About some fixed point axioms and related principles in Kripke-Platek environments

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Abstract

Starting point of this article are fixed point axioms for set-bounded monotone Σ_1 definable operators in the context of Kripke-Platek set theory KP. We analyze their relationship to other principles such as maximal iterations, bounded proper injections, and Σ_1 subset-bounded sparation. One of our main results states that in $\mathsf{KP} + (V{=}L)$ all these principles are equivalent to Σ_1 separation.

Keywords: Kripke-Platek set theory, set-bounded Σ_1 operators, fixed point axioms, bounded proper Σ_1 injections, Σ_1 separation and Σ_1 subset-bounded separation.

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1 Introduction

The famous Knaster-Tarski theorem states the following: If $(L, \prec)$ is a complete lattice and if f is an order-preserving function from $(L, \prec)$ to $(L, \prec)$, then the set of fixed points of f is also a complete lattice; see Tarski [18]. Since complete lattices are not empty, this implies, in particular, that f has a least and a greatest fixed point. Actually, as observed in Fitting [5], we do not need a complete lattice for the Knaster-Tarski theorem; it is sufficient that the lattice is chain-complete in order to carry through the usual proof.

Simple though very important special cases of complete lattices are structures $(\wp(a), \subseteq)$, where $\wp(a)$ is the power set of a. Here the least fixed point of a monotone operator Γ from $\wp(a)$ to $\wp(a)$ can be defined as the intersection of all Γ-closed subsets of a and as the union of all stages I^α_Γ, with α ranging over the ordinals and $I^\alpha_\Gamma := \Gamma(\bigcup_{\xi<\alpha} I^\xi_\Gamma)$.

What is common to nearly all standard approaches to fixed point assertions of this kind is that they are discussed in fairly strong set-theoretic environments with power set axiom and strong separation principles like Zermelo-Fraenkel set theory. The situation becomes more delicate if the power set axiom is not available and separation is restricted.

Starting point of this article are fixed point assertions for set-bounded monotone Σ_1 definable operators in the context of Kripke-Platek set theory

KP. Given a set a and a Σ_1 definable operator Γ that maps any set x to a subset $\Gamma(x)$ of a and that is monotone in the sense of

$$x \subseteq y \quad \Longrightarrow \quad \Gamma(x) \subseteq \Gamma(y)$$

for all sets x and y, then KP does <u>not</u> prove in general that Γ has a fixed point, let alone a least fixed point.

We study the effect of adding fixed point axioms for set-bounded Σ_1 definable operators and iteration principles for (possibly non-monotone) such operators to KP. In addition, we introduce interesting principles that resemble a sort of cardinality considerations as well as a new subform of Σ_1 separation – we call it Σ_1 subset-bounded sparation – and analyze their mutual relations. One of the main results of this article is that in $\mathsf{KP} + (V{=}L)$ all these principles are equivalent to Σ_1 separation.

Fixed points of monotone operators, the general theory of inductive definitios as well as variations of these topics play an important role in mathematical logic; see, e.g., Barwise [1], Moschovakis [11, 12], Welch [19], and Curi [3]. The monograph Buchholz, Feferman, Pohlers, and Sieg [2] illustrates the importance of theories of inductive definitions for proof theory, and Rathjen [13, 14, 15] analyzes fixed point principles in second order arithmetic and explicit mathematics.

2 The general framework

All formal systems considered in this paper are based on Kripke-Platek set theory KP with infinity which is formulated in the standard language of set theory $\mathcal{L}$ containing $\in$ as the only non-logical symbol besides $=$ and countably many set variables $a, b, c, , \dots$ (possibly with subscripts). The formulas and the syntactic categories of Δ_0, Σ, Π, Σ_n, and Π_n formulas of $\mathcal{L}$ are defined as usual. We shall denote formulas by uppercase Latin letters from the beginning of the alphabet (possibly with subscripts).

The theory KP is formulated in classical first order logic with equality and comprises the following non-logical asioms: (i) extensionality, pairing, union, infinity, (ii) the schemas of Δ_0 separation and Δ_0 collection, i.e.

(Δ_0-Sep) $$\exists x \forall y (y \in x \;\leftrightarrow\; y \in a \,\land\, A[y]),$$

(Δ_0-Col) $$(\forall x \in a) \exists y B[x,y] \;\rightarrow\; \exists z (\forall x \in a)(\exists y \in z) B[x,y],$$

for arbitrary Δ_0 formulas $A[u]$ and $B[u,v]$ of $\mathcal{L}$, as well as (iii) the schema of induction on $\in$,

$$\forall x ((\forall y \in x) C[y] \;\rightarrow\; C[x]) \;\rightarrow\; \forall x C[x]$$

for arbitrary formulas $C[u]$ of $\mathcal{L}$.

From now on we assume that the reader has some familiarity with KP and refer to Barwise [1] for all details. In particular, in order to increase readability, we will freely use standard set-theoretic terminology and make use of Barwise's machinery of Δ_0 predicates and Σ function symbols. For example, $\{a, b\}$ stands for the unordered pair, $\langle a, b\rangle$ for the ordered pair of the sets a, b and 1^{st} and 2^{nd} for the Σ function symbols such that $a = \langle 1^{st}(a), 2^{nd}(a)\rangle$ iff a is an ordered pair; similarly, $a = \langle 1^{st}(a), 2^{nd}(a), 3^{rd}(a)\rangle$ iff a is an ordered triple.

In addition, $Ord[a]$ is the Δ_0 formula expressing that a is an ordinal, and we use lower case Greek letters (possibly with subscripts) to range over the ordinals. Also, if $A[u]$ is an $\mathcal{L}$ formula, then $\{x \in a : A[x]\}$ denotes the collection of all elements of a satisfying A; it may be a set, but this is not necessarily the case.

2.1 Why we study fixed points of set-bounded Σ_1 operators

Before turning to the technical part of this article we would like to say a few words about why we are interested in set-bounded Σ_1 operators. Everything began with operational set theory – see Feferman [4] and Jäger [7] for an introduction into operational set theory OST – and extensions of the basic operational systems by operational fixed point principle of various sorts. It turned out that the proof-theoretic analysis of these theories requires new sorts of model constructions.

Proof-theoretically perfectly suited frameworks for this enterprise are provided by Kripke-Platek set theory KP (with infinity) plus the fixed point principles that we will study below. It is planned for a future publication to present these model constructions and to use them for extablishing relationships between such extensions of KP and fixed point extensions of OST. However, in order to deduce proof-theoretic information from such results, we have to know the strenghts of the corresponding KP extensions. Their analysis is one aim of this article.

A second motivation for studying set-bounded Σ_1 operators over KP is inherent in our interest in understanding inductive definability. If $\mathfrak{A}[x, R^+]$ is an R-positive arithmetic formula, then KP provides a simple set-theoretic environment to study the least fixed point of the operator $\Gamma_{\mathfrak{A}}$ that maps a set of natural numbers S to the set

$$\Gamma_{\mathfrak{A}}(S) \ := \ \{u \in \omega : \mathfrak{A}[u, S]\}.$$

This is done, for example, in Jäger [6]. But what happens if we go up in the logical complexity of the operator forms and allow them to be Δ_1 definable? We may even replace positivity by a monotonicity condition.

More precisely, suppose that $C[u, x]$ is a Σ_1 formulas and $D[u, x]$ a Π_1 formula, both with the distinguished variables u, x and possibly further parameters. Given a set a, we let (C, D)-$\mathsf{M}\Delta_1\mathsf{O}[a]$ be the conjunction of the formulas

- $(\forall u \in a)\forall x(C[u, x] \leftrightarrow D[u, x])$,
- $(\forall u \in a)\forall x, y(C[u, x] \wedge x \subseteq y \ \rightarrow \ C[u, y])$.

Obviously, it states that the pair of formulas (C, D) is a monotone Δ_1 operator form on a. Then it is the most natural question to ask what it means for proof-theoretic strengths to add fixed point axioms to KP that claim that such monotone Δ_1 operator forms have fixed points or least fixed points.

For various technical reasons it is more convenient to work with what we call set-bounded Σ_1 operators (see below) rather than monotone Δ_1 operator forms. It is easy to see that both approaches lead to the same fixed points on a given set a:

- Assume (C, D)-$\mathsf{M}\Delta_1\mathsf{O}[a]$ with C and D as above. Now define $A[x, y]$ to be the formula
$$y \ = \ \{u \in a : C[u, x]\}.$$
Then $A[x, y]$ is equivalent to a Σ_1 formula and we have

  (i) $\forall x \exists! y A[x, y] \ \wedge \ \forall x, y(A[x, y] \ \rightarrow \ y \subseteq a)$,

  (ii) $\forall x_0, x_1, y_0, y_1(A[x_0, y_0] \ \wedge \ A[x_1, y_1] \ \wedge \ x_0 \subseteq x_1 \ \rightarrow \ y_0 \subseteq y_1)$,

  stating that A is functional, set-bounded, and monotone. Clearly, the (least) fixed points of A are the (least) fixed points of the operator form (C, D).

- On the other hand, assume that $A[x, y]$ is a Σ_1 formula such that (i) and (ii) hold. Now we define
$$C[u, x] \ := \ \exists y(A[x, y] \ \wedge \ u \in y) \text{ and } D[u, x] \ := \ \forall y(A[x, y] \ \rightarrow \ u \in y).$$
Thus C is (equivalent to) Σ_1 and D is Π_1. Furthermore, we have (C, D)-$\mathsf{M}\Delta_1\mathsf{O}[a]$ and the (least) fixed points of A coincide with those of (C, D).

As we will see, fixed point assertions for set-bounded Σ_1 operators lead to a considerable increase of proof-theoretic strength. They are closely related to specific separation principles and assertions about existence of injections of the universe or the ordinals to given sets; see below.

2.2 Fixed points, least fixed points and maximal iterations of set-bounded Σ_1 operators

As mentioned in the introduction, one central aspect of this article is to study the effect of adding fixed point assertions for monotone and set-bounded Σ_1

describable operators to KP. Typical such examples are the assertions that every monotone set-bounded Σ_1 definable operator has a fixed point or a least fixed point.

To formalize these assertions in KP, pick a formula $A[u,v]$ with distinguished free variables u, v and set

$$\mathcal{B}_A[a] \;:=\; \forall x \exists ! y A[x,y] \;\wedge\; \forall x, y (A[x,y] \;\rightarrow\; y \subseteq a).$$

$\mathcal{B}_A[a]$ states that $A[u,v]$ describes an operator the maps all sets to subsets of a; in this sense it is bounded by a. Keep in mind that A may contain other free variables than those displayed.

We write $\mathcal{M}_A[a]$ for the conjunction of $\mathcal{B}_A[a]$ and the monotonicity assertion

$$\forall x_0, x_1, y_0, y_1 (A[x_0, y_0] \;\wedge\; A[x_1, y_1] \;\wedge\; x_0 \subseteq x_1 \;\rightarrow\; y_0 \subseteq y_1).$$

The axioms for (least) fixed points of monotone set-bounded Σ_1 operators are then the two schemas

(Σ_1-FP) $$\mathcal{M}_A[a] \;\rightarrow\; \exists x A[x,x],$$

(Σ_1-LFP) $$\mathcal{M}_A[a] \;\rightarrow\; \exists x (A[x,x] \wedge \forall y (A[y,y] \;\rightarrow\; x \subseteq y)),$$

where $A[u,v]$ is a Σ_1 formula in both cases and, as mentioned above, may contain additional free variables besides u and v.

Next we turn to the iteration of set bounded but not necessarily monotone operators Γ, starting from the empty set,

$$\Gamma(\varnothing),\; \Gamma(\Gamma(\varnothing)),\; \Gamma(\Gamma(\varnothing) \cup \Gamma(\Gamma(\varnothing))),\; \ldots$$

and possibly continued into the transfinite. More precisely, consider again a formula $A[u,v]$ with distinguished free variables u, v and define

$$\mathcal{H}_A[f,\alpha] \;:=\; \mathit{Fun}[f, \alpha+1] \;\wedge\; (\forall \beta \leq \alpha)(A[\bigcup_{\xi<\beta} f(\xi), f(\beta)]),$$

where $\mathit{Fun}[f, \alpha+1]$ says that f is a function with domain $\alpha + 1$. If $A[u,v]$ defines an operator, then $\mathcal{H}_A[f,\alpha]$ states that f is the function obtained by iterating the application of this operator along the ordinals up to $\alpha + 1$. The existence (and uniqueness) of such a function f follows for any α by Σ recursion.

The *maximal iterations principle* (Σ_1-MI) states that for any set-bounded Σ_1 definable operator there exists an ordinal where an iteration of this sort comes to an end. In strong systems of set theory like ZFC or NBG this follows from a simple cardinality argument. However, it is not provable in KP.

(Σ_1-MI) $$\mathcal{B}_A[a] \;\rightarrow\; \exists \alpha, f(\mathcal{H}_A[f,\alpha] \;\wedge\; f(\alpha) \subseteq \bigcup_{\xi<\alpha} f(\xi)),$$

where, as before, $A[u, v]$ is a Σ_1 formula which may contain additional free variables besides u an v.

It is clear that if $A[u, v]$ describes a monotone set-bounded operator then the maximal iterations principle provides the definition of the least fixed point of this operator from below.

2.3 Class extension KP^c of KP

As one can see, this kind of formalization is rather clumsy since we cannot speak about operators directly. To overcome this syntactic limitation, we introduce a more "user-friendly" class or second order extension KP^c of KP.

The language $\mathcal{L}^c$ is the extension of $\mathcal{L}$ by countably many class variables $F, G, H, U, V, W, X, Y, Z$ (possibly with subscripts). The atomic formulas of $\mathcal{L}^c$ comprise the atomic formulas of $\mathcal{L}$ and all expressions of the form $(a \in U)$. The formulas of $\mathcal{L}^c$ are built up from these atomic formulas by use of the propositional connectives and quantification over sets and classes. Equality of classes is defined by

$$(U = V) \ := \ \forall x(x \in U \ \leftrightarrow \ x \in V)$$

and not treated as an atomic formula.

We say that an $\mathcal{L}^c$ formula is elementary iff it contains no class quantifiers. The Δ_0^c, Σ^c, Π^c, Σ_n^c, and Π_n^c formulas of $\mathcal{L}^c$ are defined in analogy to $\mathcal{L}$ but now permitting subformulas of the form $(a \in U)$.

The theory KP^c is formulated in $\mathcal{L}^c$ and also based on classical logic, now of course for sets and classes. As before we have extensionality, pairing, infinity for sets plus the extension of Δ_0 separation and Δ_0 collection to Δ_0^c formulas: For all Δ_0^c formulas $A[u]$ and $B[u, v]$,

(Δ_0^c-Sep) $$\exists x \forall y(y \in x \ \leftrightarrow \ y \in a \ \land \ A[y]),$$

(Δ_0^c-Col) $$(\forall x \in a)\exists y B[x, y] \ \rightarrow \ \exists z(\forall x \in a)(\exists y \in z)B[x, y].$$

The existence of classes is provided in KP^c by the schema of Δ_1^c comprehension: For every Σ_1^c formula $A[u]$ and every Π_1^c formula $B[u]$ we claim that

(Δ_1^c-CA) $$\forall x(A[x] \ \leftrightarrow \ B[x]) \ \rightarrow \ \exists X \forall x(x \in X \ \leftrightarrow \ A[x]).$$

In contrast to more familiar class theories like von Neumann-Bernays-Gödel, elementary formulas do not define classes in general.

Caution is also called for when formulating induction on $\in$. In KP^c we ask for

(El-$\mathsf{I}_\in$) $$\forall x((\forall y \in x)A[y] \ \rightarrow \ A[x]) \ \rightarrow \ \forall x A[x]$$

only for elementary formulas $A[u]$ of $\mathcal{L}^c$. KP^c plus induction on $\in$ for all $\mathcal{L}^c$ formulas is significantly stronger and proves the consistency of KP^c.

It is routine work to verify that central properties of KP like Σ reflection and Σ recursion can be proved in KP^c for Σ^c formulas. In particular, every Σ^c formula is provably equivalent in KP^c to a Σ^c_1 formula and every Π^c formula to a Π^c_1 formula. Consequently, (Δ^c_1-CA) can be lifted to

$$(\Delta^c\text{-}\mathsf{CA}) \qquad \forall x(A[x] \leftrightarrow B[x]) \rightarrow \exists X \forall x(x \in X \leftrightarrow A[x]),$$

where $A[u]$ is a Σ^c formula and $B[u]$ a Π^c formula.

It is obvious that KP is a subtheory of KP^c. In a next step we use a simple interpretation argument to show that KP^c is a conservative extension of KP.

Theorem 1. *Every $\mathcal{L}$ sentence provable in KP^c is already provable in KP; i.e., KP^c is a conservative extension of KP.*

Proof. There is a simple model-theoretic argument: Given a model $\mathfrak{M} = (M, \ldots)$ of KP we write $\Delta(\mathfrak{M})$ for the collection of all subsets of M that are Δ definable over $\mathfrak{M}$ with possible set parameters from M. Then $\mathfrak{M}$ is extended to the structure $\mathfrak{M}^c = (M, \Delta(\mathfrak{M}), \ldots)$ with $\Delta(\mathfrak{M})$ as the range of the class variables, and it is shown that $\mathfrak{M}^c$ is a model of KP^c.

However, we do not work out this semantic idea. For later purposes we prefer a sysntactic approach instead. Let $\Sigma\text{-}Sat_2[u, v]$ be the Σ_1 formula of $\mathcal{L}$ introduced in Barwise [1], Chapter V.1 and define

$$Delta[a] := \begin{cases} a = \langle 1^{st}(a), 2^{nd}(a), 3^{rd}(a)\rangle \ \wedge \\ \forall x(\Sigma\text{-}Sat_2[1^{st}(a), 3^{rd}(a), x] \leftrightarrow \neg\Sigma\text{-}Sat_2[2^{nd}(a), 3^{rd}(a), x]). \end{cases}$$

Intuitively, this formula states that a is the Gödel number of a Δ_1 definable collection of sets with parameter $3^{rd}(a)$. If $\vec{a}$ is the sequence $a_1, \ldots, a_n$, then $Delta[\vec{a}]$ is short for

$$Delta[a_1] \wedge \ldots \wedge Delta[a_n].$$

Until the end of this proof we make the ad hoc convention that $\vec{U}$ is a sequence of class variable $U_1, \ldots, U_n$, that $A[\vec{U}]$ is an $\mathcal{L}^c$ formula with at most the class variables $\vec{U}$ free, and that $\vec{a}$ is a sequence $a_1, \ldots, a_n$ of set variables not occurring in $A[\vec{U}]$.

We first translate every elementary $A[\vec{U}]$ into the $\mathcal{L}$ formula $A_\Sigma[\vec{a}]$ by replacing (for $1 \leq i \leq n$) all

$$\begin{array}{lll} \text{positive occurrences of } (v \in U_i) & \text{by} & \Sigma\text{-}Sat_2[1^{st}(a_i), 3^{rd}(a_i), v], \\ \text{negative occurrences of } (v \in U_i) & \text{by} & \Sigma\text{-}Sat_2[2^{nd}(a_i), 3^{rd}(a_i), v]. \end{array}$$

Similarly, $A_\Pi[\vec{a}]$ is the $\mathcal{L}$ formula obtained from $A[\vec{U}]$ by replacing (for $1 \leq i \leq n$) all

$$\begin{array}{lll} \text{positive occurrences of } (v \in U_i) & \text{by} & \neg\Sigma\text{-}Sat_2[2^{nd}(a_i), 3^{rd}(a_i), v], \\ \text{negative occurrences of } (v \in U_i) & \text{by} & \neg\Sigma\text{-}Sat_2[1^{st}(a_i), 3^{rd}(a_i), v]. \end{array}$$

Then we observe the following:

(1) If $A[\vec{U}]$ is a Σ^c formula, then $A_\Sigma[\vec{a}]$ is a Σ formula.

(2) If $A[\vec{U}]$ is a Π^c formula, then $A_\Pi[\vec{a}]$ is a Π formula.

(3) KP proves that
$$Delta[\vec{a}] \ \rightarrow \ (A_\Sigma[\vec{a}] \leftrightarrow A_\Pi[\vec{a}]).$$

Given an arbitrary $\mathcal{L}^c$ formula $A[\vec{U}]$ we obtain its translation $A_\Sigma[\vec{a}]$ into the language $\mathcal{L}$ by simply distributing the previous translation over the propositional connectives and set quantifiers and by treating class quantifiers as follows: If $A[\vec{U}]$ is the formula $\exists X B[\vec{U}, X]$ then

$$A_\Sigma[\vec{a}] := \exists x(Delta[x] \wedge B_\Sigma[\vec{a}, x]);$$

if $A[\vec{U}]$ is the formula $\forall X B[\vec{U}, X]$ then

$$A_\Sigma[\vec{a}] := \forall x(Delta[x] \rightarrow B_\Sigma[\vec{a}, x]).$$

A further trivial observation tells us that the Σ translation B_Σ of an $\mathcal{L}^c$ formula B without class variables (i.e. an $\mathcal{L}$ formula) is identical to B.

We want to show that this Σ translation provides an interpretation of KP^c into KP in the following sense:

(1) $$\mathsf{KP}^c \vdash A[\vec{U}] \quad \Longrightarrow \quad \mathsf{KP} \vdash Delta[\vec{a}] \ \rightarrow \ A_\Sigma[\vec{a}].$$

To this end we only have to establish that KP proves

(2) $$Delta[\vec{a}] \ \rightarrow \ A_\Sigma[\vec{a}]$$

for all axioms of KP^c. In view of properties (1) – (3) this is obvious for all axioms of KP^c except (Δ^c_1-CA). In order to show that it is also the case for (Δ^c_1-CA) let $B[\vec{U}, \vec{v}, w]$ and $C[\vec{U}, \vec{v}, w]$ be Σ^c_1 and Π^c_1 formulas with at most the indicated variables free. In addition, choose a sequence $\vec{a}$ of set variables of the same length as $\vec{U}$ not occurring in $B[\vec{U}, \vec{v}, w]$ and $C[\vec{U}, \vec{v}, w]$. Finally, working in KP assume that $Delta[\vec{a}]$ and

$$\forall x(B_\Sigma[\vec{a}, \vec{b}, x] \leftrightarrow C_\Sigma[\vec{a}, \vec{b}, x]).$$

Because of (3) we also have

$$\forall x(B_\Sigma[\vec{a}, \vec{b}, x] \leftrightarrow C_\Pi[\vec{a}, \vec{b}, x]).$$

Since several set parameters can be coded into one, it is clear that there exist a Σ formula $B'[u, v]$ and a Π formula $C'[u, v]$ with at most u, v free such that

$$\forall x(B'[\langle \vec{a}, \vec{b} \rangle, x] \leftrightarrow C'[\langle \vec{a}, \vec{b} \rangle, x]).$$

Making use of a Proposition 1.6 of Barwise [1], Chapter V.1, we also obtain that

$$\forall x(B'[\langle \vec{a}, \vec{b} \rangle, x] \ \leftrightarrow \ \Sigma\text{-}Sat_2[\ulcorner B'[u,v] \urcorner, \langle \vec{a}, \vec{b} \rangle, x]),$$

$$\forall x(\neg C'[\langle \vec{a}, \vec{b} \rangle, x] \ \leftrightarrow \ \Sigma\text{-}Sat_2[\ulcorner \neg C'[u,v] \urcorner, \langle \vec{a}, \vec{b} \rangle, x]),$$

Now it only remains to set

$$c \ := \ \langle \ulcorner B'[u,v] \urcorner, \ulcorner \neg C'[u,v] \urcorner, \langle \vec{a}, \vec{b} \rangle \rangle$$

and to verify that $Delta[c]$ as well as

$$\forall x(\Sigma\text{-}Sat_2[1^{st}(c), 3^{rd}(c), x] \ \leftrightarrow \ B_\Sigma[\vec{a}, \vec{b}, x]).$$

This finishes the proof of (2) for (Δ^c_1-CA) and thus also the proof of (1). However, since our Σ translation does not change $\mathcal{L}$ formulas, assertion (1) immediately yields the claimed conservativity statement. $\Box$

KP^c is a natural framework for speaking about operators. We call a class U an operator iff all its elements are ordered pairs such that it is right-unique,

$$Op[U] \ := \ \begin{cases} (\forall x \in U)\exists y, z(x = \langle y, z \rangle) \ \wedge \\ \forall y, z_0, z_1(\langle y, z_0 \rangle \in U \ \wedge \ \langle y, z_1 \rangle \in U \ \rightarrow \ z_0 = z_1). \end{cases}$$

We say that a belongs to the domain of U, in symbols $Dom[U,a]$, iff there exists an x such that $\langle a, x \rangle \in U$.

The following lemma shows that all Σ^c_1 definable operators can be represented as operational classes that are total in the sense that they assign a set to each element of the universe. But keep in mind that $Op[U]$ does in general not imply that the domain of U is a class; U may be partial.

Lemma 2. *Let $A[u,v]$ be a Σ^c_1 formula. Then KP^c proves that*

$$(\forall x \in U)\exists! y A[x,y] \ \rightarrow \ \exists X(Op[X] \wedge \forall x, y(\langle x, y \rangle \in X \ \leftrightarrow \ x \in U \wedge A[x,y])).$$

Clearly, the domain of this operator X is the class U.

Proof. Take any Σ^c_1 formula $A[u,v]$, assume that $\forall x \exists! y A[x,y]$, and consider the formulas

$$B_0[u] \ := \ u = \langle 1^{st}(u), 2^{nd}(u) \rangle \ \wedge \ 1^{st}(u) \in U \ \wedge \ A[1^{st}(u), 2^{nd}(u)],$$

$$B_1[u] \ := \ \begin{cases} u = \langle 1^{st}(u), 2^{nd}(u) \rangle \ \wedge \ 1^{st}(u) \in U \ \wedge \\ \forall z(A[1^{st}(u), z] \ \rightarrow \ z = 2^{nd}(u)). \end{cases}$$

Then $B_0[u]$ and $B_1[u]$ are Σ^c and Π^c, respectively. Since $B_0[u]$ is provably equivalent to $B_1[u]$ we obtain our assertion by (Δ^c-CA). $\Box$

In the following we often write F, G, or H when we speak about operators. If a belongs to the domain of an operator F, then $F(a)$ denotes the unique b such that $\langle a, b\rangle \in F$. The following abbreviations make this precise:

$$\begin{aligned}
F(a) = b \quad &:= \quad \langle a, b\rangle \in F, \\
F(a) \in b \quad &:= \quad (\exists x \in b)(\langle a, x\rangle \in F), \\
b \in F(a) \quad &:= \quad \exists x(\langle a, x\rangle \in F \,\wedge\, b \in x), \\
F(a) \subseteq b \quad &:= \quad \exists x(\langle a, x\rangle \in F \,\wedge\, x \subseteq b), \\
b \subseteq F(a) \quad &:= \quad \exists x(\langle a, x\rangle \in F \,\wedge\, b \subseteq x), \\
F(a) = F(b) \quad &:= \quad \exists x(\langle a, x\rangle \in F \,\wedge\, \langle b, x\rangle \in F), \\
F(a) \subseteq F(b) \quad &:= \quad \exists x, y(\langle a, x\rangle \in F \,\wedge\, \langle b, y\rangle \in F \,\wedge\, x \subseteq y), \\
F(a) \in U \quad &:= \quad (\exists x \in U)(\langle a, x\rangle \in F).
\end{aligned}$$

Clearly, the first and the second of these definitions are Δ_0^c and the others are Σ^c. But if F is an operator, we can do better. In all relevant cases, they are also Π^c.

Remark 3. In KP^c we obtain from $Op[F] \wedge Dom[F, a]$ the following equivalences:

(i) $b \in F(a) \;\leftrightarrow\; \forall x(\langle a, x\rangle \in F \;\rightarrow\; b \in x)$.

(ii) $F(a) \subseteq b \;\leftrightarrow\; \forall x(\langle a, x\rangle \in F \;\rightarrow\; x \subseteq b)$.

(iii) $b \subseteq F(a) \;\leftrightarrow\; \forall x(\langle a, x\rangle \in F \;\rightarrow\; b \subseteq x)$.

(iv) $F(a) = F(b) \;\leftrightarrow\; \forall x(\langle a, x\rangle \in F \;\rightarrow\; \langle b, x\rangle \in F)$.

(v) $F(a) \in U \;\leftrightarrow\; \forall x(\langle a, x\rangle \in F \;\rightarrow\; x \in U)$.

If in addition $Dom[F, b]$ then also

(v) $F(a) \subseteq F(b) \;\leftrightarrow\; \forall x, y(\langle a, x\rangle \in F \,\wedge\, \langle b, y\rangle \in F \;\rightarrow\; x \subseteq y)$.

3 Some fixed point principles for KP^c

In this section we introduce a series of fixed point principles, all formulated in our extended language $\mathcal{L}^c$ and above KP^c as base theory. We begin with the equivalents of (Σ_1-FP), (Σ_1-LFP), and (Σ_1-MI). Our point of departure is an operator F that is monotone and maps all sets to subsets of a given set a,

$$Mon[F, a] \;:=\; \forall x(F(x) \subseteq a) \,\wedge\, \forall x, y(x \subseteq y \;\rightarrow\; F(x) \subseteq F(y)).$$

The fixed point axiom (FP^c) then claims that such an operator has a fixed point; in case of the least fixed point axiom (LFP^c) it is even required that this fixed point is contained in all fixed points.

$$(\mathsf{FP}^c) \qquad \mathit{Mon}[F,a] \ \rightarrow\ \exists x(F(x) = x).$$

$$(\mathsf{LFP}^c) \qquad \mathit{Mon}[F,a] \ \rightarrow\ \exists x(F(x) = x \ \wedge\ \forall y(F(y) = y \ \rightarrow\ x \subseteq y)).$$

To formulate the maximal iterations principle in KP^c, let $\mathit{Hier}[F,f,\alpha]$ be the formula given by

$$\mathit{Fun}[f,\alpha+1] \ \wedge\ (\forall\beta \leq \alpha)(f(\beta) = F(\bigcup_{\xi<\beta} f(\xi))),$$

and observe that, as above, the existence (and uniqueness) of such a function f follows for any α by Σ^c recursion. Then the maximal iterations principle in KP^c is

$$(\mathsf{MI}^c) \qquad \forall x(F(x) \subseteq a) \ \rightarrow\ \exists \alpha, f(\mathit{Hier}[F,f,\alpha] \ \wedge\ f(\alpha) \subseteq \bigcup_{\xi<\alpha} f(\xi)).$$

With Theorem 1 in mind it should be obvious that (FP^c), (LFP^c), and (MI^c) are the class analogues of (Σ_1-FP), (Σ_1-LFP), and (Σ_1-MI), respectively.

Now we turn two additional fixed point principles for KP^c. In the appendix we present their equivalent formulations in $\mathcal{L}$ and above KP.

Fixed points on set-complete classes.

A class U is called *set-complete* iff the union of every subset of U belongs to U. Note that a set-complete class U is not necessarily chain-complete. Hence the interest in considering the variant of (FP^c) considering operators that map into a set-complete subclass of a given set a and whose monotonicity is restricted to elements of this class,

$$\mathit{Msc}[F,a,U] \ :=\ \begin{cases} (\forall x \in U)(x \subseteq a) \ \wedge\ (\forall x \subseteq U)(\bigcup x \in U) \ \wedge \\ \forall x(F(x) \in U) \ \wedge\ (\forall x,y \in U)(x \subseteq y \ \rightarrow\ F(x) \subseteq F(y)). \end{cases}$$

The corresponding principle postulates that such operators have fixed points,

$$(\mathsf{ScFP}^c) \qquad \mathit{Msc}[F,a,U] \ \rightarrow\ \exists x(F(x) = x).$$

Chain fixed points

If F is an operator that maps all ordinals to subsets of a given set and describes an increasing chain in the sense that $F(\alpha) \subseteq F(\beta)$ for $\alpha < \beta$, then it is postulated that there exists an α for which $F(\alpha) = F(\alpha+1)$,

$$(\mathsf{ChFP}^c) \quad \forall\alpha,\beta(\alpha < \beta \ \rightarrow\ F(\alpha) \subseteq F(\beta) \subseteq a) \ \rightarrow\ \exists\alpha(F(\alpha) = F(\alpha+1)).$$

it is clear that the existence of such an ordinal α can be proved in strong set theories by a simple cardinality argument. In KP^c it has to be added as an additional axiom.

We end this section by summarizing some first (and rather straightforward) relationships between these principles. In this connection let us fix a manner of speaking: If T is an extension of KP^c and if (P_0) and (P_1) are two of our principles, we say that (P_0) *implies* (P_1) *over* T iff every instance of (P_1) is provable in $T + (P_0)$.

Theorem 4. *Over* KP^c *we have:*

(i) (LFP^c) *implies* (FP^c).

(ii) (ScFP^c) *implies* (FP^c).

(iii) (MI^c) *implies* (LFP^c).

(iv) (ChFP^c) *implies* (MI^c).

(v) (MI^c) *implies* (ScFP^c).

Proof. The proofs of (i), (ii), and (iii) are straightforward.

(iv) Given a and F as in (MI^c) we know (see above) that for every α there exists a unique f such that $Hier[F, f, \alpha]$. Hence,

$$\forall\alpha\exists!x\exists f(Hier[F, f, \alpha] \wedge x = \bigcup_{\xi\leq\alpha} f(\xi)).$$

According to Lemma 2 we thus have an operator G such that

$$G(\alpha) = x \leftrightarrow \exists f(Hier[F, f, \alpha] \wedge x = \bigcup_{\xi\leq\alpha} f(\xi))$$

for all α and x. Also, it is clear that $G(\alpha) \subseteq G(\beta) \subseteq a$ for $\alpha < \beta$. By (ChFP^c) we have an α such that $G(\alpha) = G(\alpha + 1)$. Hence, making use of the uniqueness f once more, we conclude that there exists an f with $Hier[F, f, \alpha + 1]$ and $f(\alpha + 1) \subseteq \bigcup_{\xi<\alpha+1} f(\xi)$. This completes the proof of (MI^c).

(v) Assume that we are given F, a, U such that $Msc[F, a, U]$. We introduce an operator G defined on the universe for which

$$G(x) = \begin{cases} F(x) & \text{if } x \in U, \\ \varnothing & \text{if } x \notin U. \end{cases}$$

Then $\forall x(G(x) \subseteq a)$ and by (MI^c) there exist an α and an f for which

$$Hier[G, f, \alpha] \wedge f(\alpha) \subseteq \bigcup_{\xi<\alpha} f(\xi).$$

Since U is set-complete and F maps into U, transfinite induction shows $f(\beta) \in U$ for all $\beta \leq \alpha$. The monotonicity of F on U thus yields $f(\beta) = f(\gamma)$ for $\beta \leq \gamma \leq \alpha$. Therefore,

$$f(\alpha) \;=\; \bigcup_{\xi<\alpha} f(\xi)$$

and, as a consequence,

$$F(f(\alpha)) \;=\; F(\bigcup_{\xi<\alpha} f(\xi)) \;=\; G(\bigcup_{\xi<\alpha} f(\xi)) \;=\; f(\alpha).$$

This means that we have a fixed point of F, as requested by (ScFP^c). $\square$

4 Related principles

In strong set theories like ZFC the existence of (least) fixed points of monotone operators on complete lattices is often proved by means of a cardinality argument. But in KP and KP^c such cardinality arguments cannot be carried out. In this section we formulate several principles that have the flavor of cardinality considerations and begin to study their effect over KP^c related to the fixed point principles introduced before.

Bounded proper injections

The first such principle states that there is no proper injection of the whole universe of sets into a given set,

(BPI^c) $\qquad \forall x(F(x) \in a) \;\to\; \exists x, y(x \neq y \wedge F(x) = F(y)).$

A variant of (BPI^c) is the statement that it is not possible to properly inject the ordinals into a set,

(BPI^c_{On}) $\qquad \forall \alpha(F(\alpha) \in a) \;\to\; \exists \alpha, \beta(\alpha \neq \beta \wedge F(\alpha) = F(\beta)).$

It is easy to formulate further variants and strengthenings of (BPI^c), for example the claim that for every set a there exists a set b so large that there is no injective mapping from b to a.[†] However, in this article we confine ourselves to (BPI^c) and (BPI^c_{On}) and begin with pointing out some first connections to our fixed point principles.

Theorem 5. *Over* KP^c *we have:*

(i) (BPI^c_{On}) *implies* (BPI^c).

[†] As pointed out by the referee there is a conceptual relationship between our bounded proper injections and the notion of nonprojectibility in Barwise [1].

(ii) (ChFP^c) *implies* (BPI^c_{On})

(iii) (MI^c) *implies* (BPI^c).

Proof. (i) is obvious. For (ii) assume that $\forall\alpha(F(\alpha) \in a)$ and that G is the operator with domain On and

$$G(\alpha) \;=\; \{F(\xi) : \xi < \alpha\}$$

for all α. By definition G is monotone and $\forall\alpha(G(\alpha) \subseteq a)$. Hence, (ChFP^c) yields the existence of an α with $G(\alpha) = G(\alpha + 1)$. This means that we have $F(\alpha) \in G(\alpha)$ and thus $F(\alpha) = F(\beta)$ for some $\beta < \alpha$ as needed for establishing (BPI^c_{On}).

(iii) Suppose by contradiction that there exist a set a and an operator F such that

(a) $\forall x(F(x) \in a)$,

(b) $\forall x, y(x \neq y \;\rightarrow\; F(x) \neq F(y))$.

Then consider the operator G with

$$G(x) = \{F(x)\}$$

for any set x. By (a), G satisfies the hypothesis of (MI^c). Hence there exist α and g such that

$$Hier[G, g, \alpha] \quad \text{and} \quad g(\alpha) \subseteq \bigcup_{\xi<\alpha} g(\xi).$$

From this g we can easily define a function f with domain $\alpha + 1$ for which

$$(\forall\xi \leq \alpha)(g(\xi) = \{f(\xi)\}).$$

Indeed, simply define $f(\xi) = x$ by

$$\exists y(g(\xi) = y \,\wedge\, x \in y \,\wedge\, (\forall z \in y)(z = x)).,$$

which is equivalent to the Π^c formula

$$\forall y(g(\xi) = y \;\rightarrow\; x \in y \,\wedge\, (\forall z \in y)(z = x))).$$

We claim that

$$(*) \qquad (\forall\xi \leq \alpha)(f(\xi) \notin \bigcup_{\eta<\xi} g(\eta)).$$

However, from (*) we deduce that

$$g(\alpha) = \{f(\alpha)\} \not\subseteq \bigcup_{\eta<\alpha} g(\eta)),$$

a contradiction to the properties of g. So it only remains to prove (*), and this is done by transfinite induction. Assume, toward a contradiction, that β is the least ordinal such that $f(\beta) \in \bigcup_{\eta<\beta} g(\eta)$. Hence $f(\beta) = f(\gamma)$ for some $\gamma < \beta$. Thus we have

$$\{F(\bigcup_{\eta<\beta} g(\eta))\} \;=\; g(\beta) \;=\; \{f(\beta)\} \;=\; \{f(\gamma)\} \;=\; g(\gamma) \;=\; \{F(\bigcup_{\eta<\gamma} g(\eta))\},$$

i.e.

$$F(\bigcup_{\eta<\beta} g(\eta)) \;=\; f(\beta) \;=\; f(\gamma) \;=\; F(\bigcup_{\eta<\gamma} g(\eta)).$$

Because of the choice of β we also know that $f(\gamma) \notin \bigcup_{\eta<\gamma} g(\eta)$, whereas $f(\gamma)$ clearly belongs to $\bigcup_{\eta<\beta} g(\eta)$. Altogether we thus have

$$\bigcup_{\eta<\beta} g(\eta) \neq \bigcup_{\eta<\gamma} g(\eta) \quad \text{and} \quad F(\bigcup_{\eta<\beta} g(\eta)) = F(\bigcup_{\eta<\gamma} g(\eta)),$$

in contradiction to assumption (b) above. This completes the proof of (*) and thus also the proof of (iii). □

Strong separation principles

Over KP^c we can easily replace the cardinality argument that plays a prominent role in the standard proof of the existence of least fixed points of set-bounded monotone operators from below. A well-know such separation principle is Σ_1^c (or simply Σ_1 separation if we work in KP),

$$(\Sigma_1^c\text{-}\mathsf{Sep}) \qquad\qquad \exists y \forall x(x \in y \;\leftrightarrow\; x \in a \,\wedge\, A[x])$$

for arbitrary Σ_1^c formulas $A[u]$. It is easy to see that (Σ_1^c-Sep) implies all the other principles introduced so far. In view of Theorem 4 and Theorem 5 it suffices to show that (ChFP^c) follows from (Σ_1^c-Sep).

Theorem 6. *Over* KP^c*,* (Σ_1^c-Sep) *implies* (ChFP^c)*.*

Proof. Pick an arbitrary set a and an arbitrary operator F for which

$$\forall \alpha, \beta(\alpha < \beta \,\rightarrow\, F(\alpha) \subseteq F(\beta) \subseteq a).$$

In a first step we use (Σ_1^c-Sep) to introduce the set

$$b \;:=\; \{x \in a : \exists \xi(x \in F(\xi))\}.$$

for which we have

$$(\forall x \in b)\exists \xi(x \in F(\xi)).$$

Hence Σ^c reflection yields

$$b = \{x \in a : (\exists \xi < \alpha)(x \in F(\xi))\}$$

for some α. Since F is monotone we thus have $F(\alpha) = b = F(\alpha + 1)$. □

There is an interesting special case of Σ_1^c separation that – to the best of our knowledge – has not been discussed in the literature yet and that we call *subset-bounded separation*,

$$(\mathsf{SBS}^c) \qquad \exists z \forall x (x \in z \;\leftrightarrow\; x \in a \wedge (\exists y \subseteq b) A[x,y])$$

for arbitrary Δ_0^c formulas $A[u,v]$. Since KP^c provides $(\Delta^c\text{-}\mathsf{CA})$, it is an easy exercise to show that (SBS^c) can be syntactically extended to

$$(\mathsf{eSBS}^c) \qquad \begin{aligned} &\forall x,y(A[x,y] \leftrightarrow B[x,y]) \;\rightarrow \\ &\qquad\qquad \exists z \forall x (x \in z \;\leftrightarrow\; x \in a \wedge (\exists y \subseteq b) A[x,y]) \end{aligned}$$

for arbitrary Σ^c formulas $A[u,v]$ and Π^c formulas $B[u,v]$.

Remark 7. The notion of subset-bounded formulae is not new. For example, the class $\Delta_0^{\mathcal{P}}$ of formulae is defined to be the least collection of formulas that contains all atomic formulas of $\mathcal{L}$ and is closed under the propositional connectives and all quantifiers form $(Qx \in y)$ and $(Qx \subseteq y)$. It is considered in Mathias [10] and Rathjen [16]. As it seems this definiton goes back to Takahashi [17] where the formulas of this collection were called *quasi-bounded* formulas. The theories $\mathsf{KP}^{\mathcal{P}}$ and $\mathsf{KP}(\mathcal{P})$ due to Mathias and Rathjen have $(\Delta_0^{\mathcal{P}}\text{-}\mathsf{Sep})$ among their axioms. We will say more about the relationship between our (SBS^c) and $(\Delta_0^{\mathcal{P}}\text{-}\mathsf{Sep})$ in Appendix 1.

(SBS^c) – in its extended form (eSBS^c) – directly enables us, for example, to define the least fixed point of a monotone operator F from the powerset $\wp(a)$ of a set a to $\wp(a)$ as the intersection of all subsets of a closed under F, where a set b is called *closed under F (or F-closed)* iff $F(b) \subseteq b$.

Theorem 8. *Over KP^c we have:*

(i) (SBS^c) *implies* (LFP^c).

(ii) (SBS^c) *implies* (ScFP^c).

Proof. (i) Given a and F as in (LFP^c), we use (eSBS^c) to introduce the set

$$b \;:=\; \{x \in a : (\exists y \subseteq a)(F(y) \subseteq y \;\wedge\; x \notin y)\}.$$

For $c := a \backslash b$ we then have

$$c \;=\; \{x \in a : (\forall y \subseteq a)(F(y) \subseteq y \;\rightarrow\; x \in y)\},$$

i.e. c is the intersection of all F-closed subsets of a. The proof that c is the least fixed point of F is as usual.

(ii) Assume that a, F, and U are such that $Msc[F,a,U]$. Now we use (eSBS^c) to introduce the set

$$b \;:=\; \{x \in a : (\exists y \subseteq a)(y \in U \;\wedge\; y \subseteq F(y) \;\wedge\; x \in y)\}.$$

We claim that $b \in U$ and $F(b) = b$. Indeed,

$$(\forall x \in b)\exists y(y \subseteq a \wedge y \in U \wedge y \subseteq F(y) \wedge x \in y).$$

Hence Σ^c collection provides us with a set c such that

$$(\forall x \in b)(\exists y \in c)(y \subseteq a \wedge y \in U \wedge y \subseteq F(y) \wedge x \in y),$$
$$(\forall y \in c)(\exists x \in b)(y \subseteq a \wedge y \in U \wedge y \subseteq F(y) \wedge x \in y).$$

From that we get $c \subset U$ and $b = \cup c$. So b is the union of a subset of U, hence $b \in U$. By the monotonicity of F on U we further obtain

$$b = \bigcup c = \bigcup_{y \in c} y \subseteq \bigcup_{y \in c} F(y) \subseteq F(b).$$

It remains to show that $F(b) \subseteq b$. But if $x \in b' := F(b)$, then the properties of F and the previous inclusion yield

$$b' \subseteq a \wedge b' \in U \wedge b' \subseteq F(b') \wedge x \in b'.$$

Hence $x \in b$ by the definition of b, finishing the proof of (ii). $\square$

We end this section with pointing out that (SBS^c) is equivalent to a fairly strong replacement property. The exact formulation is

(SRep^c) $\qquad \forall x(F(x) \in a) \rightarrow \exists z \forall x(x \in z \leftrightarrow (\exists y \subseteq b)(x = F(y)).$

Informally speaking, if F is an operator that maps the universe into a given set a, then for any set b the image of $\wp(b)$ under F is a set.

Theorem 9. *Over* KP^c, (SBS^c) *and* (SRep^c) *are equivalent.*

Proof. It is clear that (SRep^c) follows from (SBS^c) via (eSBS^c). For the converse direction, fix sets a, b and a Δ_0^c formula $A[u, v]$. Suppose, for a contradiction, that $\{x \in a : (\exists y \subseteq b)A[x, y]\}$ is not a set. Then there exists a $c \in a$ for which $(\exists y \subseteq b)A[c, y]$.

As ad hoc abbreviation write $good[u]$ for

$$u = \langle 1^{st}(u), 2^{nd}(u)\rangle \wedge 1^{st}(u) \in a \wedge 2^{nd}(u) \subseteq b.$$

and set

$$B[u, v] := \begin{cases} (\neg good[u] \wedge v = c) \vee \\ (good[u] \wedge \neg A[2^{nd}(u), 1^{st}(u)] \wedge v = c) \vee \\ (good[u] \wedge A[2^{nd}(u), 1^{st}(u)] \wedge v = 1^{st}(u)). \end{cases}$$

$B[u, v]$ is a Δ_0^c formula and

$$\forall x \exists! y B[x, y] \wedge \forall x, y(B[x, y] \rightarrow y \in a).$$

Finally, let F be the operator associated with $B[u, v]$ according to Lemma 2. Now (SRep^c) comes into play and yields the existence of a set d with

$$\forall x(x \in d \leftrightarrow (\exists z \subseteq b \times a)(x = F(z))).$$

Clearly, $d = \{x \in a : (\exists y \subseteq b)A[x, y]\}$, a contradiction. $\square$

From this equivalence we can immmediately deduce that our subset-bounded separation implies that there are no proper injections of the universe into a given set.

Theorem 10. *Over* KP^c, (SBS^c) *implies* (BPI^c).

Proof. Let F be an operator with $\forall x(F(x) \in a)$ where a is given set. Then (SBS^c) in the form of (SRep^c) tells us that there exists a set b for which

$$b = \{F(x) : x \subseteq a\}.$$

Suppose we had

$$(*) \qquad (\forall x, y \subseteq a)(x \neq y \to F(x) \neq F(y)).$$

Then we can easily define a function g with domain b such that, for $x \in b$,

$$g(x) = \text{the unique } y \subseteq a \text{ for which } x = F(y).$$

This function g is so that

$$(\forall y \subseteq a)(\exists x \in b)(y = g(x)).$$

Namely, if $y \subset a$ then $F(y) \in b$ and g maps $F(y)$ to y. Now define the set

$$c := \{x \in b : x \notin g(x)\}.$$

Since c is a subset of a we have a $d \in b$ such that $c = g(d)$. However, this implies that $d \in g(d)$ iff $d \notin g(d)$; a contradiction.

Hence (*) is false, and thus there are different sets x and y – even subsets of a – for which $F(x) = F(y)$, as needed for finishing our proof. $\square$

The axiom (β)

The axiom (β) will play an important role in the article Jäger and Steila [8]; it discusses the principles we have introduced here in the context of $\mathsf{KP}^c+(\beta)$. Now we confine ourselves to one specific result showing that (β) is provable in $\mathsf{KP}^c + (\mathsf{MI}^c)$.

A relation b is *well-founded* on a set a, in symbols $Wf[a, b]$, iff $b \subseteq a \times a$ and

$$(\forall x \subseteq a)(x \neq \varnothing \to (\exists y \in x)(\forall z \in x)(\langle z, y\rangle \notin b)).$$

This is a Π_1 formula. The axiom (β) has the effect of making well-foundedness a Δ_1 predicate,

$$(\beta) \qquad Wf[a,b] \;\rightarrow\; \exists f(Fun[f,a] \wedge (\forall x \in a)(f(x) = \{f(y) : \langle y, x\rangle \in b\})),$$

where $Fun[f,a]$ abbreviates (as earlier) that f is a function with domain a.

In Barwise [1] it is mentioned that (β) cannot be proved in KP and it is shown how to derive it in $\mathsf{KP} + (\Sigma_1\text{-}\mathsf{Sep})$. More or less the same argument goes already through in $\mathsf{KP}^c + (\mathsf{MI}^c)$.

Theorem 11. $\mathsf{KP}^c + (\mathsf{MI}^c)$ *proves* (β).

Proof. Given a well-founded relation b on a set a we define an operator F such that

$$F(x) \;=\; \{y \in a : (\forall z \in a)(\langle z, y\rangle \in b \;\rightarrow\; z \in x)\}.$$

We have $\forall x(F(x) \subseteq a)$. Hence, by (MI^c), there exist an ordinal α and a function f such that

- $f(\beta) = \{x \in a : (\forall y \in a)(\langle y, x\rangle \in b \;\rightarrow\; y \in \bigcup_{\xi<\beta} f(\xi))\}$ for all $\beta \leq \alpha$,
- $f(\alpha) \subseteq \bigcup_{\xi<\alpha} f(\xi)$.

By exploiting $Wf[a,b]$ we obtain $f(\alpha) = a$. The rest of the proof is as in Barwise [1], Chapter I.9. $\square$

The results achieved so far can be summarized in the following graphic.

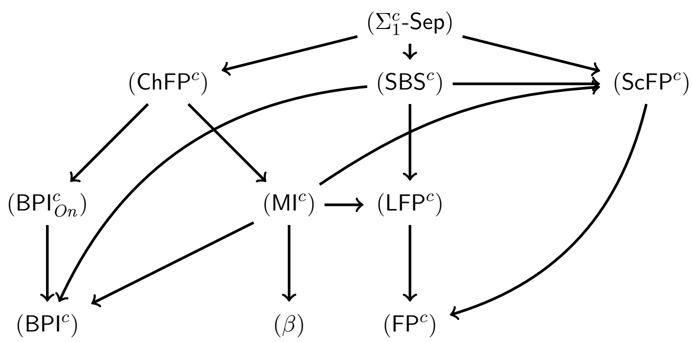

Figure 1: Over KP^c

5 Adding the axiom of constructibility

After the basic observations above the present article shows that our principles are equivalent over $\mathsf{KP} + (V{=}L)$. This is the main achievement of the present article. In the last section of this paper we say a few word about future work concerning our principles in the context of $\mathsf{KP} + (\beta)$ and over KP alone.

In the following we will be working within a universe where all sets are constructible, but we cannot introduce the constructible hierarchy here. Most relevant details can be found, for example, in Barwise [1] or Kunen [9]. Very briefly, $(a \in L_\alpha)$ means that the set a is an element of the αth level L_α of the constructible hierarchy and $(a \in L)$ is short for $\exists\alpha(a \in L_\alpha)$. We write $(a <_L b)$ to state that a is smaller than b according to the well-ordering $<_L$ of the constructible universe. The *axiom of constructibility* is the statement $(V{=}L)$, i.e. $\forall x \exists \alpha(x \in L_\alpha)$. It is well-known that the assertions $(a \in L_\alpha)$ and $(a <_L b)$ are Δ over $\mathsf{KP} + (V{=}L)$.

Remark 12. It follows from the standard properties of the well-ordering $<_L$ that in $\mathsf{KP}^c + (V{=}L)$ the collections of Σ^c and Π^c formulas are closed under restricted quantifiers $(\exists x <_L s)$ and $(\forall x <_L s)$.

The first result of this section states that (BPI^c) implies $(\Sigma^c_1\text{-}\mathsf{Sep})$ over $\mathsf{KP}^c + (V{=}L)$. This is achieved in two steps:

- We first show that over $\mathsf{KP}^c + (V{=}L)$ the non-existence of proper injections of the universe into a set is equivalent to the non-existence of proper injections of the ordinals into this set.
- Afterwards we demonstrate that all instances of $(\Sigma^c_1\text{-}\mathsf{Sep})$ are provable in $\mathsf{KP}^c + (V{=}L) + (\mathsf{BPI}^c_{On})$.

The first of these two assertions is a direct consequence of a combinatorial propery of L, formulated in the following lemma.

Lemma 13. *There exists an operator H_L such that $\mathsf{KP}^c + (V{=}L)$ proves*

$$\forall x \exists \xi (H_L(x) = \xi) \ \wedge \ \forall x, y(x \neq y \ \rightarrow \ H_L(x) \neq H_L(y)).$$

Since the proof of this lemma has nothing to do with the central questions of this article, we defer it to Appendix 2. Instead, we immediately make use of this lemma to obtain the following result.

Theorem 14. *Over $\mathsf{KP}^c + (V{=}L)$, (BPI^c) implies (BPI^c_{On}).*

Proof. Assume $\forall \xi(F(\xi) \in a)$ and let H_L be the operator of the previous lemma. Then we have

$$\forall x \exists! y \exists \xi (H_L(x) = \xi \ \wedge \ F(\xi) = y).$$

According to Lemma 2 and the assumption about F there exists an operator G such that $\forall x(G(x) \in a)$ and

$$\forall x, y(G(x) = y) \ \leftrightarrow \ \exists \xi (H_L(x) = \xi \ \wedge \ F(\xi) = y).$$

By (BPI^c) there exist x, y for which $x \neq y$ and $G(x) = G(y)$. For $\alpha := H_L(x)$ and $\beta := H_L(y)$ we thus have $\alpha \neq \beta$ and $F(\alpha) = F(\beta)$. $\square$

Theorem 15. *In* $\mathsf{KP}^c + (V{=}L) + (\mathsf{BPI}^c_{On})$ *every instance of* $(\Sigma^c_1\text{-}\mathsf{Sep})$ *is provable.*

Proof. Suppose to the contrary that there exist a set a and a Δ^c_0 formula $A[u, v]$ for which

$$\mathcal{R} := \{x \in a : \exists y A[x, y]\}$$

is not a set. For every ordinal α we introduce the set

$$G(\alpha) \ := \ \{x \in a : (\exists y \in L_\alpha)A[x, y]\}$$

and conclude that

$$(*) \qquad \mathcal{R} \ = \ \bigcup_\alpha G(\alpha).$$

Now the idea of the proof is to use induction on the ordinals to define an operator F from the ordinals to $\mathcal{R}$: If F has been defined for all ordinals less than α, then $\{F(\xi) \in a : \xi < \alpha\}$ is a set. Since $\mathcal{R}$ is not a set and because of (*) there exists a least β such that

$$\{F(\xi) \in a : \xi < \alpha\} \ \subsetneqq \ G(\beta).$$

Let x_0 be the $<_L$-least element of $G(\beta) \setminus \{F(\xi) \in a : \xi < \alpha\}$ and set $F(\alpha) := x_0$. This F is a one-to-one operator from the ordinals to a, thus violating (BPI^c_{On}).

In more detail, if f is a function whose domain is a superset of β we write $least[\gamma, \beta, f]$ for

$$\{f(\xi) \in a : \xi < \beta\} \subsetneqq G(\gamma) \ \wedge \ (\forall \delta < \gamma)(G(\delta) \subseteq \{f(\xi) \in a : \xi < \beta\},$$

stating that γ is the least ordinal such that $G(\gamma)$ is a proper superset of the set $\{f(\xi) \in a : \xi < \beta\}$. Then $B[\alpha, f, x]$ is defined to be the conjunction of the following formulas:

(1) $Fun[f, \alpha + 1]$,

(2) $(\forall \beta \leq \alpha)\exists\gamma(least[\gamma, \beta, f] \ \wedge \ f(\beta) = least_{<_L}(G(\gamma) \setminus \{f(\xi) \in a : \xi < \beta\})$,

(3) $x = f(\alpha)$.

From this definition (and the informal explanation above) we immediately deduce that

$$B[\alpha, f, x] \wedge B[\alpha, g, y] \ \rightarrow \ x = y,$$

$$B[\alpha, f, x] \wedge B[\beta, g, y] \wedge \alpha \neq \beta \ \rightarrow \ x \neq y,$$

$$\forall\alpha\exists! x\exists f B[\alpha, f, x].$$

Also, $\exists f B[\alpha, f, x]$ is provably equivalent in $\mathsf{KP}^c + (V{=}L)$ to a Σ^c_1 formula. In view of Lemma 2, this means that there exists an operator F satisfying

- $\forall\alpha(F(\alpha) \in a)$,
- $\forall\alpha,\beta(\alpha \neq \beta \to F(\alpha) \neq F(\beta))$.

This contradicts (BPI^c_{On}). $\square$

A second result of this section is that (FP^c) implies (SBS^c) over $\mathsf{KP}^c + (V{=}L)$. This closes the circle and is the last step in showing that all our fixed point principles, our statements about bounded proper injections, and subset-bounded separation are equivalent, over $\mathsf{KP}^c + (V{=}L)$, to Σ_1 separation.

Theorem 16. *Over* $\mathsf{KP}^c + (V{=}L)$*,* (FP^c) *implies* (SBS^c)*.*

Proof. We proceed indirectly and assume that there exists a Δ^c_0 formula $\varphi[u,v]$ and sets a,b such that

$$\mathcal{R} \;:=\; \{x \in a : (\exists y \subseteq b)\varphi[x,y]\}$$

is not a set. Before producing a contradiction, we introduce some auxiliary notation and begin with setting

$$\psi[u,v] \;:=\; \varphi[u,v] \wedge (\forall z <_L v)(z \subseteq b \to \neg\varphi[u,z]).$$

By Remark 12 there exist a Σ^c fomula $\psi_1[u,v]$ and a Π^c formula $\psi_2[u,v]$, both with the same free variables as $\psi[u,v]$, such that $\mathsf{KP}^c + (V{=}L)$ proves

$$\psi[u,v] \;\leftrightarrow\; \psi_1[u,v] \quad \text{and} \quad \psi[u,v] \;\leftrightarrow\; \psi_2[u,v],$$

i.e. $\psi[u,v]$ is Δ^c with respect to $\mathsf{KP}^c + (V{=}L)$. Furthermore, $\psi[u,v]$ satisfies the uniqueness condition that for any u there exists at most one v such that $\psi[u,v]$ and

$$\mathcal{R} \;=\; \{x \in a : (\exists y \subseteq b)\psi[x,y]\}.$$

For any subset s of $a \times b$ and any $x \in a$ we set

$$(s)_x \;:=\; \{y \in b : \langle x,y\rangle \in s\},$$

and for any $c \subseteq b$ we define

$$\mathcal{R}_c \;:=\; \{x \in a : \psi[x,c]\}.$$

It is obvious that $(s)_x$ and $\mathcal{R}_c$ are sets, uniformly definable in their respective parameters by Δ^c separation. Also,

$$v <_b w \;:=\; v \subseteq b \wedge w \subseteq b \wedge v <_L w$$

and $v \leq_b w$ is short for $(v <_b w \vee v = w)$. Finally, given a set $s \subseteq a \times b$, we call a set y *a critical point of* s, in symbols $Cr[s,y]$, iff

$$y \subseteq b \;\wedge\; (\exists x \in \mathcal{R}_y)((s)_x \neq y) \;\wedge\; (\forall z <_b y)(\forall x \in \mathcal{R}_z)((s)_x = z).$$

In view of Remark 12, $Cr[s,t]$ is Δ^c. Clearly, if y is a critical point of s, then it is uniquely determined. Moreover, since $\mathcal{R}$ is not a set, we will prove that every subset of $a \times b$ has a critical point. In the following we list this and further properties of critical points; s and s' range over subsets of $a \times b$:

(C1) $\exists y \, Cr[s,y]$.

Proof of (C1). Assume that s has no critical point. Then $(s)_x = y$ for all $y \subseteq b$ and all $x \in \mathcal{R}_y$. Hence, if $x \in \mathcal{R}$, then $x \in \mathcal{R}_y$ for some $y \subseteq b$ and thus

$$\mathcal{R} = \{x \in a : \psi[x,(s)_x]\}.$$

However, this is a contradiction since $\{x \in a : \psi[x,(s)_x]\}$ is a set.

(C2) $Cr[s,y] \land z <_b y \rightarrow \mathcal{R}_z \times z \subseteq s$.

Proof of (C2). Assume $Cr[s,y]$ and $z <_b y$. For any element $\langle u,v\rangle$ of $\mathcal{R}_z \times z$ we then have $(s)_u = z$, hence $\langle u,v\rangle \in s$.

(C3) $Cr[s,y] \land \mathcal{R}_y \times y \subseteq s \rightarrow (\exists x \in \mathcal{R}_y)(y \subsetneq (s)_x)$.

Proof of (C3). From $\mathcal{R}_y \times y \subseteq s$ we obtain $y \subseteq (s)_x$ for all $x \in \mathcal{R}_y$. Hence $Cr[s,y]$ implies $y \subsetneq (s)_x$ for at least one $x \in \mathcal{R}_y$.

(C4) $Cr[s,y] \land Cr[s',y'] \land s \subseteq s' \land y <_b y' \rightarrow \mathcal{R}_y \times y \not\subseteq s$.

Proof of (C4). Assume the left hand side of this implication and $\mathcal{R}_y \times y \subseteq s$. By (C3) there exists $x \in \mathcal{R}_y$ such that $y \subsetneq (s)_x$. Hence we also have $y \subsetneq (s')_x$. This implies $y' \leq_b y$; a contradiction.

(C5) $Cr[s,y] \land \psi[x,z] \land z <_b y \rightarrow (s)_x = z$.

Proof of (C5). From $\psi[x,z]$ we obtain $x \in \mathcal{R}_z$. Therefore, $Cr[s,y]$ and $z <_b y$ yield $(s)_x = z$.

(C6) $Cr[s,y] \land x \in \mathcal{R}_z \land z <_b y \rightarrow \psi[x,(s)_x] \land (s)_x = z$.

Proof of (C6). Obvious from the previous assertion.

This finishes our preliminary remarks. Now let $\theta[s,t]$ be the formula stating that

- $s \subseteq a \times b \land t \subseteq a \times b$,
- there exists a y such that $Cr[s,y]$, i.e. y is the critical point of s,
- for all $x \in a$,

$$(t)_x = \begin{cases} y & \text{if } x \in \mathcal{R}_y, \\ (s)_x & \text{if } \psi[x,(s)_x] \land (s)_x <_b y, \\ b & \text{if } x \notin \mathcal{R}_y \land (\neg\psi[x,(s)_x] \lor y \leq_b (s)_x) \land \mathcal{R}_y \times y \subseteq s, \\ \varnothing & \text{if } x \notin \mathcal{R}_y \land (\neg\psi[x,(s)_x] \lor y \leq_b (s)_x) \land \mathcal{R}_y \times y \not\subseteq s. \end{cases}$$

Because of the uniqueness of $\psi[u, v]$ these four cases are mutually exclusive, and we immediately observe that $\theta[s, t]$ is equivalent to a Σ^c formula with the properties

$$(\forall s \subseteq a \times b)\exists! t\theta[s, t], \tag{1}$$

$$(\forall s \subseteq a \times b)\neg\theta[s, s]. \tag{2}$$

To establish the monotonicity of $\theta[s, t]$, we assume that

$$\theta[s, t] \wedge \theta[s', t'] \wedge s \subseteq s'$$

and that y and y' are the critical points of s and s', respectively. First we consider the following two special cases.

(S1) $y' \leq_b y \wedge \mathcal{R}_{y'} \times y' \subseteq s' \rightarrow (\forall x \in a)((t)_x \subseteq (t')_x)$.

Proof of (S1). Assume $y' \leq_b y$ and $\mathcal{R}_{y'} \times y' \subseteq s'$. For all $x \in a$ we have

$$(t')_x = \begin{cases} y' & \text{if } x \in \mathcal{R}_{y'}, \\ (s')_x & \text{if } \psi[x, (s')_x] \wedge (s')_x <_b y', \\ b & \text{if } x \notin \mathcal{R}_{y'} \wedge (\neg\psi[x, (s')_x] \vee y' \leq_b (s')_x) \end{cases}$$

and show $(t)_x \subseteq (t')_x$ by the following case distinction:

(S1.1) If $x \in \mathcal{R}_{y'}$ and $y' <_b y$, then (C6) yields $\psi[x, (s)_x]$ and $(s)_x = y'$. Hence $(t)_x = (s)_x = y' = (t')_x$.

(S1.2) If $x \in \mathcal{R}_{y'}$ and $y' = y$, then $x \in \mathcal{R}_y$ and $(t)_x = y = y' = (t')_x$.

(S1.3) If $\psi[x, (s')_x]$ and $(s')_x <_b y'$, then $(s)_x = (s')_x$ according to (C5). Hence $(t)_x = (s)_x = (s')_x = (t')_x$.

(S1.4) If $x \notin \mathcal{R}_{y'}$ and $(\neg\psi[x, (s')_x] \vee y' \leq_b (s')_x)$, then $(t')_x = b$. This implies $(t)_x \subseteq (t')_x$.

(S2) $y \leq_b y' \wedge \mathcal{R}_y \times y \not\subseteq s \rightarrow (\forall x \in a)((t)_x \subseteq (t')_x)$.

Proof of (S2). Assume $y \leq_b y'$ and $\mathcal{R}_y \times y \not\subseteq s$. Now for all $x \in a$,

$$(t)_x = \begin{cases} y & \text{if } x \in \mathcal{R}_y, \\ (s)_x & \text{if } \psi[x, (s)_x] \wedge (s)_x <_b y, \\ \varnothing & \text{if } x \notin \mathcal{R}_y \wedge (\neg\theta[x, (s)_x] \vee y \leq_b (s)_x) \end{cases}$$

and $(t)_x \subseteq (t')_x$ is obtained as follows:

(S2.1) If $x \in \mathcal{R}_y$ and $y <_b y'$, then $\psi[x, (s')_x]$ and $(s')_x = y$ because of (C6). Therefore, $(t)_x = (t')_x$.

(S2.2) If $x \in \mathcal{R}_y$ and $y = y'$, then $x \in \mathcal{R}_{y'}$ and $(t)_x = y = y' = (t')_x$.

(S2.3) If $\psi[x,(s)_x]$ and $(s)_x <_b y$, then $(s')_x = (s)_x$ follows from (C5). Consequently, we have $(t)_x = (s)_x = (s')_x = (t')_x$.

(S2.4) If $x \notin \mathcal{R}_y$ and $(\neg\psi[x,(s)_x] \vee y \leq_b (s)_x)$, then $(t)_x = \varnothing$. Therefore, $(t)_x \subseteq (t')_x$.

We claim that $t \subseteq t'$ is an easy consequence of (S1) and (S2). Indeed, simply consider:

- If $y' <_b y$, then $\mathcal{R}_{y'} \times y' \subseteq s$ according to (C2). Since $s \subseteq s'$ we also have $\mathcal{R}_{y'} \times y' \subseteq s'$ and thus (S1) implies $t \subseteq t'$.
- It $y' = y$ and $\mathcal{R}_y \times y \subseteq s$, then $\mathcal{R}_{y'} \times y' \subseteq s \subseteq s'$ and (S1) implies $t \subseteq t'$.
- It $y' = y$ and $\mathcal{R}_y \times y \not\subseteq s$, then $t \subseteq t'$ follows from (S2).
- If $y <_b y'$, then (C4) yields $\mathcal{R}_y \times y \not\subseteq s$. Hence $t \subseteq t'$ by (S2).

Summing up we observe that θ describes a monotone operator on $a \times b$. However, it is obvious how to extend it to the full universe: Simply define for all sets u and v,

$$\theta^*[u,v] \;:=\; \theta[u \cap (a \times b), v].$$

In view of (1) we then have $\forall x \exists! y \theta^*[x,y]$. Moreover, if F is the operator defined by (a Σ_1^c formula equivalent to) $\theta^*[u,v]$, then our previous considerations give us:

- $\forall x(F(x) \subseteq a \times b)$,
- $\forall x, y(x \subseteq y \;\rightarrow\; F(x) \subseteq F(y))$,
- $\forall x(F(x) \neq x)$.

But this is a contradiction to (FP^c), finishing our proof. □

Summarizing what we have obtained so far, we see that all the principles introduced in this article are equivalent over $\mathsf{KP}^c + (V{=}L)$. The following assertion about the equivalence of these principles is an immediate consequence of Theorems 4–10 and Theorems 14–16. That (β) is their consequence then follows from Theorem 11.

Corollary 17. *Over* $\mathsf{KP}^c + (V{=}L)$ *our principles* (FP^c)*,* (LFP^c)*,* (ScFP^c)*,* (ChFP^c)*,* (MI^c)*,* (BPI^c)*,* (BPI^c_{On})*, and* (SBS^c) *are equivalent to* $(\Sigma_1^c\text{-}\mathsf{Sep})$ *and, therefore, not provable in* KP^c*. They all imply* (β)*.*

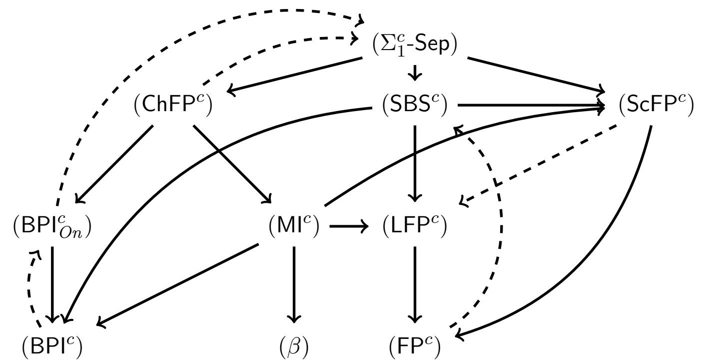

Figure 2: Over $\mathsf{KP}^c + (V{=}L)$. The arrow "$\dashrightarrow$" means that the proof of the corresponding implication uses $(V{=}L)$.

Although we already know the equivalence of our principles, it may be interesting to look at some more direct proofs of two implications. We sketch them below. Recall that given an operator F on the universe, we say that a set b is closed under F iff $F(b) \subseteq b$.

Lemma 18. *Working in KP^c, assume that F is an operator with*

$$\forall x, y(x \subseteq y \ \rightarrow\ F(x) \subseteq F(y) \subseteq a)$$

for some set a. If a set b is closed under F and a subset of all F-closed sets, then b is the least fixed point of F.

Proof. Since b is closed under F, the monotonicity of F implies that $F(b)$ is also closed under F. By assumption we thus have $b = F(b)$. Since every fixed point of F is closed under F, b is the least fixed point of F. $\square$

Theorem 19. *Over $\mathsf{KP}^c + (V{=}L)$, (ScFP^c) implies (LFP^c).*

Proof. Let F be an operator such that

$$\forall x, y(x \subseteq y \ \rightarrow\ F(x) \subseteq F(y) \subseteq a)$$

for some set a. Now we define

$$U \ :=\ \{x : a\backslash x \text{ is closed under } F\}.$$

We first observe that U is set-complete. Indeed, if $b \subseteq U$ then

$$F(a\backslash \bigcup b)\ \subseteq\ \bigcap\{F(a\backslash x) : x \in b\}\ \subseteq\ \bigcap\{a\backslash x : x \in b\}\ =\ a\backslash \bigcup b.$$

This means that $\bigcup b$ is closed under F, hence an element of U. To reach a contradiction we now assume that F has no least fixed point. Given an $x \in U$, we know that $a \backslash x$ is closed under F. We set

$$\begin{aligned} D_1[u,v] &:= F(v) \subseteq v \ \wedge \ a \backslash u \not\subseteq v, \\ D_2[u,v] &:= D_1[u,v] \ \wedge \ (\forall z <_L y) \neg D_1[u,z] \end{aligned}$$

and observe that $D_2[u,v]$ is equivalent to a Σ_1^c formula. We obtain from the previous lemma that

$$(\forall x \in U) \exists y D_1[x,y].$$

Thus we also have

$$(\forall x \in U) \exists ! y D_2[x,y].$$

Now we let G be the operator associated with this formula according to Lemma 2. With the help of G we define a further operator H, now action on the universe,

$$H(x) := \begin{cases} a \setminus ((a \backslash x) \cap \bigcap \{z \leq_L G(x) : F(z) \subseteq z\}) & \text{if } x \in U, \\ \varnothing & \text{if } x \notin U. \end{cases}$$

We claim that H is monotone on U. To see why, pick $x, y \in U$ such that $x \subseteq y$. Then $a \backslash y \subseteq a \backslash x$ and $G(x) \leq_L G(y)$. This implies that

$$\bigcap \{z \leq_L G(y) : F(z) \subseteq z\} \ \subseteq \ \bigcap \{z \leq_L G(x) : F(z) \subseteq z\}.$$

From that we obtain

$$(a \backslash y) \cap \bigcap \{z \leq_L G(y) : F(z) \subseteq z\} \ \subseteq \ (a \backslash x) \cap \bigcap \{z \leq_L G(x) : F(z) \subseteq z\}$$

and, therefore, $H(x) \subseteq H(y)$, establishing the monotonicity of H.

Since $\varnothing \in U$ and the intersection of F-closed sets is F-closed, we have $H(x) \in U$ for all x. Thus it follows that $Msc[H, a, U]$. Now we apply (ScFP^c) and conclude that H has a fixed point b.

However, the following argument shows that this is a contradiction: By the definition of G we have $(a \backslash b) \cap G(b) \subsetneq a \setminus b$, hence also

$$(a \backslash b) \cap \bigcap \{z \leq_L G(b) : F(z) \subseteq z\} \ \subsetneq \ a \backslash b,$$

and thus, as a consequence of the definition of H, we have $H(b) \neq b$. $\square$

Theorem 20. *In* $\mathsf{KP}^c + (V{=}L) + (\mathsf{ChFP}^c)$ *every instance of* (Σ_1^c-Sep) *is provable.*

Proof. Assume that there exist a set a and a Δ_0^c formula $A[u, v]$ such that

$$\mathcal{R} \ := \ \{x \in a : \exists y A[x, y]\}$$

is not a set. We will show that this leads to a contradiction. We define an operator F on the universe by

$$F(u) \ := \ \{x \in a : (\exists y \in u) A[x, y]\}$$

for arbitrary u. Obviously,

$$\forall u, v(u \subseteq v \ \to \ F(u) \subseteq F(v)).$$

Furthermore, since $\mathcal{R}$ is not a set and $(V{=}L)$ is available, we have

$$\forall u \exists ! v(F(u) \subsetneq F(v) \ \wedge \ (\forall z <_L v) \neg (F(u) \subsetneq F(z))).$$

Let G be the operator such that $G(u)$ is the respective witness; such an operator exists according to Lemma 2. Now we simply have to iterate the operator G along the ordinals. Simply set

$$B[\alpha, f, u] \ := \ \begin{cases} Fun[f, \alpha + 1] \ \wedge \ u = f(\alpha) \ \wedge \\ (\forall \beta \leq \alpha)(f(\beta) = G(\bigcup\limits_{\xi < \beta} f(\xi))) \end{cases}$$

By the properties of G we then have

$$\forall \alpha \exists ! u \exists f B[\alpha, f, u],$$

and thus, again in view of Lemma 2, there exists an operator H that maps every ordinal to the witness according to the previous line. Hence,

$$\forall \alpha (H(\alpha) \in a) \ \wedge \ \forall \alpha, \beta(\alpha < \beta \ \to \ H(\alpha) \subsetneq H(\eta)).$$

However, according to (ChFP^c) this is not possible, and so we have reached the desired contradiction. □

6 Future work

This article introduces a collection of fixed point axioms and a series of principles closely related to those. We present a few basic results about the mutual relations between these systems in the basic systems KP and its class extension KP^c and study particularly what happens if the axiom of constructibility is added, the main result being that they are all equivalent to Σ_1 separation then.

In the meantime we also know a lot about the mutual relationships of our principles over KP plus axiom beta (β). However, the methods of proof are very different and including them would have been beyond the scope of this article. A publication [8] dealing with these questions is in preparation.

Further work in preparation addresses the proof-theoretic relationship of our principles over KP alone and deals with the question when adding $(V{=}L)$ leads to an increase in proof-theoretic strength.

7 Appendix 1

In this first appendix we list the first order versions of (ScFP^c), (ChFP^c), (BPI^c), (BPI^c_{On}), (SBS^c), and $(\Sigma^c_1\text{-}\mathsf{Sep})$. Following the strategy of the proof of Theorem 1 the reader can easily verify that for every of our second order principles $(P2)$ and its corresponding first order version $(P1)$, the theory $\mathsf{KP}^c + (P2)$ is a conservative extension of $\mathsf{KP} + (P1)$.

$(\Sigma_1\text{-}\mathsf{ScFP})$ :: corresponding to (ScFP^c)

For all Σ_1 formulas $A[u, v]$, all Σ formulas $B[u]$, and all Π formulas $C[u]$: The conjunction of

(1) $\forall x(B[x] \leftrightarrow C[x])$,

(2) $\forall x \exists! y A[x, y]$,

(3) $\forall x, y(A[x, y] \;\rightarrow\; y \subseteq a \;\wedge\; B[y])$,

(4) $\forall z((\forall x \in z)B[x] \;\rightarrow\; B[\bigcup z])$,

(5) $\forall x_0, x_1, y_0, y_1(B[x_0] \wedge B[x_1] \wedge A[x_0, y_0] \wedge A[x_1, y_1] \wedge x_0 \subseteq x_1 \;\rightarrow\; y_0 \subseteq y_1)$

implies that there exists an x such that $A[x, x]$.

$(\Sigma_1\text{-}\mathsf{ChFP})$ :: corresponding to (ChFP^c)

For all Σ_1 formulas $A[\alpha, x]$: The conjunction of

(1) $\forall \alpha \exists! x A[\alpha, x]$,

(2) $\forall \alpha_0, \alpha_1, x_0, x_1(A[\alpha_0, x_0] \;\wedge\; A[\alpha_1, x_1] \;\wedge\; \alpha_0 < \alpha_1 \;\rightarrow\; x_0 \subseteq x_1 \subseteq a)$,

implies that there exists an α such that $\exists x(A[\alpha, x] \wedge A[\alpha + 1, x])$.

$(\Sigma_1\text{-}\mathsf{BPI})$ :: corresponding to (BPI^c)

For all Σ_1 formulas $A[u, v]$: The conjunction of

(1) $\forall x \exists! y A[x, y]$,

(2) $\forall x, y(A[x, y] \;\rightarrow\; y \in a)$,

implies that there exist x, y, z such that $x \neq y \;\wedge\; A[x, z] \;\wedge\; A[y, z]$.

$(\Sigma_1\text{-}\mathsf{BPI}_{On})$ :: corresponding to (BPI^c_{On})

For all Σ_1 formulas $A[u, v]$: The conjunction of

(1) $\forall \alpha \exists! x A[\alpha, x]$,

(2) $\forall \alpha, x(A[\alpha, x] \;\rightarrow\; x \in a)$,

implies that there exist α, β, z such that $\alpha \neq \beta \;\wedge\; A[\alpha, z] \;\wedge\; A[\beta, z]$.

(Σ_1-SBS) **:: corresponding to** (SBS^c)

For all Σ formulas $A[u,v]$ and all Π formulas $B[u,v]$:

$$\forall x, y(A[x,y] \leftrightarrow B[x,y]) \;\rightarrow\; \exists z(z = \{x \in a : (\exists y \subseteq b)A[x,y]\}).$$

Remark 21. What immediately catches the eye is that the theories $\mathsf{KP}^{\mathcal{P}}$ and $\mathsf{KP}(\mathcal{P})$ of Mathias and Rathjen comprise – in contrast to $\mathsf{KP}+(\Sigma_1\text{-}\mathsf{SBS})$ - the power set axiom. But apart from that, even $\mathsf{KP}+(\Delta_0^{\mathcal{P}}\text{-}\mathsf{Sep})$ is significantly stronger then $\mathsf{KP}+(\Sigma_1\text{-}\mathsf{SBS})$. The theory $\mathsf{KP}+(\Delta_0^{\mathcal{P}}\text{-}\mathsf{Sep})$ clearly comprises full second order arithmetic, whereas $\mathsf{KP}+(\Sigma_1\text{-}\mathsf{SBS})$ clearly is contained in $\mathsf{KP}+(\Sigma_1\text{-}\mathsf{Sep})$. Therefore, $\mathsf{KP}+(\Sigma_1\text{-}\mathsf{SBS})$ is proof-theoretically much weaker that $\mathsf{KP}+(\Delta_0^{\mathcal{P}}\text{-}\mathsf{Sep})$. On the other hand, it seems not so clear whether all instances of (Σ_1-SBS) can be proved in $\mathsf{KP}+(\Delta_0^{\mathcal{P}}\text{-}\mathsf{Sep})$. What makes ($\Sigma_1$-$\mathsf{SBS}$) comparatively strong is that it provides for separation with respect to a subset-bounded existential formula with a Δ_1 kernel. According to our knowledge such principles have not be studied elesewhere.

(Σ_1-Sep) **:: corresponding to** (Σ_1^c-Sep)

For all Σ_1 formulas $A[u]$:

$$\exists y(y = \{x \in a : A[x]\}).$$

8 Appendix 2

In this second appendix we sketch the proof of Lemma 13. There are different ways to introduce the constructible hierearchy L. Here we follow Barwise [1] and make use of the notation and terminology used there, without further explanations. We only recall that

$$\mathscr{D}(a) \;:=\; a \cup \bigcup_{1\le i\le N} \{\mathscr{F}_i(x,y) : x, y \in a\},$$

where $\mathscr{F}_1, \dots, \mathscr{F}_N$ are the Σ_1 Gödel operations used in [1]. However, we begin with a preliminary observation, whose proof in KP^c is left to the reader.

Claim 1. *Suppose that f is an injective mapping from a set a to an ordinal α. Then there exist an ordinal $\beta \le \alpha$ and an bijective mapping g from a to β such that*

$$(\forall x, y \in a)(g(x) < g(y) \;\leftrightarrow\; f(x) < f(y)).$$

We next introduce a notation for expressing that a subset a of L is order-preservingly embedded into an ordinal. For $a \subseteq L$ we write

$$f : a \simeq_L \alpha$$

to state that f is a bijective mapping from a to α such that

$$(\forall x, y \in a)(x <_L y \leftrightarrow f(x) < f(y)).$$

It should be obvious that this notion fulfills the following uniqueness condition:

Claim 2. *In* KP^c *we can prove that*

$$\alpha \le \beta \wedge f : L_\alpha \simeq_L \gamma \wedge g : L_\beta \simeq_L \delta \rightarrow (\forall x \in L_\alpha)(f(x) = g(x)).$$

Claim 3. *We can prove in* KP^c *that*

$$f : L_\alpha \simeq_L \beta \rightarrow \exists \gamma, g(g : \mathscr{D}(L_\alpha \cup \{L_\alpha\}) \simeq_L \gamma).$$

Proof. Assume $f : L_\alpha \simeq_L \beta$ and extend f to a bijection f^+ from $L_\alpha \cup \{L_\alpha\}$ to $\delta := \beta + 1$ by

$$f^+(x) := \begin{cases} f(x) & \text{if } x \in L_\alpha, \\ \beta & \text{if } x = L_\alpha. \end{cases}$$

We obtain an injective function h from $\mathscr{D}(L_\alpha \cup \{L_\alpha\})$ to $N \times \delta \times \delta$ by setting

$$h(x) := \langle 0, 0, f^+(x) \rangle$$

if $x \in L_\alpha \cup \{L_\alpha\}$ and

$$h(x) := \langle i, f^+(y), f^+(z) \rangle$$

if $x \in \mathscr{D}(L_\alpha \cup \{L_\alpha\}) \setminus (L_\alpha \cup \{L_\alpha\})$ and $x = \mathscr{F}_i(y, z)$ where $\langle i, y, z \rangle$ is the triple that represents x in the definition of the well-ordering $<_L$ on L. From that it is not difficult to see that there exist an ordinal δ' and injective function h' from $\mathscr{D}(L_\alpha \cup \{L_\alpha\})$ to δ' such that

$$(\forall x, y \in \mathscr{D}(L_\alpha \cup \{L_\alpha\}))(x <_L y \rightarrow h'(x) < h'(y)).$$

Now our assertion is immediate from Claim 1. □

Claim 4. *We can prove in* KP *and* KP^c *that*

$$\forall \alpha \exists \beta \exists f(f : L_\alpha \simeq_L \beta).$$

Proof. This assertion is show by induction on α. Clear for $\alpha = 0$. If α is a successor ordinal, then we simply have to use Claim 3. Now let α be a limit ordinal. Then the induction hypothesis implies

$$(\forall \xi < \alpha) \exists \eta, g(g : L_\xi \simeq_L \eta).$$

By Σ^c reflection there exist an ordinal β and a set b such that

$$(\forall \xi < \alpha)(\exists \eta < \beta)(\exists g \in b)(g : L_\xi \simeq_L \eta).$$

We define

$$f := \bigcup \{g \in b : (\exists \xi < \alpha)(\exists \eta < \beta)(g : L_\xi \simeq_L \eta)\}.$$

Claim 2 insures that f is the required function. □

To finish the proof of Lemma 13, consider the Σ_1^c formula

$$A[x,\xi] \ := \ \exists\alpha,\beta,f(x \in L_\alpha \ \wedge \ f : L_\alpha \simeq_L \beta \ \wedge \ x = f(\xi)).$$

From Claim 2 and Claim 4 we conclude that

$$\forall x \exists! \xi A[x,\xi] \ \wedge \ \forall x,y,\xi(A[x,\xi] \ \wedge \ A[y,\xi] \ \to \ x = y).$$

Therefore, Lemma 2 implies the existence of the requested operator H_L.

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