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**Integral Equations and Operator Theory**

# The numerical range of positive operators on Hilbert lattices

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**Abstract.** We show symmetry properties of the numerical range of positive operators on Hilbert lattices. These results generalise the respective properties for positive matrices shown in Li et al. (Linear Algebra Appl 350:1–23, 2002) and Maroulas et al. (Linear Algebra Appl 348:49–62, 2002). Similar assertions are also valid for the block numerical range of positive operators.

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## 1. Introduction

In [5,8], the numerical range of positive matrices was investigated based on the unpublished PhD thesis [4]. The authors prove analogues of the results from Perron–Frobenius theory. They can easily show that the numerical radius of a positive matrix is always contained in its numerical range. This is parallel to the well-known fact that the spectral radius of a positive matrix is always in its spectrum. Moreover, it turns out that the numerical range of positive matrices with irreducible real part exhibits a rotational symmetry. To be more precise, in [5, Prop. 3.11] it is stated that for such a nonnegative matrix $A$ and any unimodular complex number $\xi$ the following equivalence holds:

$$\xi\,\mathrm{W}(A) = \mathrm{W}(A) \quad \Longleftrightarrow \quad \xi\,\mathrm{w}(A) \in \mathrm{W}(A) \tag{1.1}$$

where $\mathrm{W}(A)$ and $\mathrm{w}(A)$ denote the numerical range and the numerical radius of $A$, respectively.

One of the main tools to prove these results is the Perron–Frobenius theory itself. Since this theory has an important extension to Banach lattices, see the monograph [9], this technique is also available in the infinite dimensional situation. However, since the numerical range need not be closed in this case, we encounter new obstacles. Our results on the numerical range of positive operators can be found in Sect. 2. We show that the implication "$\Leftarrow$" in (1.1) still holds (Theorem 2.8); under some additional assumptions we again obtain equivalence (Theorem 2.9).

In Sect. 3 we consider the block numerical range introduced in [15]. It gives a better localisation of the spectrum, since, roughly speaking, it lies between the spectrum and the numerical range. Motivated by results in [2] for the matrix case, we use the results of Sect. 2 to derive symmetry properties for the block numerical range of positive operators.

In this paper we work in complex Hilbert lattices and keep to the notation and terminology from [9]. In particular, for a complex Hilbert lattice $H$ the underlying real lattice is denoted by $H_{\mathbb{R}}$ and the positive cone by $H_+$. For $x \in H$ we write $x \geq 0$ if $x \in H_+$, and $x > 0$ if $x \in H_+$ and $x \neq 0$. Moreover, $\sup M$ is the least upper bound of $M \subseteq H$ (if it exists). If $x \in H_{\mathbb{R}}$, then $x^+ := \sup\{x, 0\}, x^- := \sup\{-x, 0\}$, and $|x| := x^+ + x^-$. If $z = x + \mathrm{i}\, y \in H, x, y \in H_{\mathbb{R}}$, we define $|z| := \sup_{0 \leq \theta < 2\pi} |(\cos\theta)x + (\sin\theta)y|$. For $x, y \in H$ the set $[x, y] := \{z \in H \ : \ x \leq z \leq y\}$ is called the *order interval* between $x$ and $y$. Then, $x \in H_+$ is a *quasi-interior* point of $H_{\mathbb{R}}$ if

$$H_x := \bigcup_{n \in \mathbb{N}} [-nx, nx]$$

is dense in $H_{\mathbb{R}}$. An operator $A \in \mathcal{L}(H)$ is said to be *positive*, in symbols $A \geq 0$, if $AH_+ \subseteq H_+$. Observe that any operator $A \in \mathcal{L}(H)$ can be decomposed into $A = A_1 + \mathrm{i}\, A_2$ where $A_1, A_2 \in \mathcal{L}(H_{\mathbb{R}})$. The operator $A$ is *regular* if both $A_1$ and $A_2$ can be written as the difference of two positive operators. In this case

$$|A| := \sup\{(\cos\theta)A_1 + (\sin\theta)A_2 : 0 \leq \theta \leq 2\pi\}$$

exists, see [9, Prop. IV.1.2]. Finally, $A$ is called *irreducible* if there exists no closed non-trivial lattice ideal of $H$ that is invariant under $A$, see [9, p. 341].

Note that every complex Hilbert lattice $H$ is isometrically lattice isomorphic to $L^2(\Omega, \mu)$ for some measure space $(\Omega, \Sigma, \mu)$ where $\Omega$ is a locally compact space and $\mu$ is a strictly positive Radon measure, see [7, Cor. 2.7.5] or [9, Thm. IV.6.7].

The subsequent properties always hold and are often used without reference. Here and in the following, $A^*$ denotes the Hilbert space adjoint of $A \in \mathcal{L}(H)$.

**Proposition 1.1.** *Let $H$ be a (complex) Hilbert lattice and let $A \in \mathcal{L}(H)$. Then the following statements hold.*

(i) *If $x \in H$, then $x \in H_+$ if and only if $\langle x, y\rangle \geq 0$ for every $y \in H_+$,*
(ii) *$\langle x^+, x^-\rangle = 0$ for every $x \in H_{\mathbb{R}}$,*
(iii) *$\|x\|^2 = \|x^+\|^2 + \|x^-\|^2$ for every $x \in H_{\mathbb{R}}$,*
(iv) *$A \geq 0 \iff \forall x \geq 0\ \forall y \geq 0\ \langle Ax, y\rangle \geq 0$,*
(v) *$A \geq 0 \iff A^* \geq 0$,*
(vi) *$|\langle x, y\rangle| \leq \langle |x|, |y|\rangle$ for every $x, y \in H$,*
(vii) *if $A$ is regular, then $|A^*| = |A|^*$.*

*Proof.* We only show part (vii). Using (v) one can easily see that $A^*$ is regular if $A$ is regular. Thus $|A^*|$ exists. Let $A_1, A_2 \in \mathcal{L}(H_\mathbb{R})$ such that $A = A_1 + \mathrm{i}\, A_2$ and let $\theta \in [0, 2\pi]$. It is clear from the definition of $|A|$ that

$$|A| - ((\cos\theta)A_1 + (\sin\theta)A_2) \geq 0.$$

By (v) we obtain

$$(|A| - ((\cos\theta)A_1 + (\sin\theta)A_2))^* = |A|^* - ((\cos\theta)A_1^* + (\sin\theta)A_2^*)) \geq 0,$$

and thus

$$|A|^* \geq \sup\{\cos\theta)A_1^* + (\sin\theta)A_2^* : \theta \in [0, 2\pi]\} = |A^*| \,. \tag{1.2}$$

The assertion then follows from

$$|A|^* = |A^{**}|^* \overset{(1.2)}{\leq} |A^*|^{**} = |A^*| \,.$$

□

## 2. The numerical range of positive operators

Our object of interest is the numerical range of positive operators on a complex Hilbert lattice $H$. The goal is to derive symmetry properties similar to those obtained for positive matrices on $\mathbb{C}^n$ in [5,8].

We first recall some basic definitions and results valid for bounded linear operators on an arbitrary complex Hilbert space $H$. For $A \in \mathcal{L}(H)$ the *numerical range* is defined as

$$W(A) := \{\langle Ax, x\rangle \,:\, x \in H,\ \|x\| = 1\}.$$

Its *numerical radius* is

$$\mathrm{w}(A) := \sup\{|\lambda| \,:\, \lambda \in \mathrm{W}(A)\}.$$

Moreover, the spectrum of $A$ is denoted by $\sigma(A)$, while the point spectrum (or set of eigenvalues) of $A$ is $\sigma_p(A)$, and the spectral radius is $\mathrm{r}(A)$. An eigenvalue $\lambda$ of $A$ is called a *peripheral eigenvalue* if $|\lambda| = \mathrm{r}(A)$. Finally, the complex unit circle is denoted by $\Gamma$, i.e.,

$$\Gamma = \{\lambda \in \mathbb{C} \,:\, |\lambda| = 1\}.$$

Then for the numerical radius the following properties hold, see [3, p. 8] and [3, Thm. 1.4-2].

**Lemma 2.1.** *Let $H$ be a complex Hilbert space.*

(i) *For any $A \in \mathcal{L}(H)$ we have*

$$|\langle Ax, x\rangle| \leq \mathrm{w}(A)\,\langle x, x\rangle \,.$$

(ii) *If $A \in \mathcal{L}(H)$ is self-adjoint or normal, then its norm, its spectral radius and its numerical radius coincide, i.e.,*

$$\|A\| = \mathrm{r}(A) = \mathrm{w}(A).$$

We are now ready to derive a first symmetry property for a positive operator on $H$. In fact, this property is true for any operator leaving the underlying real space $H_\mathbb{R}$ invariant.

**Proposition 2.2.** *If* $0 \leq A \in \mathcal{L}(H)$*, then* $\mathrm{W}(A)$ *is symmetric with respect to the real axis.*

*Proof.* Let $\lambda \in \mathrm{W}(A)$. Then there exists $z = x + \mathrm{i}\,y \in H = H_{\mathbb{R}} \oplus \mathrm{i}\,H_{\mathbb{R}}$ such that $\|z\| = 1$ and $\langle Az, z\rangle = \lambda$. Then also $\|x - \mathrm{i}\,y\| = 1$, and one obtains that

$$\langle A(x - \mathrm{i}\,y), x - \mathrm{i}\,y\rangle = \overline{\langle A(x + \mathrm{i}\,y), x + \mathrm{i}\,y\rangle} = \overline{\langle Az, z\rangle} = \overline{\lambda},$$

and thus $\overline{\lambda} \in \mathrm{W}(A)$. □

In the following, an important role is played by the *real* or *Hermitian part*

$$\mathrm{R}(A) := \tfrac{1}{2}(A + A^*)$$

of a bounded linear operator $A$ on a Hilbert lattice $H$. Clearly, if $A$ is irreducible, then also $\mathrm{R}(A)$ is irreducible. By a straightforward calculation the following result can be verified for arbitrary Hilbert spaces.

**Lemma 2.3.** *Let* $A \in \mathcal{L}(H)$*. Then*

$$\langle \mathrm{R}(A)x, x\rangle = \mathrm{Re}(\langle Ax, x\rangle)$$

*for every* $x \in H$*.*

Next we state the relation between spectral properties of $\mathrm{R}(\xi A), \xi \in \Gamma$, and the numerical range of $A$. This enables us to apply results from the theory of positive operators to $\mathrm{R}(A)$ and then draw conclusions for the numerical range.

**Proposition 2.4.** *Let* $H$ *be a Hilbert space and* $A \in \mathcal{L}(H)$*.*

(i) *For all* $\xi \in \Gamma$ *we have*

$$\mathrm{w}(A) \geq \mathrm{w}(\mathrm{R}(\xi A)). \tag{2.1}$$

(ii) *If* $\overline{\xi}\,\mathrm{w}(A) \in \overline{\mathrm{W}(A)}$ *for some* $\xi \in \Gamma$*, then*

$$\mathrm{w}(\mathrm{R}(\xi A)) = \mathrm{w}(A).$$

(iii) *For all* $\xi \in \Gamma$ *we have*

$$\{x \in H \;:\; \xi\,\langle Ax, x\rangle = \mathrm{w}(A)\,\|x\|^2\} = \ker(\mathrm{w}(A) - \mathrm{R}(\xi A)).$$

*Proof.* (i) For $x \in H$ we compute

$$\langle(\mathrm{w}(\xi A) - \mathrm{R}(\xi A))x, x\rangle \overset{\text{Lemma 2.3}}{=} \mathrm{w}(\xi A)\,\|x\|^2 - \underbrace{\mathrm{Re}\,\langle \xi Ax, x\rangle}_{\leq \mathrm{w}(\xi A)\|x\|^2} \geq 0.$$

Thus,

$$\mathrm{w}(\xi A) \geq \left\langle \mathrm{R}(\xi A)\tfrac{x}{\|x\|}, \tfrac{x}{\|x\|}\right\rangle \quad \text{for every } x \in H\backslash\{0\}.$$

Since $\mathrm{w}(\xi A) = \mathrm{w}(A)$, the assertion follows.

(ii) Let $(u_n)_{n\in\mathbb{N}} \subseteq H, \|u_n\| = 1$, such that $\langle Au_n, u_n\rangle \to \overline{\xi}\,\mathrm{w}(A)$ as $n\to\infty$. Then

$$\begin{aligned}\langle \mathrm{R}(\xi A)u_n, u_n\rangle &= \tfrac{1}{2}\left(\langle \xi A u_n, u_n\rangle + \langle \overline{\xi} A^* u_n, u_n\rangle\right)\\ &= \tfrac{1}{2}\big(\xi \underbrace{\langle Au_n, u_n\rangle}_{\to\overline{\xi}\,\mathrm{w}(A)} + \overline{\xi}\, \underbrace{\overline{\langle Au_n, u_n\rangle}}_{\to \xi\,\mathrm{w}(A)}\big)\end{aligned}$$

which converges to $w(A)$ as $n\to\infty$. This implies that

$$\mathrm{w}(\mathrm{R}(\xi A)) \geq \sup_{n\in\mathbb{N}} |\langle \mathrm{R}(\xi A)u_n, u_n\rangle| \geq \mathrm{w}(A). \tag{2.2}$$

The other inequality was already shown in part (i).

(iii) This follows from [1, Lemma 1.2] after renorming the operators. □

Next we prove some immediate numerical range analogues of the Perron–Frobenius theory for positive operators, which generalises results in [5,8] for the matrix case.

In the following $H$ will always denote a complex Hilbert lattice. Note that in the infinite dimensional case the numerical range need not be closed. Thus, in assertion (ii) of the following proposition the closure cannot be omitted.

**Proposition 2.5.** *Let $A\in\mathcal{L}(H)$ and suppose that $A\geq 0$. Then*

(i) $\mathrm{w}(A) = \sup\{\langle Ax, x\rangle \;:\; x\in H_+,\; \|x\| = 1\}$.

(ii) $\mathrm{w}(A)\in\overline{\mathrm{W}(A)}$.

(iii) *If* $\xi\,\mathrm{w}(A)\in\mathrm{W}(A)$ *for some* $\xi\in\Gamma$, *then also* $\mathrm{w}(A)\in\mathrm{W}(A)$.

(iv) *If* $\mathrm{w}(A)\in\mathrm{W}(A)$, *then there exists* $x\in H_+, \|x\| = 1$, *such that*

$$\mathrm{w}(A) = \langle Ax, x\rangle\,;$$

*if, in addition,* $\mathrm{R}(A)$ *is irreducible, then $x$ is a quasi-interior point of* $H_+$.

(v) *If* $|B|\leq A$ *for some regular operator* $B\in\mathcal{L}(H)$, *then*

$$\mathrm{w}(B)\leq\mathrm{w}(A).$$

*Proof.* Assertions (i), (ii), (iii) and the first part of (iv) immediately follow from the estimate

$$|\langle Ax, x\rangle| \leq \langle |Ax|, |x|\rangle \leq \langle A\,|x|, |x|\rangle$$

and the fact that $\|x\| = \||x|\|$ for every $x\in H$. Similarly, (v) follows from

$$|\langle Bx, x\rangle| \leq \langle |B|\,|x|, |x|\rangle \leq \langle A\,|x|, |x|\rangle\,.$$

If $\mathrm{w}(A)\in\mathrm{W}(A)$, then by Proposition 2.4 $\mathrm{w}(A)$ is a peripheral eigenvalue of $\mathrm{R}(A)$. If $\mathrm{R}(A)$ is irreducible, then we know from [9, Thm. V.5.2] that the corresponding eigenspace is one-dimensional and spanned by a quasi-interior point of $H_+$. This shows the second part of (iv). □

By means of Proposition 2.5 (ii) and 2.4 (ii) and Lemma 2.1, we immediately obtain the following.

**Corollary 2.6.** *For* $0 \leq A \in \mathcal{L}(H)$ *we have*

$$\mathrm{r}(\mathrm{R}(A)) = \mathrm{w}(A).$$

Next we show a numerical range analogue of Wielandt's lemma. The key tool is an infinite dimensional version Wielandt's lemma for matrices, see [7, Prop. 4.2.12]. In the following, the identity operator is denoted by Id.

**Lemma 2.7.** *Let* $B, C \in \mathcal{L}(H), B \geq 0, C$ *regular,* $|C| \leq B$ *and* $\mathrm{R}(B)$ *irreducible. If there exists* $\xi \in \Gamma$ *such that* $\xi\,\mathrm{w}(B) \in \mathrm{W}(C)$*, then*

$$C = \xi D B D^*$$

*for a unitary operator* $D$ *such that* $|D| = \mathrm{Id}$.

*Proof.* The proof is similar to the finite dimensional version in [5, Lemma 3.8]. However, we have to use the terminology from the theory of positive operators. If $\xi\,\mathrm{w}(B) \in \mathrm{W}(C)$, then there exists $y \in H$, $\|y\| = 1$, such that

$$\xi\,\mathrm{w}(B) = \langle Cy, y\rangle \in \mathrm{W}(C). \tag{2.3}$$

By the monotonicity of the numerical radius (Proposition 2.5 (v)) we immediately see that

$$\mathrm{w}(C) = \mathrm{w}(B).$$

From

$$\mathrm{w}(B) = |\xi\,\mathrm{w}(B)| = |\langle Cy, y\rangle| \leq \langle B\,|y|\,, |y|\rangle \leq \mathrm{w}(B)$$

it follows that $\mathrm{w}(B) \in \mathrm{W}(B)$. Using Proposition 2.5 (iv) we conclude that $|y|$ is a quasi-interior point of $H$. Moreover, using Proposition 2.4 we see that $\mathrm{w}(B)$ is a peripheral eigenvalue of $\mathrm{R}(B)$ and of $\mathrm{R}(\overline{\xi}C)$, respectively. Since $\left|\mathrm{R}(\overline{\xi}C)\right| \leq \mathrm{R}(B)$, all the assumptions of [7, Prop. 4.2.12] are satisfied (consider $\frac{1}{\mathrm{w}(B)}\,\mathrm{R}(B)$ and $\frac{1}{\mathrm{w}(B)}\,\mathrm{R}(\overline{\xi}C)$), and we obtain that there exists a unitary operator $D \in \mathcal{L}(H)$, such that $|D| = |D^*| = \mathrm{Id}$ and

$$\mathrm{R}(B) = D^*\,\mathrm{R}(\overline{\xi}C)D.$$

The estimate

$$\begin{aligned} 0 &\leq \langle B\,|y|\,, |y|\rangle = \langle \mathrm{R}(B)\,|y|\,, |y|\rangle = \left\langle D^*\,\mathrm{R}(\overline{\xi}C)D\,|y|\,, |y|\right\rangle \\ &= \mathrm{Re}(\overline{\xi}\,\langle D^*CD\,|y|\,, |y|\rangle) \leq |\langle D^*CD\,|y|\,, |y|\rangle| \leq \langle |C|\,|y|\,, |y|\rangle \leq \langle B\,|y|\,, |y|\rangle \end{aligned}$$

implies that

$$\mathrm{Re}(\overline{\xi}\,\langle D^*CD\,|y|\,, |y|\rangle) = \left\langle \overline{\xi}D^*CD\,|y|\,, |y|\right\rangle) = \langle B\,|y|\,, |y|\rangle\,.$$

By [9, Sect. II.11, p. 135] there exist operators $T_1, T_2 \in \mathcal{L}(H_{\mathbb{R}})$ such that

$$\overline{\xi}D^*CD = T_1 + \mathrm{i}\,T_2.$$

Then,

$$\underbrace{\langle B\,|y|\,, |y|\rangle}_{\in\mathbb{R}} = \underbrace{\langle T_1\,|y|\,, |y|\rangle}_{\in\mathbb{R}} + \mathrm{i}\,\underbrace{\langle T_2\,|y|\,, |y|\rangle}_{\in\mathbb{R}},$$

and thus $\langle T_2\,|y|\,, |y|\rangle = 0$. Since $T_1 \leq |D^*CD| \leq |C| \leq B$, we have $B - T_1 \geq 0$. Take $n \in \mathbb{N}$ and $x \in [0, n\,|y|]$. Then

$$0 \leq \langle (B - T_1)x, |y|\rangle \leq n\,\langle (B - T_1)\,|y|\,, |y|\rangle = 0.$$

Since $|y|$ is a quasi-interior point and since $(B - T_1)x \geq 0$, we conclude that $(B - T_1)x = 0$. So we obtain

$$Bx = T_1 x \quad \text{ for every } x \in \bigcup_{n\in\mathbb{N}} [-n\,|y|\,, n\,|y|] =: H_{|y|}.$$

Since $H_{|y|}$ is dense in $H_{\mathbb{R}}$ as $|y|$ is quasi-interior, we have $B = T_1$. Moreover, $T_2 = 0$ because $|T_1 + \mathrm{i}\,T_2| = B$, and thus

$$B = \overline{\xi} D^* C D. \qquad \square$$

Next, we consider the case that the *numerical circle*, i.e. the circle centered at 0 with radius $\mathrm{w}(A)$, contains a point from the numerical range of $A$. The main result for this situation is the following theorem.

**Theorem 2.8.** *Let $0 \leq A \in \mathcal{L}(H)$, such that $\mathrm{R}(A)$ is irreducible. Then, for each $\xi \in \Gamma$ the implication*

$$\xi\,\mathrm{w}(A) \in \mathrm{W}(A) \quad \Longrightarrow \quad \xi\,\mathrm{W}(A) = \mathrm{W}(A) \tag{2.4}$$

*holds. In this case, the space*

$$V_\xi := \{x \in H \ : \ \xi w(A)\,\langle x, x\rangle = \langle Ax, x\rangle\}$$

*is one-dimensional. Moreover, $V_1$ is spanned by a quasi-interior point of $H_+$, and if $x \in V_\xi$, then $|x| \in V_1$.*

*Proof.* Suppose that $\xi\,\mathrm{w}(A) \in \mathrm{W}(A)$. Lemma 2.7 with $C = B = A$ yields

$$A = \xi DAD^* \tag{2.5}$$

for some unitary operator $D \in \mathcal{L}(H)$ such that $|D| = |D^*| = \mathrm{Id}$. By the invariance of the numerical range under unitary transformations we obtain

$$\mathrm{W}(A) = \xi\,\mathrm{W}(DAD^*) = \xi\,\mathrm{W}(A).$$

Clearly, it follows from (2.5) that

$$\mathrm{R}(A) = D\,\mathrm{R}(\xi A) D^*.$$

In view of Proposition 2.4 (iii), this implies that the spaces $V_\xi$ and $V_1$ have the same dimension. By [9, Thm. V.5.2], $V_1$ is one-dimensional and spanned by a quasi-interior point $y \in H_+$. From (2.5) we also see that if $x \in V_\xi$, then $D^*x \in V_1$. However, since $D^*x$ is a multiple of $y$, we conclude that $|D^*x| \in V_1$. On the other hand, $|D^*x| = |x|$ which shows the last assertion of the theorem. $\square$

The reverse implication in (2.4) is not true in general. Consider for example the left shift operator $L$ on $\ell^2$. It is well-known that its numerical range is the open unit disk. Thus, $\xi\,\mathrm{W}(L) = \mathrm{W}(L)$ is fulfilled for any $\xi \in \Gamma$. However, $\xi\,\mathrm{w}(L) = \xi$ is not contained in the numerical range.

In the next theorem we establish conditions on the Hilbert lattice $\ell^2$ ensuring that the intersection of the numerical circle with the numerical range is the same as the intersection with the closure of the numerical range.

To prove this we make use of an embedding procedure turning the approximate spectrum of an operator into the point spectrum of the embedded operator. Such embeddings occur frequently in various contexts, see [11,12]. Here, we want the order structure to be preserved as well as positivity and irreducibility of the operators involved. Such a construction can be found in [9, Section V.1]. We will briefly summarise the main points but we refer to the reference above for details.

We start from the space

$$B := \ell^\infty(\ell^2) := \{(x_n)_{n\in\mathbb{N}} : x_n \in \ell^2, n \in \mathbb{N}, (x_n)_{n\in\mathbb{N}} \text{ is bounded}\}$$

of bounded sequences in $\ell^2 = \ell^2(\mathbb{N})$. We fix a free ultra filter $U$ on $\mathbb{N}$ and define

$$c_U := \{(x_n)_{n\in\mathbb{N}} \in B \ : \ \lim_U \|x_n\| = 0\},$$

where $\lim_U$ means the limit with respect to the ultra filter $U$. The quotient space of $B$ by $c_U$ is denoted by

$$M = B/c_U,$$

and it can be endowed with an ordering in a canonical way, see [9, Prop. II.5.4]. The space $\ell^2$ can be embedded into $M$ via

$$x \in \ell^2 \mapsto \widehat{x} := (x, x, x, \ldots) + c_U \in M.$$

Moreover, to an operator $C \in \mathcal{L}(\ell^2)$ we associate its extension $\widehat{C} \in \mathcal{L}(M)$ by

$$\widehat{C}((x_1, x_2, x_3, \ldots) + c_U) = (Cx_1, Cx_2, Cx_3, \ldots) + c_U.$$

Clearly, if $C \geq 0$, then also $\widehat{C} \geq 0$.

**Theorem 2.9.** *Let $0 \leq A \in \mathcal{L}(\ell^2)$ such that $\mathrm{R}(A)$ is irreducible and let $\mathrm{r}(\mathrm{R}(A))$ be a pole of the resolvent of $\mathrm{R}(A)$. Then, for each $\xi \in \Gamma$ the following are equivalent.*

(i) $\xi\,\mathrm{w}(A) \in \overline{\mathrm{W}(A)}$,
(ii) $\xi\,\mathrm{w}(A) \in \mathrm{W}(A)$,
(iii) $\xi\,\mathrm{W}(A) = \mathrm{W}(A)$.

*In this case, the space*

$$V_\xi := \{x \in \ell^2 \ : \ \xi w(A)\,\langle x, x\rangle = \langle Ax, x\rangle\}$$

*is one-dimensional. Moreover, $V_1$ is spanned by a quasi-interior point of $\ell^2_+$, and if $x \in V_\xi$, then $|x| \in V_1$.*

*Proof.* "(i) ⇒ (ii)" If $\xi\,\mathrm{w}(A) \in \overline{\mathrm{W}(A)}$, then there exists a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \ell^2, \|u_n\| = 1$, such that

$$\overline{\xi}\,\langle Au_n, u_n\rangle \to \mathrm{w}(A) \quad \text{ as } n \to \infty. \tag{2.6}$$

As the unit ball in $\ell^2$ is weakly sequentially compact, we can extract a weakly convergent subsequence from $(u_n)$ called $(u_n)$ again. By [3, Thm. 1.5-4] either

$$(u_n)_{n\in\mathbb{N}} \text{converges weakly to } 0, \tag{$*$}$$

or

$$(u_n)_{n\in\mathbb{N}} \text{ converges weakly to some } z \in V_\xi\setminus\{0\}. \tag{$**$}$$

Clearly, ($**$) implies that $\xi\,\mathrm{w}(A) \in \mathrm{W}(A)$. Observe that in the space $\ell^2$ ($*$) is satisfied if and only if

$$(|u_n|)_{n\in\mathbb{N}} \text{ converges weakly to } 0, \tag{$*'$}$$

as one can check directly or use [7, Prop. 2.5.23]. So our goal in the following is to exclude ($*'$).

Since the limit in (2.6) is real, we have

$$\left\langle \mathrm{R}(\overline{\xi}A)u_n, u_n\right\rangle = \mathrm{Re}(\overline{\xi}\left\langle Au_n, u_n\right\rangle) \to \mathrm{w}(A) \quad \text{ as } n \to \infty.$$

From the estimate

$$\left|\left\langle \mathrm{R}(\overline{\xi}A)u_n, u_n\right\rangle\right| \le \left\langle \left|\mathrm{R}(\overline{\xi}A)u_n\right|, |u_n|\right\rangle \le \left\langle \mathrm{R}(A)\,|u_n|\,, |u_n|\right\rangle \le \mathrm{w}(\mathrm{R}(A)) = \mathrm{w}(A)$$

we see that also

$$\lim_{n\to\infty} \left\langle \mathrm{R}(A)\,|u_n|\,, |u_n|\right\rangle = \mathrm{w}(\mathrm{R}(A)).$$

To exclude ($*'$) we return to the embedding procedure sketched above. Without loss of generality we may assume that $\mathrm{r}(\mathrm{R}(A)) = 1$ (otherwise consider $\frac{1}{\mathrm{r}(\mathrm{R}(A))}\,\mathrm{R}(A)$). By [9, Cor. V.5.2], $\mathrm{r}(A)$ is a first order pole. Let $y$ be the normalised strictly positive vector spanning the eigenspace of $\mathrm{R}(A)$. Then the residue $P$ is of the form

$$P : \ell^2 \to \ell^2, \quad x \mapsto \varphi(x)y,$$

for some strictly positive linear form $\varphi$ such that $\varphi(y) = 1$. Thus $P$ is a strictly positive projection of rank 1. It follows that also the embedded operator $\widehat{\mathrm{R}}(A)$ has a first order pole at 1 with residue $\widehat{P} = \widehat{\varphi}(\cdot)\widehat{y}$ where

$$\widehat{\varphi}((x_n)_{n\in\mathbb{N}} + c_U) = \lim_U \varphi(x_n),$$

see the proof of [9, Thm. V.5.4].

An elementary computation shows that

$$\mathrm{R}(A)\,|u_n| - |u_n| \to 0, \quad n \to \infty.$$

Hence, $(|u_n|)_{n\in\mathbb{N}} + c_U$ is an eigenvector of $\widehat{\mathrm{R}}(A)$, and thus $(|u_n|)_{n\in\mathbb{N}} + c_U = \widehat{P}((|u_n|)_{n\in\mathbb{N}} + c_U)$. Now

$$0 < (|u_n|)_{n\in\mathbb{N}} + c_U = \widehat{P}((|u_n|)_{n\in\mathbb{N}} + c_U) = \lim_U \varphi(|u_n|)\widehat{y}. \tag{2.7}$$

If $(|u_n|)_{n\in\mathbb{N}}$ converges weakly to 0, then also $\lim_U \varphi(|u_n|) = 0$ which contradicts the positivity of $\lim_U \varphi(|u_n|)\widehat{y}$ in (2.7).

"(ii) ⇒ (iii)" See Theorem 2.8.

"(iii) ⇒ (i)" We know from Proposition 2.5 (ii) that $\mathrm{w}(A) \in \overline{\mathrm{W}(A)}$. Then

$$\xi\, \mathrm{w}(A) \in \xi \overline{\mathrm{W}(A)} = \overline{\xi\, \mathrm{W}(A)} \overset{\text{ass.}}{=} \overline{\mathrm{W}(A)}. \qquad \square$$

The requirement that $\mathrm{r}(\mathrm{R}(A))$ is a pole of the resolvent is, for example, satisfied for any compact or quasi-compact operator $\mathrm{R}(A)$.

*Example.* Let $L_w$ be a compact weighted shift operator on $\ell^2$ with positive weights. It is well-known that $\mathrm{W}(L_w)$ is a closed disk, see [10, Cor. 8]. Since $\mathrm{R}(L_w)$ is positive, irreducible and compact, we have that $\mathrm{r}(\mathrm{R}(L_w)) \in \sigma_p(\mathrm{R}(L_w))$. Moreover, we know that for a compact operator the eigenvalues are poles of the resolvent, see [13, Thm. 5.8-E]. Thus, all the assumptions of Theorem 2.9 are satisfied. Hence, for every $\xi \in \Gamma$ the space $V_\xi$ from Theorem 2.9 is one-dimensional, see also [16, Prop. 2.1].

## 3. The block numerical range of positive operators

In this section we study symmetry properties of the block numerical range of positive operators. Concerning the block numerical range of bounded operators, which was introduced in [15], we refer to the monograph [14] and [15]. The block numerical range of positive matrices has already been investigated in [2]. We briefly recall some of the basic definitions. Suppose that $H$ is decomposed into the orthogonal direct sum

$$H = H_1 \oplus \cdots \oplus H_n$$

of $n$ Hilbert spaces $H_1, \ldots, H_n$. Then an operator $A \in \mathcal{L}(H)$ can be represented by an operator matrix

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

where $A_{ij} \in \mathcal{L}(H_j, H_i)$. To every $x = (x_1, \ldots, x_n) \in H_1 \times \cdots \times H_n$ we associate a scalar $n \times n$-matrix

$$A_x := \begin{pmatrix} \langle A_{11}x_1, x_1\rangle & \cdots & \langle A_{1n}x_n, x_1\rangle \\ \vdots & & \vdots \\ \langle A_{n1}x_1, x_n\rangle & \cdots & \langle A_{nn}x_n, x_n\rangle \end{pmatrix}.$$

The set

$$\mathrm{W^n}(A) = \bigcup_{x \in \mathcal{S}^n} \sigma(A_x)$$

where $S^n = \{(x_1, \ldots, x_n) \in H_1 \times \cdots \times H_n \ : \ \|x_i\| = 1, i = 1, \ldots, n\}$ is called the *block numerical range* of $A$. In analogy to the numerical radius we define the *block numerical radius* as

$$\mathrm{w_n}(A) := \sup_{\lambda \in \mathrm{W^n}(A)} |\lambda|.$$

Note that in the case $n = 1$ the block numerical range and radius reduce to the numerical range and radius, respectively. In general, the block numerical range and radius depend on the particular decomposition of $H$. In the following we fix such a decomposition and omit this dependence in the notation, writing $\mathrm{W}^{\mathrm{n}}(A)$ instead of $\mathrm{W}^{\mathrm{n}}_{H_1\oplus\cdots\oplus H_n}(A)$.

For a Hilbert lattice $H$, we admit only *positive* orthogonal decompositions of the form

$$H = H_1 \oplus \cdots \oplus H_n$$

where each $H_k, k = 1, \ldots, n$, is a closed lattice ideal of $H$. Note that for a positive decomposition of $H$ and a positive operator $A \in \mathcal{L}(H)$ the operators $A_{ij}$ in the matrix representation are positive.

As in Proposition 2.2 we immediately obtain symmetry with respect to the real axis.

**Proposition 3.1.** *For a positive decomposition of $H$ the block numerical range of an operator $0 \le A \in \mathcal{L}(H)$, is symmetric with respect to the real axis.*

*Proof.* Any $y \in H$ is of the form $a + \mathrm{i}\, b$, where $a, b \in H_{\mathbb{R}}$ Define $\overline{y} := a - \mathrm{i}\, b$. Then it is easy to see that $\lambda \in \sigma(A_{(x_1,\ldots,x_n)})$ if and only if $\overline{\lambda} \in \sigma(A_{(\overline{x_1},\ldots,\overline{x_n})})$. □

**Lemma 3.2.** *Let $0 \le A \in \mathcal{L}(H)$, be irreducible and consider a positive decomposition $H = H_1 \oplus \cdots \oplus H_n$. If $(x_1, \ldots, x_n) \in \mathcal{S}^n$, where each $x_i$ is a quasi-interior element of $H_i$, then also the matrix $A_x$ is irreducible.*

*Proof.* The idea is to replace vectors with positive entries in the proof of [2, Prop. 4.1] by quasi-interior points. Suppose that under the given assumptions $A_x$ is reducible. Then there exists $B \subseteq \{1, \ldots, n\}, B \neq \emptyset$ and $B \neq \{1, \ldots, n\}$, such that the space

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^n \ : \ z_i = 0 \text{ for every } i \in B\}$$

is invariant under $A_x$. Since $x_i$ is quasi-interior for every $i \in \{1, \ldots, n\}$, it follows that $\langle A_{ij}x_j, x_i\rangle = 0$ if and only if $A_{ij}x_j = 0$. Since $x_j$ is a quasi-interior point of $H_j$, this implies $A_{ij} = 0$. Hence, the closed ideal

$$\{(y_1, \ldots, y_n) \in H_1 \oplus \cdots \oplus H_n \ : \ y_i = 0 \text{ for every } i \in B\}$$

is invariant under $A$, and thus $A$ is not irreducible contradicting our assumption. □

Next, we generalise Proposition 2.5 to the block numerical range.

**Proposition 3.3.** *Consider a positive decomposition $H_1 \oplus \cdots \oplus H_n$ of $H$. Let $0 \le A \in \mathcal{L}(H)$ and set*

$$\mathrm{W}^{\mathrm{n}}_{+}(A) := \bigcup_{\substack{(x_1,\ldots,x_n)\in\mathcal{S}^n,\\ x_i\ge 0,\ i=1,\ldots,n}} \sigma(A_x).$$

*Then the following statements hold.*

(i) $\mathrm{w}_{\mathrm{n}}(A) = \sup_{z\in \mathrm{W}^{\mathrm{n}}_{+}(A)} |z|$.

(ii) $\mathrm{w}_{\mathrm{n}}(A) \in \overline{\mathrm{W}^{\mathrm{n}}(A)}$.

(iii) *If* $\xi\,\mathrm{w_n}(A) \in \mathrm{W^n}(A)$ *for some* $\xi \in \Gamma$, *then also* $\mathrm{w_n}(A) \in \mathrm{W^n}(A)$.
(iv) *If* $\mathrm{w_n}(A) \in \mathrm{W^n}(A)$, *then there exists* $x = (x_1, \ldots, x_n) \in \mathcal{S}^n, x_i \geq 0, i = 1, \ldots, n$, *such that*

$$\mathrm{w_n}(A) = \mathrm{r}(A_x);$$

*if, in addition,* $A$ *is irreducible, then* $x_i$ *is a quasi-interior point of* $H_i$ *for every* $i \in \{1, \ldots, n\}$.
(v) *If* $|B| \leq A$ *for some regular operator* $B \in \mathcal{L}(H)$, *then*

$$\mathrm{w_n}(B) \leq \mathrm{w_n}(A).$$

*Proof.* (i) Note that for $(x_1, \ldots, x_n) \in \mathcal{S}^n$ we have

$$\left|A_{(x_1,\ldots,x_n)}\right| \leq A_{(|x_1|,\ldots,|x_n|)}.$$

By the monotonicity of the spectral radius for matrices (see [9, p. 21]) it follows that

$$\mathrm{r}(A_{(|x_1|,\ldots,|x_n|)}) \geq \mathrm{r}(A_{(x_1,\ldots,x_n)}). \tag{3.1}$$

Since $\mathrm{r}(A_{(|x_1|,\ldots,|x_n|)}) \in \sigma(A_{(|x_1|,\ldots,|x_n|)})$ we conclude that

$$\mathrm{w_n}(A) = \sup_{z \in \mathrm{W^n_+}(A)} |z|\,.$$

(ii) This follows from (3.1).

(iii) If $\xi\,\mathrm{w_n}(A) \in \mathrm{W^n}(A)$, then there exists $x \in \mathcal{S}^n$ such that

$$\xi\,\mathrm{w_n}(A) \in \sigma(A_x). \tag{3.2}$$

Moreover, using again the monotonicity of the spectral radius we have

$$\mathrm{w_n}(A) = |\xi\,\mathrm{w_n}(A)| \overset{(3.2)}{\leq} \mathrm{r}(A_x) \leq \mathrm{r}(|A_x|) \leq \mathrm{r}(A_{|x|}) \leq \mathrm{w_n}(A).$$

and thus $\mathrm{w_n}(A) = \mathrm{r}(A_{|x|}) \in \sigma(A_{|x|}) \subseteq \mathrm{W^n}(A)$.

(iv) The first assertion is immediate from (3.1) and the fact that

$$\mathrm{r}(A_{(|x_1|,\ldots,|x_n|)}) \in \sigma(A_{(|x_1|,\ldots,|x_n|)}).$$

For the second part we use the idea from the proof of [2, Prop. 4.1] Let $x := (x_1, \ldots, x_n) \in \mathcal{S}^n$ such that $x_i \geq 0$, and $\mathrm{w_n}(A) = \mathrm{r}(A_x)$ and suppose that there exists an index $k \in \{1, \ldots, n\}$ such that $x_k$ is not quasi-interior in $H_k$. Without loss of generality we may assume that $k = n$ and that all other $x_i$ are quasi-interior points of $H_i$. Denote by $I$ the closure of the principal ideal generated by $x_n$. Then the orthogonal complement $I^\perp$ of $I$ is again a non-trivial closed ideal in $H_n$, see [9, Thm. II.2.10, Thm. II.5.14]. Thus,

$$H = H_1 \oplus \cdots \oplus H_{n-1} \oplus I \oplus I^\perp$$

is a positive decomposition of $H$ refining the original decomposition. By our assumption there exists a quasi-interior point $y, y \geq 0, \|y\| = 1$ of $I^\perp$. Then for $\tilde{x} := (x_1, \ldots, x_n, y)$ the matrix $A_{\tilde{x}}$ is irreducible by Lemma 3.2. Moreover, it contains $A_x$ as a principal submatrix. Thus, by [6, Thm. I.5.1] we have $\mathrm{r}(A_x) < \mathrm{r}(A_{\tilde{x}})$. On the other hand for the block numerical radius of our refinement we have $\mathrm{w_{n+1}}(A) \leq \mathrm{w_n}(A)$, see [15, Thm.3.5], and therefore we obtain the contradiction

$$\mathrm{w_n}(A) = \mathrm{r}(A_x) < \mathrm{r}(A_{\tilde{x}}) \leq \mathrm{w_{n+1}}(A) \leq \mathrm{w_n}(A).$$

(v) The claim is immediate from the monotonicity of the spectral radius. □

**Theorem 3.4.** *Let* $0 \leq A \in \mathcal{L}(H)$ *such that* $\mathrm{R}(A)$ *is irreducible. Then, for* $\xi \in \Gamma$ *and a positive decomposition of* $H$ *we have the implication*

$$\xi\,\mathrm{w}(A) \in \mathrm{W}(A) \quad \Longrightarrow \quad \xi\,\mathrm{W^n}(A) = \mathrm{W^n}(A).$$

*Proof.* From the proof of Theorem 2.8 we obtain that there exists a unitary operator $D \in \mathcal{L}(H)$ such that $|D| = \mathrm{Id}$ and

$$\xi A = DAD^*.$$

Observe that the spaces $H_1, \ldots, H_n$ are invariant under $D$. Hence, there exist operators $D_i \in \mathcal{L}(H_i), i = 1, \ldots, n$, such that $D$ has an operator matrix representation in diagonal form

$$D = \begin{pmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_n \end{pmatrix}.$$

Moreover, each $D_i$ is a unitary operator on $H_i$. Hence,

$$\mathrm{W^n}(DAD^*) = \mathrm{W^n}(A),$$

see [14, Prop. 1.1.7]. □

**Theorem 3.5.** *Let* $0 \leq A \in \mathcal{L}(\ell^2)$ *be such that the conditions of Theorem* 2.9 *are satisfied and consider a positive decomposition* $H_1 \oplus \cdots \oplus H_n$ *of* $\ell^2$. *Then for* $\xi \in \Gamma$ *we have the implication*

$$\xi\,\mathrm{w}(A) \in \overline{\mathrm{W}(A)} \quad \Longrightarrow \quad \xi\,\mathrm{W^n}(A) = \mathrm{W^n}(A).$$

*Proof.* By Theorem 2.9 we conclude that $\xi\,\mathrm{w}(A) \in \mathrm{W}(A)$. Then the claim follows directly from Theorem 3.4. □

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