 1)$. Given

$$M_{W_t}(v) = \frac{1}{\Lambda(t)} \int_0^t \frac{1}{1 - \nu^{-1} v e^{r(t-y)}} (\alpha_0 + \alpha_1 y^{a-1} e^{-by}) dy, \tag{12}$$

we note that the ratio inside the integral is bounded if for no $y \in [0, t]$, $\nu^{-1} v e^{r(t-y)} = 1$. This is implied by $v < \nu e^{-rt}$, when $r > 0$, and by $v < \nu$, when $r < 0$, i.e. by $v < \gamma_t$. Hence the integral in Eq. 12 converges if $v < \gamma_t$. We can expand integral in Eq. 12 as

$$M_{W_t}(v) = \frac{1}{\Lambda(t)} \left(\frac{\alpha_0}{r} \log\left(\frac{\nu - v}{\nu e^{-rt} - v}\right) + \alpha_1 \sum_{n=0}^{\infty} \frac{(\nu^{-1} v e^{rt})^n}{(nr + b)^a} \Gamma(a, [nr + b]t) \right), \tag{13}$$

when $r > 0$ and $|v| < \gamma_t$. Equation 10 follows after applying Lemma 2.1 to Eq. 13, given that $\Lambda(t) = \alpha_0 t + \alpha_1 b^{-a} \Gamma(a, bt)$.

Example 2.5 Polynomial intensity and exponential claim amounts.

Consider the polynomial intensity $\lambda(s) = \alpha_0 + \sum_{j=1}^{k} \alpha_j s^k$, for all $s \in [0, t]$, and the exponential claim amount density $f_X(x) = \nu e^{-\nu x}$, for all $x, \nu > 0$. Then we have

$$K_{Z(N_t)}(v) = \frac{\alpha_0}{r} \log\left(\frac{\nu - v}{\nu - v e^{rt}}\right) + \sum_{j=1}^{k} \alpha_j \left(\sum_{n=0}^{\infty} \frac{(\nu^{-1} v e^{rt})^n}{n^{j+1}} \Gamma(j+1, nt) - \frac{t^{j+1}}{j+1} \right), \tag{14}$$

for all v such that $|v| < \gamma_t = \nu \min\{1, e^{-rt}\}$ and $r \neq 0$ and where $\Gamma(\cdot, \cdot)$ is the incomplete gamma function (Eq. 11).

For the proof of Eq. 14, we first compute

$$\begin{aligned} M_{W_t}(v) &= \frac{1}{\Lambda(t)} \int_0^t \frac{1}{1 - \nu^{-1} v e^{r(t-y)}} \left(\alpha_0 + \sum_{j=1}^{k} \alpha_j y^j \right) dy \\ &= \frac{1}{\Lambda(t)} \left(\frac{\alpha_0}{r} \log\left(\frac{\nu - v}{\nu e^{-rt} - v}\right) + \sum_{j=1}^{k} \alpha_j \sum_{n=0}^{\infty} (\nu^{-1} v e^{rt})^n \int_0^t e^{-ny} y^j dy \right) \\ &= \frac{1}{\Lambda(t)} \left(\frac{\alpha_0}{r} \log\left(\frac{\nu - v}{\nu e^{-rt} - v}\right) + \sum_{j=1}^{k} \alpha_j \sum_{n=0}^{\infty} (\nu^{-1} v e^{rt})^n n^{-(j+1)} \Gamma(j+1, nt) \right) \end{aligned} \tag{15}$$

when $r \neq 0$ and $|v| < \gamma_t$. Equation 14 follows after applying Lemma 2.1 to Eq. 15 with the substitution $\Lambda(t) = \alpha_0 t + \sum_{j=1}^{k} \alpha_j t^{j+1}/(j+1)$.

3 Saddlepoint Approximation

The saddlepoint approximation yields accurate approximations to quite general complex integrals. When these integrals are Fourier transforms, the saddlepoint method leads to approximations to densities and tail probabilities. Since its introduction in statistics by Daniels (1954), there have been several extensions and applications in both statistics and applied probability. Some general references are Barndorff-Nielsen and Cox (1989), Field and Ronchetti (1990) and Jensen (1995). Result 3.1 below gives the Lugannani and Rice (1980) saddlepoint approximation to the upper tail probabilities of the inhomogeneous discounted compound Poisson process. The conditions for the existence of this saddlepoint approximation are given later under Lemma 3.1.

Result 3.1 *Saddlepoint approximation of the discounted total claim amount.*

Let us define

$$K(v) = \log\{\exp\{K_{Z(N_t)}(v)\} - e^{-\Lambda(t)}\} - \log\{1 - e^{-\Lambda(t)}\}, \tag{16}$$

and let us denote $K'(v) = (d/dv)K(v)$ and $K''(v) = (d/dv)^2 K(v)$. Then the Lugannani and Rice saddlepoint approximation to $\bar{F}_t(x) = \mathrm{P}[Z(N_t) > x]$ is given by

$$\bar{G}_t(x) = \left[1 - \Phi(r_x) + \phi(r_x)\{s_x^{-1} - r_x^{-1}\}\right][1 - e^{-\Lambda(t)}], \tag{17}$$

where $s_x = v_x\{K''(v_x)\}^{1/2}$, $r_x = \mathrm{sgn}(v_x)\{2[v_x x - K(v_x)]\}^{1/2}$, $v_x \in \mathrm{R}$ is the saddlepoint implicitly defined as the solution in v of

$$K'(v) = x, \tag{18}$$

ϕ and Φ are the standard normal density and distribution functions and $x > 0$ satisfies the conditions of Lemma 3.1.

Proof Lugannani and Rice (1980) provided saddlepoint approximations of the tail probabilities of the sum of absolutely continuous random variables. In order to use their approximation for our problem, we first need to deal with the probability mass at zero of $Z(N_t)$. From $\mathrm{E}[e^{vZ(N_t)}] = \mathrm{E}[e^{vZ(N_t)}|N_t > 0]\mathrm{P}[N_t > 0] + \mathrm{P}[N_t = 0]$ it follows that the cumulant generating function of the absolutely continuous random variable $Z^*(N_t) \stackrel{\text{def}}{=} Z(N_t)|(N_t > 0)$ is given by Eq. 16. Let us define the distribution functions $F_t(x) = \mathrm{P}[Z(N_t) \leq x]$ and $F_t^*(x) = \mathrm{P}[Z^*(N_t) \leq x]$, for all $x > 0$, and the upper tail probability function $\bar{F}_t^* = 1 - F_t^*$. Then, for all $x \geq 0$, $\mathrm{P}[Z(N_t) > x] = \mathrm{P}[Z(N_t) > x|N_t > 0]\mathrm{P}[N_t > 0] = \mathrm{P}[Z^*(N_t) > x]\mathrm{P}[N_t > 0]$, i.e.

$$\bar{F}_t(x) = \bar{F}_t^*(x)(1 - e^{-\Lambda(t)}). \tag{19}$$

The Lugannani and Rice (1980) saddlepoint approximation to the integral

$$\bar{F}_t^*(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \exp\{K(iv) - ivx\}\frac{dv}{v} \tag{20}$$

is given by

$$\bar{G}_t^*(x) = 1 - \Phi(r_x) + \phi(r_x)\{s_x^{-1} - r_x^{-1}\}. \tag{21}$$

The standard saddlepoint technique, as used by Daniels (1954), cannot be applied to approximate the integral in Eq. 20 because of the presence of a pole at $v = 0$ in the integrand. Lugannani and Rice (1980) used a particular technique for dealing with this pole; the interested reader can also refer to Daniels (1987, Section 4) for precise explanations. By transforming Eq. 21 in analogy with Eq. 19 we find

$$\bar{G}_t(x) = \bar{G}_t^*(x)(1 - e^{-\Lambda(t)}) \tag{22}$$

as saddlepoint approximation to $\bar{F}_t(x)$, as given in Eq. 17. □

We now give a few short remarks. First, the convexity of K implies that r_x and $s_x \in \mathrm{R}$. Second, the derivatives of orders one or higher of K are identical to the ones of $K_{Z(N_t)}$. Next, if we define $z_x = r_x + r_x^{-1}\log(r_x^{-1}s_x)$, for $x > 0$, then the approximation to $\bar{F}_t^*(x)$ given by

$$\bar{H}_t^*(x) = 1 - \Phi(z_x)$$

is known to have the same accuracy as Eq. 21 and hence

$$\bar{H}_t(x) = \bar{H}_t^*(x)(1 - e^{-\Lambda(t)}) \tag{23}$$

provides an alternative formula to Eq. 17. One advantage of Eq. 23 is that it yields values in $[0, 1]$ only. The next remark is that when $\Lambda(t)$ is large, the transform of $K_{Z(N_t)}$ to K given by Eq. 16 together with the related re-centering of the tail probability given by Eq. 22 are inessential. Finally, although we avoid a rigorous analysis of the asymptotic validity of the saddlepoint approximation, we can briefly mention that $K_{Z(N_t)}(v) = \Lambda(t)\kappa_t(v)$, with $\kappa_t(v) = \int_0^t [M_X(ve^{r(t-y)}) - 1] f_Y(y)dy$ and f_Y given by Eq. 1, has the form of the cumulant generating function of the sum of $\Lambda(t)$ i.i.d. random variables with cumulant generating function κ_t, supposing $\Lambda(t)$ to be a positive integer. Hence we are in the same setting of Lugannani and Rice (1980) and therefore we can expect our saddlepoint approximation to have bounded relative error as $\Lambda(t) \to \infty$, provided that κ_t behaves asymptotically like a constant.

In Lemma 3.1 below we analyze the existence of the saddlepoint. More precisely, we provide conditions under which Eq. 18, the saddlepoint equation, can be solved. It can happen that the saddlepoint v_x ceases to exist when x belongs to a particular interval of the domain of F_t and we call this interval the "cemetery" of the saddlepoint.

Lemma 3.1 *Existence of the saddlepoint approximation.*

Suppose that $M_{W_t}(v)$ exists for all $v \in (-\infty, \gamma_t)$, $0 \le \gamma_t \le \infty$.

(a) If $\gamma_t = \infty$, then the cemetery of the saddlepoint is empty, precisely the empty set. This happens if $M_X(v)$ exists for all $v \in \mathrm{R}$.
(b) If $\gamma_t < \infty$ and $K'(\gamma_t) < \infty$, then the cemetery is the interval

$$(x^\dagger, \infty) = (K'(\gamma_t), \infty) = \left(\frac{\Lambda(t)M'_{W_t}(\gamma_t)}{1 - e^{-\Lambda(t)M_{W_t}(\gamma_t)}}, \infty\right), \tag{24}$$

where $M'_{W_t}(v) = (d/dv)M_{W_t}(v)$.

(c) If $\gamma_t < \infty$, then the cemetery is empty iff

$$\lim_{v \to \gamma_t, v < \gamma_t} M'_{W_t}(v) = \infty. \tag{25}$$

A sufficient condition to have an empty cemetery is

$$\lim_{v \to \gamma_t, v < \gamma_t} M_{W_t}(v) = \infty. \tag{26}$$

Proof This proof is based on Daniels (1954, Section 6). Suppose for the moment that K is the cumulant generating function of some distribution function F (which may not have a density). For the formal validity of the saddlepoint approximation we require that $M(v) = \exp\{K(v)\}$ exists for $v \in (-c_1, c_2)$, where $0 \leq c_1, c_2 \leq \infty$ and $c_1 + c_2 > 0$. Assume that $F(x) = 0$, if $x < a$, $F(x) \in (0, 1)$, if $x \in (a, b)$ and $F(x) = 1$, if $x > b$. If $|a|, |b| < \infty$ then $c_1 = c_2 = \infty$. In this case we can prove the following facts: $K'(v) = x$ has no real root for $x < a$ or $x > b$; it has a unique simple real root for $x \in (a, b)$; and as v increases continuously from $-c_1 = -\infty$ to $c_2 = \infty$, $K'(v)$ increases continuously from a to b. If $|a| < \infty$ and $b = \infty$ then $c_1 = \infty$ and $c_2 \leq \infty$. In the case that $c_2 = \infty$ the previous facts hold again. In the case that $c_2 < \infty$, the validity of the previous facts requires the assumption

$$\lim_{v \to c_2, v < c_2} K'(v) = \infty. \tag{27}$$

Clearly, if $K'(c_2) = x^\dagger < \infty$, then $K'(v)$ increases continuously from a to $x^\dagger$, as v increases continuously from $-\infty$ to c_2, and jumps to ∞, as $v > c_2$. Thus, in this case $(x^\dagger, \infty)$ is the cemetery of the saddlepoint. Note that although the existence of the moment generating function within a neighborhood of zero is sometimes assumed for the saddlepoint approximation, this is neither a sufficient nor a necessary condition. This analysis leads to the following facts.

Regarding part (a) of Lemma 3.1, from $M(v) \stackrel{\text{def}}{=} M_{Z^*(N_t)}(v) = (\exp\{\Lambda(t) M_{W_t}(v)\} - 1)/(\exp\{\Lambda(t)\} - 1)$, we can say that the cemetery is empty if $\gamma_t = \infty$. Indeed, $\gamma_t = \infty$ leads to $c_2 = \infty$ in the previous analysis. Moreover, we can see from Eq. 3 that $\gamma_t = \infty$ holds whenever $M_X(v)$ exists for all $v \in \mathrm{R}$.

For part (b), if $\gamma_t < \infty$ and $K'(\gamma_t) < \infty$, then a direct computation shows that the cemetery is given by Eq. 24.

For part (c), the form of $x^\dagger$ in Eq. 24 indicates directly that the empty cemetery condition given by Eq. 27 can be simplified to Eq. 25. In addition to this, a sufficient condition to have an empty cemetery is given by Eq. 26, which is however not necessary, because $\lim_{v \to \gamma_t, v < \gamma_t} M_{W_t}(v) < \infty$ could be true when Eq. 25 holds. Hence, the finiteness of the left-hand-side of Eq. 26 is not implied by an empty cemetery. □

4 Implementations and Numerical Accuracies

In this section we test the effectiveness of the methods proposed and illustrate their numerical accuracies by comparisons with Monte Carlo methods. We focus on Examples 2.1, 2.2 and 2.4. We also briefly analyze the additional example of constant intensity and inverse-Gaussian claim amounts, which motivates a short comparison of the saddlepoint with the adjustment coefficient of actuarial risk theory.

Example 2.1 (continued) Constant intensity and linear combination of exponentials claim amount.

We consider the setting of Example 2.1 and we first show that the saddlepoint cemetery is empty. By differentiating Eq. 5 we obtain

$$M'_{W_t}(v) = \frac{1}{\Lambda(t)} \frac{\lambda}{r} \sum_{j=1}^{k} \alpha_j \left(\frac{1}{\nu_j e^{-rt} - v} - \frac{1}{\nu_j - v} \right). \tag{28}$$

The empty cemetery condition given by Eq. 25 of Lemma 3.1 is satisfied by $\gamma_t = \min\{\nu_1, \ldots, \nu_k\} \min\{1, e^{-rt}\}$: if $r > 0$, then there is a $j \in \{1, \ldots, k\}$ such that the first ratio in the parenthesis of the sum of Eq. 28 tends to ∞ as $v \to \gamma_t$, and so does the whole expression; else if $r < 0$, then there is a $j \in \{1, \ldots, k\}$ such that the second ratio in the parenthesis tends to $-\infty$ as $v \to \gamma_t$, but because the coefficient r^{-1} is negative, the whole expression tends again to ∞. Alternatively, we can see from Eq. 5 that $\lim_{v \to \gamma_t} M_{W_t}(v) = \infty$, so that the sufficient condition given by Eq. 26 of Lemma 3.1 for having an empty cemetery is satisfied, which is in accordance with the above justification. We now consider $\lambda = 1$, $\nu_1 = 1$, $\nu_2 = 2$, $\nu_3 = 3$, $\alpha_1 = 3$, $\alpha_2 = -3$, $\alpha_3 = 1$, $r = 0.1$ and $t = 10$. This linear combination of exponential distributions is the distribution of $E_1 + E_2 + E_3$, where the summands are independent and E_j has a density je^{-jx}, $x > 0$, $j = 1, 2, 3$. The numerical accuracy of the saddlepoint approximation to the upper tail probability of $Z(N_t)$ is shown in Fig. 1. The first graph of Fig. 1 shows the error $\bar{G}_t - \bar{F}_t$ of the Lugannani and Rice approximation $\bar{G}_t$, given by Eq. 17 in Result 3.1, when the exact upper tail probabilities $\bar{F}_t$ are actually computed by 10^6 Monte Carlo generations of the total claim amount. The second graph of Fig. 1 shows the corresponding relative error computed by $(\bar{G}_t - \bar{F}_t)/\min\{\bar{F}_t, 1 - \bar{F}_t\}$, which acts similarly for both left and right tail probabilities. The third graph of Fig. 1 shows the difference $\bar{H}_t - \bar{G}_t$ between the Barndorff-Nielsen formula $\bar{H}_t$, given by Eq. 23, and the Lugannani and Rice formula. All three graphs are plotted with respect to the values of $\bar{F}_t$. They show that the Lugannani and Rice formula is unstable near to the center, otherwise it is very accurate. Indeed, both r_x^{-1} and s_x^{-1} appearing in Eq. 21 become large when x is near to $\mathrm{E}[Z^*(N_t)]$ and this causes the observed erratic behavior of the saddlepoint approximation around the center. This instability at the center may be unimportant if one considers reserving problems in actuarial science, in which quantile-based risk measures are often used in setting up reserves and the quantiles are taken from the right tails of loss distributions. However, in other actuarial applications, like premium calculations, the central part of a loss distribution may be more important. In these cases, one may control that the desired saddlepoint approximation does not coincide with the typical central oscillation, as seen in Fig. 1 for example. If this is the case, one could either move to the closest grid-point to the left or to the right from the center, or consider a linear interpolation of the surrounding probabilities. The relative errors seen in Fig. 1 are very small and the peaks at the extremities are presumably due to the Monte Carlo approximation to the exact probabilities. The third graph shows both the Lugannani and Rice and the Barndorff-Nielsen formulas are numerically very close, except in the central region.

Example 2.2 (continued) Constant intensity and gamma claim amounts.

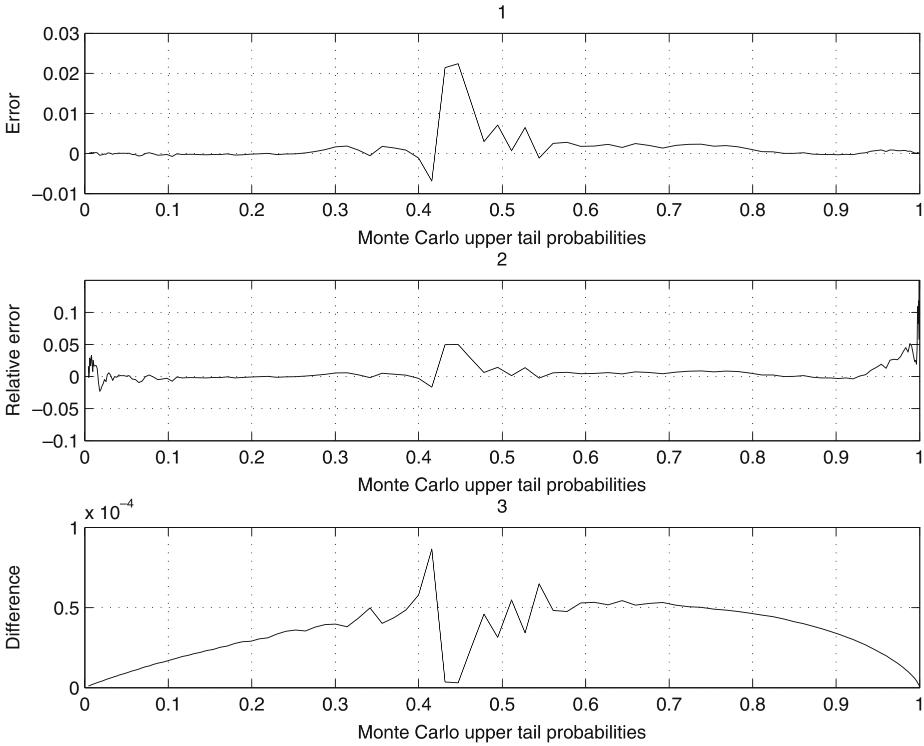

Fig. 1 Accuracy of the saddlepoint approximation to the upper tail probabilities $\bar{F}_t(x) = \mathrm{P}[Z(N_t) > x]$ with constant intensity function and linear combination of exponentials claim amount. Graph 1: error, $\bar{G}_t - \bar{F}_t$ versus $\bar{F}_t$. Graph 2: relative error, $(\bar{G}_t - \bar{F}_t)/\min\{\bar{F}_t, 1 - \bar{F}_t\}$ versus $\bar{F}_t$. Graph 3: difference between Barndorff-Nielsen and Lugannani and Rice, $\bar{H}_t - \bar{G}_t$ versus $\bar{F}_t$. $\bar{G}_t$: Lugannani and Rice. $\bar{H}_t$: Barndorff-Nielsen. $\bar{F}_t$: Monte Carlo

We consider the setting of Example 2.2 and we first justify that the cemetery is empty. By differentiating Eq. 7 we obtain, after some manipulations,

$$M'_{W_t}(v) = \frac{1}{\Lambda(t)} \frac{\lambda}{r} \left[\frac{1}{\nu e^{-rt} - v} - \frac{1}{\nu - v} + \sum_{i=1}^{\alpha-1} \binom{\alpha - 1}{i} \frac{1}{i} a(v; i) \right], \tag{29}$$

where

$$a(v; i) = i\nu[v^{i-1}(\nu - v)^{-i-1} - (\nu - v)^{-2i-1}\{e^{-rt}(\nu + v) - 2v\}\{v(\nu e^{-rt} - v)\}^{i-1}],$$

with $i = 1, \ldots, \alpha - 1$. If $r > 0$, then as $v \to \nu e^{-rt}$ the first ratio in the squared parenthesis of Eq. 29 tends to ∞, the second ratio tends to $\{\nu(1 - e^{-rt})\}^{-1}$ and $a(v; i)$ tends to $i\nu^{-i}e^{-rt(i-1)}(1 - e^{-rt})^{-i-1}$, hence $\lim_{v \to \nu e^{-rt}} M'_{W_t}(v) = \infty$. If $r < 0$, then as $v \to \nu$ the first ratio tends to $\{\nu(e^{-rt} - 1)\}^{-1}$, the second ratio tends to $-\infty$ and $a(v; i)$ tends to $-\infty$ as well, but because the coefficient r^{-1} is negative, $\lim_{v \to \nu} M'_{W_t}(v) = \infty$. Consequently, the empty cemetery condition given by Eq. 25 of Lemma 3.1 holds with $\gamma_t = \nu \min\{1, e^{-rt}\}$. We arbitrarily fix $\lambda = 1$, $\nu = 5$, $\alpha = 4$, $r = 0.1$ and $t = 10$. The numerical accuracy of the saddlepoint approximation to the upper tail probability of $Z(N_t)$ is shown in Fig. 2. The first graph of Fig. 2 shows the error $\bar{G}_t - \bar{F}_t$ of the Lugannani and Rice approximation $\bar{G}_t$, given by Eq. 17, when the exact upper tail probabilities $\bar{F}_t$ are again computed with 10^6 Monte Carlo generations of the total claim amount. The second graph of Fig. 2 shows the corresponding relative error computed by $(\bar{G}_t - \bar{F}_t)/\min\{\bar{F}_t, 1 - \bar{F}_t\}$. Both graphs are plotted versus $\bar{F}_t$. The third graph of Fig. 2 shows the Lugannani and Rice approximation $\bar{G}_t$, as a solid line, together with the saddlepoints obtained by solving Eq. 18, as dashed line. The solid line refers to the left axis and the dashed line to the right axis. In this situation, with the exception of the unstable behavior around the center, the saddlepoint approximation is very accurate.

Example 2.4 (continued) Gamma intensity and exponential claim amounts.

We consider the setting of Example 2.4. We can see from Eq. 13 that M'_{W_t} is the sum of two parts: the derivative of the logarithmic part, which tends to ∞ as $v \to \gamma_t = \nu e^{-rt}$, as already justified, and the derivative of the series, which is positive as $v \to \gamma_t$. Hence the empty cemetery condition given by Eq. 25 of Lemma 3.1 is satisfied. Note that the non-validity of Eq. 13 for values $v < -\gamma_t$ is only due to the expansion of the integral in power series. Hence, small lower tail probabilities

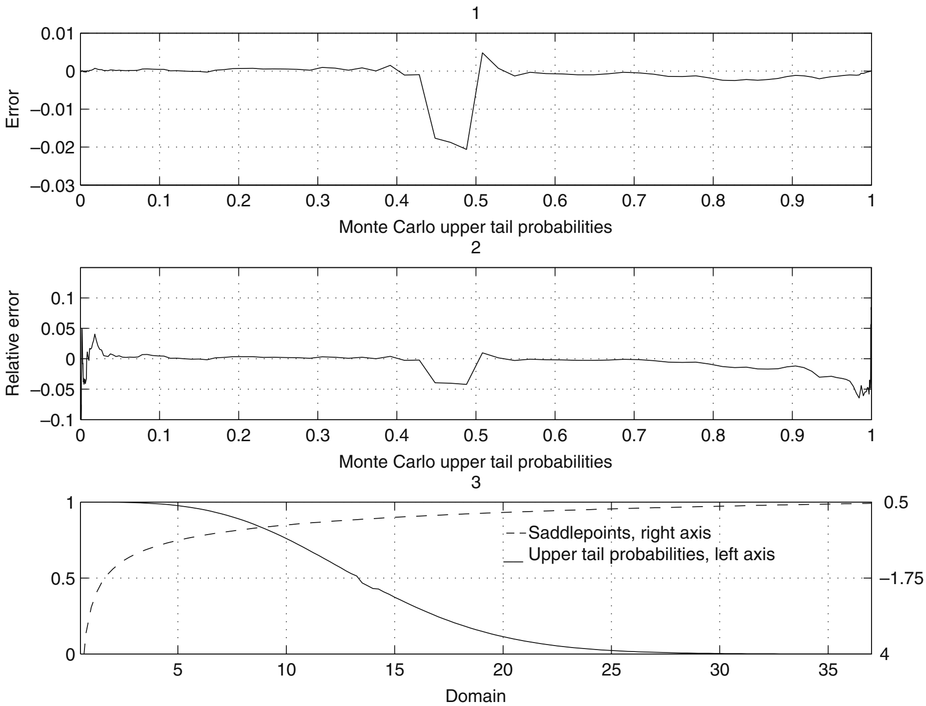

Fig. 2 Accuracy of the saddlepoint approximation to the upper tail probabilities $\bar{F}_t(x) = \mathrm{P}[Z(N_t) > x]$ with constant intensity function and gamma claim amounts. Graph 1: error, $\bar{G}_t - \bar{F}_t$ versus $\bar{F}_t$. Graph 2: relative error, $(\bar{G}_t - \bar{F}_t)/\min\{\bar{F}_t, 1 - \bar{F}_t\}$ versus $\bar{F}_t$. Graph 3: upper tail probabilities $\bar{G}_t$ and saddlepoints. $\bar{G}_t$: Lugannani and Rice. $\bar{F}_t$: Monte Carlo

may not be obtained when using this series expansion of the integral. This is not a drawback in several actuarial applications where only small upper tail probabilities are relevant. Regarding the choice of the model parameters, we fix $\alpha_0 = 0$ and arbitrary fix $\alpha_1 = 1$, $a = 2$, $b = 0.1$, $\nu = 2$, $r = 0.1$ and $t = 10$. The resulting intensity function is increasing over $[0, 10]$ (as the maximum is at $(a-1)/b$). When $\alpha_0 = 0$, the inhomogeneous Poisson process can be simulated by the inversion method, which is described in the Appendix. Let us denote $P(a, x) = \Gamma(a, x)/\Gamma(a)$ and its inverse function with respect to x as $P^{(-1)}(a, u)$, for all $u \in [0, 1]$ and $a, x > 0$. This inverse function exists in *Matlab*, for example. Then $\Lambda(s) = \alpha_1 \Gamma(a) b^{-a} P(a, bs)$ can be inverted to $s = \Lambda^{(-1)}(u) = P^{(-1)}(b^a u/(\alpha_1 \Gamma(a)), a)/b$, $u \geq 0$. The numerical accuracy of the saddlepoint approximation to a small upper tail probability of $Z(N_t)$ is shown in Table 1. The second column of Table 1 shows the exact upper tail probabilities $\bar{F}_t$, computed by 10^6 Monte Carlo replications, by the inversion method. The third column shows the Lugannani and Rice approximation $\bar{G}_t$, see Eq. 17. The fourth and fifth columns of Table 1 show the error $\bar{G}_t - \bar{F}_t$ and the upper tail relative error $(\bar{G}_t - \bar{F}_t)/\bar{F}_t$ of the Lugannani and Rice approximation. The saddlepoint approximation is again fast to obtain and, as we see, accurate.

We now consider the more general case that $\alpha_0 \geq 0$. Because here the inversion method alone is not an efficient simulation method, we now consider the two following solutions: the thinning method and the inversion method in conjunction with the decomposition method, which are summarized in the Appendix. For the first solution we generate arrival times of an homogeneous Poisson process with constant intensity $\sup_{s \in [0,t]} \lambda(s) = \lambda(\max\{0, (a-1)/b\}) = 4.6788$ and we accept and reject them as described in the Appendix. For the second solution, we merge a first Poisson process with constant intensity α_0 with a second independent Poisson process with intensity function $\alpha_1 s^{a-1} e^{-bs}$, for all $s \in [0, t]$. The simulation is obvious for the first homogeneous process and for the second process it relies on the inversion

Table 1 Accuracy of the saddlepoint approximation to small upper tail probabilities $\bar{F}_t(x) = \mathrm{P}[Z(N_t) > x]$ with gamma intensity function exponential claim amounts

x	$\bar{F}_t(x)$	$\bar{G}_t(x)$	$(\bar{G}_t - \bar{F}_t)(x)$	$(\{\bar{G}_t - \bar{F}_t\}/\bar{F}_t)(x)$
21	0.4155	0.4157	0.0002	0.0006
22	0.3518	0.3507	−0.0011	−0.0031
23	0.2927	0.2919	−0.0008	−0.0028
24	0.2395	0.2391	−0.0004	−0.0016
25	0.1938	0.1934	−0.0003	−0.0016
26	0.1541	0.1542	0.0000	0.0003
27	0.1211	0.1212	0.0000	0.0003
28	0.0936	0.0940	0.0005	0.0048
29	0.0722	0.0720	−0.0002	−0.0029
30	0.0546	0.0544	−0.0001	−0.0025
31	0.0411	0.0407	−0.0004	−0.0109
32	0.0307	0.0301	−0.0006	−0.0191
33	0.0225	0.0220	−0.0006	−0.0247
34	0.0163	0.0159	−0.0003	−0.0215
35	0.0118	0.0114	−0.0004	−0.0370
36	0.0081	0.0081	−0.0001	−0.0102
37	0.0056	0.0057	0.0000	0.0059
38	0.0040	0.0039	−0.0001	−0.0274
39	0.0028	0.0027	−0.0001	−0.0435
40	0.0019	0.0019	−0.0000	−0.0122

Fourth and fifth columns: error and upper tail relative error. $\bar{G}_t$: Lugannani and Rice. $\bar{F}_t$: Monte Carlo method. For both the thinning and the decomposition with inversion solutions, we generate 10^6 compound processes and we display, in the fourth column of Table 2, the small upper tail probabilities $\mathrm{P}[Z(N_t) > x]$ based on the decomposition with inversion method, denoted $\bar{F}_t$, and in the fifth column the probabilities based on the thinning method, denoted $\bar{F}_t^{\dagger}$. In the second and third column of Table 2 we display the saddlepoint approximations given by Eqs. 17 and 23, $\bar{G}_t$ and $\bar{H}_t$, respectively. In Table 2 we first see that the two saddlepoint approximations are almost everywhere equal, up to shown decimals, and that both saddlepoint approximations are close to the decomposition with inversion Monte Carlo probabilities. The thinning Monte Carlo probabilities are quite distant from all other probabilities and we can therefore conjecture, that the thinning method does not lead to accurate results in this case. To conclude, using the saddlepoint approximation with $\alpha_0 > 0$ does not bring any additional difficulty, whereas the Monte Carlo method, besides being substantially more computationally intensive, can even fail to give accurate results.

As mentioned precisely, the existence of the moment generating function of the individual claim amounts in a neighborhood of zero is not a sufficient condition to guarantee that the cemetery is empty. We illustrate this fact with the following example.

Example 4.1 Constant intensity and inverse-Gaussian claim amounts.

Let us consider the homogeneous Poisson process with intensity $\lambda > 0$ and the inverse-Gaussian claim amount density $f_X(x) = \sqrt{\theta/(2\pi x^3)}\exp\{-\theta/(2x)(x/\mu - 1)^2\}$,

Table 2 Accuracy of the saddlepoint and the Monte Carlo approximations to small upper tail probabilities $\mathrm{P}[Z(N_t) > x]$ with gamma intensity function exponential claim amounts

x	$\bar{G}_t(x)$	$\bar{H}_t(x)$	$\bar{F}_t(x)$	$\bar{F}_t^{\dagger}$
30	0.4069	0.4069	0.4041	0.3766
31	0.3542	0.3542	0.3533	0.3209
32	0.3039	0.3039	0.3045	0.2693
33	0.2590	0.2590	0.2596	0.2235
34	0.2184	0.2184	0.2188	0.1820
35	0.1822	0.1822	0.1826	0.1467
36	0.1503	0.1504	0.1516	0.1160
37	0.1230	0.1230	0.1243	0.0911
38	0.0995	0.0995	0.0998	0.0711
39	0.0798	0.0798	0.0797	0.0544
40	0.0634	0.0634	0.0635	0.0414
41	0.0500	0.0500	0.0501	0.0308
42	0.0390	0.0390	0.0393	0.0228
43	0.0302	0.0302	0.0308	0.0168
44	0.0232	0.0232	0.0236	0.0119
45	0.0176	0.0176	0.0176	0.0088
46	0.0133	0.0133	0.0133	0.0063
47	0.0100	0.0100	0.0098	0.0043
48	0.0074	0.0074	0.0072	0.0032
49	0.0055	0.0055	0.0051	0.0024
50	0.0040	0.0040	0.0037	0.0017

$\bar{G}_t$: Lugannani and Rice.
$\bar{H}_t$: Barndorff-Nielsen.
$\bar{F}_t$: decomposition and inversion Monte Carlo.
$\bar{F}_t^{\dagger}$: thinning Monte Carlo

for all $x, \mu, \theta > 0$. Because the moment generating function of the individual claim amounts is $M_X(v) = \exp\{\theta/\mu(1 - \sqrt{1 - 2\mu^2/\theta\, v})\}$, for all $v \leq \theta/(2\mu^2)$, we have

$$M_{W_t}(v) = \frac{1}{\Lambda(t)} \lambda e^{\frac{\theta}{\mu}} \int_0^t \exp\left\{-\frac{\theta}{\mu}\sqrt{1 - 2\frac{\mu^2}{\theta} e^{r(t-y)} v}\right\} dy,$$

which converges for all $v \leq \gamma_t = \theta/(2\mu^2) \min\{1, e^{-rt}\}$. We now see that the cemetery is non-empty. By differentiating we obtain

$$M'_{W_t}(\gamma_t) = \frac{1}{\Lambda(t)} \lambda \mu e^{\frac{\theta}{\mu}} \int_0^t \delta^{-\frac{1}{2}} e^{-\frac{\theta}{\mu}\delta^{\frac{1}{2}}} dy,$$

where $\delta = 1 - e^{r(t-y)} \min\{1, e^{-rt}\}$. For $r > 0$, We have $M'_{W_t}(\gamma_t) \leq \int_0^t \sqrt{1 - e^{-rt}} dy = 2r^{-1} \operatorname{arctanh}\sqrt{1 - e^{-rt}}$, because (d/dx) $\operatorname{arctanh}\sqrt{1 - e^{-rx}} = r(1 - e^{-rx})^{-1/2}/2$. The same justification can be given for $r < 0$. Hence by Lemma 3.1 (b) the cemetery is the non-empty interval $(x^\dagger, \infty)$, where

$$x^\dagger = \frac{\lambda \mu e^{\frac{\theta}{\mu}} \int_0^t \delta^{-\frac{1}{2}} e^{-\frac{\theta}{\mu}\delta^{\frac{1}{2}}} dy}{1 - \exp\left\{-\lambda e^{\frac{\theta}{\mu}} \int_0^t e^{-\frac{\theta}{\mu}\delta^{\frac{1}{2}}} dy\right\}},$$

which has no trivial behavior with respect to the model parameters. So we are not able to compute saddlepoint approximations to the upper tail probabilities $\mathrm{P}[Z(N_t) > x]\ \forall x > x^\dagger$. We can still obtain these probabilities for $x \leq x^\dagger$, but these lower tail probabilities are not the most interesting in actuarial practice.

It is interesting to note that an analog problem to the one found in Example 4.1 arises in the determination of the adjustment coefficient, defined below, in the homogeneous compound Poisson risk process without force of interest and with inverse-Gaussian individual claim amounts, see Bowers et al. (1997, Example 13.4.3). This similarity is not surprising because both the saddlepoint and the adjustment coefficient are parameters of exponential tilts, or of Esscher transforms. Let us recall that the exponential tilt of a distribution function F with density f, with respect to the Lebesgue measure, is $e^{vx} f(x)/\int_{-\infty}^{\infty} e^{vy} dF(y)$, where v is the tilting, or Esscher, parameter. Let us denote the insurer surplus process at time $t \geq 0$ by $S(N_t) = s_0 + ct - Z(N_t)$, where $Z(N_t)$ is here a non-discounted homogeneous compound Poisson process, $c > 0$ is a constant premium rate and $s_0 \geq 0$ is a fixed initial surplus. Let us also define the insurer aggregate loss process $L(N_t) = Z(N_t) - ct$, for all $t \geq 0$, and $K_{L(N_t)}(v) = \log(\mathrm{E}[e^{vL(N_t)}])$. The adjustment coefficient α is defined as the smallest positive solution in v of the equation $K_{L(N_t)}(v) = 0 \Leftrightarrow K_{Z(N_t)}(v) = ctv$. On the other side, the saddlepoint equation at $x > 0$ is given by Eq. 18, which is equivalent to $K'_{Z(N_t)}(v) \stackrel{\text{def}}{=} (d/dv)K_{Z(N_t)}(v) = x$, as mentioned by the second remark following the proof of Result 2.1. In fact we have here two exponential tilts of the same distribution. In the context of the saddlepoint approximation, the saddlepoint Eq. 18 provides the parameter v_x in the exponential tilt of F_t^* which gives expectation x to F_t^* tilted. This allows for an accurate local approximation around the new expectation x, see Daniels (1954). In the context of the risk theory, the cumulant generating function of the distribution of $L(N_t)$ under the exponential tilt with tilting parameter τ is given by $K^{(\tau)}_{L(N_t)}(v) = K_{L(N_t)}(v + \tau) - K_{L(N_t)}(\tau)$. We assume $K'_{L(N_t)}(0) < 0$, where $K'_{L(N_t)}(v) \stackrel{\text{def}}{=} (d/dv)K_{L(N_t)}(v)$. From this and from the convexity of $K_{L(N_t)}$, $K_{L(N_t)}$ has two distinct roots, $v = 0$ and $v = \alpha > 0$, and we have $K'_{L(N_t)}(\alpha) > 0$. So the expectation of $L(N_t)$ under the exponential tilt with tilting parameter α is

$$\mathrm{E}_\alpha[L(N_t)] = \frac{d}{dv} K^{(\alpha)}_{L(N_t)}(v) \mid_{v=0} = K'_{L(N_t)}(\alpha) > 0,$$

for all $t > 0$. Hence $\lim_{t\to\infty} S(N_t) = -\mathrm{sgn}(\mathrm{E}_\alpha[L(N_1)]) \cdot \infty = -\infty$ with probability one under the tilted distribution, which implies $\{S(N_t)\}_{t>0}$ crosses the null line with probability one under the tilted distribution, or, in other words, that ruin is certain under the tilted distribution. This leads to an exact formula and to an asymptotic approximation for the probability of ruin under the original distribution. Note further that the surplus process under the exponential tilt with parameter τ is a compound Poisson process with intensity $\lambda M_X(\tau)$, individual claim amount density $e^{\tau x} f_X(x) M_X^{-1}(\tau)$ and with same premium rate c. More details concerning the adjustment coefficient can be found in Asmussen (2000), for example. The adjustment coefficient allows for nice interpretations in martingale and renewal theories, refer to Asmussen (2000) and Gerber (1979) respectively.

5 Final Remarks

In this article we propose computing the distributions of various discounted compound Poisson processes by the saddlepoint approximation. We show the effectiveness and the high accuracy of the approximation.

Analogue results could be given when there is an additional Wiener process, which expresses additional uncertainty regarding the aggregate claims or investment uncertainty, or when considering other counting processes with the order statistics property (mentioned in the proof of Lemma 2.1), like the mixed Poisson process. Analogue results could also be obtained for total claim amount with delayed claim settlement, namely to the process with the shot-noise form $Z(N_t) = \sum_{i=0}^{N_t} h(t - T_i) X_i$, where h is a nondecreasing function such that $h(s) = 0$, if $s < 0$, and $\lim_{s\to\infty} h(s) = 1$, see Mikosch (2004, p. 33). For example, $h(s) = I\{\delta \le s\}$ is the full payment of the insurer at the fixed delay $\delta > 0$ from the claim arrival. There are also many stochastic processes in other applied fields, like physics, astronomy, biology, hydrology, queuing theory, etc., which are basically shot-noise processes and for which analogue approximations can be developed.

A more challenging open problem is the extension of the results obtained here to the situation where the interest rate is no longer fixed but a stochastic process. In recent years, we have seen some important works on risk processes with stochastic interest rate, see Paulsen and Gjessing (1997) for example. We could generalize $Z(N_t)$ to

$$\tilde{Z}(N_t) = \sum_{i=0}^{Nt} \exp\left\{\int_{T_i}^{t} R_s ds\right\} X_i,$$

where $\{R_s\}_{0\le s\le t}$ is a stochastic process representing the interest rate over the time interval $[0, t]$. If $R_s = r$ for all $s \in [0, t]$, then $\tilde{Z}(N_t) = Z(Nt)$. It is difficult to obtain a saddlepoint approximation to the distribution of $\tilde{Z}(N_t)$ in general. However, if the interest rate in $\tilde{Z}(N_t)$ is replaced by a deterministic integrable function over the time $q : [0, t] \to \mathrm{R}$, then following the steps of the proof of Lemma 2.1 we obtain

$$K_{\tilde{Z}(N_t)}(v) = \int_0^t M_X(ve^{Q(t)-Q(y)})\lambda(y)dy - \Lambda(t), \tag{30}$$

for all $t > 0$ and $v < \gamma_t \stackrel{\text{def}}{=} c \exp\{\inf_{y\in[0,t]} Q(y) - Q(t)\}$, where Q denotes the primitive of the function q. Thus Eq. 30 generalizes Lemma 2.1.

It is interesting to note that for the homogeneous Poisson process, the discounted total claim amount can be represented by the stochastic differential equation

$$\frac{d}{dt}Z(N_t) - rZ(N_t) = \frac{d}{dt}S(N_t), \tag{31}$$

where $S(N_t) = \sum_{i=0}^{N(t)} X_i$, see Novikov et al. (2005) for example. Solving the first-order linear differential Eq. 31 yields

$$Z(N_t) = e^{rt}\int_0^t e^{rs} dS(N_s),$$

which shows that the shot-noise process $\{Z(N_t)\}_{t\geq 0}$ is an Ornstein-Uhlenbeck process, as considered for example by Paulsen and Gjessing (1997).

The computer programs used for this article are written in *Matlab* and can be obtained under http://www.staff.unibe.ch/gatto.

Appendix

In this appendix we briefly summarize the three methods used in Section 4 for generating a Poisson process with intensity and expectation functions $\lambda(s)$ and $\Lambda(s)$ respectively, $s \geq 0$. For more details about this appendix, refer e.g. to Devroye (1986).

The first method for generating a Poisson process is the inversion method. Given $S_1, S_2, \ldots$ arrival times of an homogeneous Poisson process with unitary rate, the desired process is generated by $T_k = \Lambda^{(-1)}(S_k)$, $k = 1, 2, \ldots$, where $\Lambda^{(-1)}(s) = \inf\{y \geq 0 | \Lambda(y) \geq s\}$, $s \geq 0$. Hence we mainly need to generate i.i.d. exponential first-order differences with mean one.

The second method could be called a decomposition method and it follows from the result that given $m \geq 1$ independent Poisson processes with intensity functions $\lambda_1(s), \ldots, \lambda_m(s)$, $s \geq 0$, the merged ordered arrival times form a Poisson process with intensity function $\sum_{k=1}^m \lambda_k(s)$, $s \geq 0$. Suppose the decomposition $\lambda(s) = \sum_{k=1}^m \lambda_k(s)$, $s \geq 0$ holds. We first generate the set $\mathcal{S}$ of first arrival times of the m Poisson processes, we define $T_1 = \min(\mathcal{S})$, we replace $\min(\mathcal{S})$ in $\mathcal{S}$ by the second arrival time of the process that gave $\min(\mathcal{S})$, we iterate and obtain the desired process. Alternatively, we simply sort in an increasing order the merged simulated arrival times of the m processes.

The third method is the thinning method, which is an acceptance-rejection type method. Suppose $\mu(s) \geq \lambda(s)$, $\forall s \geq 0$. We generate $S_1, S_2, \ldots$ the arrival times of a Poisson process with intensity function $\mu(s)$, $s \geq 0$, we generate $U_1, U_2, \ldots$ independent uniform random variables on $[0, 1)$ and we retain the indexes k which satisfy $U_k \leq \lambda(S_k)/\mu(S_k), k = 1, 2, \ldots$, which we denote as $k_1 < k_2 < \ldots$. The desired process is generated by $T_1 = S_{k_1}, T_2 = S_{k_2}, \ldots$.

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