Arch. Hist. Exact Sci. (2013) 67:477–551
DOI 10.1007/s00407-013-0118-0

Leonhard Euler's early lunar theories 1725–1752

Part 2: developing the methods, 1730–1744

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Received: 24 February 2013 / Published online: 4 April 2013

Abstract The analysis of unpublished manuscripts and of the published textbook on mechanics written between about 1730 and 1744 by Euler reveals the invention, application, and establishment of important physical and mathematical principles and procedures. They became central ingredients of an "embryonic" lunar theory that he developed in 1744/1745. The increasing use of equations of motion, although still parametrized by length, became a standard procedure. The principle of the transference of forces was established to set up such equations. Trigonometric series expansions together with the method of undetermined coefficients were introduced to solve these equations approximatively. These insights constitute the milestones achieved in this phase of research, which thus may be characterized as "developing the methods". The documents reveal the problems Euler was confronted with when setting up the equations of motion. They show why and where he was forced to introduce trigonometric functions and their series expansions into lunar theory. Furthermore, they prove Euler's early recognition and formulation of the variability of the orbital elements by differential equations, which he previously anticipated with the concept of the osculating ellipse. One may conclude that by 1744 almost all components needed for a technically mature and successful lunar theory were available to Euler.

Communicated by : N. Guicciardini.

In memoriam Curtis A. Wilson (1921–2012).

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1 Introduction

In the first part of this trilogy of papers devoted to the development of Euler's lunar theories (cf. Verdun 2012), I analyzed unpublished manuscripts and notebook records written by him between 1725 and 1730 and containing his earliest tentative approaches to lunar theory. I presented evidence that Euler learned the basic empirical facts about the Moon's motion from the book *Astronomia philolaica* of 1645 by Ismael Boulliau (1605–1694) and from the books by David Gregory (1659–1708) and Charles Leadbetter (1681–1744). Euler was most probably motivated to engage with lunar theory by his careful study of the statements on the motion of the lunar apses in the three editions of the *Principia* published by Isaac Newton (1643–1727). The documents we have reveal that he struggled considerably with different approaches to find a method or strategy by means of which a sound lunar theory could be constructed. Most of his early methods were not successful, because he formulated his very first thoughts on the motion of the Moon in terms of central forces, radius of the osculating circle or curvature radius, Huygens' centrifugal principle, de Moivre's and Keill's theorems, and of multiple force centers (represented by the Earth and the Sun) acting simultaneously on the Moon. I found that Euler, in that early phase of the development of his research, did not yet realize the importance neither of the usefulness of the equations of motion nor of the need to parametrize them by time. His first approaches were dominated by the problem of central force motion, which at that time (about 1710) played an important role in the proof of the inverse problem of central forces by Johann I Bernoulli (1667–1748) and Jacob Hermann (1678–1733). Pierre Varignon (1654–1722) published a series of papers on that topic using Leibnizian calculus between 1703 and 1712. When Euler was transforming Newton's *Principia* into analytical language resulting in what we now call rational mechanics, he was following a tradition of mechanical and mathematical methods prepared by Bernoulli, Hermann, and Varignon. The fact that Euler's first steps emerged in this context may be appraised as quite unspectacular with respect to innovative ideas. His approaches may be judged as inadequate to solve the problem of multiple central forces, on which he based his first theoretical constituents to cope with the lunar problem. Nevertheless, his first trials and ideas contained the nuclei which some time later led him to the concept of the osculating ellipse and to the principle of the transference of forces. I argued that his grappling with multiple force centers and associated curvature radii led him to the concept of the osculating ellipse. The conflict how to deal with multiple force centers actually concealed a fundamental problem, namely the choice of an appropriate (origin of) reference frame, that Euler disentangled some time later in his "*Mechanica*" by the discovery of what he called "genuina methodus" (genuine method), but what I call the principle of the transference of forces. This principle turned out to be one of the most important steps towards a mature lunar theory which he took between 1730 and 1744. I will present in this part of my trilogy both the path that has taken him to this principle and to the use of the equations of motion (although still parametrized by length). By the use of equations of motion he was, for the first time, confronted with approximate solutions in the form of series expansions.

The manuscripts presented in this part have not been analyzed before in the context of the development of Euler's lunar theory. A first approximate analysis was presented in Verdun (2010). That study showed the potential inherent in these documents. Although some of the manuscripts are unfinished or only fragmentarily preserved, they may be used to put together the pieces for the development of Euler's early lunar theories (Verdun 2010). The reconstruction presented in this paper, although incomplete, will reveal the development of the principal methods invented and applied by Euler to cope with the lunar problem. A general overview of Euler's unpublished and published documents related to lunar theory is given in my first article (cf. Verdun 2012).

Let me also repeat some caveats of my first part that are valid for this part as well. For this part, four manuscripts and Euler's first textbook on mechanics are relevant. It is impossible to date the manuscripts exactly to within, e.g., 1 or 2 years. Consequently, the order I present them here is mainly for didactic reasons. Not every derivation of mathematical formulae and not every explanation and comment by Euler can be reproduced here. This will be the goal of a critical edition of these works, which is also being prepared.[1] Figures (sketches) are reproduced as exact copies of Euler's original drawings, including all inconsistencies and inaccuracies (e.g., faultily drawn tangential, parallel or perpendicular lines). When reconstructing intermediate results and formulae which Euler did not derive explicitly, I often will refer to Volume I, Chapter V, of his "*Mechanica*" (cf. Euler 1736), which Euler finished in 1734 and which contains much of the results developed by him already until that time.

The structure and organization of this second part is similar to the first one. In Sect. 2, I will start with short descriptions of the documents and with my attempt to date them as exactly as possible due to indications that were not yet considered up to now. In Sect. 3, I will summarize and review those parts of them, which are relevant both for the time interval under consideration and for the development of Euler's lunar theories. The whole body of these summaries may be regarded as a reconstruction of the development of Euler's methods invented and used for lunar theory during the relevant period of time. This reconstruction serves for the final assessment of Euler's achievements given in Sect. 4. It is followed by the conclusions given in Sect. 5. As in the first part, I relegate more detailed descriptions of the documents into the Appendix to relieve the main body of this article from the more technical and mathematical contents of these documents.

2 Description and dating of the documents

There are four unpublished manuscripts by Euler which were written most probably between 1730 and 1744, when he established his position at the Academy of St. Petersburg and then moved on to become a member of the Berlin Academy in 1741

[1] It is part of my granted project to edit and publish Euler's manuscripts and notebook records related to lunar theory as an online edition in the context of the "Euler Heritage Project" planned by the Bernoulli-Euler-Center in Basel, Switzerland.

(cf. Fellmann 2007). They are preserved in the Archives of the Russian Academy of Science in St. Petersburg and referred to here as Ms 167, Ms 271, Ms 273, and Ms 276. In addition, early in this time period, Euler composed his two-volume textbook on mechanics, which was published in 1736. At least one chapter of the first volume—the shortcut of which is E 15—is directly related to lunar theory. Due to the uncertainty in the exact dating of these documents, I will present them (as I have already done in my first paper) in a hypothetical, didactically motivated order thus emphasizing the evolutionary character of the development of Euler's methods used for lunar theory.

2.1 Euler's textbook manuscript Ms 167 on *Mechanics*

This document is listed as number 167 in the catalogue of Euler's unpublished manuscripts (see Kopelevič et al. 1962, pp. 58–60). It consists of 99 folios written on both sides in a carefully executed clean copy style handwriting. Kopelevič et al. (1962) dates this manuscript to the [early] 1730s, perhaps because it may be regarded as a draft version of his "*Mechanica*" (E 15, cf. Euler 1736), which was finished by 1734. It is entitled "Mechanica seu scientia motus" (Mechanics or the science of motion) and must have been written before Ms 271 and before the final version of the first volume of his "Mechanica sive motus scientia analytice exposita" (Mechanics, or the analytical exposition of the science of motion), because it contains neither the lunar problem (as presented in Chapter V in E 15, cf. Euler 1736) nor the principle of the transference of forces (which will be addressed below). This manuscript, intended to be composed in at least two parts, is only fragmentarily preserved, because only the first part exists and several folios from it were lost in the course of time. The whole manuscript was published in 1965 by Gleb K. Mikhailov (cf. Mikhailov 1965, pp. [93]–224). It is not identical with the published version, but differs considerably with respect to its content and structure, which is why the editors of Kopelevič et al. (1962) called it a "variant" version of the "*Mechanica*". Part I "De motu a potentiis producto" (On the motion produced by forces) contains three sections: Section I "De motu a potentiis in punctum liberum agentibus producto" (On the motion produced by forces acting on a free pointlike body) deals with the free motion of point masses. It contains four chapters. Section II "De motu a potentiis in punctum non liberum agentibus producto" (On the motion produced by forces acting on a pointlike body which is not free) treats the constrained motion of point masses and consists of three chapters. Section III "De motu corporum rigidorum a potentiis utcunque sollicitatorum" (On the motion of rigid bodies driven by any forces) is devoted to the free motion of rigid bodies and contains only one chapter, thus indicating the incompleteness of this manuscript. This section is of special importance due to the fact that it contains the very first derivation of the law of angular momentum for motions of rigid bodies around space fixed axes. This derivation corresponds to Euler's records in his third notebook Ms 399, fol. 75v–76r, written between 1736 and 1740 (cf. Verdun 2010, pp. 497–500), in his "*Scientia navalis*" (cf. E 110, Eneström 1910, p. 30f, finished in 1738, cf. Fellmann and Mikhailov 1998, letter no. 21, p. 264, dated December 20, 1738), and in his "Dissertation sur la meilleure construction du cabestan" (cf. E 78, Eneström

1910, p. 19f, presented to the St. Petersburg Academy on Juli 3, 1738, cf. Kopelevič et al. 1962, No. 246, p. 78).

2.2 Euler's unpublished manuscript Ms 271

This document is registered as number 271 in the catalogue of Euler's unpublished manuscripts (cf. Kopelevič et al. 1962, p. 85). It consists of three folios written on both sides in a carefully executed clean copy style handwriting. It is entitled "De Motu Lunæ in Ellipsin" (On the motion of the Moon in an ellipsis) and contains 16 paragraphs consisting of text, formulae, and one figure. There are some corrections, overwritings, and marginal notes by Euler. The mathematical formulae are integrated into the text and are written in the old-fashioned geometric-synthetic Newtonian style without any use of the calculus. It may be considered an unfinished first approach by Euler to construct lunar tables using elementary physical and geometrical methods. This is probably why Kopelevič et al. (1962) dated the origin of this manuscript to the years 1725–1726, but also perhaps due to its characteristic style (ductus litterarum) of Euler's early handwriting. There is evidence given by the notebooks Ms 397 and Ms 398 (cf. Kopelevič et al. 1962, pp. 114–115), that this manuscript could also have been written a few years later (between 1727–1729). It is, indeed, quite possible, that it was written even around 1730 due to the fact that it contains the very early formulation of the *Principle of the transference of forces*, as it will be presented below. This assumption is supported by the following argument: In Proposition 97 of his "*Mechanica*", wherein Euler formulated this principle, he wrote (cf. Euler 1736, Prop. 97, Schol. 1):

> […] Interim tamen certum est, si huius propositionis solutio posset evolvi ex eaque tabula confici, hoc in astronomia maximam habiturum esse utilitatem.
> ([…] Yet meanwhile it is certain that, if the solution of this proposition can be evolved and from that a table constructed, then it would be of the most use in astronomy.)

Ms 271 is, in fact, Euler's first trial to construct lunar tables using the principle of the transference of forces. He introduced his treatise by stating:

> §.1. In præcedenti dissertatione, qua motus lunæ tabulas exhibui, orbitam lunæ pro circulo habui, et ex hoc tabulas computavi. Cum autem orbita lunæ multo propius ad ellipsin accedat, statui hic motum lunæ in ellipsi examini subjicere, et inde tabulas supputare, in quibus omnes motus lunæ anomaliæ, quæ hactenus sunt observatæ, continerentur, quatenus eæ non ab aberratione orbitæ ab ellipsi oriuntur, qui error hinc ortus valde exiguus est, et de quo alibi tractabo.
> (In the preceding treatise, in which I have given the tables of the Moon's motion, I have supposed the Moon's orbit to be a circle and calculated the tables on that assumption. But because the lunar orbit resembles much more an ellipse, I have here investigated the Moon's motion in an ellipse and computed tables, which contain all irregularities of the Moon's motion that have been observed until now, as far as they are not caused by the deviation of the orbit from an ellipse; the error emerging from that is very small, and I will treat it elsewhere.)

The "preceding treatise" Euler mentions in this statement, in which he computed lunar tables on the basis of a *circular* lunar orbit, is not preserved. Furthermore, it becomes not clear from this statement whether Euler planned to give new lunar tables but did not calculate them (for any reasons or because he did not finish his dissertation), or whether he in fact constructed lunar tables due to the recipe derived in this treatise from the principle of the transference of forces but that these tables have also gone lost (there are no such tables known or preserved). From the above statement of the "*Mechanica*" one may assume that the former case happened, which means, that he did not calculated lunar tables on the basis of an *elliptic* lunar orbit incorporating the principle of the transference of forces (thus considering the action of the Sun) before the time he was composing and publishing his "*Mechanica*", that is by 1734–1736.

2.3 Euler's unpublished manuscript Ms 273

This document is registered as number 273 in the catalogue of Euler's unpublished manuscripts (see Kopelevič et al. 1962, p. 86). It is only fragmentarily preserved and consists of 11 folios written on both sides. Due to the missing title and due to its structure, dividing it into propositions, corollaries, and scholia, it might have been part of a larger treatise on celestial mechanics or lunar theory. It is peppered with mathematical derivations and formulae presented in a text book style manner. What stands out are the varying characteristics of Euler's hand writing, which implies that this manuscript presumably was not written at one and the same time, but its composition was taken up anew time and again. In the margins, Euler referred to 13 Figures which all got lost. The manuscript was dated by Kopelevič et al. (1962) to the years 1727–1730. It is most likely the earliest document in which Euler clearly formulated the principle of the transference of forces. This is why it was most probably written after Ms 167 and after Ms 271, and not before 1730.

2.4 Euler's published textbook on Mechanics E 15

Euler's textbook on mechanics, the so-called "first mechanics", was published in two volumes in 1736 (cf. E 15 and E 16 in Eneström 1910, p. 4). Only the first volume is relevant for the development of the lunar theory (cf. Euler 1736). It is his first monumental work that made him widely famous in the scientific community. Although a milestone in the history of mechanics, this book has never been analyzed and appreciated in depth from a modern historiographical point of view. There are only two translations, one in German by Jakob Philipp Wolfers (cf. Wolfers 1848), and one in English newly made by Ian Bruce (cf. http://www.math.dartmouth.edu/euler/). Euler finished the first volume already during the year 1734. In a letter written probably in November 1734 to Daniel Bernoulli, Euler announced that he had completed the first volume of his "*Mechanica*":

> Von meiner *Mechanica* ist der erste Tomus auch ganz fertig, habe aber wenig Hoffnung, dass man denselben allhier drucken werde. [2]
> (The first volume of my *Mechanica* is also quite done, but I have only little hope that it will be printed here.)

Actually, Euler probably planned to write a textbook on celestial mechanics or lunar theory (cf. Fellmann 1983, p. 65), but only Chapter V addresses clearly these topics and thus gives evidence for this assumption, although the preliminary chapters are relevant to such a program with regard to their concepts and foundations as well. Nevertheless, Chapter V may be considered Euler's first published investigations into lunar theory.

2.5 Euler's unpublished manuscript Ms 276

This document is registered as number 276 in the catalogue of Euler's unpublished manuscripts (see Kopelevič et al. 1962, p. 86), where its composition is dated to the 1740s. This fragmentarily preserved manuscript is entitled "De Motu Lunæ" (On the motion of the Moon) and consists of four double-side written folios. The text is written in a carefully executed clean copy style handwriting. There are no references to figures, but it is most likely that Euler intended to illustrate this manuscript. Due to its content and state of the used methods one may conclude that is was written between 1738 and 1743. The text is carefully written out in full length, from which one may assume that this manuscript was ready for publication. Unfortunately, the main body of the manuscript is lost. It would have contained most probably Euler's first elaborated lunar theory. Evidence for this assumption is given by the mathematical procedure developed by Euler in the beginning of the manuscript which has been advanced to such a high degree that it would have led to a first approximate solution of the lunar problem.

3 Reconstruction of the development of Euler's methods used for lunar theory

Similar to part 1 (cf. Verdun 2012), this reconstruction is given by way of summaries of the contents of Euler's documents, which are presented in more detail in the Appendix to this part. These documents disclose Euler's efforts to treat the Moon's motion as a three-body-problem, defined by the Earth–Moon–Sun system, and as a dynamical problem. On the one hand, this approach led Euler to equations of motion that demanded new mathematical methods for their solution. Most important are the central issues from Ms 167: his considerations on relative forces and "scalas" of forces, interpreted as "variational curves". They probably set the theoretical framework which enabled him, in Ms 271, to formulate the principle of the transference of forces. On the other hand, his approaches in Ms 167 are still linked to geometric-kinematic considerations. Euler derived the equation of motion for a pointlike body moving in a

[2] Cited according to Eneström (1906), p. 139.

straight line in the notation of Leibnizian calculus parametrized by time, which he immediately re-parametrized by length, apparently still believing in the advantage of finding the geometric nature of the curve described by the Moon much easier. We observe Euler cutting this cord gradually in Ms 273, in which he established the principle of the transference of forces definitely and stating it explicitly. In addition, he became aware of the significance of terms in the equation of motion depending reciprocally on the cube of the distance between Sun and Moon. The treatment of this problem became an important issue in Ms 276. Ms 273 also proves Euler's virtuosic mastering of the two-body-problem in all its facets, including the explanation of the motion of the lunar apses by a force law which is inversely proportional but not to the square of the distance. In this manuscript, Euler became fully aware of the variation of the orbital parameters due to the perturbational effects caused by the Sun. To solve the differential equations for the motion of the Moon that take into account the simultaneous actions of Sun and Earth, he was forced to introduce a novelty: the approximation by series expansions, allowing him to integrate the equations term by term. Euler published most of these achievements in Chapter V of his textbook on mechanics, the shortcut of which is E 15. It deals mainly with the mobile ellipse rotating around the central body located in one focus, and Euler's attempts to treat it as an osculating ellipse whose characteristic parameters are variable with time. His achievements in E 15 are dominated by the definite establishment of the principle of the transference of forces, that became an important cornerstone for his further developments. Using this principle, he also derived the motion of the nodes of the lunar orbit probably for the first time. The last manuscript, Ms 276, Euler wrote most probably before 1744. It contains nearly all ingredients needed for a successful and technically mature lunar theory. Here, he clearly recognized that approximations by series expansions are unavoidable. Moreover, the determination of the inverse cube of the distance between the Sun and the Moon in terms of the distances between the Earth and the Sun and between the Earth and the Moon as well as the geocentric angular distance between the Sun and the Moon involves a formula for the inverse cube of the Sun's distance that has to be put into the power of $-3/2$. The solution implied the expansion of the resulting formula into an infinite series that contains the cosines of angular arguments defined by the geocentric angular distance between the Sun and the Moon. At that point series expansions by trigonometric functions found its entrance into celestial mechanics, at least with Euler. At the same time, he began to recognize the importance of the angular arguments for the interpretation of the perturbational effects. Last but not least, the manuscript contains Euler's very first application of the then well known method of undetermined coefficients for solving the differential equations. Up until 1744, there remained only one important step leading to the final breakthrough that opened the door to a powerful lunar theory. This will be the topic of part 3.

3.1 Summary of Ms 167

Inspired by Jacob Hermann's *Phoronomia* (cf. Hermann 1716), Euler introduced the concepts "scala celeritatum" and "scala potentiarum", which express the graphical

representation of the point-to-point dependency of a parameter (such as the velocity or the force) from another one (e.g., from the distance between two bodies or the distance covered by a body). From the way Euler used and applied these notions in the presented examples and problems one may suppose that the "scala" does not represent just one but any curve or family of curves, similar to "variational curves". If this assumption holds, as some problems treated by Euler in this manuscript suggest, he might also have asked the following question within that context: how can the whole system of forces acting in the system Sun–Earth–Moon be transformed in such a way that there are no forces left acting on the Earth? In other words, is there any transformation of forces operating in such a way as to keep the Earth at rest with respect to inertial space? The consequence of such kind of transformation would imply that the Moon's motion could be described as observed from the Earth considered to be at rest. Euler treated the problem concerning the relative motions of bodies already in his second notebook Ms 398 (cf. Verdun 2012, Chapter D.2). The solution of this question involves the choice of an appropriate reference frame and the concept of relative motion (cf. Maltese 2000). This approach is probably motivated by Euler's definitions and considerations on absolute and relative forces, which appear neither in Newton's nor in Hermann's work (cf. Jammer 1957; Westfall 1971; Kutschmann 1983).

Before starting with some "variational-type" problems, Euler introduced the equation of motion (or "Newton's" second law of motion) as formulated by Hermann in 1716 (cf. Hermann 1716, p. 57). It seems that Euler in that time assumed that the equation of motion is applicable only for the case of rectilinear motions. He substituted—and this is even more striking—the time element with the path or line element, thus deriving the equation of motion parametrized by length. This kind of parametrization is more convenient if the geometry of the curve covered by a body has to be figured out. This is why Euler kept this parametrization in his "*Mechanica*" (cf. Euler 1736, Prop. 20) as well as in his further investigations until about 1744. The integration of the equation of motion (parametrized by length) produces a factor 2, which Euler (e.g. in his "*Mechanica*") kept in the equation of motion by choosing the physical units in such a way that this factor would not cancel out (cf. Euler 1736, Prop. 25 and Explanation 15). Then Euler treated some problems which are closely related to what we call today "calculus of variations" in terms of the concepts of "scala potentiarum" and "scala temporum". This type of problems concerns the question to find all possible "scalas potentiarum" causing the body either to cover a given space in a given time interval or to reach a given position with a given velocity. We observe here Euler's acquaintance with a then quite new but steadily growing branch of mechanics and applied mathematics which later became known as "variational calculus". However, its relation to Euler's Ms 167 and to his "*Mechanica*" has not been recognized and analyzed yet by historians of science (cf. Goldstine 1980; Fraser 1994).

Having introduced the terms "normal" and "tangential" forces (thus following Johann Bernoulli and Varignon, cf. Bernoulli 1712; Varignon 1703a,b,c, 1704, 1705, 1707a,b, 1712) and using variants of the equation of motion, Euler treated the two-body-problem in the usual way as he had already done in the previous manuscripts in terms of curvature radius and Huygens' theorem (cf. Verdun 2012). It is remarkable that, in this context, the concept "central force" does not appear at all. Euler further

derived Keill's theorem. Finally, he clearly recognized that this kind of approach is limited and not suitable for solving the three-body-problem, i.e., in cases where the trajectories become more complicated. In retrospect, this may be seen as a first important insight by Euler. It may be the reason for his insisting in trying to solve the problem of two or multiple force centers. This problem he now tackled in a new approach by reducing the forces acting on the considered body into the resulting normal and tangential force components. With that he was able to derive an integral equation determining the nature of the body's trajectory in terms only of the central forces and the distances of the test body to the force centers. Moreover, he found an equation defining the resulting (instantaneous) curvature radius expressed by those terms. This fact proves the central role that the curvature radius plays in Euler's theory of that time. Finally, he considered two cases, where one of the force centers is infinitely far away. This situation approximates the Earth-Moon-Sun system, where the Sun may be regarded as infinitely further away from the Moon than the Earth.

3.2 Summary of Ms 271

The main goal of this manuscript is to develop a formula or recipe useable for easy construction of lunar tables. Euler based his derivation on the equation of motion as a function of the curvature radius. In this form it was given by Varignon (1707a,b) and can easily be derived from the equation of motion, parametrized by length, as presented in Eulers Ms 167 as well as in his "*Mechanica*". Euler started by determining the forces acting on the Moon by the Sun and the Earth, using the principle of the transference of forces, without stating it explicitly. Here, we observe Euler applying this principle for the very first time. Evidence for its novelty is given by some cancellations, insertions, and marginal notes added by Euler, which indicate his uncertainty as to how to apply this principle correctly. He found an equation for the Moon's velocity in terms of a set of "orbital parameters", as we would call them today. This equation makes transparent the dependencies of the relevant parameters: the first term depends only on the Earth's attraction, the second and third on the Sun's gravitational force. In addition, the second term depends on the Moon's position in its orbit and on the Earth's distance from the Sun, while the third term depends on the lunar aspect, namely on the angle between Moon and Sun. Euler recognized that the excess of the terms depending on the Sun over the terms not depending on it is always reciprocal to the cube of the Sun's distance from the Earth. This is why the Moon's velocity produced by the Sun is much smaller than that generated by the Earth. In the 1740s, it turned out that the structure of such equations became important for the correct interpretation of the perturbational terms represented by the relevant angular arguments and its linear combinations. Euler then simplified this equation by approximations and made it applicable for the numerical calculation of lunar tables.

3.3 Summary of Ms 273

This fragmentarily preserved manuscript contains six propositions, in which Euler launched a first serious and comprehensive attack to master the lunar problem. In

Proposition I he determined the Keplerian motion (represented by orbital elements characterizing the two-body-problem) starting from a dynamic relation that replaces the equation of motion. The derivations and corollaries are thoroughly worked out and contain all standard aspects characterizing the problem. Proposition II deals with the "mobile orbit", i.e., the problem of the apsidal motion of the Moon's orbit, which Euler solved using the results gained from the first proposition, combined with the substitution of the Moon's distance y from the Earth by $y + \mathrm{d}y$. Again, this proposition is followed by several corollaries, in which different aspects and results are discussed. In one of them Euler generalized the law of gravitation by replacing the exponent 2 by an arbitrary number n and discussed different cases for the values $n > 2$ and $n < 2$. Both cases produce an apsidal motion, decreasing and increasing in either case. A similar discussion became important in the late 1740s, when the problem of the lunar apse developed into a touchstone for the inverse square law of gravitation. Proposition III is innovative with respect to two aspects: 1. The statement of the problem (Invenire Vires Solis ad Lunæ motum perturbandum) contains for the very first time in Euler's work on celestial mechanics the word "perturbation", and 2. Euler puts Newton's Proposition XXV into a rule, thus stating "expressis verbis" the principle of the transference of forces. Using this principle Euler was able to confirm a result already found by Newton (cf. Newton 1687, Lib. III, Prop. XXV). What Euler in his Proposition actually demonstrated was the essential seed hidden in what Newton showed in a rather intransparent way in the *Principia*. In Proposition IV Euler analyzed the effects of the Earth's force component acting on the Moon resulting in equations representing the change or variation of orbital elements (Euler called them "incrementum", without assigning them any symbols). This kind of differential equations, relating the change of orbital parameters to any other differentials of orbital elements, never before appeared in celestial mechanics. It marks the beginning of perturbation theory in terms of orbital elements and their variations. However, these "increments" of orbital parameters still do not depend on the time element, but on certain distance elements. As will be shown in part 3 of this study, Euler's recognition of the importance of the dependency of the equation of motion and of the variation of orbital parameters on the time element means an essential part of his breakthrough not only in lunar theory but in celestial mechanics and mechanics in general. Proposition V deals with the investigation of the effect of an additional force acting on the Moon by the Sun, thus explaining qualitatively the form of the lunar orbit and its motion. We observe here Euler's geometric interpretation of the theoretical results gained by his new "perturbational approach", which led him to qualitatively correct conclusions. This interplay became a cornerstone of general perturbation theory. Finally, in Proposition VI, Euler derived differential equations for the motion of the Moon considering the simultaneous actions of Sun and Earth. To solve them, he again introduced a novelty that coined celestial mechanics for the next three hundred years: approximations by series expansions. Euler was forced to use such an approach to be able to integrate the equations term by term. As will be shown below in Euler's manuscript Ms 276, he combined this approach with his development of trigonometric series, thus gaining one of the most powerful methods introduced in celestial mechanics.

3.4 Summary of topics in E 15 related to lunar theory

Most of the "mechanical" terms Euler defined in the "explanations" of his "*Mechanica*", which are also related to lunar theory, were already mentioned and discussed in the manuscripts previously presented in this and the first part (cf. Verdun 2012). They concern, in particular, 1. the introduction and definition of concepts and technical terms as, e.g. absolute and relative motion, various "scalas", and special forces (absolute, relative, normal, tangential, centripetal), and 2. Euler's derivation of the equation of motion (Newton's second law of motion), parametrized by time and by length. Some of them were addressed in his manuscript Ms 167 (see above). It is, however, most important to note, that Euler—during the time interval until about 1744—used the equation of motion always in one dimension and parametrized by length. The structure of its derivation is visualized in Verdun (2010), Fig. 2.33. It is striking that Euler herein did not refer to Newton nor to Hermann. The difference between Hermann's (cf. Hermann 1716, p. 57) and Euler's equation of motion (cf. Euler 1736, Prop. 20, § 155) consists in Euler's constant of proportionality, expressed by the number n, that "depends neither on the [size of the] force, nor on the element of the time, nor on the quantity of matter." The integration of this equation of motion, parametrized by length, generates a factor 2, which Euler kept in the equation by choosing the units of mass and force in such a way that $n = 1/2$ (cf. Euler 1736, Explanation 15 and Scholia). In Ms 167 he did not fix this number yet. Propositions 20, 25 and 26, together with the accompanied corollaries and scholia, as well as the explanations 15 and 16 in Euler's "*Mechanica*" were of fundamental importance for his lunar theories at that time. However, the central ideas related to planetary and lunar theory that Euler published in that time are presented in Chapter V concerning the curvilinear motion of free points acted on by absolute forces of any kind (De motu puncti curvilineo libero a quibuscunque potentiis absolutis sollicitati) of his first *opus magnum*, which is also the most voluminous chapter of the "*Mechanica*". It contains results that are important for planetary theory, which may be used for and applied to lunar theory as well, e.g. the definition of tangential and normal force components, of the curvature radius, and the derivation of the theorems of de Moivre and Keill (cf. Euler 1736, Prop. 74, and Ms 180, Ms 397, and Ms 398 in Verdun 2012; see also Moivre 1717; Keill 1708).

One of the central concepts Euler introduced into celestial mechanics in E 15 is the "osculating ellipse", which is of crucial importance particularly for lunar theory. His definition of this term differs from Newton's: to explain the motion of the lunar apse, Newton introduced a "moving" ellipse, i.e., an ellipse rotating around the central body's focus (cf. Newton 1687, Lib. I, Sec. IX). Euler, on the other hand, considered the parameters defining the ellipse to be subject of variations (cf. Euler 1736, Prop. 83, Schol. 2, already cited in Verdun 2012). Moreover, he already sketched ways to approach the three-body-problem (cf. Newton 1687, Lib. I, Prop. XVII, Coroll. 3 and 4; Lib. I, Prop. LXVI).[3] However, Euler formulated the problem in a way that matches the real situation of the Moon's motion as close as possible: If the centripetal force does not disagree much with the ratio of the inverse square with the distances,

[3] I thank Professor Niccolò Guicciardini, University of Bergamo, Italy, for bringing these references to my attention.

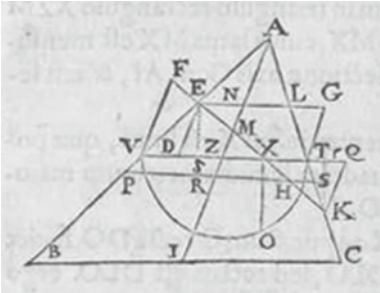

Fig. 1 Boulliau's figure illustrating the scalene cone as a model for circular and uniform planetary motion (*Source*: ETHZ)

one has to determine the motion of the ellipse, the continual change of its form, and the motion of the body following this variable ellipse (cf. Euler 1736, Prop. 84). The determination of the osculating ellipse demands, according to Euler, not only the knowledge of the nature of the curve, but also of the body's velocity and of the centripetal force as well (cf. Euler 1736, Prop. 84, Schol. 1):

> The ellipse determined in this way deserves to be called the osculating curve of the ellipse from the likeness of the circles of osculation, by which the curvatures of lines are measured. Truly this consideration is not purely geometrical, for by finding the osculating ellipse, besides the nature of the curve, it is also necessary to know the speed of the body and the centripetal force. (translated by Ian Bruce)

This statement proves Euler's understanding of the problem involved with the concept of the osculating ellipse: Given the resulting force causing a certain state of motion, one has to determine the parameters defining the osculating ellipse for each instant of time (cf. Euler 1736, Prop. 84, Schol. 2, already cited in Verdun 2012). In this respect, the notion "osculating ellipse" is an innovative element that may not to be found in Newton (or anywhere else). It is an original idea by Euler and has the meaning as it is still used today (in terms of "osculating orbital elements").

One of the central issues of Verdun 2012 was to show the importance of the concepts associated with the "curvature radius". Euler's transition from this term to the notion of an "osculating ellipse" might have been motivated by Boulliau's model of planetary motion, which most probably was studied by Euler very carefully (cf. Boulliau 1645, Lib. I., Cap. XV, pp. 29–38). Using this model Boulliau tried to reduce the non-circular and non-uniform motions of the planets to circular and uniform motions. For that purpose he considered a scalene cone ABC, a plane section of which through the axis AI perpendicular to the plane of the circular base, whose trace is BC, is illustrated in Fig. 1. This cone consists of an infinite number of circles centered on the axis AI with radii linearly decreasing from I to A (an example of such a circle in upright projection is indicated in Fig. 1 with VOT). Let EK be the trace of the cutting plane, which is perpendicular to ABC, representing the major axis of the planet's elliptic orbit, the empty focus of which being in M. The cone is supposed to be rotating around its axis AI with a given angular velocity ω, thus defining the orbital velocity by $\omega \cdot r$, where r is the radius of the circle at any position on the axis AI. The planet's motion in the elliptic orbit EK is now considered to be composed of all instantaneous circular

motions between E and K (cf. Wilson 1970, pp. 112–113; Wilson 1989, pp. 172–173.). These circles may thus be interpreted as "osculating circles".

Euler's introduction of the osculating ellipse was mainly motivated by his insights gained in Ms 273 concerning the variation of orbital parameters. If it is true that these parameters are changing continuously in time, then the resulting curve described by the body may become very complicated and a solution of the problem will only approximatively be possible. Therefore, in order to get out of this difficulty, Euler tried to find methods which reverse the problem. The reason for the variability of the orbital parameters are the multiple force centers being either at rest or in motion. So, Euler tried to reduce the problem by considering the motion around the common center of gravity, assuming first that the individual forces are proportional to the distances of the body from these forces (cf. Euler 1736, Prop. 85). Then he concluded (cf. Euler 1736, Prop. 85, Schol. 2):

> If the centers of forces are attracting in some other ratio besides the simple ratio of the distances, a reduction of this kind to a single central position of the forces cannot be had in a straightforward manner, and I can hardly calculate the motion of the body, nor indeed can hardly anything be determined about the motion. Therefore, in these cases, it is necessary to flee to approximations, which are set up in different ways according to the various conditions. And on this account, Newton was unable to determine the true motion of the moon, which arises from two attractions, but he tried only to describe it as nearest as possible. To do this, however, it is necessary both to give this problem the most singular consideration and to apply the inverse method, according to which one has to determine the attracting forces from the curve that the body describes. On this account, we will explain in the following what tools can be put in place, when we are to investigate the force acting as the unknown in the inverse order. Therefore, as we are progressing through this discussion, which can be established in two ways. Firstly, we take besides the curve described also the direction of the forces acting at individual points as known, and from these quantities the forces acting, and the motion of the body itself is to be found. In the other way, by considering the curve and the motion of the body on that curve is taken as given, from which it is required to extract the force acting. (translation by Ian Bruce, modified by AV)

Using this strategy, Euler derives the centripetal force causing the body to move in a mobile orbit, i.e., a rotating ellipse (cf. Euler 1736, Prop. 89 and Schol. 1, already cited in Verdun 2012). This situation is illustrated in Fig. 2. While the body M' is moving in the ellipse $A'M'B'$, this ellipse is rotating around the central body C being in its focus with a given angular velocity. After a given time interval will the ellipse be situated in AMB, while the body will be located in M. The principal ideas to solve this problem may be found in Ms 397 (cf. Verdun 2012). Euler mentioned, that this approach—already used by Newton (cf. Guicciardini 1999, Chap. 3.6, pp. 60–65)—is nevertheless insufficient to describe the real motion of the lunar apses adequately (cf. Euler 1736, Prop. 89, Schol. 2):

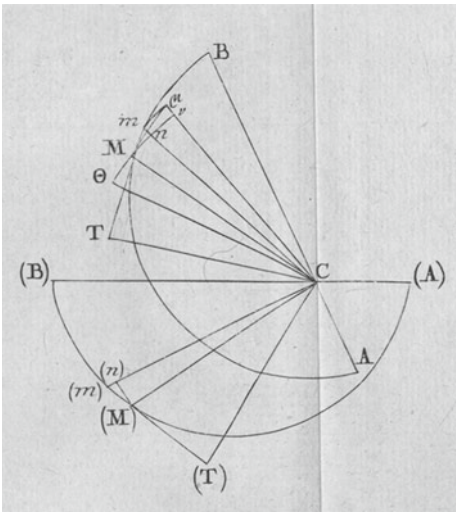

Fig. 2 Euler's figure in his "*Mechanica*" illustrating Proposition 89 (*Source*: ECHO)

> [...] Newton has explored this problem in the Principia, Book I, in the whole of Section IX, and he applied that theory to determine the motion of the apsidal line of the Moon's orbit. But this examination is applied with less accuracy to the Moon, since the force acting on the Moon is not directed to a certain fixed point, as we have put here, but to a continuously moving point. Therefore we will take pains, after having explained remaining relevant matters associated with that topic, to set up other more suitable propositions, which can be transferred to the motion of the Moon. (translation by Ian Bruce, modified by AV)

This is why Euler tried to describe the motion of the apsidal line by using variant centripetal forces which differ from the inverse square law, as he had already done so in Ms 273, Prop. II. He discusses centripetal forces that are, e.g., proportional to $V + C/y^3$, where $V = f^2/y^2$ and y denotes the distance between the resulting force center and the Moon (cf. Euler 1736, Prop. 90 and 91). However, this approach of one single centripetal force turned out not to be convenient in the case of the Earth–Sun–Moon system. Therefore, Euler reversed the problem again (see Fig. 3) and derived the force components P and Q causing the body M (Moon) to move freely in any given curve AMB rotating around the force center C (Earth), where P is acting along MC according to the inverse square law, and Q is acting (along the Sun's direction MP) normal to an arbitrarily defined reference line PCD through C. Euler remarks, that the resulting formulae may be considerably simplified if the orbit of M differs not much from an ellipse (cf. Euler 1736, Prop. 94 and Schol. 1):

> These formulae for the curve of the ellipse can be made simpler in various ways, if the curve in which the body is moving is approximately circular. And in this case it is of some use in the motion of the Moon to be theoretically defined. For the Earth is put at rest at C and the Sun on the line CP perpendicular at C is considered equally as being at rest; with which put in place and with these forces compared both with the forces of the sun and the earth, the synodal motion of the

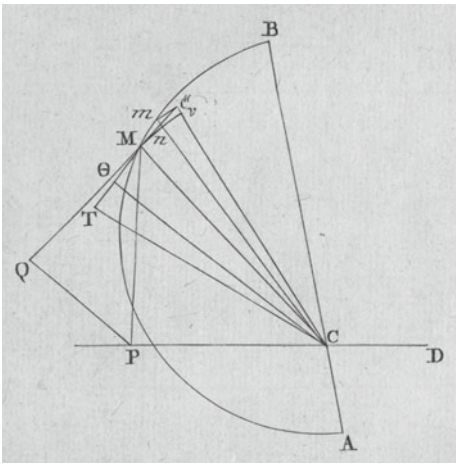

Fig. 3 Euler's figure in his "*Mechanica*" illustrating Proposition 94 (*Source*: ECHO)

> Moon is elicited for some position of the apsidal line and likewise the motion of the apsidal line, which only differs slightly from the true motion of the Moon. (translation by Ian Bruce, modified by AV)

He continued approaching to the main problem (cf. Euler 1736, Prop. 94 and Schol. 2):

> This proposition certainly appears of greater extent than the above, in which all the force was directed towards the center of rotation of the orbit; indeed the former is included with the force Q vanishing. Yet it cannot account perfectly for the explanation of the Moon's motion because one term of the force P varies inversely with the cube of the distance MC. Because of this we have to consider other orbits [besides rotating ellipses], which appear wider and agree more with the physics questions. Of this kind are motions of orbits along certain curves which happen in such a way that the orbit is always parallel to itself, which on contemplation deserve to be preferred from others, since the forces acting are both easier to find and simpler to express by formulas. This is why the following theorem is needed. (translation by Ian Bruce, modified by AV)

This theorem concerns the relative motion of the body M with respect to the center C: The body M is moving along the curve AM, by some force, around the point C, and in addition both the body M and the point C are acted on by an equal force in the same direction; consequently, the relative motion of the body M with respect to the point C, or the motion of the body M such as is seen from C, and likewise, will be such as if this new force is not to be added (cf. Euler 1736, Prop. 95). The proof is straightforward.

In the preceding propositions and statements, we observe Euler preparing a new strategy, which he will fully develop in Proposition 97. He motivated it by the following proposition (cf. Euler 1736, Prop. 96, and Fig. 4):

> Let the body M be revolving around the center of force L at rest in the curve BM; to determine the force causing the body to move in the same orbit

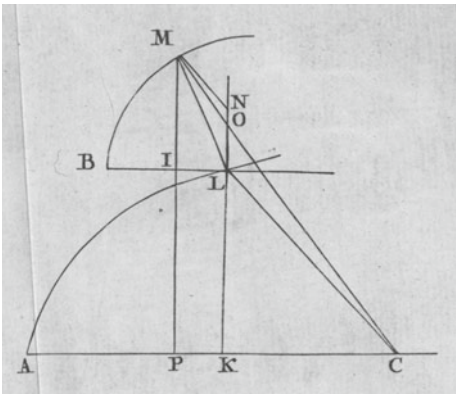

Fig. 4 Euler's figure in his "*Mechanica*" illustrating Proposition 96 (*Source*: ECHO)

> while it is following the curve *AL* in such a way that it is always parallel to itself.

Euler's proof of this proposition is based on the previous one and on the transformation of the force components into a new set of components, the characterizing parameters of which allow the motion along the curve *AL*. In the corollaries following this proposition Euler discusses the cases where the curves *AL* and *BM* are assumed to be circles and ellipses. In particular, Corollary 3 deals with the situation supposed by John Machin (1680–1751), published in the appendix of the English edition of Newton's *Principia* (cf. Machin 1729). Euler claims, however, that this approach is not sufficient enough for an accurate lunar theory and proposes another "genuine" method (cf. Euler 1736, Prop. 96, Schol. 2):

> I have therefore especially reported on this Proposition, because in an appendix to the new edition of Newton's Principia in English, the most distinguished Machin asserts that the motion of the Moon can be considered as in an ellipse, the transverse axis of which shall be in the ratio of 2:1 to the conjugate axis made around the center of the ellipse, while meanwhile with that ellipse itself moving parallel to the periphery of a circle, on which it progresses freely, as I have explained in Cor. 3. For my part, I do not deny that this motion is extremely similar to the motion that the Moon can show, but I would doubt very much that it was an exact ratio. Moreover in the following proposition I have decided to determine, what needs to be indicated to determine the motion of the Moon. Even if indeed this proposition pertains to astronomy, yet it is assumed that the genuine method reported here can be used to examine fundamental questions of this kind that are to be resolved. (translation by Ian Bruce, modified by AV)

What Euler here calls a "genuine method" is the *principle of the transference of forces*, which he formulated in Proposition 97 (cf. Euler 1736, Prop. 97, and Fig. 5) and which he introduced here by stating the following problem:

> With the Sun at rest at *S* and with the Earth *T* moving around it uniformly in the circle *TD* while the Moon *L* is attracted both to the Earth *T* as to the Sun

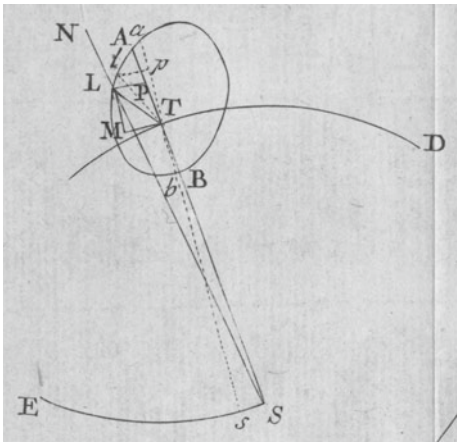

Fig. 5 Euler's figure in his "*Mechanica*" illustrating Proposition 97 (*Source*: ECHO)

> S in the inverse square of the distances; with which put in place it is required to determine the motion of the Moon, such as can be seen from the Earth T.

Then Euler explains clearly the very principle that he had prepared and applied already in Ms 271 and Ms 273 (cf. Euler 1736, Prop. 97, pp. 333–334):

> Ab his igitur viribus lunam sollicitantibus, qualis motus producatur, est investigandum. At quia lunae motus, qualis a spectatore in terra constituto observatur, definiri debet, terra tanquam quiescens est consideranda; id quod fit, dum toti systemati motus ei, quem terra habet, aequalis et contrarius imprimitur, simulque sollicitationes, quas terra a sole recipit, contrario modo in lunam et solem cogitatione transferuntur.
> (Therefore it is to be investigated what kind of motion is being produced by these forces that are acting on the Moon. But since this motion has to be determined in such a way as it is observed from the Earth, then the Earth has to be considered at rest. This is done by assigning the Earth's motion to the whole system equally and in opposite direction, together with transferring fictitiously the accelerations impressed on the Earth by the Sun onto the Moon and the Sun in reverse manner.)

To the procedure explaining this principle Euler added a Scholion in which he addmits that the proposition still contains some serious simplifications (cf. Euler 1736, Prop. 97, Schol. 1):

> The equations which hence are deduced for the motion of the Moon, become so complex that from them neither its velocity nor its orbit nor the position of the apsides and their motion can be exactly determined. Moreover truly from the same calculation by neglecting very small quantities in a certain way approximate conclusions for the use of astronomy can be drawn, as the great Newton did in Book III of the *Principia*. But even if the calculation would not suffer from this inconvenience, yet this proposition would not describe the Moon's motion without a great deal of rigor. For we have put the Sun forward again as being

> at rest, which in a short while disagrees with the truth; then we considered the Earth moving in a circle, and the Moon's orbit placed in the same plane with the Earth's orbit, which likewise they have otherwise. Yet meanwhile it is certain, if the solution of this proposition can evolve and from that a table constructed, then it would be of the most use in astronomy. (translation by Ian Bruce, modified by AV)

As already mentioned above, this last statement by Euler concerning the construction of lunar tables probably refers to his unsuccessful trial in this direction made in Ms 271. In the corollaries, he discusses some appropriate approximations and some consequences deduced from them. Finally, Propositions 102 and 103 contain Euler's first approach to derive the motion of the nodes of the lunar orbit, i.e., the line of intersection between the orbital planes of the Earth (ecliptic) and the Moon (which are inclined by about 5° to each other). For that purpose, he introduced in Propositions 98–101 three force components acting in the three-dimensional space, which we would call *Frenet frame* today, and discussed the motion of a body driven by these forces. Actually, Euler was forced by this problem, namely to find the orbit's position in space ("situs orbitae in universum cognoscatur"), to consider three force components that are linearly independent from each other. This kind of generalization from two to three spatial dimensions appears in this context probably for the first time. It became an important issue in the 1740s, when Euler formulated the equations of motion in three dimensions exactly due to this reason as well (to be discussed in part 3). Using the results derived from the previous propositions, Euler studied in Proposition 103 the motion of the nodes in the case of forces being directly proportional to the distances between the force center A and the body on which the force components are acting. He found a retrograde motion of the nodes even in this case and judges the results by comparing them with the real situation and with observations:

> This proposition seems to have no use in astronomy, because the force, by which the body is attracted to the fixed point A, we assumed proportional to the distance, whereas for celestial bodies that have in place a force that is inversely proportional to the square of the distance. Yet this [proposition] is extremely useful, if the orbits of bodies do not depart much from circles; for if the orbit becomes more and more circular it depends not on the way how the centripetal force depends on the distance. On account of which, when the orbits of planets do not depart greatly with circular orbits, this proposition is able to be adapted successfully to their motions […] In lunar motion the motion of the nodes merits special attention, because it is retrograde according to our determination. It is observed from the opposition of the preceding node that the nodes differ by nearly 43′. Thus we may conclude that
>
> $$\frac{180(m-1)}{m} = \frac{43}{60} \qquad \text{or} \qquad m = 1 + \frac{43}{10{,}757} = 1 + \frac{1}{250}.$$
>
> From which the force is known a posteriori to be always pulling the Moon towards the plane of the ecliptic. (translation by Ian Bruce, modified by AV)

3.5 Summary of Ms 276

In the introduction of this unfinished manuscript, Euler considered the Newtonian hypothesis, i.e., the inverse square law of attraction, as being proved both by observation and by the conclusions taken from this hypothesis, although this principle of attraction is burdened with difficulties. The first one concerns physics and demands a mechanical explanation of the mutual attraction of celestial bodies. The second one concerns the determination of the motions due to this attracting force, which—in the case of the Moon's motion—is so difficult, that nothing else than approximations could be done. The power of calculus seemed for Euler not able to provide confident rules to describe such kind of motion sufficiently. It would not have been possible to accomplish anything about the lunar motion, if one would not have called the approximation for help. Nevertheless, the Moon's motion makes—Euler remarked—the approximation easier and more accurate, because it does not much deviate from a circle. This is why it becomes possible to treat initially the Moon's motion in such a way, as if it is driven only by the Earth, but then one has also to take into account the Sun's force. This is the strategy Euler is following to find out how much could be achieved by means of the state-of-the-art calculus of that time.

Euler treated first the two-body-problem of the Earth–Moon system, then (in the remainder) the three-body-problem of the Sun–Earth–Moon system. The former is carried out along the line of thought as presented in the corresponding chapters of Euler's "*Mechanica*". Let us skip this part and turn to the latter problem. From the beginning Euler applied the principle of the transference of forces thus proving that this principle had been fully established by him already at that time. This task implies the determination of the inverse cube of the distance between the Sun and the Moon in terms of the distances between the Earth and the Sun and between the Earth and the Moon as well as the geocentric angular distance between the Sun and the Moon. In doing so, the crucial point turns out to be the fact, that the resulting formula for the inverse cube has to be put into the power of $-3/2$. At this point Euler was forced, probably for the first time in celestial mechanics, to expand the resulting formula into an infinite series. Moreover, this series involves the cosines of the angular argument, which means that we observe here the very first occurrence of trigonometric series expansions to solve the lunar problem approximately. This represents a novel approach in celestial mechanics. Euler became fully aware of the difficulty inherent in the development of an accurate lunar theory and recognized the meaning and importance of the use of trigonometric series expansions and their angular arguments for the approximate solution of the differential equations. He nevertheless derived trigonometric series for the resulting tangential and normal force components acting on the Moon, which he used to formulate the differential equation of the Moon's velocity. This differential equation is, consequently, also a trigonometric series consisting of sine and cosine terms, which Euler now was able to integrate easily term by term. At that point he might have recognized the advantage of the use of trigonometric series, which are easy to integrate and therefore became so important for celestial mechanics.

In the remaining folios of the manuscript, Euler tried to prepare for integration the equation of motion due to the normal force. For this purpose he used the solution gained from the two-body-problem and "adjusted" or "corrected" it in such a way as

to consider the additional action of the Sun. The results already obtained gave him the idea about the structure of the equation and thus for a general ansatz containing coefficients still to be determined. We observe here Euler's very first use of the method of undetermined coefficients for solving differential equations in lunar theory. He used this simple but robust method throughout his works concerned with celestial mechanics (cf. Verdun 2010). Unfortunately, the manuscript is only fragmentarily preserved and ends abrupt with Euler's records dealing with the solution of the problem by using this method. Therefore it is not quite clear whether he already applied the method of undetermined coefficients in full length and whether he finally was able to solve the differential equation successfully.

4 Assessment of the development of Euler's methods invented and used for lunar theory

The study and careful analysis of Euler's documents written between 1730 and 1744 allow us to infer to some crucial steps that happened when Euler indeed was concerned mainly with the development of mechanics. During this period he developed and introduced some fundamental methods in the context of lunar theory, that developed into standard procedures in the course of his further development of celestial mechanics in general.

4.1 Equations of motion

Although formulated already by Jacob Hermann in terms of Leibnizian notation of calculus, Euler derived the equation of motion (also called "Newton's second law of motion") in his "*Mechanica*" from scratch without any reference to Newton. Remarkable and important is the fact that he used the equation of motion not parametrized by time but by length. The change of the parameter is documented already in Euler's Ms 167 and in his "*Mechanica*" (cf. Euler 1736, Prop. 20, Schol. 3). Euler used the equation of motion indeed in different forms, but always with the length or distance as independent or "free" parameter (cf. Euler 1736, Prop. 25, Explanation 15, Schol. 4 and 5). The reason for this is clear: the central issue for Euler at that time was to figure out the nature of the curve a body is following due to any forces acting on it. The "nature of the curve" means its analytic representation in the sense of a well defined geometric trajectory that could be described in mathematically closed form. For that purpose Euler regarded a parametrization by length as most advantageous and most straightforward. In the 1730s, the focus of interest was still more put on the kinematic aspects, i.e., the geometric shapes of orbits and trajectories and the temporal behavior of a body's motion, than on the dynamics. Moreover, this is why the equation of motion is always applied in one dimension, i.e., according to the length or distance as independent parameter. This did not change even when Euler discussed the Moon's motion in three dimensional space in Chapter V of his "*Mechanica*". The problem there instead was reduced to the one dimensional case. The central issue emerging during the considered time period, however, was the steadily increasing use of the equation of motion, although in variants, to solve problems in mechanics and celestial

mechanics, which finally progressed into a standard procedure. It will be shown in part 3, that this development was a necessary prerequisite for the breakthrough that happened around 1744. At that time Euler recognized the importance of the parametrization by time as well as the formulation of the equations of motion in three dimensions, thus describing the body's motion in space from the dynamical point of view. The goal was no longer the determination of "the nature of the curve", but of the body's position in three dimensional space with respect to any reference frame at any given instant of time due to the forces acting at that epoch on the body.

4.2 Principle of the transference of forces

This principle is one of the major achievements by Euler made during the considered period of time. It states that if the motion of a body M has to be described as seen from a body T at rest, then all forces acting on body T have to be transferred in reverse direction onto the body M. Today, this seems to be obvious and naturally. But the analysis of Euler's documents showed that a real process was needed to achieve this insight. There might have been many reasons that led to the recognition and formulation of this principle. According to Euler's unpublished manuscripts presented above, it was most probably motivated by his previous considerations on absolute and relative forces, on absolute and relative motions, and on the "variational problems" treated in terms of the different "scalas" that had already been introduced by Hermann. The real problem underlying these concepts is associated with the definition and choice of an appropriate reference frame (cf. Bertoloni Meli 1993). That this principle is far from being self-evident is proved by the fact that even in 1747, when Daniel Bernoulli (1700–1782) wrote his Memoir on the theory of Saturn and Jupiter, with which he competed for the prize of the Paris Academy, he either did not know anything about this principle or he did not understand it when Euler wrote him about it (cf. Verdun 2010, Chap. C.2). The documents analyzed above reveal, in addition, that this principle has been established as standard procedure during the focused period of time by Euler.

4.3 Approximations by series expansions

Actually, there was just one reason to introduce series expansions into lunar theory, namely to make differential equations integrable. Due to the practical fact that series can not be expanded infinitely, the substitution by series means that the exact solution can only be approximated. This is the price one has to pay for the convenience that the series can be integrated term by term. This is why approximations by series expansions became a standard tool in celestial mechanics. In Ms 273 we observe its very first appearance in this context. However, in Ms 276, this approach turned out to be important and successful not only in lunar theory, but in celestial mechanics at all. In this manuscript Euler expressed the distance z between the Sun and the Moon in terms of trigonometric functions. He recognized that the term $1/z^3$, which previously arose from the principle of the transference of force, induces an elliptic integral that can be solved only approximatively by series expansions. At this point trigonometric series found their way into celestial mechanics and, by their own rights, became

important for analytic perturbation theory through their potential to "explain" perturbational effects by certain terms and its angular arguments. The history evolving from the different approaches by Euler and others to solve elliptic integrals approximately by using trigonometric series expansions was comprehensively described by Heinrich Burkhardt (1861–1914), cf. Burkhardt 1908, Chap. III.

4.4 Trigonometric functions and the importance of their arguments

The successive use and establishment of the equations of motion induced the demand for mathematical methods to solve, although approximately, the differential equations resulting from them. Progress in mathematical analysis became synonym with progress in the development of integration methods. In the course of Euler's works on mechanics and celestial mechanics, two constituents of such methods turned out to be very prolific, efficient, and robust: trigonometric functions combined with the method of undetermined coefficients. Although trigonometric functions were introduced already by Newton in 1669, and although the method of undetermined coefficients used to solve linear differential equations with constant coefficients may probably be traced back to Leibniz (cf. Katz 1987, pp. 312–313), it was Euler's achievement to place them at the foundation in mathematical analysis and to standardize and establish them as parts of powerful integration methods. Let us address the latter aspect in the next section and focus first on the former.

Even until the late 1730s trigonometric functions such as sine, cosine, or tangent were considered not as *functions* in the modern sense, but as certain ratios of lines inscribed into a circle of given radius (cf. Katz 1987, pp. 315–316):

> […] though sine tables existed in abundance, the sine was not considered as a "function", even to the extent that logarithms or exponentials were. It was thought of geometrically as a certain line in a circle of a given radius, one did not, in general, draw a graph of such a function so there was no question of finding tangent lines or areas.

Therefore, it is not astonishing that a true calculus of such functions could not emerge for some time. The definition in the sense of function theory and, consequently, the analytic treatment in the context of the calculus was initiated only by Euler in two treatises, which he presented to the Petersburg Academy on December 4, 1738, and on March 30, 1739. In the Memoir entitled *De novo genere oscillationum* (On a new type of oscillation) he deals with the oscillation in the motion of the tides (cf. Euler 1750a, E 126). The Memoir entitled *Methodus facilis computandi angulorum sinus ac tangentes tam naturales quam artificiales* (An easy method for computing the natural and artificial sines and tangents of angles) is devoted to give procedures useable to calculate easily trigonometric functions by series expansions (cf. Euler 1750b, E 128). The preparatory works and earliest records associated with these two memoirs may be found in Euler's third notebook Ms 399, fol. 140r–148v, 157v–159r, and 193r, which was written between 1736 and 1740 (cf. Kopelevič et al. 1962, p. 115). This notebook contains, in particular, his earliest notes about the integration and differentiation of trigonometric functions as well as about the integration of second order differential

equations by using such functions. It gives evidence for Euler's recognition of the importance of trigonometric functions to solve differential equations resulting from the equations of motion that describe the periodic motion of celestial bodies, as well. These functions have the advantage of being easily integrable and of being accurately computable for given angular arguments by rapidly converging series expansions, as Euler showed in the second Memoir (E 128) which is contained nearly complete in his notebook Ms 399. Moreover, in Ms 276, Euler must have begun to note the meaning and importance of the angular arguments occurring in the trigonometric functions for the interpretation of the perturbational effects produced by the geometric constellation or situation of the participating bodies.

4.5 Method of undetermined coefficients

The well-known method of undetermined coefficients was commonly used for various problems since its invention by René Descartes (1596–1650), in particular, to determine reciprocal series. Euler applied this method also for solving differential equations. He established this method as a standard procedure in celestial mechanics and lunar theory. According to Moritz Cantor (1829–1920), it was Euler who first formulated the principal idea of this method in a Memoir (cf. Euler 1750c, E 130), presented to the Petersburg Academy at October 22, 1739 (cf. Cantor 1901, p. 676f.):

> Beiläufig bemerkt, dürfte dieses die erste Stelle sein, an welcher der der Methode der unbestimmten Coefficienten zu Grunde liegende Gedanke deutlich ausgesprochen ist, so vielfach die Methode auch seit ihrer Erfindung durch Descartes Anwendung gefunden hatte.
> (By the way, this may be the first place where the idea forming the basis of the method of undetermined coefficients is clearly articulated, notwithstanding the many times this method has been applied since its discovery by Descartes.)

The method of undetermined coefficients consists of setting an appropriate ansatz which formally is similar to the analytic structure (e.g., order) of the expected function that solves the differential equation. In general, this will be a series expansion, the terms of which containing yet undetermined coefficients that can be determined by comparison of the corresponding terms when this ansatz is substituted into the differential equation and integrated term by term. This method is very simple and robust, and may be applied to any equation of motion however difficult. The price to pay for this advantage is the inconvenience, that this kind of integration may become extremely time consuming, depending on the series' order of the ansatz and on the complexity of the differential equation (system). Each term has to be calculated algebraically in order to carry out the comparison of the coefficients. However, it is just this method that Euler used throughout in his works on celestial mechanics since its first appearance in Ms 276. For example, when he competed for the prize of the Paris Academy in 1752, he was forced to calculate over 10,000 terms algebraically (cf. Verdun 2010). This was caused by the ansatz, which was a trigonometric series from which the first and second derivatives had to be put into the original second order differential equations system. The resulting products of sine and cosine functions had to be resolved

by the addition theorem, thus excessively multiplying the number of terms to be integrated.

4.6 Establishment of orbital elements

A separate study would be required to reconstruct the rise and development of those parameters we call orbital elements today. They characterize form and position of an ellipse in the Keplerian two-body-problem. The definition of such orbital elements actually may be done in an arbitrary manner. However, the elements that have been established until now were published as one set of orbital parameters by Euler in a Memoir on planetary motion that he presented to the Petersburg Academy on March 28, 1740 (cf. Euler 1750d, E 131, p. 122 and Fig. 9). There are also two manuscripts, Ms 252 and Ms 253, preserved in the Archive of the St. Petersburg Academy of Sciences that are associated with this Memoire (cf. in Kopelevič et al. 1962, pp. 79–80). Ms 252 is a fragment or draft version, dated to the years 1738–1740. These six independent orbital parameters are not constant in the case if more than two celestial bodies interact gravitationally with each other. Euler became aware of that fact due to his studies on lunar theory during the considered time interval in his "*Mechanica*" (E 15) and in his unpublished manuscripts Ms 273 and Ms 276 (see above). In these two manuscripts, he derived differential equations for the "increments" or "variation" of some of these orbital parameters. It turned out (and will be shown in part 3) that this is a new kind and very important approach in lunar theory and celestial mechanics, which many years later became famous with the label "Gaussian" or "Lagrangian" (planetary) equations of motion. However, it was Euler's achievement to be the first to formulate differential equations for the orbital elements (cf. Verdun 2010).

5 Conclusions

While Euler, before 1730, still struggled within his search for the right track to cope with multiple force centers (cf. Verdun 2012), it was shown in this part that he found a way out of the entangled situation with the principle of the transference of forces. This principle immediately became a first solid standard procedure within his still sparse repertoire of methods that were potentially useful for solving the lunar problem. Another method that Euler developed, in combination with the former, into a standard procedure is the use of the equation of motion, albeit still in variant forms and still parametrized by length. The application of both procedures to the lunar problem confronted him with new serious problems, which triggered the introduction of trigonometric series expansions to solve the resulting differential equations for the Moon's motion approximately. To integrate them Euler resorted to the well known method of undetermined coefficients, which he used for that purpose probably for the first time. By 1744 almost all ingredients needed for a successful and technically mature lunar theory were available to Euler: The principle of the transference of forces, trigonometric functions to expand certain terms into trigonometric series, the representation of the variation of orbital elements by differential equations, and the method of undetermined coefficients to solve them. However, up until that time there was only

one important step missing which was needed for the final breakthrough leading to a powerful lunar theory: the equations of motions, formulated in three spatial dimensions and parametrized by time. This step allowed Euler to formulate what I call his "embryonic" lunar theory of 1744/45, which will be presented in part 3 of this series of articles. The numerous lunar tables Euler constructed between 1745 and 1750 (cf. Verdun 2011) most probably are based on that early theory, which he tried to bring into closer agreement with observations during that time, as will be subject of the next part as well.

Acknowledgments This study was supported by the Swiss National Science Foundation, which approved a grant for a project to investigate and edit Euler's unpublished manuscripts related to his early lunar theories. This project is performed in the context of the activities of the recently founded Bernoulli-Euler-Center at the University of Basel. I thank Prof. Dr. Hanspeter Kraft (Director of the Mathematical Institute, University of Basel, and main applicant of the project) for the project's administration, Dipl.-Math. Martin Mattmüller (Director of the Bernoulli-Euler-Center), Dr. Fritz Nagel and Lic.-Phil. Sulamith Gehr (Bernoulli-Euler-Center) for assistance and support, Prof. Dr. Gleb K. Mikhailov (University of Moscow) and the St. Petersburg Archive of the Russian Academy of Sciences for providing copies of Euler's manuscripts, PD Dr. Tilman Sauer (Einstein Papers Project, CalTech, Pasadena and University of Bern) for proofreading the manuscript, and the Astronomical Institute of the University of Bern for using its facilities and infrastructure.

Appendix A: The contents of Ms 167 relevant for the development of Euler's lunar theory

In the "Introductio" of Ms 167 Euler defines the concept of "scala celeritatum", which he adopted from Jacob Hermann's *Phoronomia* (cf. Hermann 1716, p. 54), a work that Euler studied carefully and "in which the whole science [of mechanics] has been enriched by the treatment of so many selected topics to be found within", as Euler mentioned in the preface to his "*Mechanica*" (cf. Euler 1736, Prefatio). The concept "scala" means the graphical representation of values or shortly "graph" or "run of the curve". It is not to be confused with the concept of a function in the modern analytical sense, although Euler already uses the term "functio" in Ms 167 (cf., e.g., §§ 69, 82, 92–96, 106) and later in his "*Mechanica*" (cf. Euler 1736), expressing the dependence of the value of a parameter from another one. In that time (about 1730) the term "scala" meant the point-to-point correlation between the values of a certain parameter (force, velocity, time) and the values on which they depend (distances). In a diagram, the latter constitutes the values of the abscissa, the former of the ordinate, thus forming a discrete steplike representation (which is the meaning of the word "scala"), which was considered to be a continuously progressing curve and expressed—in this sense for our context—the "progression of the velocity as a function of the distance covered by the considered body". Later on, in Chapter I of Section I, Euler also introduces the concepts of "scala potentiarum" and "scala temporum" (cf. Ms 167, §§ 62, 64). He adopted the former from the *Phoronomia* as well. Hermann introduced not only the term "scala potentiarum", but other "scalae", e.g., "scala solicitationum centralium" (cf. Hermann 1716, p. 28 and p. 52). Euler defined the concepts of "scala celeritatum" and "scala temporum" in his "*Mechanica*" (cf. Euler 1736, Prop. 5), he missed, however, to define the term "scala potentiarum" and used it without explanation in the solution of Proposition 41 (cf. Euler 1736, Prop. 41). The term "scala temporum" does not appear

in Hermann's *Phoronomia*, but it already occurs in the letters of Johann I Bernoulli to Jacob Hermann written on December 21, 1715, and on May 20, 1716, from which one may conclude that Euler became acquainted with it from them as well. According to the way Euler uses the "scala potentiarum" in Ms 167, one may suppose that it does not represent just one but *any* curve in the sense of "variational curves", which is why the term "scala" sometimes may be associated with a whole family or series of curves. The significance of the concept of "scala potentiarum" for the development of lunar theory concerns the fact that Euler learned to find and to determine different "scalas of forces" associated with different given initial conditions or properties defined by the problem. This actually involves not only variational principles, which were still to be developed at that time, but has—by the initial conditions—implications on the choice of appropriate reference frames as well. I will briefly address this topic below.

Euler opens the first part of his treatise, entitled "De motu a potentiis producto" (On the motion produced by forces), with definitions of *absolute* and *relative* forces, which are of equal importance with respect to the development of reference frames:

> Potentiæ quarum actiones non pendent a celeritate corporis patientis, vocentur absolutæ, Quæ vero aliter agunt, si corpus alia feratur celeritate relativæ vocentur. (Forces whose actions do not depend on the velocity of the body experiencing them are called *absolute* forces, whereas those that act in a different way when the body moves with a different velocity are called *relative*.)

Euler has used first the terms "potentiae purae" (pure forces) and "potentiae impurae" (impure forces) for absolute and relative forces, respectively, which corroborates the conjecture that these concepts are his own neologisms with respect to the meaning defined and used by him (cf. Fig. 6).

> Dato autem effectu potentiæ absolutæ in corpus quiescens sequenti modo invenietur effectus in motum. Valeat potentia quædam corpus A quiescens tempusculo infinite parvo ex A in P transferre. Nunc vero habeat corpus A celeritatem z secundum directionem AB quæritur quomodo eadem potentia eodem tempusculo motum corporis perturbabit.
> (The effect of an absolute force on a body at rest being given, the effect on the moving body will be determined as follows: Let some force be such that it sends a body A at rest in an infinitely small time from A to P. Now let the body A have the velocity z in the direction AB. How will the same force now disturb the motion of the body during the same moment of time?)

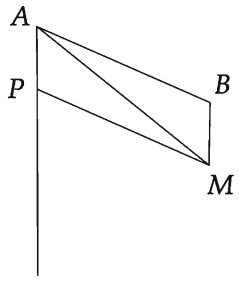

Fig. 6 Euler's sketch in Ms 167 to illustrate the concepts of absolute and relative forces

> Quia potentia æque in motum corpus ac in quiescens agere ponitur, concipiatur corpus A motum suum amisisse seu id super plano positum, quod motum habet aequalem et contrarium ei quem habet corpus transferetur id igitur in P. Restituo autem motu perveniat id interea in B motu ante concepto. Quamobrem post hoc tempusculum non in P sed in M reperietur, ducta recta PM parallela et æquali rectæ AB. Quoniam enim planum in partes contrarias motum concipiebatur, ut corpus in locum debitum restituatur, oportet plano motum ei, quem ante habere ponebatur, contrarium tribuere, hoc modo punctum P in M transferetur. Quamobrem corpus A interea diagonalem AM descripsisse putandum est. A potentia ergo angulo BAM a sua semita deflectere coactum est, et celeritatem acquisivit, quae se habet ad pristinam ut AM ad AB.
> (Since the force is assumed to act equally on a moving body as on one at rest, let us suppose the body A to have lost its motion or to be set on a plane which has a uniform motion opposite to that of the body. Thus it will be carried to P. The motion being restored, let it arrive at B by the motion conceived before. It will therefore be found after this moment of time not at P but at M, where the line PM is drawn parallel and equal to AB. For since the plane was supposed to move in the opposite direction, in order to return the body to its due place, one needs to attribute to the plane a motion opposite to that which it was supposed to have before: thus the point P will be carried to M. The body A must therefore be thought to have covered in the meantime the diagonal AM. Consequently it has been compelled by the force to deviate from its path by the angle BAM and has obtained a velocity that is to the original one as AM to AB.)

In the first section, entitled "De motu a potentiis in punctum liberum agentibus producto" (On the motion produced by forces acting in a free pointlike body) Euler treats uniform rectilinear motions. To be able to solve non-uniform rectilinear motions, he states what we now call "Newton's second law" or "equation of motion":

> Porro si tempora sunt inæqualia, quia tum incrementa celeritatum sunt ut tempora, habebimus hanc legem incrementum celeritatis esse directe ut tempusculum et potentiam atque ut corpus ipsum inverse.
> (If the time intervals are not equal, the following law holds (since then the increments of velocity are proportional to the times): *the increment of velocity is directly proportional to the time element and to the force, and inversely proportional to the body itself* [i.e., its mass].)(emphasis added)

Euler adopted this law most probably from the *Phoronomia*, because Hermann presented it in the same context as Euler did, namely when dealing with the transition from uniform to non-uniform motions (cf. Hermann 1716, pp. 55–57).

Euler continues his investigations on rectilinear motions of pointlike bodies in Chapter I of this section, entitled "De motu puncti a potentiis absolutis tracti rectilineo" (On the rectilinear motion of pointlike bodies due to absolute forces). Here he formulates the law of motion in Leibnizian notation:

$$\mathrm{d}z = mp\,\mathrm{d}t : A, \tag{1}$$

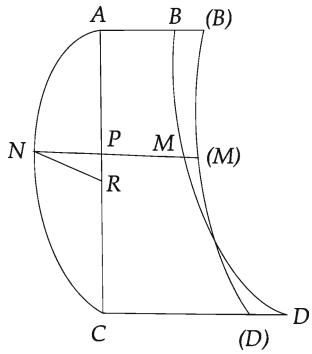

Fig. 7 Euler's setch in Ms 167 to illustrate the concepts of "scala" or function

where z denotes the body's velocity, A its mass, p the force acting on it, and m a proportionality constant. He substitutes the time element $\mathrm{d}t = n\,\mathrm{d}x : \mathrm{d}z$, n being a constant, to obtain

$$\mathrm{d}z = mnp\,\mathrm{d}x : Az \quad \text{or} \quad Az\,\mathrm{d}z = mnp\,\mathrm{d}x. \tag{2}$$

Its integration gives

$$Azz = 2mnpx, \tag{3}$$

setting the integration constant equal zero. Here we can observe the very origin of the factor 2 occurring in Euler's equation of motion, which he maintained over many decades of years by choosing appropriate units. Therefore

$$z = \sqrt{\frac{2mnpx}{A}}. \tag{4}$$

According to Euler, it is more convenient to express the quantity of velocity by the corresponding height or altitude of free fall, which is a distance and therefore easier to measure than the velocity. In the sequel Euler defines and determines the units and constants in Rhinelandian (Prussian) feet, thus obtaining $n = \frac{1}{250}$, which corresponds to the value of gravity on Earth at an altitude of 15,625 Rhinelandian feet.

The next topic concerns the motion of a body which is attracted in any multiple ratio of its distance from the center. In this context Euler introduces the terms "scala potentiarum" (progression of forces) and "scala temporum" (progression of time) and applies these concepts to solve problems closely related to what we call today "calculus of variations" such as (cf. Ms 167, § 80, and Fig. 7):

> Inveniamus nunc omnes possibiles scalas potentiarum, quæ faciant, ut corpus dictum spatium AC vel eodem tempore percurrat, vel ut in C eundem celeritas gradum assequatur.
> (Find now all possible progressions of forces which cause the body either to cover a given space AC in a given time or to reach in C with a given degree of velocity.)

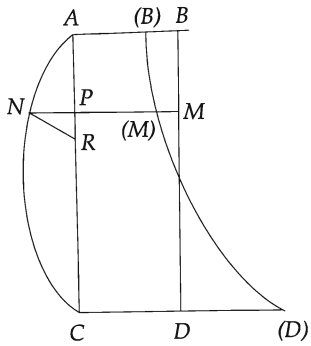

Fig. 8 Euler's sketch in Ms 167 to illustrate the construction of a "scala potentiarum" $(B)(D)$, which is equivalent to a constant force BD

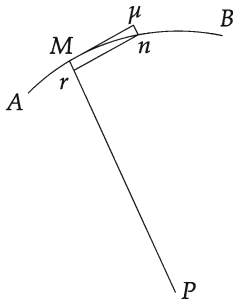

Fig. 9 Euler's sketch in Ms 167 explaining the curvature radius

Euler reformulated this kind of "variational" problem in his "*Mechanica*" (cf. Euler 1736, Prop. 47 and 48). In Ms 167, he also treats the problem of finding a special "scala potentiarum" that causes a falling body to move in a certain way according to a given time schedule (cf. Ms 167, §§ 88–102, and Fig. 8).

Chapter III deals with the curved motion of a pointlike body which is driven by absolute forces. Euler decomposes the force acting on the body into the tangential and normal components with respect to the point of the curve the body is currently located in and notes, that the former can change only the body's speed, the latter only its direction (cf. Ms 167, §§ 176–177). He defines the curvature radius ("radius osculi") or curvature ("radius curvedinis"), which he wants to determine now using the normal force: Let (cf. Fig. 9) Mr and $\mathrm{d}s = M\mu$ be the normal and tangential line elements associated with the element Mm of the curve defining the curvature radius r in M, and p be the normal force component (assuming the ratio of the normal to the gravitational force as p to 1). The altitude v that corresponds to the body's velocity is equal to $p \cdot Mr$ (cf. Euler 1736, Prop. 25, Coroll. 5). This velocity corresponds to a distance $2Mr$ covered by the body during the time element $\frac{2Mr}{\sqrt{Mr \cdot p}}$. This time element is equal to that one used by the body to cover the distance $M\mu$, which is $\frac{\mathrm{d}s}{\sqrt{v}}$. By equating these two time elements Euler derives the equation

$$Mr = \frac{p\,\mathrm{d}s^2}{4v}. \tag{5}$$

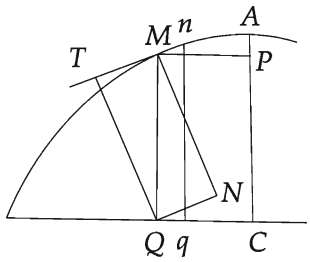

Fig. 10 Euler's sketch in Ms 167 explaining the tangential and normal force components

Using the equivalence $Mr : Mm = Mm : 2r$ and assuming $Mm \approx M\mu = \mathrm{d}s$, Euler solves this for Mr and substitutes it in Eq. (5), obtaining (cf. Ms 167, §178)

$$r = \frac{2v}{p}, \tag{6}$$

which agrees with the result of Ms 180 (cf. Verdun 2012, Appendix B, Eq. (1), considering that $p : 1 = N : A$).

Let P be the force acting on the body, $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$ the line element of its curved path that it covers in the time element (cf. Ms 167, §179, and Fig. 10). Then the normal and tangential components are given by $\frac{P\,\mathrm{d}x}{\mathrm{d}s}$ and $\frac{P\,\mathrm{d}y}{\mathrm{d}s}$, respectively. Again using the "law" relating the increment of velocity with tangential force (cf. Euler 1736, Prop. 25, Coroll. 5), Euler obtains

$$\mathrm{d}v = \frac{-P\,\mathrm{d}y}{\mathrm{d}s}\mathrm{d}s = -P\,\mathrm{d}y. \tag{7}$$

He substitutes the normal force $\frac{P\,\mathrm{d}x}{\mathrm{d}s}$ into Eq. (6) and equates the result with the formula for the curvature radius $r = \frac{\mathrm{d}s\,\mathrm{d}y}{\mathrm{d}\mathrm{d}x}$ derived from differential geometry, obtaining

$$P\,\mathrm{d}x\,\mathrm{d}y = 2v\,\mathrm{d}\mathrm{d}x. \tag{8}$$

The substitution of Eq. (7) into the last one gives

$$\frac{\mathrm{d}v}{v} + \frac{2\,\mathrm{d}\mathrm{d}x}{\mathrm{d}x} = 0, \tag{9}$$

whose integral is

$$v\,\mathrm{d}x^2 = C\,\mathrm{d}s^2, \tag{10}$$

where C denotes the integration constant. If $x = 0$ then $v = b$ and $\mathrm{d}y : \mathrm{d}s = f : 1$, therefore $C = bg^2$, where $g = \sqrt{1 - ff}$. From Eq. (10) Euler obtains for the altitude

$$v = \frac{bg^2\,\mathrm{d}s^2}{\mathrm{d}x^2} \tag{11}$$

and for the corresponding velocity

$$\sqrt{v} = \sqrt{\frac{bg^2\,\mathrm{d}s^2}{\mathrm{d}x^2}}. \tag{12}$$

Euler considers first parabolic and circular orbits using this result (cf. Ms 167, §§ 183–189), before he treats the general central force problem (cf. Ms 167, §190f, and Fig. 11). Let BMA be the curve described by a body M due to a central force located in C. Let the distance between M and C be y and the central force be P, supposing the gravitational force $= 1$. Let further the length of the perpendicular CT to the tangent MT be p and $MT = \sqrt{yy - pp} = t$. The normal and tangential components of the central force are thus given by $\frac{Pp}{y}$ and $\frac{Pt}{y}$, respectively. Let v be the altitude corresponding to the velocity of M and $v + \mathrm{d}v$ that of m. Let r be the curvature radius in M. Due to $mr = \mathrm{d}y$ and the equivalence of the triangles CMT and Mmr, it follows (cf. Ms 167, §191) $Mr = \frac{p\,\mathrm{d}y}{t}$ and $Mm = \frac{y\,\mathrm{d}y}{t}$. Therefore, due to the tangential force which acts in reverse direction to the motion of M, the increment of its velocity becomes

$$\mathrm{d}v = -\frac{Pt}{y}\frac{y\,\mathrm{d}y}{t} = -P\,\mathrm{d}y. \tag{13}$$

According to Eq. (6), the curvature radius is given by $r = \frac{2vy}{Pp}$. On the other hand, it is also defined by $r = \frac{y\,\mathrm{d}y}{\mathrm{d}p}$. Therefore,

$$\frac{y\,\mathrm{d}y}{\mathrm{d}p} = \frac{2vy}{Pp} \quad \text{or} \quad Pp\,\mathrm{d}y = 2v\,\mathrm{d}p. \tag{14}$$

The substitution of Eq. (13) into Eq. (14) gives $ppv = C$. Euler sets $p = f$ and $v = b$ to obtain $C = bff$ and thus $ppv = bff$ and $v = \frac{bff}{pp}$. Assuming bff being constant, this result gives the velocity $\sqrt{v}$ of the body M defined by the parameter p (cf. Ms 167, §192). Let T be the time needed by the body to cover the arc BM, which is given by the known velocity of the body. The time element $\mathrm{d}T$ needed to cover the arc Mm is defined by

$$\frac{Mm}{\sqrt{v}} = \frac{py\,\mathrm{d}y}{tf\sqrt{b}} = \frac{y \cdot Mr}{f\sqrt{b}}, \tag{15}$$

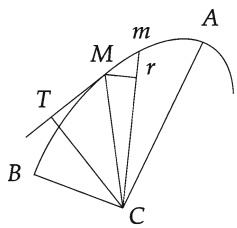

Fig. 11 Euler's sketch in Ms 167 used to derive the force components

whereby the relations $Mm = \frac{y\,\mathrm{d}y}{t}$, $Mr = \frac{p\,\mathrm{d}y}{t}$, and $v = \frac{bff}{pp}$ were used. Therefore,

$$\mathrm{d}T = \frac{2 \cdot [\text{area}]MCm}{f\sqrt{b}}, \tag{16}$$

and the time needed to cover the arc BM is thus given by

$$dT = \frac{2 \cdot \text{area}BCM}{f\sqrt{b}}. \tag{17}$$

Euler concludes that the time intervals are proportional to the areas covered by the associated radius vectors, which is Kepler's area law (cf. Ms 167, §193). He substitutes $v = \frac{bff}{pp}$ into Eq. (14) to obtain Keill's theorem (cf. Ms 167, §194):

$$P = \frac{2bff\,\mathrm{d}p}{p^3\,\mathrm{d}y}. \tag{18}$$

He derived this theorem already in Ms 398, fol. 26v (see also Euler 1736, Prop. 74, Coroll. 4, § 592; Keill 1708). Euler admits that the application of this theorem is so difficult if the body's curve, either algebraic or transcendent, can not be reduced to an equation which is general in a way that the curve may be expressed by an equation interrelating orthogonal coordinates (cf. Ms 167, §195):

> Aequatio, quæ hoc modo ad curvam invenitur a corpore prolubitu projecto et a vi data P ubique sollicitato [descriptam], est inter distantiam corporis a centro y et inter perpendiculum ad tangentem ex centro p. Cum vero ut plurimum difficile sit hinc judicare, qualis sit curva utrum algebraica an transcendens, reducam æquationem hanc generalem ad æquationem inter coordinatas orthogonales quemadmodum curvæ exprimi solent.
> (This equation found in this way for the curve described by a body that is propelled in any manner and acted anywhere by a given force P, depends on the distance y between the body and the center, and on the perpendicular p from the center to the tangent. But because it is extremely difficult to assess from it what kind of curve – algebraic or transcendent – it is, I will reduce here this general equation to an equation in terms of rectangular coordinates in the same way like curves usually are described.)

We skip Euler's application of this result to special cases, which is not relevant for the development of his lunar theory, and turn to his investigations on the motion of a body M being subject of two force centers C and D. Let the ratio between the central force in C acting on M and the gravitational force as the ratio between the distances CM and the constant straight line a, and let the ratio between the central force in D acting on M and the gravitational force as the ratio between the distances DM and the constant straight line b. Therefore $\frac{CM}{AM} = a$ and $\frac{DM}{BM} = b$, assuming the gravitational force being $= 1$. Let F be the common center of mass of the attracting bodies C and D with masses $\frac{1}{a}$ and $\frac{1}{b}$, respectively, where $CF : DF = a : b$. These two force centers

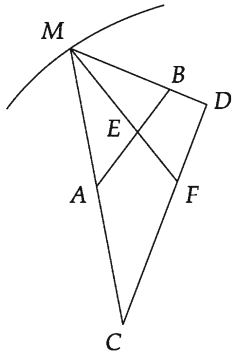

Fig. 12 Reconstruction of the missing figure in Ms 167 to illustrate two simultaneously acting force centers C and D

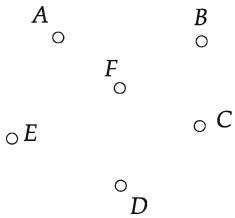

Fig. 13 Euler's sketch in Ms 167 to illustrate multiple acting force centers

thus act as one single force center being their common center of mass (cf. Ms 167, §253, and Fig. 12).

Let $2\,ME$ be the force acting on M by F. Using the law of sines, Euler derives (cf. Ms 167, §254)

$$\sin CMF : \sin CMD = MB : 2\,ME = (CF \cdot MD) : (CD \cdot MF), \tag{19}$$

therefore

$$ME : MF = (CD \cdot MB) : 2\,(CF \cdot MD) \tag{20}$$

and

$$(CF + DF)(CD) : CF = (a + b) : a. \tag{21}$$

He concludes, that

$$ME : MF = (a + b)MB : (2a \cdot MD) \tag{22}$$

and

$$2\,ME = \frac{(a + b)MF \cdot MB}{a \cdot MD} = \frac{(a + b)MF}{ab} = MF\left(\frac{1}{a} + \frac{1}{b}\right). \tag{23}$$

Later, Euler considers the more general case of two acting force centers, where the centripetal forces are no longer proportional to the distances and where one unique common center of force is not determinable (cf. Ms 167, §256, and Fig. 13). However,

he investigates only the coplanar case, where the whole orbit BM of M is situated in a plane together with the force centers C and D, which attract M with any ratio of the distance (cf. Ms 167, §257). Let v be the altitude corresponding to the velocity of M in the point M of its curve, and $v + \mathrm{d}v$ its velocity in the point m, which is situated infinitesimally close to M. Let this line element Mm of the curve be $\mathrm{d}s$. Euler denotes $CM = \mathrm{y}$, $DM = \mathrm{Y}$, $CT = p$, $DV = \pi$, $MT = \sqrt{yy - pp} = q$, and $MV = \sqrt{YY - \pi\pi} = \rho$. Let further the curvature radius in M be $= r$ and the distance CD between the two force centers c. From the geometry of the two perpendiculars CT and DV to the tangent line TV through M it is easily seen that $TV = q - \rho$ and $DV - CT = \pi - p$. Therefore (cf. Ms 167, §258)

$$cc = qq - 2q\rho + \rho\rho + pp - 2p\pi + \pi\pi = yy + YY - 2q\rho - 2\rho\pi. \tag{24}$$

Due to the equivalence of the triangles CMT and DVM with their infinitesimal small counterparts it follows $\mathrm{d}s = \frac{y\,\mathrm{d}y}{q}$ and $\mathrm{d}s = \frac{Y\,\mathrm{d}Y}{\rho}$, respectively, and therefore

$$\mathrm{d}s = \frac{y\,\mathrm{d}y}{q} = \frac{Y\,\mathrm{d}Y}{\rho}. \tag{25}$$

From § 191 we know that the curvature radii are defined either by $r = \frac{y\,\mathrm{d}y}{\mathrm{d}p}$ or by $r = \frac{Y\,\mathrm{d}Y}{\mathrm{d}\pi}$, giving the relation

$$\frac{y\,\mathrm{d}y}{\mathrm{d}p} = \frac{Y\,\mathrm{d}Y}{\mathrm{d}\pi}. \tag{26}$$

By differentiation of Eq. (24) Euler obtains

$$y\,\mathrm{d}y + Y\,\mathrm{d}Y = q\,\mathrm{d}\rho + \rho\,\mathrm{d}q + p\,\mathrm{d}\pi + \pi\,\mathrm{d}p. \tag{27}$$

Eqn. (24) may also be written as $\rho = q - \sqrt{cc - (p - \pi)^2}$, and from the combination of the Eqs. (25) and (26) one obtains $\rho = \frac{q\,\mathrm{d}\pi}{\mathrm{d}p}$. Therefore

$$q = \frac{\mathrm{d}p\sqrt{cc - (p - \pi)^2}}{\mathrm{d}p - \mathrm{d}\pi} \quad \text{and} \quad \rho = \frac{\mathrm{d}\pi\sqrt{cc - (p - \pi)^2}}{\mathrm{d}p - \mathrm{d}\pi}. \tag{28}$$

This result for q, substituted into $y = \sqrt{pp + qq}$, gives (cf. Ms 167, §259)

$$y = \frac{\sqrt{cc\,\mathrm{d}p^2 + pp(\mathrm{d}p - \mathrm{d}\pi)^2 - (p - \pi)^2\,\mathrm{d}p^2}}{\mathrm{d}p - \mathrm{d}\pi} \tag{29}$$

and analogously

$$Y = \frac{\sqrt{cc\,\mathrm{d}\pi^2 + \pi\pi(\mathrm{d}p - \mathrm{d}\pi)^2 - (p - \pi)^2\,\mathrm{d}\pi^2}}{\mathrm{d}p - \mathrm{d}\pi}, \tag{30}$$

representing the two distances of the body M from the two force centers C and D used to determine the resulting force acting on M. Euler denotes these two forces with P and Q, respectively, and decomposes them into the normal and tangential components (cf. Ms 167, §260). Let $\frac{Pp}{y}$ and $\frac{Q\pi}{Y}$ be the normal components of P and Q, and its sum $\frac{Pp}{y} + \frac{Q\pi}{Y}$ be their resulting normal force. According to § 178 and assuming the gravitational force = 1, this normal force becomes

$$\frac{Pp}{y} + \frac{Q\pi}{Y} = \frac{2v}{r}, \tag{31}$$

from which it follows

$$v = \frac{Ppr}{2y} + \frac{Q\pi r}{2Y}. \tag{32}$$

Substituting Eq. (26) into this result, Euler obtains

$$v = \frac{Pp\,\mathrm{d}y}{2\,\mathrm{d}p} + \frac{Q\pi\,\mathrm{d}Y}{2\,\mathrm{d}\pi}. \tag{33}$$

Let $\frac{Pq}{y}$ and $\frac{Q\rho}{Y}$ be the tangential components of P and Q, respectively, and its sum $\frac{Pq}{y} + \frac{Q\rho}{Y}$ their resulting tangential force (cf. Ms 167, §261). Using the result derived in § 176, the increment (taken negative against the force direction) of the body's velocity becomes

$$\mathrm{d}v = -\frac{Pq\,\mathrm{d}s}{y} - \frac{Q\rho\,\mathrm{d}s}{Y}. \tag{34}$$

Combining this result with Eq. (25) gives

$$\mathrm{d}v = -P\,\mathrm{d}y - Q\,\mathrm{d}Y, \tag{35}$$

whose integral is

$$v = C - \int P\,\mathrm{d}y - \int Q\,\mathrm{d}Y, \tag{36}$$

where C denotes the integration constant, which has to be determined using the given initial velocity in a given location of the body's orbit. Euler derived this result already in Ms 180, fol. 3r (cf. Verdun 2012). Equating the Eqs. (33) and (36) allows him to determine the two forces P and Q. Without demonstration Euler presents the result (cf. Ms 167, §262):

$$P = \frac{\pi\,\mathrm{d}p\,\mathrm{d}v + 2v\,\mathrm{d}p\,\mathrm{d}\pi}{\mathrm{d}y\,(p\,\mathrm{d}\pi - \pi\,\mathrm{d}p)} \quad \text{and} \quad Q = \frac{p\,\mathrm{d}\pi\,\mathrm{d}v + 2v\,\mathrm{d}\pi\,\mathrm{d}p}{\mathrm{d}Y\,(\pi\,\mathrm{d}p - p\,\mathrm{d}\pi)}. \tag{37}$$

In Ms 180 Euler left the integro-differential equation leading to this result still unsolved, which provides evidence for the conjecture, that Ms 167 must have been written some time later. Euler comments this result as follows:

> Ex his apparet si detur curva quæcunque et corporis in ea moti celeritas in singulis locis, insuper duo quæcunque puncta, inveniri posse vires ad ea puncta tendentes, quæ faciant ut corpus libere eam curvam describat, et in singulis locis celeritates habeat datas.
> (Thus it is clear: if any curve and the velocity of a body moving on it is given for each point, and in addition any two points [of the curve are given], then it is possible to find forces directed on to these points, which cause the body to follow this curve freely and to assume the given velocity in each point.)

However, there still remains the problem to determine the nature of the curve supposing the two central forces P and Q are given. For that purpose the parameters p, q, π, and ρ of y and Y, which define $\mathrm{d}s$ and r, have to be determined. From Eq. (25) it follows that

$$q = \frac{y\,\mathrm{d}y}{\mathrm{d}s} \quad \text{and} \quad \rho = \frac{Y\,\mathrm{d}Y}{\mathrm{d}s}. \tag{38}$$

This substituted in $q = \sqrt{yy - pp}$ and $\rho = \sqrt{YY - \pi\pi}$ gives (cf. Ms 167, §263)

$$p = \frac{y\sqrt{\mathrm{d}s^2 - \mathrm{d}y^2}}{\mathrm{d}s} \quad \text{and} \quad \pi = \frac{Y\sqrt{\mathrm{d}s^2 - \mathrm{d}Y^2}}{\mathrm{d}s}. \tag{39}$$

From the equations $r = \frac{y\,\mathrm{d}y}{\mathrm{d}p}$ and $r = \frac{Y\,\mathrm{d}Y}{\mathrm{d}\pi}$ Euler has the relations

$$\mathrm{d}p = \frac{y\,\mathrm{d}y}{r} \quad \text{and} \quad \pi = \frac{Y\,\mathrm{d}Y}{r}. \tag{40}$$

This substituted in the first derivatives of $q = \sqrt{yy - pp}$ and $\rho = \sqrt{YY - \pi\pi}$ and considering Eq. (25) gives

$$\mathrm{d}p = \mathrm{d}s - \frac{y\sqrt{\mathrm{d}s^2 - \mathrm{d}y^2}}{r} \quad \text{and} \quad \mathrm{d}\rho = \mathrm{d}s - \frac{Y\sqrt{\mathrm{d}s^2 - \mathrm{d}Y^2}}{r}. \tag{41}$$

Substituting Eqs. (38) and (39) into $cc = (q - \rho)^2 - (p - \pi)^2$, Euler obtains (cf. Ms 167, §264)

$$\mathrm{d}s^2 = \frac{4yY(cc\,\mathrm{d}y\,\mathrm{d}Y + yY\,\mathrm{d}y^2 + yY\,\mathrm{d}Y^2 - yy\,\mathrm{d}y\,\mathrm{d}Y - YY\,\mathrm{d}y\,\mathrm{d}Y)}{2ccyy + 2ccYY + 2y^2Y^2 - c^4 - y^4 - Y^4}. \tag{42}$$

If this element of the body's curve is known, one can find the associated osculting radius r in terms of y and Y together with its differentials, using $r = \frac{y\,\mathrm{d}y}{\mathrm{d}p}$ and

Eq. (39). Assuming $\mathrm{d}s$ to be constant, the first derivative of p in Eq. (39) gives

$$\mathrm{d}p = \frac{\mathrm{d}s^2\,\mathrm{d}y - \mathrm{d}y^3 - y\,\mathrm{d}y\,\mathrm{d}\mathrm{d}y}{\mathrm{d}s\sqrt{\mathrm{d}s^2 - \mathrm{d}y^2}}. \tag{43}$$

Therefore, the resulting curvature radius becomes

$$r = \frac{y\,\mathrm{d}s\sqrt{\mathrm{d}s^2 - \mathrm{d}y^2}}{\mathrm{d}s^2 - \mathrm{d}y^2 - y\,\mathrm{d}\mathrm{d}y} = \frac{Y\,\mathrm{d}s\sqrt{\mathrm{d}s^2 - \mathrm{d}Y^2}}{\mathrm{d}s^2 - \mathrm{d}Y^2 - Y\,\mathrm{d}\mathrm{d}Y}. \tag{44}$$

It may also be expressed in terms of the acting central forces P and Q by equating (36) and (32) (cf. Ms 167, §265):

$$2C - 2\int P\,\mathrm{d}y - 2\int Q\,\mathrm{d}Y = \frac{Ppr}{y} + \frac{Q\pi r}{Y}, \tag{45}$$

from which the nature of the curve can be described using Eq. (39):

$$2C - 2\int P\,\mathrm{d}y - 2\int Q\,\mathrm{d}Y = \frac{Pp\sqrt{\mathrm{d}s^2 - \mathrm{d}y^2} + Qr\sqrt{\mathrm{d}s^2 - \mathrm{d}Y^2}}{\mathrm{d}s} \tag{46}$$

and thus the resulting curvature radius becomes

$$r = \frac{2C\,\mathrm{d}s - 2\,\mathrm{d}s\int P\,\mathrm{d}y - 2\,\mathrm{d}s\int Q\,\mathrm{d}Y}{P\sqrt{\mathrm{d}s^2 - \mathrm{d}y^2} + Q\sqrt{\mathrm{d}s^2 - \mathrm{d}Y^2}}. \tag{47}$$

If the curve that the body describes and one of the central forces, say P, are given, then the other central force Q can be determined as well (cf. Ms 167, §266). Let be $\frac{pr}{2y} = x$ and $\frac{\pi r}{2Y} = z$. Therefore, $v = Px + Qz$ and

$$\mathrm{d}v = P\,\mathrm{d}x + x\,\mathrm{d}P + Q\,\mathrm{d}z + z\,\mathrm{d}Q = -P\,\mathrm{d}y - Q\,\mathrm{d}Y \tag{48}$$

or

$$\mathrm{d}Q + Q\frac{\mathrm{d}Y + \mathrm{d}z}{z} + \frac{P\,\mathrm{d}y + x\,\mathrm{d}P + P\,\mathrm{d}x}{z} = 0. \tag{49}$$

Multiplying this by $\mathrm{e}^{\int \frac{\mathrm{d}Y+\mathrm{d}z}{z}}$ Euler obtains

$$\mathrm{e}^{\int \frac{\mathrm{d}Y+\mathrm{d}z}{z}} Q = D - \int \frac{\mathrm{e}^{\int \frac{\mathrm{d}Y+\mathrm{d}z}{z}}(P\,\mathrm{d}y + x\,\mathrm{d}P + P\,\mathrm{d}x)}{z}, \tag{50}$$

where the integration constant D can be determined from the initial conditions.

Finally, Euler treats the case where one of the force centers may be considered as infinitely far away from the other (cf. Ms 167, §267, and Figs. 14, 15). (This situation

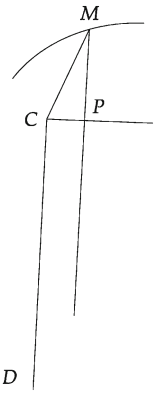

Fig. 14 Euler's sketch in Ms 167 illustrating the first case where the force center is infinitely far away

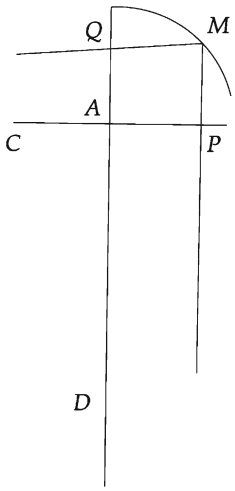

Fig. 15 Euler's sketch in Ms 167 illustrating the second case where the force center is infinitely far away

may, in fact, be assumed for first order approximations in the case of the Earth-Moon-Sun system, where the Sun is regarded as infinitely further away from the Moon than the Earth.) Supposing the force center D situated in infinity, or $c = \infty$. Therefore MD is parallel to CD. Let CP be a perpendicular to CD and denote $PM = z$. Thus $Y = c + z$ and $\mathrm{d}Y = \mathrm{d}z$ holds. Let Q be a function of z. From these assumptions, substituted into Eqs. (42) and (44), Euler obtains

$$\mathrm{d}s = \frac{\sqrt{yy\,\mathrm{d}y^2 + yy\,\mathrm{d}z^2 - 2yz\,\mathrm{d}y\,\mathrm{d}z}}{\sqrt{yy - zz}} \tag{51}$$

and

$$r = \frac{\mathrm{d}s\sqrt{\mathrm{d}s^2 - \mathrm{d}z^2}}{-\mathrm{d}\mathrm{d}z}, \tag{52}$$

where $\mathrm{d}z$ is supposed to be constant. The other equation for the curvature radius, Eq. (47), becomes

$$r = \frac{2C\,\mathrm{d}s - 2\,\mathrm{d}s \int P\,\mathrm{d}y - 2\,\mathrm{d}s \int Q\,\mathrm{d}z}{P\sqrt{\mathrm{d}s^2 - \mathrm{d}y^2} + Q\sqrt{\mathrm{d}s^2 - \mathrm{d}z^2}}. \tag{53}$$

Equating these results (Eqs. 52 and 53) gives

$$\frac{2\,\mathrm{d}\mathrm{d}z}{\sqrt{\mathrm{d}s^2 - \mathrm{d}z^2}} = \frac{P\sqrt{\mathrm{d}s^2 - \mathrm{d}y^2} + Q\sqrt{\mathrm{d}s^2 - \mathrm{d}z^2}}{\int P\,\mathrm{d}y + \int Q\,\mathrm{d}z - C}. \tag{54}$$

The second possible situation concerns the case where the force center C is also regarded as infinitely far away from the body M. Euler considers this case as well, which is, however, of minor importance for the development of his lunar theory and thus may be skipped here.

Appendix B: The content of Ms 271

In this small treatise entitled "De Motu Lunæ in Ellipsin" Euler develops a formula to construct lunar tables based on the Moon's elliptic motion represented by its orbital velocity v at any point of the trajectory. He uses two physical principles, one of them implicitly, the other one explicitly. The first one concerns the equation of motion and the balance of centrifugal and centripetal (gravitational) force F allowing him to determine the Moon's velocity v by Huygens' centrifugal formula (cf. Euler 1736, §630; see also Bomie 1708):

$$F \propto \frac{v^2}{r}, \tag{55}$$

where r designates the curvature radius or radius of the osculating circle at each point of the trajectory. This formula may be reconstructed using Euler's "*Mechanica*" which he completed already in 1734. From the equation of motion, given by (cf. §157)

$$c\,\mathrm{d}c = \frac{np\,\mathrm{d}s}{A}, \tag{56}$$

where c is the velocity of the body with mass A, $\mathrm{d}s$ the line element covered by the body, p is the force acting on this body and n is a constant, Euler derives the formula (cf. §163)

$$npr\mathrm{d}x = Ac^2\,\mathrm{d}s, \tag{57}$$

where r is the curvature radius and $\mathrm{d}x$ the line element covered by the body perpendicular to r, regarding $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. This is exactly Huygens formula, because $\mathrm{d}s = c\mathrm{d}t$ and

$$np\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{Ac^2}{r}. \tag{58}$$

If p is collinear with r as it is the case in central force motions, then $\mathrm{d}y = 0$ and $\mathrm{d}x = \mathrm{d}s$, so that this relation becomes (cf. §165)

$$r = \frac{Ac^2}{np}. \tag{59}$$

Euler always expressed the velocity c by its corresponding height of fall v, so that $v = c^2$ (cf. §202). This implies $n = \frac{1}{2}$ (cf. §206) and thus $r = \frac{2Av}{p}$. From this formula, Euler extracts (cf. §552)

$$v = \frac{r}{2}\frac{p}{A}. \tag{60}$$

Although this derivation is not included in Euler's treatise Ms 271, it seems to be part of the standard repertoire of physical principles belonging at Euler's disposal in that time (cf. Varignon 1707a, §XXIII, Corol. 2, p. 198). The second principle makes sure that the first one holds only if the center of force (i.e., the center of the Earth) is at rest with respect to inertial space, or in other words, that the Moon's motion is described by this approach in an earth fixed reference frame. This condition requires a principle which had never been used before in that context and which was developed as innovative element and applied here for the first time by Euler: the principle of the *transference of forces*. It became a standard method already in Euler's "*Mechanica*" (cf. Euler 1736, Prop. 97, §795). This principle, applied to Euler's determination of the Moon's motion, means that the force component acting on the Earth by the Sun has to be transferred in contrary direction to the Moon. Evidence for its first appearance here is given by the peculiarity that Euler had to state more precisely the way how to apply the inverse force by some deletions and by inserting a note referring to another marginal note.

Let (see Fig. 16) C, M, and N be the centers of the Earth, Moon, and Sun, respectively, which are assumed to be situated in one and the same ecliptic plane. $AMBA$ is the Moon's elliptical orbit coplanar with the ecliptic. AB is the apsidal line, A the apogee, and B the perigee. The Moon is situated in M. The straight line $GMTR$ is the tangent in M. MP is the perpendicular through M on the apsidal line AB, CO the perpendicular through C on the straight line joining M and N, TC and RN are the perpendicular lines through C and N on the tangent line $GMTR$, respectively. Euler denotes the masses of Earth and Sun, respectively, by A and S, and defines the lengths of the major and minor axes of the Moon's orbit $= 2a$ and $= 2c$, respectively, and the length of the major axis of the Sun's orbit $= 2f$. He sets the Earth's global radius $= 1$ and denotes the distances $CM = \mathrm{y}$ and $CN = \mathrm{z}$, and designates the curvature or curvature radius in M with r. He first determines the resulting normal components to the tangent line $GMTR$ of the gravitational forces acting on the Moon and the Earth. The gravitational forces acting on the Moon by the Earth and the Sun are given by $A : yy$ and $S : MN^2$, respectively, the one acting on the Earth by the Sun is given by $S : zz$. The normal components of the first two forces are given by $A \cdot CT : y^3$

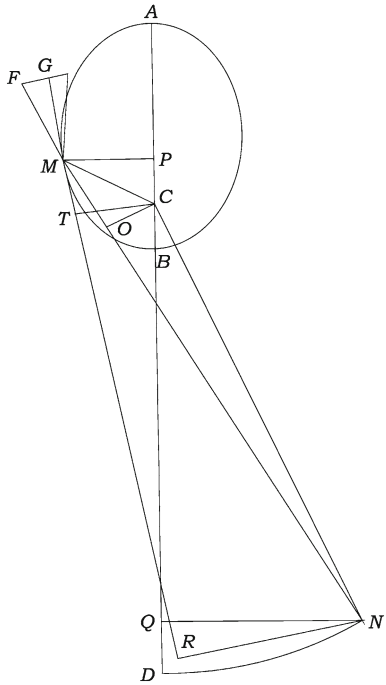

Fig. 16 The figure of Ms 271 explaining the geometry in the System Earth (C), Moon (M), and Sun (N), $AMBA$ being the elliptic lunar orbit with C in one focus

and $S\cdot NR : MN^3$. The normal component of the latter is given by $S \cdot FG : MF \cdot zz$, where the relation $MF : FG = CN : NR - CT$ is defined by the equivalence of the correspondent triangles. This normal component becomes thus $= S \cdot (NR - CT) : z^3$. Now Euler applies his principle of the transference of forces when summing up these normal components to the resulting centripetal forces acting on the Moon, giving $= A \cdot CT : y^3 + S \cdot NR : MN^3 - S(NR - CT) : z^3$. Note that the latter force actually is the normal component of the gravitational force acting on the Earth by the Sun, applied in the reverse direction on the Moon, thus making sure that its motion refers to the Earth resting in inertial space.

In the next step Euler determines the orbital velocity v of the Moon as a function of the curvature radius r in M. Using the first principle (cf. eq. 60) described in words by

> "Est vero vis gravitatis $= A$ et cum sit vis normalis ad vim gravitatis ut altitudes velocitatem generans ad dimidium radii osculi […]",

Euler derives

$$v = r \cdot CT : 2y^3 + S \cdot r \cdot NR : 2A \cdot MN^3 - Sr(NR - CT) : 2Az^3. \tag{61}$$

Without giving any derivation Euler substitutes

$$CT = c\sqrt{y} : (a - y) \tag{62}$$

and the curvature radius (cf. Varignon 1707b)

$$r = (2ay - yy)^{\frac{3}{2}} : ac, \tag{63}$$

and sets approximately $MN = CN + MO = z + MO$, thus yielding

$$v = \frac{2a - y}{2ay} + \frac{Syy(2a - y)}{2Aaz^3} - 3S \cdot r \cdot NR \cdot MO : 2Az^4. \tag{64}$$

Finally, Euler substitutes the product $NR \cdot MO$ by trigonometric functions, resulting in

$$\begin{aligned} v = {} & (2a - y) : 2ay + Syy(2a - y) : 2Aaz^3 \\ & -3Syr\,(\sin CMT + \sin[CMT + 2MCN]) : 4Az^3, \end{aligned} \tag{65}$$

and the mass ratio $S : A$ by $S : A = 2f^3 : (8795)^2 = 2f^3 \cdot n$, leading to

$$\begin{aligned} v = {} & (2a - y) : 2ay + f^3yy(2a - y) : naz^3 \\ & -3f^3yr\,(\sin CMT + \sin[CMT + 2MCN]) : 2nz^3. \end{aligned} \tag{66}$$

Euler remarks that the first term depends only on the Earth's attraction, but that the second and third depend on the Sun's gravitational force. In addition, the second term depends on the Moon's position in its orbit and on the Earth's distance from the Sun, while the third term depends on the lunar aspect, namely on the angle between Moon and Sun. He further notes that in v the excess of the terms depending on the Sun over the terms not depending on it is always reciprocal to the cube of the Sun's distance from the Earth. This is why the Moon's velocity produced by the Sun is much smaller than that generated by the Earth. The difference in the lunar time period due to the Earth's and the Sun's action is thus reciprocal to the cube of the Earth's distance from the Sun, as well.

Due to the fact that $z \gg y$, Euler approximates $z = f$, resulting in

$$\begin{aligned} v = {} & (2a - y) : 2ay + yy(2a - y) : na \\ & -3yr \cdot (\sin CMT + \sin[CMT + 2MCN]) : 2n. \end{aligned} \tag{67}$$

If the lunar orbit is considered as circle with radius a, his velocity is then given by

$$v = 1 : 2a - aa : 2n - 3aa \cdot \cos 2MCN : 2n. \tag{68}$$

Euler derives some corollaries for the elliptic orbit case, defined by the distance $\sqrt{aa - cc} = b$ of the focus from the center of the ellipse, the distance of the apogee $= a + b$ and the perigee $= a - b$. If the Moon is located in the transverse axis or apsidal line, there are two cases: In apogee, $y = a + b$, $r = cc : a$, and $\sin CMT = 1$, resulting in

$$v = (a - b) : 2a(a + b) - cc(a + b) : 2an - 3cc(a + b)\cos 2MCN : 2an. \tag{69}$$

In perigee, $y = a - b$ and $r = cc : a$, leading to

$$v = (a + b) : 2a(a - b) - cc(a - b) : 2an - 3cc(a - b) \cos 2MCN : 2na. \quad (70)$$

If oppositions or conjunctions happen to be in the apogee (upper sign) and in the perigee (lower sign), then

$$v = (a \mp b) : 2a(a \pm b) - 2cc(a \pm b) : na, \quad (71)$$

but if quadratures happen to be in the apogee (upper sign) and in the perigee (lower sign), then

$$v = (a \mp b) : 2a(a \pm b) + cc(a \pm b) : an. \quad (72)$$

If the Moon is located in the conjugate axis, then $y = a$, $r = aa : c$, and $\sin CMT = c : a$, yielding

$$v = 1 : 2a - aa : 2n - 3aa\,(c \cos 2MCN + b \sin 2MCN) : 2cn. \quad (73)$$

If oppositions or conjunctions happen to be in the conjugate axis, the sine and cosine of the angle $2MCN$ become $\sin 2MCN = 0$ and $\cos 2MCN = 1$, respectively, and therefore

$$v = 1 : 2a - 2aa : n. \quad (74)$$

But if quadratures happen to be in the conjugate axis, then $\sin 2MCN = 0$ and $\cos 2MCN = -1$, therefore

$$v = 1 : aa + aa : n. \quad (75)$$

This is an important result derived by Euler, proving that in syzygies the Moon's velocity is always smaller, in the quadratures always larger than it's mean motion. Moreover, every time when the Moon in its orbit is in conjunction or opposition, then—due to $\angle MCN = 0°$ resp. $\angle MCN = 180°$—its velocity becomes

$$v = (2a - y) : 2ay + yy(2a - y) : na - 3yr \sin CMT : n. \quad (76)$$

But $\sin CMT = CT : CM = c : \sqrt{(2ay - yy)}$, so that in syzygies

$$v = (2a - y) : 2ay - 2yy(2a - y) : na, \quad (77)$$

and in quadratures

$$v = (2a - y) : 2ay + yy(2a - y) : na. \quad (78)$$

Euler reduces the general equation (67) by using the relations $\sin CMT = c : \sqrt{2ay - yy}$ and $r \sin CMT = (2ay - yy) : a$, as well as the addition theorem, gaining

$$\begin{aligned} v = {} & (2a - y) : 2ay - yy(2a - y) : 2na \\ & -3yy(2a - y) \cos 2MCN : 2na \\ & -3yy(2a - y)\sqrt{2ay - yy - cc} \cdot \sin 2MCN : 2nac. \end{aligned} \tag{79}$$

Now he tries to make this formula more handy for the calculation of lunar tables. Using the geometry of the ellipse and introducing the ratio $T : t$ of the periods of revolution (i.e., the mean motions of the sidereal periods) of Sun and Moon, respectively, by $\overline{T - t} : T = 559 : 605$, he is able to substitute

$$\sin 2MCN = \sin(2D - 1{,}118 \cdot ACM : 605), \tag{80}$$

where D designates the Sun's elongation from the apsidal line AD as reference line of these motions. Furthermore, he substitutes $\cos ACM = q$, $y = cc : (a - bq)$, $cc : b = g$, and $a : b = K$, yielding $y = g : (K - q)$ and $2a - y = 2a - g : (K - q)$. He sets $\sin ACM = p = \sqrt{1 - yy}$, having $\sqrt{2ay - yy - cc} =$ and $bcp : (a - bq) = cp : (K - q)$. Finally, Euler determines the Moon's angular velocity ("velocitates angulares lunæ") u expressed in terms of equivalent altitude of fall corresponding to the circular motion in M and defined by the curvature radius r in that point:

$$\begin{aligned} u = {} & \frac{cc\,v}{y^3(2a - y)} \\ = {} & cc : 2ay^4 - cc : 2nay - 3cc \cos 2MCN : 2nay \\ & -3c\sqrt{2ay - yy - cc} \cdot \sin 2MCN : 2nay \\ = {} & (K - q)^4 : 2g^3K - (K - q) : 2nK - 3(K - q) \cos 2MCN : 2nK \\ & -3p \sin 2MCN : 2nK \\ = {} & (K - q)^4 : 2g^3K - (K - q) : 2nK \\ & -3 \cos 2MCN : 2n + 3 \cos(2MCN + ACM) : 2nK \\ = {} & (K - q)^4 : 2g^3K - (K - q) : 2nK - 3 \cos(2D - 1118 \cdot ACM : 605) : 2n \\ & +3 \cos(2D - 513 \cdot ACM : 605) : 2n. \end{aligned} \tag{81}$$

In this result Euler inserts the values $a = 60$ and $b = 3$, yielding $c = \sqrt{3{,}591}$, $g = 1{,}197$, and $K = 20$. In order to gain practical numbers for u, he scales u,

$$\begin{aligned} 1{,}000{,}000{,}000{,}000\,u = {} & (1{,}000{,}000)^2\Big((20 - q)^4 : 40 \cdot 1197^3 - (20 - q) : 40n \\ & -3 \cos(2D - 1{,}118 \cdot ACM : 605) : 2n \\ & -3 \cos(2D - 513\,ACM : 605) : 40n\Big), \end{aligned} \tag{82}$$

so that the final result becomes

$$1{,}000{,}000\sqrt{u} = 3.817949\,(20-q)^2 - \frac{42.3261}{(20-q)} - \frac{2{,}539.566}{(20-q)^2}\cos\left(2D - \frac{1{,}118\cdot ACM}{605}\right) + \frac{126.9783}{(20-q)^2}\cos\left(2D - \frac{513\,ACM}{605}\right). \tag{83}$$

Euler remarks that the parameter q must successively be substituted with the cosine of the angles 1°, 2°, 3°, 4° etc. In this way one may construct lunar tables depending on the angle ACM as table argument and on the angle D between the Sun and the lunar apogee, measured from the apogee, which may be varied by, e.g, 30° or 45°, represented by different tables, as well.

Appendix C: The content of Ms 273

This manuscript fragment consists of six Propositions, each of them followed by Corollaries. Unfortunately, all thirteen figures to which Euler refers in the text margins are missing.

Proposition I: Given the mass or active quantity ("quantitas activa") of a body being at rest, find the motion of another body rotating around the former.

This proposition concerns the determination of the orbital (Keplerian) elements of the two-body-problem. Let M be the mass of the Earth, r its radius, S the mass of the central body (i.e., the Sun) located in one focus of the elliptic orbit (cf. Fig. 17, which was adopted from Ms 397, fol. 121v). Let A be the aphelion and $AS = f$ the distance between A and S. Let K and v be the altitudes corresponding to the velocities of the body in A and in M, respectively. Let further be $SM = y$ and $Mn = \mathrm{d}y$, where n denotes a point infinitesimally close to M on the curve, and $SP = p$, where P is the intersection between the tangent line through M and the perpendicular through S to this tangent. Let $v + \mathrm{d}v$ be the altitude corresponding to the velocity of the body in m infinitesimally close to the point M on the curve. Due to the dynamic relation

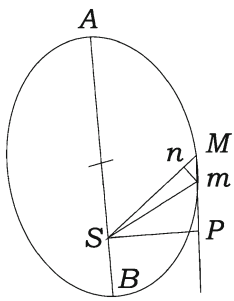

Fig. 17 Euler's sketch of Ms 397 explaining the central force motion of the body M around the central body S, A and B being the apo- and pericenter, respectively

$$\mathrm{d}y : \mathrm{d}v = \frac{M}{rr} : \frac{S}{yy}, \tag{84}$$

and due to the fact that $\mathrm{d}v$ decreases when $\mathrm{d}y$ increases, Euler starts with the equivalence

$$\frac{M\,\mathrm{d}v}{rr} = \frac{-S\,\mathrm{d}y}{yy}, \tag{85}$$

whose integral is

$$\frac{Mv}{rr} = \frac{S}{y} + A, \tag{86}$$

where A denotes the integration constant (not to be confused with the aphelion). If the body M is located in the aphelion, then $v = K$ and $y = f$ holds, so that

$$A = \frac{MK}{rr} - \frac{S}{f}. \tag{87}$$

Substituting Eq. (87) into (86) gives

$$\frac{Mv - MK}{rr} = \frac{Sf - Sy}{fy} \quad \text{or} \quad v = \frac{Sfrr - Srry}{Mfy} + K. \tag{88}$$

Using the relation $K : v = pp : ff$, Euler eliminates v in Eq. (88) and solves the result for p:

$$pp = \frac{Mf^3Ky}{Sfrr - Srry + MfyK}. \tag{89}$$

If the body is situated in the perihelion, then $p = y$, and Eq. (89) becomes

$$Mf^3K - MfKy^2 - Sfrry + Srry^2 = 0, \tag{90}$$

or, after dividing by $f - y$,

$$MffK + MfKy - Srry = 0. \tag{91}$$

Solving this equation for the perihelion distance $y \equiv BS$ gives

$$BS = \frac{MffK}{Srr - MfK}. \tag{92}$$

Thus, the major axis $a \equiv AB$ ("axis transversus") becomes

$$AB = \frac{Sfrr}{Srr - MfK}. \tag{93}$$

Substituting this result into Eq. (89) Euler obtains

$$pp = \frac{Maf^3Ky}{Safrr - Sfrry} = \frac{MaffKy}{Sarr - Srry}. \tag{94}$$

The distance between the two foci is given by $= \frac{Sfrr-2MffK}{Srr-MfK}$, which Euler uses to determine the length of the minor axis c ("axis conjugatur orbitae"):

$$c = 2f\sqrt{\frac{MfK}{Srr - MfK}} = 2f\sqrt{\frac{MaK}{Srr}}. \tag{95}$$

Therefore

$$\frac{Srrcc}{4ff} = MaK \tag{96}$$

and Eq. (94) becomes

$$pp = \frac{ccffy}{4ff(a-y)} = \frac{ccy}{4a-4y}. \tag{97}$$

The time element used to cover the infinitesimally small distance Mm is given by

$$\frac{Mm}{\sqrt{v}} = \frac{p \cdot Mm}{f\sqrt{K}}. \tag{98}$$

Using the relations $p : y = mn : Mm$ or $p \cdot Mm = y \cdot mn$ and thus $mn = \frac{p\,\mathrm{d}y}{\sqrt{yy-pp}}$, Euler obtains for the time element $= \frac{py\,\mathrm{d}y}{f\sqrt{K(yy-pp}}$. But

$$p = \frac{c}{2}\sqrt{\frac{y}{a-y}} \tag{99}$$

and

$$\sqrt{yy - pp} = \frac{1}{2}\sqrt{\frac{4ayy - 4y - ccy}{a-y}}, \tag{100}$$

therefore

$$\frac{py\,\mathrm{d}y}{f\sqrt{K(yy-pp)}} = \frac{cy\,\mathrm{d}y}{f\sqrt{4aKy - 4Kyy - ccK}}, \tag{101}$$

whose integral is the time of revolution. The time element is defined by the uniform straight line motion of the body from m to n, and thus given by $\frac{y \cdot mn}{f \cdot \sqrt{K}}$. The infinitesimally small area covered in this time is $y \cdot mn = 2\,MSm$. Let the total aera

of the orbit's ellipsis be A. The time of revolution ("tempus periodicum") T is then defined by

$$T = \frac{2A}{f\sqrt{K}}. \tag{102}$$

Let $1 : \pi$ be the ratio between the radius of a circle and its perimeter. [4] Hence the area of a circle with radius a becomes $\frac{aa\pi}{8}$. The ratio $a : c$ between the major and minor axes of the ellipse may also be expressed by the equivalent fractions $\frac{aa\pi}{8} : \frac{ac\pi}{8}$, from which Euler infers to the area of the ellipsis $A = \frac{ac\pi}{8}$. This substituted into Eq. (102) gives

$$T = \frac{\pi ac}{4f\sqrt{K}}. \tag{103}$$

Substituting Eq. (96) into this result, one obtains

$$T = \frac{\pi a\sqrt{Ma}}{2r\sqrt{S}}. \tag{104}$$

When asking for the absolute time of revolution, the variables a and r in this formula have to be expressed in Rhinelandian (Prussian) feet (cf. Ms 167, Section A above), so that the time of revolution in seconds of time is given by

$$T = \frac{\pi a\sqrt{10\,Ma}}{50r \cdot \sqrt{S}}. \tag{105}$$

This proposition is followed by nine corollaries. In the first one Euler solves Eq. (93) for K and substitutes the result into Eq. (88), which gives

$$v = \frac{Srr(a-y)}{May}. \tag{106}$$

Naming the moving body M by "the mobile", Euler concludes from this result, that the square of the mobile's velocity is proportional to the distance MF from the non-attracting focus F and reciprocal to the distance MS from the focus S, being the center of force. He notes in a second corollary, that $v = 0$ if $y = a$. In the third corollary he solves Eq. (93) for f and substitutes the result into Eq. (96), obtaining thus

$$c = \frac{2ar\sqrt{MSaK}}{Srr - MaK}. \tag{107}$$

[4] Note that Euler's definition of "π" implies that it is twice the value of Ludolph's number used in the modern sense, which is the ratio between the *diameter* of a circle and its perimeter. He used that peculiar kind of definition already in Ms 397, fol. 122r, and in Ms 167, § 371. This is one of the earliest appearance of the symbol π used in this meaning in Euler's work (cf. Mattmüller 2010, p. 185). Some times later, however, Euler defined and used this symbol already in the modern sense in his "*Mechanica*" (cf. Euler 1736, Prop. 76, Coroll. 2, § 613).

In the forth corollary Euler denotes the time of revolution by T, which is—according to Eq. (104)—proportional to $\frac{a\sqrt{a}}{\sqrt{S}}$. He interprets this result in the fifth corollary for the cases where several bodies move around one and the same center and around multiple centers. In the remaining corollaries he evaluates Eq. (105) numerically for the motions of the Earth and of Venus, and determines the mass ratio $S : M$ between the Sun and Earth, obtaining 1,053,531:1. Euler adopted the time of revolution for Venus, "224.d.17.h.44′.55″." (cf. Ms 273, fol. 2v), most probably from the 1726 edition of Gregory's book, from which he possessed a copy in his own library (cf. Gregory 1726, p. IV from the "Editoris præfatio").

Proposition II: Let S be a [central] body attracting in any way, and M a mobile thus describing any curve located in any point M; determine the ellipsis, in which the mobile proceeds, if the formers force immediately decreases according to the squared ratio of the distances.

Let v be the altitude [of free fall] corresponding to the velocity of M and $SM = y$. Let P be the force acting on M by S, and a be the major axis ("axis principalis ellipsis"), of which the arc element Mm is an infinitesimally small part. Let the attracting central force P be $= \frac{S}{yy}$ and c be the minor axis of the ellipsis. From Eq. (106) of Corollary I it follows

$$v = \frac{Prry(a-y)}{Ma}. \tag{108}$$

Euler reformulates Eq. (96) using the relation $Kff = vpp$ and obtains

$$Srrcc = 4Maffk. \tag{109}$$

He substitutes this result into Eq. (93), so that the major axis becomes

$$a = \frac{Prryy}{Prry - Mv}. \tag{110}$$

When y changes into $y + \mathrm{d}y$, the major axis increases by the element

$$\frac{PPr^4yy\,\mathrm{d}y + PMrryy\,\mathrm{d}v - 2PMrrvy\,\mathrm{d}y - Mrryyv\,\mathrm{d}P}{(Prry - Mv)^2}. \tag{111}$$

Euler inserts the relation $Kff = vpp$ into Eq. (109) and obtains the parameter of the orbit ("parametro orbitæ"):

$$\frac{cc}{a} = \frac{4Mvpp}{Prryy}. \tag{112}$$

If y increases, then this parameter will also increase by the element (as first derivative of $\frac{4Mvpp}{Prryy}$, where vpp is assumed to be constant)

$$\frac{-4Mrrvppyy\,\mathrm{d}P - 8MPrrvppy\,\mathrm{d}y}{PPr^4y^4}. \tag{113}$$

Euler concludes from this result, that the apsidal line will move according to the order of the signs ("in consequentia"), if the major axis a increases by this element. Analogously, if this element decreases, the major axis decreases as well. But if the major axis either increases or decreases, this element decreases or increases, respectively, and the apsidal line will retrograde against the order of the signs ("in antecedentia"). This is why a motion of the apsidal line can be recognized in any case.

This proposition is followed by seven corollaries. In the first one Euler inserts the relation $\mathrm{d}y : \mathrm{d}v = \frac{M}{rr} : P$ or $\mathrm{d}v = \frac{-Prr\,\mathrm{d}y}{M}$ into Eq. (111) and obtains

$$\frac{-2PMrrvy\,\mathrm{d}y - Mrryyv\,\mathrm{d}P}{(Prry - Mv)^2}. \tag{114}$$

This element increases the major axis when y increases its element $\mathrm{d}y$. In the second corollary he relates the increments of the parameter and the major axis to each other. The ratio of these increments is as $\frac{4pp}{PPr^4y^4}$ to $\frac{1}{(Prry-Mv)^2}$, which is as $4pp(Prry - Mv)^2$ to PPr^4y^4 or as $4pp$ to aa. In the third corollary he introduces—probably for the first time in his works on celestial mechanics—a "generalized law of gravitation", defining

$$P = \frac{N}{y^n}, \tag{115}$$

where N denotes an arbitrary constant of proportionality. The first derivative of Eq. (115) is given by

$$\mathrm{d}P = \frac{-Nn\,\mathrm{d}y}{y^{n+1}}. \tag{116}$$

Euler substitutes these Eqs. (115) and (116) into the increment of the major axis (Eq. 114) and of the parameter (Eq. 113), obtaining

$$\frac{\overline{n-2}\cdot MNrry\,\mathrm{d}y}{y^{n-1}(Prry - Mv)^2} = \frac{\overline{n-2}\cdot PMrrvy\,\mathrm{d}y}{(Prry - Mv)^2} = \frac{\overline{n-2}\cdot Maav\,\mathrm{d}y}{Pr^2y^3} \tag{117}$$

and

$$\frac{\overline{n-2}\cdot 4Mppv\,\mathrm{d}y}{Pr^2y^3}, \tag{118}$$

respectively. He is thus able to infer important consequences from these results, which are also relevant for the motion of the lunar apses. He concludes in the forth and fifth corollary that, if $n > 2$ and y increases or decreases, then the major axis will increase or decrease as well, respectively. If the force P decreases in a ratio larger than the square of the distances, then the major axis of the ellipsis increases with increasing distances of the mobile from the focus, and decreases with decreasing distances from it. The contrary is the case if $n < 2$. If therefore the central force P decreases in

a superior ratio than in the squared, then—so Euler continues in corollary VI—the apsidal line moves according to the order of the signs ("in consequentia"). In the other case, it moves against the signs ("in antecedentia"). In corollary VII he determines the ratio between the orbit's velocity, i.e., the velocity of the apsidal line, and the angular velocity of the mobile. Let PMQ be the tangent line through the mobile M, SP and FQ perpendiculars through the foci S and F to this tangent. Let m be a point of the ellipsis infinitesimally close to M achieved by the body while proceeding in his motion on the curve and while the major axis is moving from F to f by the element Ff. Thus the apsidal line progresses around S by the angle FSf. Let Fg and MR be the perpendiculars through F and M to the apse SF. Therefore, the ratio between the angular motion of the body and the progressive motion of the apsidal line is given by

$$\frac{Mn}{SM} : \frac{Fg}{SF} = \frac{PS \cdot mn}{PM \cdot SM} : \frac{MR \cdot Ff}{FM \cdot SF}. \tag{119}$$

Let $SM + FM = a$ be the length of the major axis, $SM = y$, $SP = p$, $PM = q$, and thus $FM = a - y$. From Eq. (117) Euler concludes

$$Ff = \frac{\overline{n-2} \cdot Maav\,\mathrm{d}y}{Pr^2y^3} = \frac{\overline{n-2} \cdot aab\,\mathrm{d}y}{4p^2y}, \tag{120}$$

where b denotes the parameter defined by $b = \frac{4Mvpp}{Prryy}$. He expresses the ratio of Eq. (119) by

$$\begin{aligned}\frac{Mn}{SM} : \frac{Fg}{SF} &= \frac{PS}{PM \cdot SM} : \frac{\overline{n-2} \cdot aab \cdot MR}{4\,SP^2 \cdot PM \cdot FM \cdot SF} \\ &= 4\,PS^2 \cdot SF \cdot FM : \overline{n-2} \cdot aab \cdot SM \cdot MR.\end{aligned} \tag{121}$$

Using the relation $SF : SP = 2MQ : MR$, the ratio between the angular velocity of the mobile and the angular velocity of the apsidal line becomes (regarding $mn = \mathrm{d}y$)

$$\begin{aligned}&4\,PS^3 \cdot SF^2 \cdot FM : \overline{n-2} \cdot 2aab \cdot SM \cdot MQ \cdot SP \\ &= 2PS^2 \cdot SF^2 \cdot FM : \overline{n-2} \cdot aab \cdot SM \cdot MQ.\end{aligned} \tag{122}$$

Considering the fact that

$$ab = \frac{4pp(a-y)}{y} = \frac{4SP^2 \cdot FM}{SM}, \tag{123}$$

the ratio of Eq. (122) finally becomes

$$\begin{aligned}&= 2PS^2 \cdot SF^2 \cdot FM : \overline{n-2} \cdot 4a\,SP^2 \cdot FM \cdot MQ \\ &= SF^2 : \overline{2n-4} \cdot aMQ = SF^2 : \overline{2n-4} \cdot FM \cdot PQ.\end{aligned} \tag{124}$$

Proposition III: One asks for the force of the Sun causing the perturbation of the Moon's motion.

In order to determine the Moon's motion in an adequate manner, it has to be described with respect to an immobile location. Thus suppose the Earth to be immobile, and therefore consider the total motion impressed to the system of bodies always to be equal and inverse directed to the Earth's motion. The forces acting on the Earth have to be considered as acting in reverse direction on the whole system. Actually, Euler neglects the Moon's mass and does not consider the force acting on the Sun by the Earth. Let T be the Earth, ALB a part of the Moon's orbit, S the Sun, L the Moon, and LT, LS its distances from the Earth and Sun, respectively. Let TS be the distance between the Earth and the Sun. Let M and S be the masses of the Earth and the Sun, respectively. The forces acting on the Moon L by the Earth T and the Sun S are given by $\frac{M}{TL^2}$ and $\frac{S}{SL^2}$, respectively. The force acting on the Earth by the Sun is given by $\frac{S}{TS^2}$. To consider the Earth to be at rest means that this force $\frac{S}{TS^2}$ has to be applied to the Moon in the reverse sense along the direction parallel to TS (Euler wrote erroneously TL). The force, with which the Moon is attracted to the Sun along the direction LS has to be decomposed into the two directions LT and TS. Therefore, the forces acting on the Moon by the Sun along LT and along TS (Euler wrote erroneously TL) become $\frac{S \cdot TL}{(LS)^3}$ and $\frac{S \cdot TS}{LS^3}$, respectively. Taking into account all these force components, the resulting forces acting on the Moon along LT and TS becomes

$$\frac{M}{TL^2} + \frac{S \cdot TL}{LS^3} = \frac{M \cdot LS^3 + S \cdot TL^3}{TL^2 \cdot LS^3} \tag{125}$$

and

$$\frac{S \cdot TS}{LS^3} - \frac{S}{TS^2} = \frac{S \cdot TS^3 - S \cdot LS^3}{TS^2 \cdot LS^3}. \tag{126}$$

Due to Corollary IX of Proposition I, the ratio of the masses between the Sun and the Earth is $S : M = 1{,}053{,}531 : 1$ or $S = 1{,}053{,}531 \cdot M$. Euler substitutes this relation into Eq. (125) and obtains the force acting on the Moon along the Earth's direction $= \frac{M}{TL^2 \cdot LS^3}(LS^3 + 1{,}053{,}531 \cdot TL^3)$. He knew approximately the ratio between the distances as $TS : LT = 572 : 1$. As an approximation he assumes $LS \approx TS$ and obtains for this force

$$\frac{M}{TL^2}\left(1 + \frac{1{,}053{,}531}{(572)^2}\right) = \frac{M}{TL^2}\left(\frac{178}{177}\right). \tag{127}$$

The ratio between the force acting on the Moon by the Earth when disregarding the Sun's action and the force when regarding the Sun's action is as 177–178.

Euler adds four corollaries to this proposition which, however, are of minor importance and thus will be skipped here, because they concern only numerical estimations and comparisons of the forces determined for different characteristic points of the lunar orbit.

Proposition IV: Suppose the force acting on the Moon by the Sun is decomposed into two components: one component which attracts the Moon towards the Earth, and one which attracts the Moon along the direction parallel to the conjugation line

between the Sun's and the Earth's center. One asks for the perturbation of the Moon's motion caused by the force component attracting towards the Earth.

Let the Moon be attracted towards the Earth by the force $\frac{M\cdot LS^3+S\cdot TL^3}{TL^2\cdot LS^3}$ according to Eq. (125). Euler denotes LT=y and LS=z. Thus this force becomes

$$P = \frac{Mz^3 + Sy^3}{y^2z^3}. \tag{128}$$

If y goes over into $y + \mathrm{d}y$ and z into $z + \mathrm{d}z$, the increment of the major axis—let us call it da—may be determined using Corollary I of Proposition II. The first derivative of Eq. (128) is given by

$$\mathrm{d}P = \frac{Sz^3y^4\,\mathrm{d}y - 2Mz^6y\,\mathrm{d}y - 3Sy^5z^2\,\mathrm{d}z}{y^4z^6} = \frac{Szy^3\,\mathrm{d}y - 2Mz^4\,\mathrm{d}y - 3Sy^4\,\mathrm{d}z}{y^3z^4}. \tag{129}$$

This result may now be substituted into Eq. (114) to determine the variation of the major axis. In addition, Eq. (108) may be written as

$$Prry - Mv = \frac{Prryy}{a}, \tag{130}$$

so that the increment or variation of the major axis da becomes

$$[\mathrm{d}a] = \frac{3Saay - 3Sayy}{Mz^4 + Sy^3z}(y\,\mathrm{d}z - z\,\mathrm{d}y). \tag{131}$$

Euler denotes the major axis of the orbit by a, and the orbit's parameter by b. Due to Corollary II of Proposition II the ratio between the increment of the parameter (let us call it db) and the increment of the major axis da is $4TP^2$ to aa, i.e., by to $aa - ay$. He substitutes this relation into Eq. (131) and obtains for the increment of the parameter

$$[db] = \frac{3Sby^2}{Mz^4 + Sy^3z}(y\,\mathrm{d}z - z\,\mathrm{d}y). \tag{132}$$

To find out how the apsidal line moves when y increases by the element dy, one has to investigate the power of y defining the decrease of the (gravitational) force. Let n be this exponent. According to Eq. (117) the increment of the major axis becomes

$$[\mathrm{d}a] = \frac{\overline{n-2}\cdot Maav\,\mathrm{d}y}{Pr^2y^3} = \frac{\overline{n-2}\cdot\overline{aa-ay}\cdot\mathrm{d}y}{y^2} = \frac{3Sy\cdot(aa-ay)}{Mz^4+Sy^3z}(y\,\mathrm{d}z - z\,\mathrm{d}y), \tag{133}$$

from which it follows

$$\overline{n-2} = \frac{3Sy^3(y\,\mathrm{d}z - z\,\mathrm{d}y)}{z\,\mathrm{d}y(Mz^3 + Sy^3)}. \tag{134}$$

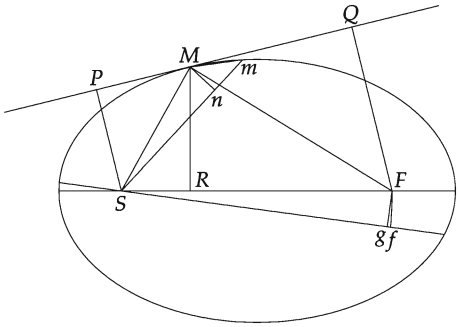

Fig. 18 Reconstruction of Fig. 2 in Ms 273

In Proposition II, Corollary VII, Eq. (124), it was already shown that the ratio between the angular velocity of the Moon and the angular velocity of the apsidal line is given by SF^2 to $\overline{2n-4} \cdot a \cdot MQ$. The distance between the foci of the ellipse is given by $SF^2 = aa - ab$. From the geometry of Fig. 18 Euler concludes that

$$\begin{aligned} MQ &= \frac{PM \cdot MF}{SM} = \frac{\overline{a-y} \cdot PM}{y} \\ &= \frac{\overline{a-y} \cdot \sqrt{4ayy - 4y^3 - aby}}{y\sqrt{4a-4y}} \\ &= \frac{\sqrt{(2ay-2yy)^2} - aby \cdot \overline{a-y}}{2y}. \end{aligned} \tag{135}$$

Therefore,

$$\begin{aligned} SF^2 : \overline{n-2} \cdot aMQ &= aa - ab : \frac{\overline{n-2} \cdot a}{y}\sqrt{(2ay-2y^2) - by\,\overline{a-y}} \\ &= ay - by : \overline{n-2} \cdot \sqrt{(2ay-2yy)^2 - aby \cdot \overline{a-y}}. \end{aligned} \tag{136}$$

Using these results, $(n-2)$ can be substituted to obtain the ratio between the Moon's angular velocity and the angular velocity of the apsidal line, representing—according to Euler—the solution of the problem:

$$\overline{a-b} \cdot z\,\mathrm{d}y(Mz^3 + Sy^3) : 3Sy^2(y\,\mathrm{d}z - z\,\mathrm{d}y)\sqrt{(2ay-2yy)^2 - aby \cdot \overline{a-y}}. \tag{137}$$

This proposition is followed by eleven corollaries and one scholium. In Corollaries I–III Euler discusses the location and motion of the major axis according to the result achieved in Eq. (131) for special cases, depending on the values of $\mathrm{d}y$ and $\mathrm{d}z$. He uses the results obtained from these conclusions in Corollaries IV–VI to quantify them, assuming the ratio between the masses of Sun and Earth as $S = 1{,}053{,}531\,M$, which

was derived in Corollary IX of Proposition I. In addition, he supposes $z = 572\,y$ as a result gained in Corollary IV of Proposition III, and uses the force ratio 177 : 178 as main result of that Proposition. He approximates the distance between the Earth T and the empty focus F of the lunar orbit by $FT = 5$ in units of the Earth's radius. The numerical results are not an important issue for the reconstruction of the development of Euler's lunar theory and may be skipped here. In the remaining four corollaries Euler analyzes the consequences which may be drawn from the ratio between the Moon's angular velocity in any place of its orbit and the angular velocity of the lunar apses, using and interpreting Eq. (137), again in terms of the values of $\mathrm{d}y$, $\mathrm{d}z$, y, and z. For this purpose he considers the infinitesimally small angular sector defined by

$$MSm = \frac{SP \cdot \mathrm{d}y}{MP \cdot SM}, \tag{138}$$

The first factor of the denominator is given by

$$MP = \frac{\sqrt{\overline{2ay - 2yy}^2 - aby \cdot \overline{a - y}}}{2a - 2y}, \tag{139}$$

and of the nominator by

$$SP = \frac{\sqrt{aby}}{2\sqrt{a - y}}. \tag{140}$$

He substitutes the last two equations into Eq. (138) and obtains

$$MSm = \frac{\mathrm{d}y\,\sqrt{aby \cdot \overline{a - y}}}{y\sqrt{\overline{2ay - 2yy}^2 - aby \cdot \overline{a - y}}}. \tag{141}$$

Using the relation (137), the angular element covered by the apsidal line—let us call it $\mathrm{d}\omega$—thus becomes

$$[\mathrm{d}\omega] = \frac{3Sy(y\,\mathrm{d}z - z\,\mathrm{d}y)\sqrt{aby \cdot \overline{a - y}}}{\overline{a - b} \cdot z(Mz^3 + Sy^3)}, \tag{142}$$

whose integral yields the motion of the lunar apse. To prepare this integration, Euler reformulates Eq. (142) to obtain

$$[\mathrm{d}\omega] = \frac{3Syz\sqrt{aby \cdot \overline{a - y}}}{\overline{a - b}(Mz^3 + Sy^3)}\left(\frac{y\,\mathrm{d}z - z\,\mathrm{d}y}{zz}\right). \tag{143}$$

Assuming the first factor to be constant, the integration gives

$$[\omega] = \frac{3Syz\sqrt{aby \cdot \overline{a - y}}}{\overline{a - y}(Mz^3 + Sy^3)}\left(\frac{BS}{e} - \frac{y}{z}\right), \tag{144}$$

where $\frac{BS}{e}$ is the constant of integration with e denoting the distance between Sun and Moon, measured when the latter is in perigee. If the direction of the apsidal line happens to be in quadrature, so he concludes from Eq. (144), then $[\omega] = 41'$ per month. The apsidal line thus regresses with $82' = 1°22'$ per anomalistic month against the series of the signs ("contra signorum seriem"). Euler does not comment on this result with regard to that one obtained by Newton (cf. Newton 1687, Lib. I, Prop. XLV, Coroll. II). In the Scholion that finishes this proposition, he states:

> Vis hæc de qua egi in nodos nullum habet influxum sed eos immutatos et immotos sinit. directior [sic!] enim ejus in ipso plano orbitæ lunæ sita est, efficitque ut Luna in eadem semper plano immoto moveatur. Motus autem nodorum ut et motus lineæ absidum in consequentia debetur alteri vi partiali derivatæ a vi solis, agenti secundum parallelas rectæ Solem et terram jungentis.
> (This force I have mentioned has no influence on the nodes, but leaves them unchanged and immobile. Its direction coincides with the plane of the Moon's orbit and causes the Moon to move always in one and the same immobile plane. Consequently, the motion of the nodes and the motion of the apsidal line is due to the other force component, which comes from the Sun's force acting along the parallel to the straight line conjoining the Sun and the Earth.)

Proposition V: Suppose the Moon moving in an ellipse and another force is acting on it, which may be derived from the Sun acting along the straight line conjoining the centers of the Sun and Earth; find the perturbation of the Moon's orbit due to this force.

Let S be the Sun, T the Earth, and L the Moon. Let CT be the line conjoining the Earth and the quadrature points of the lunar orbit. Let further N be the intersection between the lines CT and LS. The force repelling the Moon along the line CN is, according to Eq. (126) of Proposition III, given by

$$\frac{S \cdot (LS^3 - TS^3)}{TS^2 \cdot LS^3}. \tag{145}$$

Denoting $LS - TS = LN$, Euler approximates $LS^3 - TS^3$ by $3 \cdot TS^2 \cdot LN$, so that the force becomes

$$\frac{3S \cdot LN}{LS^3}. \tag{146}$$

Because LS may be considered constant with respect to LN, the force acting on the Moon varies everywhere with its distance from the straight line conjoining the quadratures. And consequently, the Moon is steadily departing from its orbit when moving on the upper side of the quadratures, which is why the orbit becomes prolonged in opposition; in the same way it is pulled away from the line conjoining the quadratures and attracted towards the Sun in the lower part of the orbit, which is why the orbit is also prolonged in conjunction. These prolongations cause the apsidal line to change its position and to retrograde continuously; however, its progression dominates the retrogression easily, as will be shown later on. From the fact that all points of the

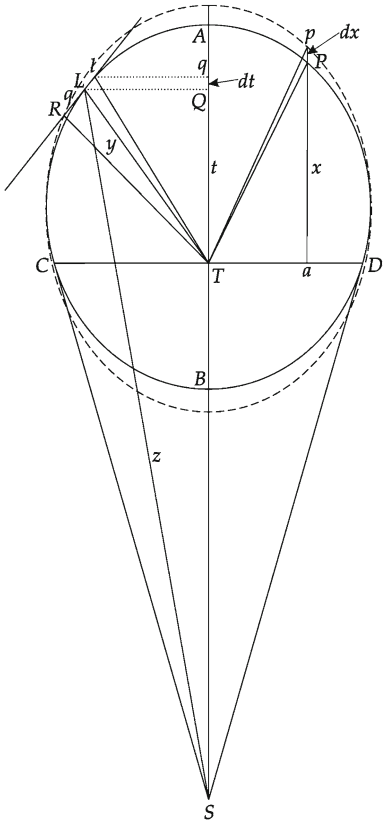

Fig. 19 Reconstruction of Fig. 11 in Ms 273

orbit are pushed away from the line conjoining the quadratures and attracted along the straight line conjoining the centers of Earth and Sun, one may conclude that the line conjoining the nodes has to change, as well as the inclination of the Moon's orbit with respect to the ecliptic. The angle of inclination must decrease when the Moon is located in the syzygies, because the Moon is then retracted by the Earth and its distance increases while the arc measuring the angle of inclination remains unchanged. This proposition is followed by eleven corollaries which are skipped here.

Proposition VI: Due to the Sun's force the Moon is everywhere repelled away from the straight line connecting the quadratures; find how far the Moon is at any point dislocated from its orbit.

Suppose this force to act on a third body (which we would call today a "massless test body") and investigate the effect which is produced on it. Let us begin by assuming, for the sake of simplicity and facility, that the apsidal line coincides with the syzygies (cf. Fig. 19). While the Moon proceeds from C to A, the Sun's force begins to repel the Moon away from the straight line CD due to the transference of the force acting on the Earth by the Sun to the Moon as explained in Proposition III. The resulting forces act along the parallels to the apsidal line and vanish in the line CD due to the Earth considered at rest by the transference of forces. Euler assumes that one of these forces

act not on the Moon but on another body a and attract it along the straight line aP. Suppose the Moon has arrived at L and the body at P, and while the Moon proceeds to l, the body—starting from P—arrives at p. Euler denotes $LT = y$ and $aP = x$. Let w be the altitude (i.e., the height of fall) corresponding to the velocity of the body in P. Then $Pp = \mathrm{d}x$. Let z be the distance between the Sun and the Moon; from L, l take the perpendiculars LQ, lq to the (major) axis, defining thus $TQ = t$ and $Qq = \mathrm{d}t$. The force moving the body in P is given by Eq. (145). Starting again with Eq. (84) and using the same meaning for the symbols M and r as at the beginning of his treatise, Euler obtains

$$\frac{M}{rr} : \mathrm{d}x = \frac{3St}{z^3} : dw \quad \text{or} \quad 3Srrt\,\mathrm{d}x = Mz^3\,dw. \tag{147}$$

He assumes that the distances Pp and Ll are covered simultaneously, hence

$$Ll : Pp = \sqrt{v} : \sqrt{w} \quad \text{or} \quad Ll^2 : \mathrm{d}x^2 = v : w. \tag{148}$$

Euler denotes $LR = q$ (not to confuse with the point p on the major axis), and concludes from the equivalence of triangles that $Ll = \frac{y\,\mathrm{d}y}{q}$. Therefore,

$$v\,\mathrm{d}x^2 = \frac{wyy\,\mathrm{d}y^2}{qq} \quad \text{or} \quad vqq\,\mathrm{d}x^2 = wyy\,\mathrm{d}y^2. \tag{149}$$

Let a and c be the length of the major and minor axes (not to confuse with the test body a), b the parameter of the ellipse or "latus rectum", and f the distance between the foci. From properties of the geometry of the ellipsis, he derives

$$TQ = \frac{2ay - aa + ff}{2f} \equiv t \quad \text{and} \quad LQ = \sqrt{\frac{4accy - 4ccy^2 - c^4}{4aa - 4cc}}. \tag{150}$$

Euler rewrites Eq. (100) as

$$q = \sqrt{\frac{4ayy - 4y^3 - ccy}{4a - 4y}} \tag{151}$$

to obtain for Eq. (149) the equation

$$v\,\mathrm{d}x^2 = \frac{(4a - 4y)wy^2\,\mathrm{d}y}{4ayy - 4y^3 - ccy} = \frac{Srr \cdot (a - y)\,\mathrm{d}x^2}{May}, \tag{152}$$

and consequently

$$\frac{Srr\,\mathrm{d}x^2}{Ma} = \frac{4wyy\,\mathrm{d}y^2}{4ay - 4yy - cc} \quad \text{or} \quad r\,\mathrm{d}x\sqrt{\frac{S}{Ma}} = \frac{2y\,\mathrm{d}y\sqrt{w}}{\sqrt{4ay - 4yy - cc}}. \tag{153}$$

Euler then substitutes Eq. (150) for t into Eq. (147) and solves the result for x to obtain

$$\begin{aligned} \mathrm{d}x &= \frac{2Mfz^3\,dw}{3Srr(2ay-cc)} \\ &= \frac{2Mz^3\,dw\sqrt{aa-ab}}{3Srr(2ay-ab)} \\ &= \frac{2y\,\mathrm{d}y\sqrt{Maw}}{r\sqrt{S}\cdot(4ay-4yy-ab)}, \end{aligned} \tag{154}$$

from which he concludes

$$\frac{2Mz^3\,dw\sqrt{a-b}}{3Sr\cdot\sqrt{Mw}} = \frac{2y\,\mathrm{d}y(2ay-ab)}{\sqrt{S\cdot 4ay-4yy-4b}}. \tag{155}$$

He integrates this equation, assuming the denominator as constant, and obtains

$$\frac{2z^3\sqrt{Mw}\cdot\overline{a-b}}{3r\sqrt{S}} = \frac{\frac{2}{3}ay^3-\frac{1}{2}aby^2-\frac{2}{3}aCT^3+\frac{1}{2}ab\cdot CT^2}{\sqrt{aa-ab}}, \tag{156}$$

which is, when setting $CT=\frac{b}{2}$, equal to

$$\frac{2z^3\cdot\overline{a-b}}{3r}\cdot\sqrt{\frac{Mw}{Sa}} = \frac{2}{3}y^3-\frac{1}{2}by^2-\frac{1}{12}b^3+\frac{1}{8}b^3 = \frac{2}{3}y^3-\frac{1}{2}by^2+\frac{1}{24}b^3. \tag{157}$$

He derives from Eq. (153)

$$\sqrt{\frac{Mw}{Sa}} = \frac{r\,\mathrm{d}x}{y\,\mathrm{d}y}\sqrt{\frac{a-b}{a}}, \tag{158}$$

which may be substituted into Eq. (157) to obtain the differential equation for $\mathrm{d}x$,

$$\frac{2z^3\,\mathrm{d}x(a-b)^{\frac{3}{2}}}{3y\,\mathrm{d}y\sqrt{a}} = \frac{2}{3}y^3-\frac{1}{2}by^2+\frac{1}{24}b^3, \tag{159}$$

the integral of which is

$$\frac{2z^3x(a-b)^{\frac{3}{2}}}{3\sqrt{a}} = \frac{2}{15}y^5-\frac{1}{8}by^4+\frac{1}{48}b^3y^2-\frac{1}{640}b^5 \tag{160}$$

and therefore

$$x = \frac{3\sqrt{a}}{2z^3\cdot(a-b)^{\frac{3}{2}}}\left(\frac{2}{15}y^5-\frac{1}{8}by^4+\frac{1}{48}b^3y^2-\frac{1}{640}b^5\right). \tag{161}$$

Euler sets $y = \frac{1}{2}b + s$, where s denotes the exzess over $\frac{1}{2}b$, thus obtaining

$$\frac{2}{15}y^5 - \frac{1}{8}by^4 + \frac{1}{48}b^3y^2 - \frac{1}{640}b^5 = \frac{1}{12}bbs^3, \tag{162}$$

which may be substituted into Eq. (161) to obtain the rectangular distance of the dislocated position of p from the line CD:

$$x = \frac{bbs^3\sqrt{a}}{8z^3(a-b)^{\frac{3}{2}}}, \tag{163}$$

where s contains the accumulation of all "dy", taken either as positive or negative. If it is introduced by $Ll \cdot \mathrm{d}y$, it has always to be taken as affirmative due to Ll.

Euler provides an alternative solution based upon the distance $LQ = t$. He reformulates the right hand side of Eq. (155) using Eq. (150): from the first one he derives

$$2y = \frac{ac + f\sqrt{cc - 4tt}}{c}, \tag{164}$$

and from the second one

$$\sqrt{4ay - 4yy - cc} = \frac{2ft}{c}. \tag{165}$$

The first derivative of Eq. (164) gives

$$\mathrm{d}y = \frac{-2ft\,\mathrm{d}t}{c\sqrt{cc - 4tt}}. \tag{166}$$

Euler solves Eq. (165) for y to obtain

$$2ay - ab = \frac{ffc + af\sqrt{cc - 4tt}}{c}. \tag{167}$$

All these results are substituted into Eq. (155), which becomes

$$\frac{2Mz^3\,\mathrm{d}w\sqrt{a-b}}{3Sr\sqrt{Mw}} = \frac{-f\,\mathrm{d}t(fc + a\sqrt{cc - 4tt})(ac + f\sqrt{cc - 4tt})}{cc\sqrt{S} \cdot \overline{cc - 4tt}}. \tag{168}$$

He introduces p and sets

$$\sqrt{cc - 4tt} = p \quad \text{or} \quad 2t = \sqrt{cc - pp}, \tag{169}$$

from which he gains the first derivative

$$\mathrm{d}t = \frac{-p\,\mathrm{d}p}{2\sqrt{cc - pp}}. \tag{170}$$

He substitutes Eqs. (169) and (171) into Eq. (168), which becomes

$$\frac{2ccz^3\,\mathrm{d}w\sqrt{M}\cdot\overline{a-b}}{3fr\sqrt{S}w} = \frac{p\,\mathrm{d}p(fc+ap)(ac+fp)}{2p\sqrt{cc-pp}}$$
$$= \frac{accf\,\mathrm{d}p+aacp\,\mathrm{d}p+cffp\,\mathrm{d}p+afpp\,\mathrm{d}p}{2\sqrt{cc-pp}}. \tag{171}$$

The integration of the left hand side gives

$$\frac{4ccz^3\sqrt{Mw}\cdot\overline{a-b}}{3fr\sqrt{S}}. \tag{172}$$

The integral of the term $\frac{\overline{aa+ff}\cdot cp\,\mathrm{d}p}{2\sqrt{cc-pp}}$ is $= \frac{-c(aa+ff)}{2}\sqrt{cc-pp}$. Euler prepares the remaining term for integration by the reformulation

$$af\cdot\frac{cc\,\mathrm{d}p+pp\,\mathrm{d}p}{2\sqrt{cc-pp}} = af\cdot\frac{2cc\,\mathrm{d}p^2-cc\,\mathrm{d}p+pp\,\mathrm{d}p}{2\sqrt{cc-pp}}$$
$$= af\left(\frac{cc\,\mathrm{d}p}{\sqrt{cc-pp}}-\frac{\mathrm{d}p}{2}\sqrt{cc-pp}\right). \tag{173}$$

The integration is straightforward if the ellipse is approximated by a circle:

$$\int af\left(\frac{cc\,\mathrm{d}p}{\sqrt{cc-pp}}-\frac{\mathrm{d}p}{2}\sqrt{cc-pp}\right)=af\left(\frac{3}{2}DCM-\frac{1}{4}p\sqrt{cc-pp}\right), \tag{174}$$

where DCM denotes the sector composed of the straight lines DC and CM, which inclose the arc MD of the semi-circle ADB. Euler obtains thus the result

$$\frac{4ccz^3\sqrt{Mw\cdot\overline{a-b}}}{3fr\sqrt{S}} = \frac{3}{2}af\cdot DCM-\overline{\frac{1}{4}afp-\frac{1}{2}aac-\frac{1}{2}ffc}\sqrt{c^2-p^2}+\text{Const.}. \tag{175}$$

Euler tries to "construct" this result in such a way that the areas involved in Eq. (175) can easily be computed, which is equivalent with an appropriate series expansion of this result. He substitutes Eqs. (164)–(166) into Eq. (153) and obtains

$$r\,\mathrm{d}x\sqrt{\frac{S}{Maw}} = \frac{-\mathrm{d}t(ac+f\sqrt{cc-4tt}}{c\sqrt{cc-4tt}} \quad\text{or}\quad \sqrt{\frac{Mw}{S}} = \frac{-cr\,\mathrm{d}x\sqrt{cc-4tt}}{\mathrm{d}t(ac+f\sqrt{cc-4tt})\sqrt{a}}. \tag{176}$$

Using this result, the term (172) becomes

$$\frac{4ccz^3\sqrt{Mw}\cdot\overline{a-b}}{3fr\sqrt{S}} = \frac{-4c^3z^3\,\mathrm{d}x\cdot\sqrt{cc-4tt}\cdot\overline{a-b}}{3f\,\mathrm{d}t(ac+f\sqrt{cc-4tt})\sqrt{a}} \tag{177}$$

Setting $\sqrt{cc-4tt} = p$ and substituting Eq. (171), this term is ready for integration by series expansion:

$$\frac{8c^3z^3\,\mathrm{d}x\sqrt{cc-pp}}{3a\,\mathrm{d}p(ac+fp)} = \int \frac{accf\,\mathrm{d}p + aacp\,\mathrm{d}p + cffp\,\mathrm{d}p + afpp\,\mathrm{d}p}{2\sqrt{cc-pp}} \tag{178}$$

Euler develops the integrand into a series that he can integrate term by term:

$$\begin{aligned}
&\frac{1}{2}\int accf\,\mathrm{d}p + aacp\,\mathrm{d}p + cffp\,\mathrm{d}p + afpp\,\mathrm{d}p \cdot \\
&\left(\frac{1}{c} + \frac{pp}{2c^3} + \frac{1\cdot 3\cdot p^4}{2^2\cdot 1\cdot 2c^5} + \frac{1\cdot 3\cdot 5\cdot p^6}{2^3\cdot 1\cdot 2\cdot 3\cdot c^4}\ \&\mathrm{c}\right) \\
&\quad = acfp + \frac{afp^3}{3\cdot 2\cdot 1\cdot c} + \frac{1\cdot 3\cdot afp^5}{5\cdot 2^2\cdot 1\cdot 2\cdot c^3} + \frac{1\cdot 3\cdot 5\cdot afp^7}{7\cdot 2^3\cdot 1\cdot 2\cdot 3\cdot c^5}\ \&\mathrm{c} \\
&\qquad + \frac{afp^3}{3\cdot c} + \frac{afp^5}{5\cdot 2\cdot 1\cdot c^3} + \frac{af\cdot 1\cdot 3\cdot p^7}{7\cdot 2^2\cdot 1\cdot 2\cdot c^5}\ \&\mathrm{c}\,(+aac+ffc)\,\frac{pp}{2c} \\
&\qquad + \frac{cp^4}{4\cdot 2\cdot 1\cdot c^3} + \frac{1\cdot 3\cdot p^6}{6\cdot 2^2\cdot 1\cdot 2\cdot c^5} + \frac{1\cdot 3\cdot 5\cdot p^8}{8\cdot 2^3\cdot 1\cdot 2\cdot 3\cdot c^7}\ \&\mathrm{c} - \mathrm{Const.}
\end{aligned} \tag{179}$$

where the value of the integration constant has to be determined in such a way that the whole series vanishes if the Moon is located in the quadratures or if $y = \frac{1}{2}b$ or if $p = \frac{-cf}{a}$ is subtracted. The resulting series becomes thus

$$\begin{aligned}
&\frac{8c^3z^3\,\mathrm{d}x\sqrt{cc-pp}}{3a\,\mathrm{d}p(ac+fp)} = acfp \\
&\quad + \frac{3afp^3}{5\cdot 2^1\cdot 1\cdot c} + \frac{7\cdot afp^5}{5\cdot 2^2\cdot 1\cdot 1\cdot c^3} + \frac{33\cdot afp^7}{7\cdot 2^3\cdot 1\cdot 2\cdot 3\cdot c^5} + \&\mathrm{c} \\
&\quad + (aa+ff)\cdot\frac{pp}{2} + \frac{1\cdot p^4}{4\cdot 2\cdot 1\cdot c^2} + \frac{1\cdot 3\cdot p^6}{5\cdot 2^2\cdot 1\cdot 2\cdot c^4}\ \&\mathrm{c} \\
&\quad + ccff + \frac{3ccf^4}{3\cdot 2\cdot 1\cdot aa} + \frac{7ccf^6}{5\cdot 2^2\cdot 1\cdot 2\cdot a^4}\ \&\mathrm{c} \\
&\quad - (aa+ff)\frac{ccff}{2aa} + \frac{1\cdot ccf^4}{4\cdot 2^1\cdot 1\cdot a^4} + \frac{1\cdot 3\cdot ccf^6}{6\cdot 2^2\cdot 1\cdot 2\cdot a^6}\ \&\mathrm{c}
\end{aligned} \tag{180}$$

and therefore

$$\begin{aligned}
&\frac{8c^3z^3\,\mathrm{d}x\sqrt{cc-pp}}{3a\,\mathrm{d}p(ac+fp)} = acfp \\
&\quad + \frac{3afp^3}{3\cdot 2^1\cdot 1\cdot c} + \frac{7afp^5}{5\cdot 2^2\cdot 1\cdot 2\cdot c^3} + \frac{33afp^7}{7\cdot 2^3\cdot 1\cdot 2\cdot 3\cdot c^5}\ \&\mathrm{c}
\end{aligned}$$

$$+(aa+ff)\frac{pp}{2}+\frac{p^4}{4\cdot 2\cdot 1\cdot c^2}+\frac{1\cdot 3\cdot p^6}{5\cdot 2^2\cdot 1\cdot 2\cdot c^4}\ \&\text{c}$$
$$+\frac{ccff}{2}-\frac{1\cdot cc\cdot f^4}{4\cdot 2\cdot 1\cdot a^2}-\frac{ccf^6}{80\cdot a^4}-\frac{3ccf^5}{896\cdot a^6}\ \&\text{c} \tag{181}$$

Before Euler proceeds with the investigations of this equation he considers the case where the Moon is orbiting in a circle, i.e., if $f=0$ and/or $a=b=c$. In this case

$$\frac{8a^3z^3\,\mathrm{d}x\sqrt{aa-pp}}{3a^3\,\mathrm{d}p}=\frac{8z^3\,\mathrm{d}x\sqrt{aa-pp}}{3\,\mathrm{d}p}=-\frac{a^3}{2}\sqrt{aa-pp}+\text{const.} \tag{182}$$

If this constant is set to be $=\frac{a^4}{2}$, then

$$16z^3\,\mathrm{d}x=\frac{3a^4\,\mathrm{d}p}{\sqrt{aa-pp}}-3a^3\,\mathrm{d}p, \tag{183}$$

and consequently

$$16z^3x=3a^4\int\frac{\mathrm{d}p}{\sqrt{aa-pp}}-3a^3p. \tag{184}$$

Let a be the circle's radius and $\sin p=A$ in degree. Eqn. (184) becomes

$$16z^3x=\frac{71Aa^4}{1356}-3a^3p, \tag{185}$$

hence

$$x=\frac{71Aa^4}{16\cdot 1356\cdot z^3}-\frac{3a^3p}{16z^3}. \tag{186}$$

But if $z=572a$, then

$$x=\frac{71Aa}{16\cdot 1356\cdot (572)^3}-\frac{3p}{16\cdot (572)^3}=\frac{71Aa-4068p}{16\cdot 1356\cdot (572)^3}. \tag{187}$$

At this point the manuscript ends abruptly.

Appendix D: The content of Ms 276

This unfinished manuscript is well prepared for publication. Euler formulated the paragraphs carefully. The first three introductory paragraphs reveal that Euler became fully aware of the difficulty involved in developing an accurate lunar theory, which is only possible by approximation. The use of series expansions of trigonometric functions appears here probably for the first time and thus represents a further innovation. Due to the importance of Euler's statements, which reflect his insight and new approach, I will not only paraphrase but translate them in full length:

"That the motion of the Moon—however perturbed—does agree very well with the Newtonian hypothesis of attraction is proved more than sufficiently both by observation and by the conclusions drawn from this hypothesis. Even if this principle of attraction is burdened with such difficulties, that one has to keep it strictly away from a rational way of philosophy, its usefulness—particularly for astronomy—is anyway considerable when considered as phenomenon; and without its help no important things could have been achieved in the theory of celestial motions until now. Observations make clear that planets and comets move in the same way as if they are attracted by the Sun and by each other in the ratio given by Newton. From this, two completely different questions emerge, which should by no means be confused with each other: the first one concerns physics and demands a mechanical reason for this phenomenon consisting of the mutual attraction of the celestial bodies. The second question, however, concerns the determination of the motions caused by this force of attraction in order to complete the theory of astronomy itself, and to calculate and predict the particular phenomena most accurately.[5]

Concerning the second question, by which astronomy has achieved its biggest progress, it has been accomplished so far by its first discoverer Newton that hardly something remains to be added, in particular with regard to the primary planets, the theory of which is no longer fraught with difficulties, apart probably from the tables of the motion of Saturn which need some correction when it is staying near conjunction with Jupiter, because then one observes that its motion is perceptibly perturbed. The motion of the Moon, however, caused by the twofold force of the Sun's and the Earth's attraction, is so much difficult to determine, that nothing else than approximations could be done. By appraising the question very carefully Newton showed after all not only that his theory matches with all irregularities of the Moon, but he had completed the lunar theory itself prodigiously, even if the complexity of the driving forces does not allow an analytic approach to the calculation.[6]

[5] Motum lunæ quantumvis perturbatum appri me congruere cum Hypothesi attractionis Newtonniana, cum observationes tum conclusiones ex hac hypothesi deductæ satis superque testantur. Etsi autem istud attractionis principium tantis laborat difficultatibus, ut a sana philosophandi ratione longissime arceri debeat, tamen si instar phænomeni spectetur, ejus in Astronomia præcipue maximus est usus; neque sine ejus subsidio quicquam solidi in theoria motuum cœlestium adhuc est præstitum. Per observationes igitur certissimum est planetas et cometas perinde plane moveri, quasi cum a sole tum a se mutuo attraherentur, idque ea ratione quam Newtonus indicavit. Ex quo duæ prorsus diversæ quæstiones nascuntur minime inter se permiscendæ; quarum prima ad Physicam pertinet atque istius phænomeni in attractione mutua corporum cœlestium positi causam mechanicam postulat. Altera quæstio autem versatur in definitione exacta motuum, qui ab hac attractionis vi resultare debent, ut hinc ipsa theoria astronomiæ perficiatur, atque singula phænomena exactissime per calculum supputari ac prædici queant (cf. Ms 276, fol. 1r, § 1).

[6] Quod ad posteriorem quæstionem attinet, unde Astronomia maxima adepta est incrementa, ea a primo inventore Newtone simul eo usque perfecta esse videtur, ut vix quicquam addendum supersit: præcipue si ad planetas primarios respiciamus, quorum theoria nullis amplius difficultatibus involvitur, nisi forte tabulæ motuum Saturni emendatione indigent tum, cum circa conjunctionem Jovis versatur; quo tempore ejus motus notabiliter perturbari deprehenditur. Lunæ autem motus, qui a duplici vi attractiva solis et terræ oritur, tantopere fit determinatu difficilis, ut præter approximationes nihil præstare liceat. Newtonus quidem rem vero proxime æstimando non solum suam theoriam omnibus anomaliis lunæ convenire ostendit, sed etiam ipsam theoriam motuum lunæ mirifice perfecit. etsi complicatio virium sollicitantium non permittet, ut calculo analytico accessus concedatur (cf. Ms 276, fol. 1r-1v, § 2).

If we consider the question in its own rights, which demands the [determination of the] motion of a body like the Moon caused by a twofold driving force, the power of the analysis known until now seems to be insufficient to describe such a motion with confident rules; and it would not have been possible even now to accomplish anything going in that direction, if one would not have called the approximation for help. It is convenient that the force emerging from the Sun is less of a multiple than the force of the Earth driving the Moon, and this is why it becomes possible to treat initially the Moon's motion in such a way, as if it is driven only by the Earth, but then one had to investigate the deviations from this quite regular motion occurring from the Sun's force; this would not have been possible if the forces of Earth and Sun would not come close to the ratio of unity. Furthermore, the orbit of the Moon's motion makes the approximation easier and more accurate at the same time, because it does not much deviate from a circle. Using these resources I tried to find out how much could be achieved by means of calculations."[7]

In the following paragraphs 4 and 5 Euler treats first the two-body-problem of the Earth-Moon system, then (in the remaining) the three-body-problem of the Sun-Earth-Moon system. He assumes the lunar orbit to be coplanar with the ecliptic. Let A be the perigee, T the Earth, and $AT = a$. Let b be the velocity corresponding to the altitude (=hight of free fall) of the Moon at perigee directed along the straight line AT. The Earth is considered to be at rest, while the Sun is revolving around it contrary to its annual motion. Let the Moon be moving from A to any position L, thus describing the arc AL or the angle $ATL = x$ during the time interval T. Euler denotes the distance between the Earth and the Moon by $LT = y$. Let LN be the tangent through L and $TP = p$ the perpendicular from T to LN intersecting the tangent in P. Let v be the velocity corresponding to the altitude of the Moon in L, and let $lp = \mathrm{d}y$ be an infinitesimally small radial element of the distance between Earth and Moon. Due to

$$LP = \sqrt{(yy - pp)}, \tag{188}$$

and due to the equivalence of the triangles TPL and Lpl (do not confuse the point p with the distance $p = TP$), the distances Lp and Ll covered in the time element $\mathrm{d}t$ become

$$Lp = \frac{p\,\mathrm{d}y}{\sqrt{(yy - pp)}} \quad \text{and} \quad Ll = \frac{y\,\mathrm{d}y}{\sqrt{(yy - pp)}}. \tag{189}$$

[7] Quodsi enim quæstionem in se perpendamus, qu[a] motus corporis requiritur, qui a duplici vi sollicitante veluti luna producitur, vis analyseos adhuc cognita minime sufficere videtur ad ejusmodi motum certis regulis circumscribendum. neque quicquam in hac parte etiamnunc præstare licuisset, nisi approximatio in subsidium vocari potuisset. Commode scilicet usu venit ut vis a sole oriunda multoties minor sit vi terræ lunam sollicitante, ex quo motum lunæ primum ita tractare licuit, quasi a sola vi terræ oriretur, tum vero aberrationes ab hoc motu, satis regulari a vi solis oriundas investigari oportebat; quod fieri non potuisset, si vires terræ et solis magis ad æqualitatis rationem accederent. Præterea vero ipsa motus lunæ orbita ideo quod a circulo non multum recedit approximationem tum faciliorem tum accuratiorem reddit. His igitur subsidiis, quantum calculi ope effici queat, periculum faciam (cf. Ms 276, fol. 1r, § 3).

This time element $\mathrm{d}T$ itself, and the corresponding angular element $\mathrm{d}x$, are given by

$$\mathrm{d}T = \frac{y\,\mathrm{d}y}{\sqrt{v(yy - pp)}} \quad \text{and} \quad \mathrm{d}x = \frac{p\,\mathrm{d}y}{y\sqrt{(yy - pp)}}. \tag{190}$$

Euler sets both the accelerative gravitational force on the Earth's surface and the Earth's radius $= 1$. The force acting on the Moon by the Earth along the direction LT then is $= \frac{1}{yy}$, which he decomposes into the tangential and normal components. Thus the former becomes $= \frac{\sqrt{(yy-pp)}}{y^3}$, and the latter $= \frac{p}{y^3}$. To determine the Moon's motion driven only by the Earth's gravitational force, Euler applies the equation of motion as given in his "*Mechanica*" (cf. Euler 1736, Def. 15, Corol. 5, § 207: $\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{p}{A}$, and § 209: $r = \frac{2Av}{p}$, where $A = 1$), considering that the tangential component is a retarding force:

$$\frac{\mathrm{d}v}{Ll} = \frac{\mathrm{d}v\sqrt{(yy - pp)}}{y\,\mathrm{d}y} = -\frac{\sqrt{(yy - pp)}}{y^3} \tag{191}$$

and

$$\frac{2v}{\text{rad. Osc.}} = \frac{2v\,\mathrm{d}p}{y\,\mathrm{d}y} = \frac{p}{y^3}. \tag{192}$$

But Eq. (191) simply is $\mathrm{d}v = \frac{\mathrm{d}y}{yy}$, which can easily be integrated:

$$v = b - \frac{1}{a} + \frac{1}{y}. \tag{193}$$

This result, inserted into Eq. (192), gives

$$2\left(b - \frac{1}{a} + \frac{1}{y}\right)\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}y}{yy} \quad \text{or} \quad \frac{2\,\mathrm{d}p}{p} = \frac{\mathrm{d}y}{y(1 + by - \frac{y}{a})}, \tag{194}$$

which can be integrated as well:

$$2\,\ell p = \ell\frac{y}{1 + by - \frac{y}{a}} + C, \tag{195}$$

where ℓ denotes the logarithm to the base 10 and C the constant of integration. Euler determines this constant by setting $y = a$ and $p = a$, so that C becomes

$$C = 2\,\ell a - \ell\frac{a}{ab} = 2\,\ell a + \ell b. \tag{196}$$

This constant, inserted into Eq. (195), gives

$$2\,\ell p = \ell\frac{aaby}{1 + by - \frac{y}{a}}, \tag{197}$$

and therefore

$$pp = \frac{aaby}{1 + by - \frac{y}{a}}. \tag{198}$$

This equation contains the nature of the Moon's orbit. Denoting the latus rectum tentatively with $CT = c$, Euler derives the relation

$$b = \frac{1}{a} - \frac{1}{c} \tag{199}$$

and substitutes it into Eq. (198), obtaining

$$pp = \frac{a(c-a)y}{c-y}, \tag{200}$$

which represents the equation for an ellipse with major axis c. Because the Moon's orbit does not much differ from a circle, one may approximately use the value $c = 2a$, otherwise $a = \frac{1}{2}c - \frac{c}{2n}$, where n is a big number. Likewise, if y does not much deviates from $\frac{1}{2}c$, Euler finds

$$v = \frac{1}{c} + \frac{\omega}{c}, \tag{201}$$

where ω is a very small value, and thus

$$pp = \frac{(nn-1)cc}{4nn(1+\omega)}. \tag{202}$$

With these results the two-body-problem is solved, and Euler passes on to the three-body-problem considering also the Sun's force.

Let the ratio between the forces of the Sun and the Earth be as m to 1 (at equal distances). Let the distance between the Sun S and the Earth T be $ST = h$, and the angle $ATS = k$. This angular argument has to be considered as variable. The Earth is attracted towards the Sun along the direction TS by the force $= \frac{m}{hh}$, which has to be transferred with inverse direction to the Moon and parallel to the direction TS, if the Earth has to be considered at rest. This statement seemed to be quite self-evident for Euler at that time when writing this manuscript:

> Attrahetur ergo terra ad solem in directione TS vi $= \frac{m}{hh}$ quæ quia terram quiescentum spectamus transferri debet in lunam contrarie, ita ut luna sollicitata considerari debeat in directione LM parallela directioni TS vi $= \frac{m}{hh}$.

It proves that the principle of the transference of forces had been fully established by him at that time. The procedure following that statement underlines this conjecture. The force acting on the Moon by the Sun along the direction LS is given by $\frac{m}{LS^2}$. Let the geocentric angular distance of the Moon from the Sun be the argument $LTS = k - x$.

Euler derives the distance between the Sun and the Moon from the geometry of the system:

$$LS^2 = hh + yy - 2hy\cos \mathrm{A}(k - x), \tag{203}$$

where A has no mathematical meaning, but has the task to indicate that $k - x$ is the Argument or Arc ("Arcus"). Euler decomposes the force $\frac{m}{LS^2}$ into components acting along the directions LT and LV, and denotes it with the same symbols: $LT = \frac{my}{LS^3}$ and $LV = \frac{mh}{LS^3}$. The direction LV is identical with LM mentioned in the quotation above. In the following calculations he uses only LM instead of LV for the direction parallel to TS. To consider the Earth at rest means, that the force component acting on the Moon along the direction LM becomes $\frac{m}{hh} - \frac{mh}{LS^3}$. He expands $\frac{1}{LS^3}$ into a trigonometric series, probably for the first time in the context of lunar theory:

$$\begin{aligned}\frac{1}{LS^3} &= (hh - 2hy\cos \mathrm{A}(k - x) + yy)^{-\frac{3}{2}}\\ &= \frac{1}{h^3} + \frac{3y\cos \mathrm{A}(k - x)}{h^4} - \frac{3yy}{2h^5} + \frac{15yy(\cos \mathrm{A}(k - x))^2}{2k^5} \text{ etc.}\\ &= \frac{1}{h^3} + \frac{3y\cos \mathrm{A}(k - x)}{h^4} + \frac{9yy}{4h^5} + \frac{15yy\cos \mathrm{A}2(k - x)}{4h^5}.\end{aligned} \tag{204}$$

We see Euler here introducing approximations by series expansions into celestial mechanics. Using this result, the forces acting on the Moon along the directions LT and LM become, respectively,

$$= \frac{my}{h^3} + \frac{3myy\cos \mathrm{A}(k - x)}{h^4} + \frac{9my^3}{4h^5} + \frac{15my^3\cos \mathrm{A}2(k - y)}{4h^5} + \text{etc.}$$

and

$$= -\frac{3my\cos \mathrm{A}(k - x)}{h^3} - \frac{9myy}{4h^4} - \frac{15myy\cos \mathrm{A}2(k - x)}{4h^4}\ [+] \text{ etc.} \tag{205}$$

Due to the parallelism of the lines TS and LM, the angle $k - x$ is also defined by $MTL = k - x$. Furthermore, the sine and cosine of the angle TLP is given by

$$\begin{aligned} c\sin TLP &= \frac{p}{y}\\ \cos TLP &= \frac{\sqrt{(yy - pp)}}{y},\end{aligned} \tag{206}$$

respectively, and the sine and cosine of the angle $MLN = MLT - PLT$ is given by

$$\begin{aligned} c\sin MLN &= \frac{\sqrt{(yy - pp)}}{y}\sin \mathrm{A}(k - x) - \frac{p}{y}\cos \mathrm{A}(k - x)\\ \cos MLN &= \frac{p}{y}\sin \mathrm{A}(k - x) + \frac{\sqrt{(yy - pp)}}{y}\cos \mathrm{A}(k - x),\end{aligned} \tag{207}$$

respectively. From the force acting along LM results the normal force

$$\frac{3m\sqrt{(yy-pp)}}{2h^3}\sin \mathrm{A2}(k-x)-\frac{3mp}{2h^3}-\frac{3mp}{2h^3}\cos \mathrm{A2}(k-x), \tag{208}$$

neglecting small terms, and the accelerating tangential force becomes

$$\frac{3mp}{2h^3}\sin \mathrm{A2}(k-x)+\frac{3m\sqrt{(yy-pp)}}{2h^3}+\frac{3m\sqrt{(yy-pp)}}{2h^3}\cos \mathrm{A2}(k-x). \tag{209}$$

And from the force acting along LT results the normal force

$$\frac{mp}{h^3} \tag{210}$$

and the accelerating tangential force

$$-\frac{m\sqrt{(yy-pp)}}{h^3}. \tag{211}$$

But the force acting on the Moon by the Earth produces the normal component

$$\frac{p}{y^3} \tag{212}$$

and the accelerating tangential component

$$-\frac{\sqrt{(yy-pp)}}{y^3}. \tag{213}$$

The resulting tangential force accelerating the Moon thus is given by the combination of Eqs. (191), (209), (211), and (213):

$$\begin{aligned}\frac{\mathrm{d}v\sqrt{(yy-pp)}}{y\,\mathrm{d}y} &= -\frac{\sqrt{(yy-pp)}}{y^3}+\frac{m\sqrt{(yy-pp)}}{2h^3}\\ &\quad +\frac{3mp}{2h^3}\sin \mathrm{A2}(k-x)+\frac{3m\sqrt{(yy-pp)}}{2h^3}\cos \mathrm{A2}(k-x),\end{aligned} \tag{214}$$

from which Euler concludes

$$\mathrm{d}v=-\frac{\mathrm{d}y}{yy}+\frac{my\,\mathrm{d}y}{2h^3}+\frac{3mpy\,\mathrm{d}y\sin \mathrm{A2}(k-x)}{2h^3\sqrt{(yy-pp)}}+\frac{3my\,\mathrm{d}y\cos \mathrm{A2}(k-x)}{2h^3}. \tag{215}$$

The resulting normal force acting on the Moon is given by the combination of Eqs. (192), (208), (210), and (212):

$$\frac{2v\,\mathrm{d}p}{y\,\mathrm{d}y}=\frac{p}{y^3}-\frac{mp}{2h^3}$$

$$+\frac{3m\sqrt{(yy-pp)}}{2h^3}\sin \mathrm{A2}(k-x)-\frac{3mp}{2h^3}\cos \mathrm{A2}(k-x). \tag{216}$$

Euler substitutes Eq. (190) into Eq. (215) and obtains

$$\mathrm{d}v=-\frac{\mathrm{d}y}{yy}+\frac{my\,\mathrm{d}y}{2h^3}+\frac{3myy\,\mathrm{d}x\sin A2(k-x)}{2h^3}+\frac{3my\,\mathrm{d}y\cos A2(k-x)}{2h^3}, \tag{217}$$

which he can integrate immediately:

$$v=C+\frac{1}{y}+\frac{myy}{4h^3}+\frac{3myy\cos \mathrm{A2}(k-x)}{4h^3}$$

or

$$v=\frac{1}{y}-\frac{1}{c}+\frac{myy}{4h^3}+\frac{3myy\cos \mathrm{A2}(k-x)}{4h^3}. \tag{218}$$

"This equation defines the absolute velocity of the Moon at any position in its orbit, which can be determined from the Moon's distance from the Earth, from the angular separation between Sun and Moon given by $k-x$, and from the Earth's distance from the Sun, whose variation does not affect the velocity considerably."

On the remaining folios of this manuscript Euler tries to prepare the equation of motion (216) due to the normal force for the integration. For that purpose he has first to figure out the items p, v, $2v\,\mathrm{d}p$, etc. If the Moon's motion is driven only by the Earth, then Eq. (200) holds, as shown above, representing the unperturbed Keplerian motion. Using the abbreviation $a(c-a)=bb$, this equation becomes

$$p=b\sqrt{\frac{y}{c-y}}. \tag{219}$$

But due to the additional action of the Sun, this parameter needs to be corrected. The results obtained above gives Euler the idea of a general ansatz:

$$p=b\sqrt{\frac{y}{c-y}}+\frac{mP}{h^3}+\frac{mQ\cos \mathrm{A2}(k-x)}{h^3}+\frac{mR\sin \mathrm{A2}(k-x)}{h^3}, \tag{220}$$

where P, Q, and R are coefficients to be determined. We observe here Euler's very early use of the method of undetermined coefficients for solving differential equations in lunar theory. The first derivative of Eq. (220) is given by

$$\begin{aligned}\mathrm{d}p=&\frac{bc\,\mathrm{d}y}{2(c-y)\sqrt{(cy-yy)}}+\frac{m\,\mathrm{d}P}{h^3}\\&+\frac{m\,\mathrm{d}Q\cos \mathrm{A2}(k-x)}{h^3}+\frac{2mQp\,\mathrm{d}y}{h^3y\sqrt{(yy-pp)}}\sin \mathrm{A2}(k-x)\\&+\frac{m\,dR\sin \mathrm{A2}(k-x)}{h^3}-\frac{2mRp\,\mathrm{d}y\cos \mathrm{A2}(k-x)}{h^3y\sqrt{(yy-pp)}}.\end{aligned} \tag{221}$$

Using Eq. (219), the distance PL becomes

$$\sqrt{(yy-pp)} = \frac{\sqrt{(cyy-y^3-bby)}}{\sqrt{(c-y)}}, \tag{222}$$

and therefore, approximately,

$$\frac{p}{\sqrt{(yy-pp)}} \approx \frac{b}{\sqrt{cy-yy-bb}}. \tag{223}$$

Euler substitutes Eqs. (219), (220), and (223) into the equation of motion (216) to obtain

$$\begin{aligned} 2v\,\mathrm{d}p = {} & \frac{b\,\mathrm{d}y\sqrt{y}}{yy\sqrt{(c-y)}} + \frac{mP\,\mathrm{d}y}{h^3yy} \\ & + \frac{mQ\,\mathrm{d}y\cos \mathrm{A2}(k-x)}{h^3yy} + \frac{mR\,\mathrm{d}y\sin \mathrm{A2}(k-x)}{h^3yy} \\ & - \frac{mb}{2h^3}y\,\mathrm{d}y\sqrt{\frac{y}{c-y}} + \frac{3my\,\mathrm{d}y\sqrt{(cyy-y^3-bby)}}{2h^3\sqrt{(c-y)}}\sin \mathrm{A2}(k-x) \\ & - \frac{3mby\,\mathrm{d}y\sqrt{y}}{2h^3\sqrt{(c-y)}}\cos \mathrm{A2}(k-x) - \frac{2mRp\,\mathrm{d}y\cos \mathrm{A2}(k-x)}{h^3y\sqrt{(yy-pp)}}. \end{aligned} \tag{224}$$

On the other hand, the product $2v\,\mathrm{d}p$ is also given by Eqs. (218) and (221), resulting in

$$\begin{aligned} 2v\,\mathrm{d}p = {} & \frac{b\,\mathrm{d}y}{y\sqrt{(cy-yy)}} + \frac{2m(c-y)\,\mathrm{d}P}{h^3cy} + \frac{2m(c-y)\,\mathrm{d}Q\cos \mathrm{A2}(k-x)}{h^3cy} \\ & + \frac{4mb(c-y)Q\,\mathrm{d}y\sin \mathrm{A2}(k-x)}{h^3cyy\sqrt{(cy-yy-bb)}} + \frac{mbcy\,\mathrm{d}y\sqrt{y}}{4h^3(c-y)^{\frac{3}{2}}} \\ & - \frac{4mb(c-y)R\,\mathrm{d}y\cos \mathrm{A2}(k-x)}{h^3cyy\sqrt{(cy-yy-bb)}} + \frac{2m(c-y)\,\mathrm{d}R\sin \mathrm{A2}(k-x)}{h^3cy} \\ & + \frac{3mbcyy\,\mathrm{d}y\cos \mathrm{A2}(k-x)}{4h^3(c-y)^{\frac{3}{2}}\sqrt{y}}. \end{aligned} \tag{225}$$

At this point (§ 10), the fragment of the manuscript ends. It is quite possible that Euler determined the coefficients P, Q, and R on the lost folios. It is therefore not clear whether he already applied the method of undetermined coefficients in full length and whether he was already able to solve the differential equation approximately. Evidence for the former assumption is given by the way of his approach presented above.

References

Bernoulli, Johann. 1712. Extrait de la Réponse de M. Bernoulli à M. Hermann, datée de Basle le 7. Octobre 1710. *Mémoires de l'Academie Royale des Sciences (Paris)* 12(1710): 521–533.

Bertoloni Meli, Domenico. 1993. The emergence of reference frames and the transformation of mechanics in the Enlightenment. *HSPS* 23(2): 301–335.

Bomie, ?. 1708. Des forces centripetes et centrifuges, Considerées en général dans toutes sortes de Courbes, & en particulier dans le Cercle. *Mémoires de l'Academie Royale des Sciences (Paris)* 9 (1707), 477–487.

Boulliau, Ismael. 1645. *Astronomia philolaica*. Parisiis: Simeonis Piget M.DC.XLV.

Bredekamp, Horst, and Wladimir Velminski. 2010. *Mathesis & Graphé. Leonhard Euler und die Entfaltung der Wissenssysteme*. Berlin: Akademie Verlag.

Burkhardt, Heinrich. 1908. Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik. In *Mathematiker-Vereinigung*, vol. 10, ed. Jahresbericht der Deutschen. Leipzig: Teubner.

Burckhardt, Johann Jakob, Emil A. Fellmann, and Walter Habicht (eds.). 1993. *Leonhard Euler 1707–1783. Beiträge zu Leben und Werk. Gedenkband des Kantons Basel-Stadt*. Basel: Birkhäuser Verlag.

Cantor, Moritz. 1901. *Vorlesungen über Geschichte der Mathematik.* Dritter Band, von 1668–1758, Zweite ed. Leipzig: Teubner.

Eneström, Gustaf. 1906. Der Briefwechsel zwischen Leonhard Euler und Daniel Bernoulli. *Bibliotheca Mathematica* 7: 126–156.

Eneström, Gustaf. 1910. *Verzeichnis der Schriften Leonhard Eulers. Jahresbericht der Deutschen Mathematiker-Vereinigung*, Ergänzungsband IV. Leipzig: B. G. Teubner.

Euler, Leonhard. 1736. *Mechanica sive motus scientia analytice exposita*. Tomus I. Petropoli: Ex Typographia Academiae Scientiarum.

Euler, Leonhard. 1750. De novo genere oscillationum. *Comment. acad. sc. Petrop*. 11(1739): 128–149.

Euler, Leonhard. 1750. Methodus facilis computandi angulorum sinus ac tangentes tam naturales quam artificiales. *Comment. acad. sc. Petrop* 11(1739): 194–230.

Euler, Leonhard. 1750. De seriebus quibusdam considerationes. *Comment. acad. sc. Petrop*. 12(1740): 53–96.

Euler, Leonhard. 1750. Emendatio tabularum astronomicarum per loca planetarum geocentrica. *Commend. acad. sc. Petrop*. 12(1740): 109–221.

Fellmann, Emil A. 1983. Leonhard Euler–Ein Essay über Leben und Werk. In: [Burckhardt et al. 1983], 13–98.

Fellmann, Emil A., and Gleb K. Mikhailov. 1998. *Leonhardi Euleri Commercium Epistolicum. Commercium cum Johanne (I) Bernoulli et Nicolao (I) Bernoulli*. Leonhardi Euleri Opera Omnia, Series quarta A, Volumen secundum. Basileae: Birkhäuser.

Fellmann, Emil A. 2007. *Leonhard Euler*. Translated by Erika Gautschi and Walter Gautschi. Basel/Boston/Berlin: Birkhäuser.

Fraser, Craig G. 1994. The origins of Euler's variational calculus. *Archive for History of Exact Sciences* 47(2): 103–141.

Goldstine, Herman H. 1980. *A History of the Calculus of Variations from the 17th through the 19th Century*. (Studies in the History of Mathematikcs and Physical Sciences, Vol. 5). New York/Heidelberg/Berlin: Springer-Verlag.

Gregory, David. 1726. *Astronomiæ physicæ & geometricæ elementa*. Secunda Editio revisa & correcta. Genevæ: Bousquet M.DCC.XXVI.

Guicciardini, Niccolò. 1999. *Reading the Principia. The Debate on Newton's Mathematical Methods for Natural Philosophy from 1687 to 1736*. Cambridge: Cambridge University Press.

Hermann, Jacob. 1716. *Phoronomia, sive de Viribus et Motibus corporum solidorum et fluidorum libri duo*. Amstelaedami: Apud Rod. & Gerh. Wetstenios M.D.CCXVI.

Jammer, M. 1957. *Concepts of Force*. A study in the foundations of dynamics. Cambridge: Harvard University Press.

Katz, Victor J. 1987. The Calculus of Trigonometric Functions. *Historia Mathematica* 14: 311–324.

Keill, John. 1708. De legibus virium centripetarum. *Philosophical Transactions of the Royal Society of London* 26: 174–188.

Kopelevič J.C., M.V. Krutikova, G.K. Mikhailov, N.M. Raskin. 1962. *Rukopisnye materialy L. Ejlera v Archive Akademii Nauk SSSR/Manuscripta Euleriana Archivi Academiae Scientiarum URSS*, Tomus I, Descriptio scientifica. (Acta archivi academiae scientiarum URSS, Fasc. 17). Moskva/Leningrad: Akademija Nauk.

Kutschmann, Werner. 1983. *Die Newtonsche Kraft—Metamorphose eines wissenschaftlichen Begriffs*. Studia Leibnitiana, Sonderheft 12. Wiesbaden: Franz Steiner Verlag.

Machin, John. 1729. The Laws of the Moon's Motion. According to Gravity. In [Newton 1729], Vol. 2, 289–292.

Maltese, Giulio. 2000. On the Relativity of Motion in Leonhard Euler's Science. *Archive for History of Exact Sciences* 54: 319–348.

Mattmüller, Martin. 2010. Eine neue Art Mathematik zu schreiben. In [Bredekamp et al. 2010], 177–188.

Mikhailov, Gleb K. 1965. *Rukopisnye materialy L. Ejlera v Archive Akademii Nauk SSSR/Manuscripta Euleriana Archivi Academiae Scientiarum URSS*, Tomus II, Opera mechanica. (Acta archivi academiae scientiarum URSS, Fasc. 120). Moskva/Leningrad: Akademija Nauk.

Moivre, Abraham de. 1717. Proprietates quædam simplices Sectionum Conicarum ex natura Focorum deductæ; cum Theoremate generali de Viribus Centripetis; quorum ope Lex Virium Centripetarum ad Foco Sectionum tendentium, Velocitates Corporum in illis revolventium, & Descriptio Orbium facillime determinantur. *Philosophical Transactions of the Royal Society of London*: 30, 622–628.

Newton, Isaac. 1687. *Philosophiae naturalis principia mathematica*. Londini: Jussu Societatis Regiae ac Typis Josephi Streater MDCLXXXVII.

Newton, Isaac. 1729. *The Mathematical Principles of Natural Philosophy*. Translated into English by Andrew Motte. To which are added, The Laws of the Moon's Motion according to Gravity. By John Machin. In Two Volumes. London, Printed for Benjamin Motte MDCCXXIX.

Taton, René, and Curtis Wilson (eds.). 1989. *Planetary astronomy from the Renaissance to the rise of astrophysics. Part A: Tycho Brahe to Newton. (The General History of Astronomy, Vol. 2)*. Cambridge: Cambridge University Press.

Varignon, Pierre. 1703a. Maniere generale de determiner les Forces, les Vîtesses, les Espaces, & les Tems, une seule de ces quatre choses étant donnée dans toutes sortes de mouvemens rectilignes variés à discrétion. *Mémoires de l'Academie Royale des Sciences (Paris)* 2(1700): 22–27.

Varignon, Pierre. 1703b. Du mouvement en général par toutes sortes de courbes; et des forces centrales, tant centrifuges, que centripetes, nécessaires aux corps qui les décrivent. *Mémoires de l'Academie Royale des Sciences (Paris)*: 2(1700): 83–101.

Varignon, Pierre. 1703c. Des forces centrales ou des pesanteurs nécessaires aux Planetes pour leur faire décrire les orbes qu'on leur a supposés jusqu'ici. *Mémoires de l'Academie Royale des Sciences (Paris)*: 2(1700): 218–238.

Varignon, Pierre. 1704. Autre Regle Generale des Forces Centrales. Avec une maniere d'en déduire & d'en trouver une infinité d'autres à la fois, dépendemment & indépendemment des Rayons osculateurs, qu'on va trouver aussi d'une maniere infiniment generale. *Mémoires de l'Academie Royale des Sciences (Paris)* 3(1701): 20–38.

Varignon, Pierre. 1705. Des Courbes Decrites Par le concours de tant de Forces centrales qu'on voudra, placées à discretion entr'elles, & par rapport aux plans de ces mémes Courbes. *Mémoires de l'Academie Royale des Sciences (Paris)* 5(1703): 212–229.

Varignon, Pierre. 1707a. Comparaison Des Forces centrales aves les Pesanteurs absoluës des corps mûs de vîtesses variées à discrétion le long de telles Courbes qu'on voudra. *Mémoires de l'Academie Royale des Sciences (Paris)* 8(1706): 178–235.

Varignon, Pierre. 1707b. Differentes manieres infiniment generales De trouver les Rayons osculateurs de toutes sortes de Courbes, soit qu'on regarde ces Courbes sous la forme de Polygones, ou non. *Mémoires de l'Academie Royale des Sciences (Paris)* 8(1706): 490–507.

Varignon, Pierre. 1712. Des Forces Centrales Inverses. *Mémoires de l'Academie Royale des Sciences (Paris)* 12(1710): 533–544.

Verdun, Andreas 2010. *Entwicklung, Anwendung und Standardisierung mathematischer Methoden und physikalischer Prinzipien in Leonhard Eulers Arbeiten zur Himmelsmechanik. [Development, Application, and Standardization of Mathematical Methods and Physical Principles in Leonhard Euler's Works on Celestial Mechanics.]* Habilitationsschrift (written in German), Bern: University of Bern (Switzerland).

Verdun, Andreas. 2011. Die (Wieder-) Entdeckung von Eulers Mondtafeln. [The (Re-) Discovery of Euler's Lunar Tables.] *N.T.M.* 19: 271–297. doi:10.1007/s00048-011-0053-6.

Verdun, Andreas. 2012. Leonhard Euler's early lunar theories 1725–1752. Part 1: first approaches, 1725–1730. *Archive for History of Exact Sciences*, published online 02 December 2012, doi:10.1007/s00407-012-0112-y.

Westfall, Richard S. 1971. *Force in Newton's Physics. The Science of Dynamics in the Seventeenth Century*. London/New York: MacDonald/American Elsevier.

Wilson, Curtis. 1970. From Kepler's Laws, So-called, to Universal Gravitation: Empirical Factors. *Archive for History of Exact Sciences* 6: [89]–170.

Wilson, Curtis. 1989. *Predictive astronomy in the century after Kepler*. In: [Taton et al. 1989], Chap. 10, [161]–206.

Wolfers, Joseph Philipp. 1848. *Leonhard Euler's Mechanik oder analytische Darstellung der Wissenschaft von der Bewegung, mit Anmerkungen und Erläuterungen*. Erster Theil. Greifswald: C. A. Koch's Verlagshandlung.