

Low-energy dynamics of 3d $\mathcal{N} = 2$ G_2 supersymmetric gauge theory

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ABSTRACT: We study a three-dimensional $\mathcal{N} = 2$ supersymmetric G_2 gauge theory with and without fundamental matters. We find that a classical Coulomb branch of the moduli space of vacua is partly lifted by monopole-instantons and the quantum Coulomb moduli space would be described by a complex one-dimensional space. Depending on the number of the matters in a fundamental representation, the low-energy dynamics of the theory shows various phases like s-confinement or quantum merging of the Coulomb and the Higgs branches. We also investigate superconformal indices as an independent check of our analysis.

KEYWORDS: Confinement, Supersymmetric Gauge Theory, Supersymmetry and Duality, Duality in Gauge Field Theories

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1 Introduction

Supersymmetry allows us to understand the dynamics of strongly-coupled gauge theories. Especially, holomorphy and non-renormalization theorems put quite strong constraints on low-energy SUSY dynamics and we can extract non-perturbative effects exactly [1–3]. Since supersymmetry is a symmetry between bosons and fermions, SUSY theories typically contain scalar fields. The potential of these scalar modes often possesses flat directions, and this subspace is called a moduli space of vacua. Then we can introduce vacuum expectation values (vevs) to these massless modes. This fact very simplifies the SUSY dynamics because by taking the large vevs we can analyze the theory at a semi-classical limit and extrapolate the strongly-coupled region. So it is essential to understand the structure of the moduli space of vacua. In addition to this brilliant property, one can regard supersymmetry as one of the most prevalent theories for Beyond the Standard-Model. Therefore it would be exciting and relevant to investigate possible SUSY theories and their dynamics including such as SUSY breaking scenarios.

This paper aims to study non-perturbative aspects of G_2 gauge theories. G_2 gauge theories have been studied continuously for the last 15 years. For non-supersymmetric G_2

cases, see [4–10]. Since the G_2 group has a trivial center, it is interesting to investigate its confinement phenomena. A confinement phase is usually related to the center symmetry. Wilson loops become well-behaved order parameters in pure gauge theories with a nontrivial center. However, in G_2 pure YM theories, Wilson loops in any representations are not well-defined order parameters since the Wilson loops are always screened by gluons. Also in a G_2 QCD with and without Higgs fields, the confinement and Higgs phases are continuously connected, and we have no order parameter to distinguish them. These situations are similar to an ordinary $SU(3)$ QCD and SQCD with fundamental (s)quarks. Furthermore, G_2 has an $SU(3)$ group as a maximal subgroup. By breaking G_2 to $SU(3)$ via the Higgs field, the adjoint representation is decomposed into $\mathbf{8} + \mathbf{3} + \bar{\mathbf{3}}$. Therefore we can connect the G_2 dynamics to the ordinary vector-like QCD.

In 4d, a supersymmetric version of the G_2 theory was also well investigated. An $\mathcal{N} = 2$ supersymmetric G_2 gauge theory (called a Seiberg-Witten theory) was studied in [11, 12], where the Seiberg-Witten curves and their singularities are studied. A 4d $\mathcal{N} = 1$ G_2 gauge theory was investigated in [13–19], where it was found that the G_2 dynamics is similar to the 4d $\mathcal{N} = 1$ $SU(N)$ SQCD. Depending on the number of fundamental matters, there are various phases. For $N_f = 0$, there are discrete SUSY vacua while for $N_f = 1, \dots, 3$ there are no stable SUSY vacua. For $N_f = 4$, we observe the quantum-deformed moduli space and for $N_f = 5$ the theory is s-confined. For $N_f \geq 6$ we have a Seiberg dual description. Not limited to the development of the SUSY G_2 gauge theories, in 4d, other SUSY exceptional gauge theories were also well-studied [20–27].

Recently, the dynamics of the 3d $\mathcal{N} = 2$ SUSY gauge theories has been better understood. One of the most prominent developments is a localization calculation of SUSY-preserving quantities, such as partition functions, superconformal indices, supersymmetric Wilson loops and so on. Using the exact results of these quantities, we can test various conjectures such as AdS/CFT correspondence, Seiberg dualities, mirror symmetry, etc., and one can even discover unknown dualities. The other progress is a derivation of the 3d (Seiberg) dualities from 4d dualities [28, 29], where the discussion emphasized the significance of the twisted instantons which appear when putting a 4d theory on a circle. In 3d, there are some new properties which are absent in 4d. For instance, we can introduce Chern-Simons terms and real masses by background gauging the global symmetries. The 3d vector superfields supply scalar fields whose potential is (classically) flat, and this would be a new modulus absent in 4d. The 3d gauge coupling is relevant even for $U(1)$ gauge theories, and we can expect non-trivial dynamics of the $U(1)$. In 3d, there are various dualities known, including the dualities with and without Chern-Simons terms. By connecting the 3d and 4d dualities and their dynamics à la [28, 29], we can obtain a clear and unified understanding of the SUSY gauge theories in various dimensions. While these developments have been mostly achieved for the theories with classical Lie groups, little has been established in the 3d $\mathcal{N} = 2$ exceptional gauge theories.

Given the above situations, we cut into a 3d $\mathcal{N} = 2$ supersymmetric G_2 gauge theory with and without fundamental matters in detail. This attempt is the first step for understanding the 3d $\mathcal{N} = 2$ exceptional gauge theories and would be an excellent representative example since G_2 is a tractable exceptional group. We first classically analyze the moduli

	G_2	$SU(N_f)$	$U(1)$	$U(1)_R$
Q	\square (7)	\square	1	R
λ	adj. (14)	1	0	1
$\eta_{N_f} = \Lambda^b$	1	1	$2N_f$	$2N_f(R - 1) + 8$

Table 1. Quantum numbers of the 4d $\mathcal{N} = 1$ G_2 gauge theory.

space of vacua of the G_2 theory and then turn on quantum effects mostly arising from monopole-instantons. Those non-perturbative effects are exactly determined by holomorphy and through various consistency checks via deformations. Especially the connection with 3d $\mathcal{N} = 2$ $SU(3)$ SQCD would be a good test of our study. We find non-perturbative superpotentials consistent with all the symmetries as in the 4d G_2 cases [13, 14]. We will find that for $N_f \leq 2$ the theory has no stable SUSY vacua, for $N_f = 3$ the classical moduli spaces are quantum-mechanically merged and for $N_f = 4$ we will encounter so-called an “s-confinement” phase, where N_f is a number of fundamental matters. We also study superconformal indices of the theory and discuss the spectrum of the low-lying operators (or states). The superconformal index also gives us a non-perturbative check of our treatment. In particular, this will confirm a structure of the quantum Coulomb branch which would be drastically different from the classical picture.

The rest of this paper is organized as follows. In section 2, we briefly review the dynamics of the 4d $\mathcal{N} = 1$ supersymmetric G_2 gauge theories and define some notations. More complete notations used in the paper are given in appendix. In section 3, we discuss the Coulomb branch of the moduli space of vacua in the 3d $\mathcal{N} = 2$ SUSY gauge theory exclusively focusing on the G_2 case. In section 4 and 5, we investigate quantum aspects of the 3d $\mathcal{N} = 2$ G_2 gauge theory with and without fundamental matters. In section 6, the connection between the 3d and 4d G_2 gauge theories is investigated. In section 7, we compute the superconformal indices for 3d $\mathcal{N} = 2$ G_2 gauge theories, and this would be a non-trivial check of our analysis. In section 8, we will summarize our findings and discuss possible future directions.

2 Review of 4d $\mathcal{N} = 1$ G_2 SQCD

We will briefly review the results of the 4d $\mathcal{N} = 1$ supersymmetric G_2 gauge theory with N_f fundamental matters [13, 14]. Since a coefficient of the one-loop beta-function is given by $b = 12 - N_f$, the theory is asymptotically free for $N_f < 12$. The matter contents and their quantum numbers are summarized in table 1, where η_{N_f} is a dynamical scale of a G_2 gauge coupling, Q is a chiral superfield in a fundamental representation and λ is a gaugino in a vector superfield. We listed the generic R-charge in table 1 and of course, the infrared $U(1)_R$ charge is different from this value. Notice that we are listing the anomalous $U(1)$ and $U(1)_R$ symmetries, therefore the linear combination of these $U(1)$ ’s becomes a genuine $U(1)_R$ symmetry and the other global $U(1)$ is spurious in 4d due to a chiral anomaly. Since we are interested in a 3d theory, we will use this charge assignment for the rest of our paper.

	SU(N_f)	U(1)	U(1) _R
$M := QQ$	$\square\square$	2	2R
$B := Q^3$	$\begin{array}{c} \square \\ \square \\ \square \end{array}$	3	3R
$F := Q^4$	$\begin{array}{c} \square \\ \square \\ \square \\ \square \end{array}$	4	4R

Table 2. Gauge invariants of the 4d $\mathcal{N} = 1$ G_2 gauge theory.

Since a G_2 group has a (real) fundamental representation with dimension 7 and there are three independent invariant tensors; δ_{ab} , f_{abc} , \tilde{f}_{abcd} , we can construct the following gauge invariant operators from the chiral superfields (table 2). B_{ijk} is possible for $N_f \geq 3$ and F_{ijkl} is for $N_f \geq 4$.

In the following we briefly sketch the quantum dynamics depending on the number of fundamentals. For more detailed analyses, see [13–16, 30].

$N_f = 0$: discrete SUSY vacua. Let us first consider the pure G_2 Super Yang-Mills (SYM) theory. The theory supports four discrete supersymmetric vacua [5, 13, 15, 16] (see also [31–33]). The superpotential is given by

$$W \sim \pm\eta^{1/4}, \quad \pm i\eta^{1/4}, \tag{2.1}$$

where we omitted numerical factors for simplicity and only kept the fourth root of unity.

$N_f = 1, 2$: gaugino condensation. Next we move on to the G_2 SQCD with one or two fundamentals. The superpotential is dynamically generated by gaugino condensation. For $N_f = 1, 2$, the superpotential consistent with all the symmetries takes

$$W = \left(\frac{\eta_{N_f}}{\det M} \right)^{\frac{1}{4-N_f}}, \tag{2.2}$$

and there is no stable SUSY vacua.

$N_f = 3$: instanton generated superpotential. In this case, the dynamically generated superpotential is again allowed although we have the cubic baryonic branch labeled by $B \equiv \frac{1}{3!} f_{abc} Q^a Q^b Q^c$. In the case of three flavors, via the generic vacuum expectation value on the Higgs branch, the G_2 gauge group is completely broken and the semi-classical calculation of the instanton is justified. As a result, we obtain

$$W = \frac{\eta_3}{\det M_{ij} - B^2}, \tag{2.3}$$

which again has no stable SUSY vacua.

$N_f = 4$: quantum deformed moduli space. For $N_f = 4$, the quartic baryon $F \equiv \frac{1}{4!} \tilde{f}_{abcd} Q^a Q^b Q^c Q^d$ can be constructed and we classically have some constraints between

	SU($N_f - 3$)	SU(N_f)	U(1)	U(1) _R
\bar{q}	$\bar{\square}$	$\bar{\square}$	-1	$1 - R - \frac{1}{N_f - 3}$
\bar{q}_0	$\bar{\square}$	1	0	$1 - \frac{1}{N_f - 3}$
s	$\square\square$	1	0	$\frac{2}{N_f - 2}$
M	1	$\square\square$	2	$2R$
$\tilde{\eta}_{N_f} = \Lambda^{\tilde{b}}$	1	1	$-N_f$	$-N_f(R - 1) - 4$

Table 3. A magnetic dual of the 4d $\mathcal{N} = 1$ G_2 gauge theory.

the mesonic, cubic-baryonic and quartic-baryonic operators. The classical moduli space is quantum mechanically corrected and the origin of the moduli space of vacua is lifted;

$$\det M - F^2 - B^i M_{ij} B^j = \eta_4. \quad (2.4)$$

The origin of the moduli space of vacua is lifted and some of the symmetries are inevitably broken in this phase.

$N_f = 5$: s-confinement. For $N_f = 5$, the classical moduli space, including the classical relation between the gauge invariant chiral superfields, is not modified. Especially the origin of the moduli space remains as the quantum moduli space. Therefore it is called s-confinement where we have the confining phase without symmetry breaking. The symmetry, holomorphy and mass-deformation arguments lead to the superpotential

$$W = \frac{1}{\eta_5} \left(-\det M + \frac{1}{2} B^{ik} B^{jl} M_{ij} M_{kl} + F^i M_{ij} F^j + \frac{1}{4} \epsilon_{ijklm} F^i B^{jk} B^{lm} \right). \quad (2.5)$$

The classical constraints are represented via the equations of motion for the above superpotential.

$N_f \geq 6$: Seiberg duality. For $N_f \geq 6$, we expect a non-abelian Coulomb phase and the low-energy dynamics is described by the Seiberg magnetic dual [30] with an SU($N_f - 3$) gauge group with a superpotential

$$W = M \bar{q} \bar{q} s + \bar{q}_0 \bar{q}_0 s + \det s. \quad (2.6)$$

The matter content includes the anti-fundamentals, a symmetric matter and a symmetric meson which is a gauge singlet. The quantum numbers for those matters and for the dual dynamical scale $\tilde{\eta}$ of the SU($N_f - 3$) gauge group are listed in table 3.

This dual was first found by using the Seiberg duality of the 4d $\mathcal{N} = 1$ Spin(7) gauge theory with spinorial matters in [30]. By giving a vacuum expectation value to the spinorial representation, the G_2 Seiberg dual is obtained. Notice again that the U(1) global symmetry is spurious and then under the matching of the baryonic operators

$$B := Q^3 \leftrightarrow \bar{q}^{N_f - 3} \quad (2.7)$$

$$F := Q^4 \leftrightarrow \bar{q}^{N_f - 4} \bar{q}_0, \quad (2.8)$$

this U(1) is not acting properly.

3 Coulomb branch and monopole operators

In this section we explain how we define the Coulomb branch operators and calculate their global charges (quantum-mechanically we have to construct so-called monopole (creating) operators.). For the monopole in the G_2 case and the quantization of the magnetic charges, please see [34–36] and [37–39].

The moduli space of vacua in 3d $\mathcal{N} = 2$ SUSY gauge theories is described by two regions, Higgs and Coulomb branches, where chiral and vector superfields take non-zero vacuum expectation values respectively. Of course, depending on the representation of the chiral superfields and the breaking pattern of the gauge group, the Higgs branch might be called a Coulomb or confinement phase. The Higgs branch is parametrized by the gauge invariant composites of the chiral superfields with some constraints between them. This is the same as the 4d case, so we have three composites M_{ij}, B_{ijk} (for $N_f \geq 3$) and F_{ijkl} (for $N_f \geq 4$) for the G_2 case. For $N_f \geq 4$, at a generic point of the Higgs branch, G_2 can be completely higgsed.

Let us consider the classical Coulomb branch of the 3d $\mathcal{N} = 2$ supersymmetric G_2 gauge theory. We need $\text{rank}(G_2) = 2$ coordinates to describe it. At a generic point of the Coulomb branch, G_2 is broken to $U(1) \times U(1)$. For each $U(1)$ factor, a corresponding $U(1)$ vector superfield yields complex one-dimensional Coulomb branch which consists of a real scalar in a vector superfield and a dual photon. The dual photon is Hodge-dual to a gauge field and then it is compact. We would like to parametrize these Coulomb branches in the language of the UV theory. A set of operators to describe the Coulomb branch is semi-classically given by

$$V_\alpha \simeq \exp [\text{Tr} (\phi \alpha^\vee \cdot \mathbf{H})] \tag{3.1}$$

where α^\vee denotes a dual root defined as

$$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle} \tag{3.2}$$

and ϕ is an adjoint scalar in a vector superfield and is valued in the Cartan subalgebra:

$$\mathbf{H} = (H_1, H_2), \quad \phi = \left(\phi_1, \frac{\phi_2}{\sqrt{3}} \right), \tag{3.3}$$

$$\phi = \phi \cdot \mathbf{H} = \phi_1 H_1 + \frac{\phi_2}{\sqrt{3}} H_2 \tag{3.4}$$

By using the Weyl symmetry we can choose the following chamber with

$$\phi_1 \geq \phi_2 \geq 0. \tag{3.5}$$

In the definition of the Coulomb branch operator we are omitting the gauge coupling dependence for simplicity. Since the superfield completion is manifest, we are not specifying the difference between the scalar fields and chiral superfields. Furthermore, rigorously speaking, we have to dualize the gauge field to a dual photon and include this into the

above scalar field to make one complex field. But in this paper we omit this for simplicity and the dependence of the dual photon can be easily restored.

Depending on how to give a vev to each ϕ_i , there are in principle various regions in the Coulomb branch and also there are many corresponding monopole operators. For each positive root (more correctly for each positive dual root), we obtain

$$\begin{aligned} V_\alpha &\simeq \exp(2\phi_2), & V_\beta &\simeq \exp(\phi_1 - \phi_2), & V_{\alpha+\beta} &\simeq \exp(3\phi_1 - \phi_2) \\ V_{2\alpha+\beta} &\simeq \exp(3\phi_1 + \phi_2), & V_{3\alpha+\beta} &\simeq \exp(\phi_1 + \phi_2), & V_{3\alpha+2\beta} &\simeq \exp(2\phi_1) =: Z \end{aligned} \quad (3.6)$$

where $V_{3\alpha+2\beta} = \exp(2\phi_1)$ will be of special importance, so we labeled it as Z for later convenience.

3.1 Callias index theorem and zero-modes

Fermion zero-modes for each Coulomb branch operator can be counted by using Callias' index theorem [40–42]. The index theorem states that the number of zero-modes for fermions in some representations of the gauge group is given by

$$N_\alpha = \frac{1}{2} \sum_{w \in \text{all the weights}} \text{sign}(w(\phi)) w(g), \quad (3.7)$$

where $g = \boldsymbol{\alpha}^\vee \cdot \mathbf{H}$ represents the magnetic charge of the monopole we are considering and ϕ is the coordinates of the Coulomb branch. The summation is taken over all the weights in a representation.

For two roots $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, for example, using

$$\boldsymbol{\alpha}^\vee = (0, 2\sqrt{3}), \quad \boldsymbol{\beta}^\vee = (1, -\sqrt{3}), \quad (3.8)$$

the zero-modes for adjoint and fundamental fermions are computed as

$$\begin{aligned} N_\alpha^{\text{adj.}} &= \frac{1}{2} \left[-6\text{sign}(\phi_1 - \phi_2) - 2\text{sign}(3\phi_1 - \phi_2) + 4\text{sign}(\phi_2) \right. \\ &\quad \left. + 6\text{sign}(\phi_1 + \phi_2) + 2\text{sign}(3\phi_1 + \phi_2) \right] = 2 \\ N_\alpha^{\text{fund.}} &= \frac{1}{2} \left[-2\text{sign}(3\phi_1 - \phi_2) + 4\text{sign}(\phi_2) + 2\text{sign}(3\phi_1 + \phi_2) \right] = 2, \end{aligned} \quad (3.9)$$

$$\begin{aligned} N_\beta^{\text{adj.}} &= \frac{1}{2} \left[2\text{sign}\phi_1 + 4\text{sign}(\phi_1 - \phi_2) + 2\text{sign}(3\phi_1 - \phi_2) - 2\text{sign}(\phi_2) - 2\text{sign}(\phi_1 + \phi_2) \right] = 2 \\ N_\beta^{\text{fund.}} &= \frac{1}{2} \left[2\text{sign}(3\phi_1 - \phi_2) - 2\text{sign}\phi_2 \right] = 0, \end{aligned} \quad (3.10)$$

where the sign function is evaluated under the Weyl chamber. In the presence of the monopole vertex, the naive global symmetries are broken because the vertex contains the fermions corresponding to the zero-modes above. In order to recover the global symmetries, we have to transform the monopole operator in a opposite way to the fermions under the global symmetry [43]. Since the Coulomb branch operators are made from the vector

	adj.	fund.	U(1)	U(1) _R
V_α	2	2	$-2N_f$	$2N_f(1 - R) - 2$
V_β	2	0	0	-2
$V_{\alpha+\beta}$	8	2	$-2N_f$	$2N_f(1 - R) - 8$
$V_{2\alpha+\beta}$	10	4	$-4N_f$	$4N_f(1 - R) - 10$
$V_{3\alpha+\beta}$	4	2	$-2N_f$	$2N_f(1 - R) - 4$
$Z = V_{3\alpha+2\beta} = V_\alpha V_\beta^2$	6	2	$-2N_f$	$2N_f(1 - R) - 6$

Table 4. Zero-modes for Coulomb branch operators and global charges.

superfields, they are originally neutral. However, on the monopole background, they are non-trivially charged. Consequently, the operator V_α possesses a $U(1)_R$ -charge

$$R[V_\alpha] = -2 \cdot R[\lambda] - 2 \cdot N_f \cdot R[\psi_Q] = 2N_f(1 - R) - 2, \tag{3.11}$$

while for V_β and $Z = V_{3\alpha+2\beta}$

$$R[V_\beta] = -2 \cdot R[\lambda] = -2 \tag{3.12}$$

$$R[Z] = R[V_\alpha V_\beta^2] = 2N_f(1 - R) - 6. \tag{3.13}$$

The other number of zero-modes for each operator is summarized in table 4. It is remarkable to note that the $U(1)_R$ -charge of the operator V_β depends neither on N_f or R , due to the absence of zero modes in fundamental representations. This implies that the inverse of V_β will be ubiquitous in the superpotential for any number of flavors. We will discuss the uplift of the V_β -direction in the next section.

3.2 Mixed Chern-Simons terms and zero-modes

Since the number of fermionic zero-modes can be also studied via mixed Chern-Simons terms [44], we here give an alternative argument of deriving the global charges for the Coulomb branch operators. But one can easily find that this is equivalent to the above calculation.

Let us first calculate the charges of the monopole operator $Z \simeq \exp(2\phi_1)$. Along the moduli of a non-zero value of $\langle Z \rangle$, the gauge group is broken as $G_2 \rightarrow SU(2) \times U(1)$. The Z direction corresponds to a dual root of $3\alpha + 2\beta$ which is perpendicular to the root α . Therefore the $SU(2)$ with the roots $\alpha, -\alpha$ and a Cartan generator H_2 remains unbroken and a Z direction corresponds to the unbroken $U(1)$ related to H_1 . Under this breaking, the fields are decomposed as

$$\mathbf{7} \rightarrow \mathbf{3}_0 + \mathbf{2}_1 + \mathbf{2}_{-1} \tag{3.14}$$

$$\mathbf{14} \rightarrow \mathbf{3}_0 + \mathbf{1}_0 + \mathbf{4}_1 + \mathbf{4}_{-1} + \mathbf{1}_2 + \mathbf{1}_{-2} \tag{3.15}$$

In order to calculate the effective Chern-Simons terms, we have to know the sign of the masses of fermions which appear in 1-loop graphs. The mass terms for the fermions

are dictated from

$$\begin{aligned} \phi &= \phi_1 H_1 + \frac{\phi_2}{\sqrt{3}} H_2 \\ &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\phi_1 - \frac{1}{3}\phi_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{3}\phi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\phi_1 + \frac{1}{3}\phi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3}\phi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_1 + \frac{1}{3}\phi_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi_1 - \frac{1}{3}\phi_2 \end{pmatrix}. \end{aligned} \quad (3.16)$$

Then the mixed Chern-Simons terms are

$$\begin{aligned} k_{\text{eff}}^{\text{U}(1)_{\text{gauge}}\text{U}(1)_{\text{global}}} &= \frac{1}{2} N_f \left[\text{sign} \left(\phi_1 + \frac{\phi_2}{3} \right) + \text{sign} \left(\phi_1 - \frac{\phi_2}{3} \right) \right. \\ &\quad \left. - \text{sign} \left(-\phi_1 + \frac{\phi_2}{3} \right) - \text{sign} \left(-\phi_1 - \frac{\phi_2}{3} \right) \right] = 2N_f \end{aligned} \quad (3.17)$$

$$\begin{aligned} k_{\text{eff}}^{\text{U}(1)_{\text{gauge}}\text{U}(1)_R} &= 2N_f(R-1) + \frac{1}{2} \left[2\text{sign}(\phi_1) - 2\text{sign}(-\phi_1) \right. \\ &\quad \left. + \text{sign}(\phi_1 + \phi_2) + \text{sign} \left(\phi_1 + \frac{\phi_2}{3} \right) + \text{sign} \left(\phi_1 - \frac{\phi_2}{3} \right) + \text{sign}(\phi_1 - \phi_2) \right. \\ &\quad \left. - \text{sign}(-\phi_1 - \phi_2) - \text{sign} \left(-\phi_1 - \frac{\phi_2}{3} \right) - \text{sign} \left(-\phi_1 + \frac{\phi_2}{3} \right) - \text{sign}(-\phi_1 + \phi_2) \right] \\ &= 2N_f(R-1) + 6 \end{aligned} \quad (3.18)$$

Notice that the CS term for $\text{U}(1)_{\text{gauge}}$ is vanishing and this is consistent with the fact that the monopole operator Z is gauge-invariant.

Next we consider the operator $V_\alpha \simeq \exp(2\phi_2)$. Along this direction, an $\text{SU}(2)$ with $3\alpha + 2\beta$, $-(3\alpha + 2\beta)$ and H_1 remains unbroken. The operator V_α corresponds to the monopole-creating operator with a $\text{U}(1)$ from H_2 . Under the breaking $G_2 \rightarrow \text{SU}(2) \times \text{U}(1)_{H_2}$, the fundamental and adjoint fields are decomposed as

$$\mathbf{7} \rightarrow \mathbf{2}_1 + \mathbf{2}_{-1} + \mathbf{1}_2 + \mathbf{1}_0 + \mathbf{1}_{-2} \quad (3.19)$$

$$\mathbf{14} \rightarrow \mathbf{3}_0 + \mathbf{1}_0 + \mathbf{2}_{\pm 3} + \mathbf{2}_{\pm 1} + \mathbf{1}_{\pm 2}. \quad (3.20)$$

By carefully taking into account the mass term for each representation, we find

$$\begin{aligned} k_{\text{eff}}^{\text{U}(1)_{\text{gauge}}\text{U}(1)_{\text{global}}} &= \frac{1}{2} N_f \left[\text{sign} \left(\phi_1 + \frac{1}{3}\phi_2 \right) + \text{sign} \left(-\phi_1 + \frac{1}{3}\phi_2 \right) \right. \\ &\quad \left. - \text{sign} \left(-\phi_1 - \frac{1}{3}\phi_2 \right) - \text{sign} \left(\phi_1 - \frac{1}{3}\phi_2 \right) + 2\text{sign} \left(\frac{2}{3}\phi_2 \right) - 2\text{sign} \left(-\frac{2}{3}\phi_2 \right) \right] \\ &= 2N_f \end{aligned} \quad (3.21)$$

$$\begin{aligned}
k_{\text{eff}}^{\text{U}(1)_{\text{gauge}}\text{U}(1)_{\text{R-charge}}} &= 2N_f(R-1) + \frac{1}{2} \left[3\text{sign}(\phi_1 + \phi_2) + 3\text{sign}(-\phi_1 + \phi_2) \right. \\
&\quad - 3\text{sign}(-\phi_1 - \phi_2) - 3\text{sign}(\phi_1 - \phi_2) + \text{sign}\left(\phi_1 + \frac{\phi_2}{3}\right) + \text{sign}\left(-\phi_1 + \frac{\phi_2}{3}\right) \\
&\quad \left. - \text{sign}\left(\phi_1 - \frac{\phi_2}{3}\right) - \text{sign}\left(-\phi_1 - \frac{\phi_2}{3}\right) + 2\text{sign}\left(\frac{\phi_2}{3}\right) - 2\text{sign}\left(-\frac{\phi_2}{3}\right) \right] \\
&= 2N_f(R-1) + 2
\end{aligned} \tag{3.22}$$

Finally we study the Coulomb branch with $V_\beta \simeq \exp(\phi_1 - \phi_2)$. The symmetry breaking is schematically

$$G_2 \rightarrow \text{SU}(2)_{\pm(2\alpha+\beta), H_1+H_2/\sqrt{3}} \times \text{U}(1)_{H_1-\sqrt{3}H_2} \tag{3.23}$$

and the branching rules for the fundamental and adjoint representations are

$$\mathbf{7} \rightarrow \mathbf{3}_0 + \mathbf{2}_1 + \mathbf{2}_{-1} \tag{3.24}$$

$$\mathbf{14} \rightarrow \mathbf{3}_0 + \mathbf{1}_0 + \mathbf{4}_1 + \mathbf{4}_{-1} + \mathbf{1}_2 + \mathbf{1}_{-2}. \tag{3.25}$$

Then we can compute the mixed Chern-Simons terms generated along this direction:

$$\begin{aligned}
k_{\text{eff}}^{\text{U}(1)_{\text{gauge}}\text{U}(1)_{\text{global}}} &= \frac{1}{2} N_f \left[\text{sign}\left(-\frac{2\phi_2}{3}\right) + \text{sign}\left(\phi_1 - \frac{\phi_2}{3}\right) \right. \\
&\quad \left. - \text{sign}\left(-\phi_1 + \frac{\phi_2}{3}\right) - \text{sign}\left(\frac{2\phi_2}{3}\right) \right] = 0
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
k_{\text{eff}}^{\text{U}(1)_{\text{gauge}}\text{U}(1)_{\text{R-charge}}} &= \frac{1}{2} \left[\text{sign}(\phi_1) + \text{sign}\left(\phi_1 - \frac{\phi_2}{3}\right) + \text{sign}\left(-\frac{\phi_2}{3}\right) + \text{sign}(-\phi_1 - \phi_2) \right. \\
&\quad - \text{sign}(\phi_1 + \phi_2) - \text{sign}\left(\frac{\phi_2}{3}\right) - \text{sign}\left(-\phi_1 + \frac{\phi_2}{3}\right) - \text{sign}(-\phi_1) \\
&\quad \left. + 2\text{sign}(\phi_1 - \phi_2) - 2\text{sign}(-\phi_1 + \phi_2) \right] \\
&= 2
\end{aligned} \tag{3.27}$$

Therefore along the branch $\langle V_\beta \rangle \simeq \exp(\phi_1 - \phi_2)$, we have no fundamental fermion zero-mode but two gluino zero-modes should contribute.

4 3d $\mathcal{N} = 2$ G_2 pure Yang-Mills

We will start with quantum considerations of the moduli space of vacua from the pure G_2 SYM without fundamental matters. The 3d $\mathcal{N} = 2$ pure SYM theories for various gauge groups were studied in [16] with the connection to the theory in 4d and in $\mathbb{S}^1 \times \mathbb{R}^3$. Since the G_2 group has rank 2, the Coulomb branch is classically two-dimensional and these are described by two monopole operators corresponding to the simple roots. We labeled them as V_α and V_β . The symmetry argument says that the following terms are generated in the superpotential

$$W = \frac{3}{V_\alpha} + \frac{1}{V_\beta}, \tag{4.1}$$

where we are including the relative coefficient related to the length of the roots. This is consistent with [16]. These are monopole-generated superpotentials since the monopoles corresponding to the G_2 breaking,

$$G_2 \rightarrow U(1) \times U(1) \tag{4.2}$$

contains two gaugino zero-modes and they can contribute to the superpotential. These terms prevent us from giving the vacuum expectation values to these Coulomb branch directions. If we recall the relation between the monopole operators and the classical Coulomb branch, $V_\beta \simeq \exp(\phi_1 - \phi_2)$, the repulsive force is acting between ϕ_1 and ϕ_2 . So in the Weyl chamber we expect that the ϕ_1 direction can be turned on while the ϕ_2 is frozen to zero. Even if we add the fundamental matters, the direction with V_β is still lifted via the monopole superpotential since the fundamental quarks do not have any zero-mode around the V_β monopole as we have seen in section 3. Then it is natural to think that the quantum Coulomb branch is one-dimensional and this would be parametrized by an operator including only the ϕ_1 variable, namely, a $Z \simeq \exp(2\phi_1)$ operator. The validity of this candidate will be discussed by extending to the inclusion of fundamental flavors in section 5. In addition, the discussion on the theory on $\mathbb{S}_1 \times \mathbb{R}_3$ will make the operator Z more suitable for the description of the quantum Coulomb branch in section 6.

5 3d $\mathcal{N} = 2$ G_2 SQCD

Next we introduce chiral superfields in a fundamental representation to the G_2 theory discussed above. The matter contents and their representations are summarized in table 5. Notice that the global $U(1)$ symmetry is not spurious but a genuine symmetry now. Fermion zero-modes from the fundamental matters modify the zero-mode counting for the Coulomb branch operators except for V_β . Therefore, even for non-zero N_f , we can still have the superpotential

$$W = \frac{1}{V_\beta} \tag{5.1}$$

and this direction would be lifted. It is natural to regard the one-dimensional Coulomb branch of Z as quantum-mechanically massless and as a globally defined monopole-creating operator for non-zero N_f . By using $M_{ij}, B_{ijk}, F_{ijkl}$ and Z we can find the following phases and the superpotentials for $N_f \leq 4$. We will also briefly discuss the phases for $N_f \geq 5$.

$N_f = 1$: runaway vacua. When $N_f = 1$, the low-energy dynamics is similar to the 4d $\mathcal{N} = 1$ G_2 gauge theory with $N_f = 2$ fundamental matters. The superpotential below is allowed from the symmetry and holomorphy argument.

$$W = \left(\frac{1}{MZ} \right)^{\frac{1}{2}} \tag{5.2}$$

By differentiating the superpotential, we obtain the runaway potential and there is no stable SUSY vacua. The consistency can be checked by flowing to the Higgs branch. Along the

	G_2	$SU(N_f)$	$U(1)$	$U(1)_R$
Q	\square	\square	1	R
λ	adj.	1	0	1
$M_{ij} = Q_i^a Q_j^a$	1	$\square\square$	2	$2R$
$B = Q^3$	1	$\begin{matrix} \square \\ \square \\ \square \end{matrix}$	3	$3R$
$F = Q^4$	1	$\begin{matrix} \square \\ \square \\ \square \\ \square \end{matrix}$	4	$4R$
$Z = e^{2\phi_1}$	1	1	$-2N_f$	$2N_f(1 - R) - 6$

Table 5. Quantum numbers of the G_2 SQCD with N_f flavors.

Higgs branch, the G_2 gauge group is broken down to $SU(3)$ which is a maximal subgroup of G_2 . By introducing the vev $\langle M \rangle = v$, we again find no supersymmetric solution. This is consistent with the dynamics of a 3d $\mathcal{N} = 2$ $SU(3)$ without matters where the monopole corresponding to the breaking $SU(3) \rightarrow U(1) \times U(1)$ creates the runaway potential. We can also test this superpotential by introducing a complex mass to the chiral superfield. By integrating the massive modes, we have $W = Z^{-1/3}$ and there is no stable SUSY vacuum.

$N_f = 2$: runaway vacua. The dynamics of $N_f = 2$ is similar to the 4d $\mathcal{N} = 1$ G_2 theory with $N_f = 3$. We again have a runaway-type superpotential.

$$W = \frac{1}{Z \det M} \tag{5.3}$$

By introducing a vacuum expectation value with rank $\langle M \rangle = 1$, we can flow to a 3d $\mathcal{N} = 2$ $SU(3)$ with one flavor. By properly rescaling the Coulomb branch operator $V_{SU(3)} := 2vZ$ the low-energy dynamics is described by

$$W = \frac{1}{V_{SU(3)} M_{11}}, \tag{5.4}$$

which explains the dynamics of a 3d $\mathcal{N} = 2$ $SU(3)$ with one flavor. We can also deform the theory by a complex mass. Let us introduce a term mM_{22} and we obtain

$$W = \frac{1}{Z \det M} + mM_{22} \rightarrow \left(\frac{m}{Z M_{11}} \right)^{\frac{1}{2}}. \tag{5.5}$$

By properly rescaling the monopole operator, we have the superpotential of $N_f = 1$.

$N_f = 3$: quantum deformed moduli space. The dynamics of a 3d G_2 theory with $N_f = 3$ is similar to the 4d $\mathcal{N} = 1$ G_2 gauge theory with 4 fundamental matters and also to the 3d $\mathcal{N} = 2$ $SU(N_c)$ gauge theory with $N_c - 1$ flavors. We find that the following constraint is consistent with all the symmetries.

$$Z (\det M - B^2) = 1 \tag{5.6}$$

This constraint relates the large values of the Higgs branch to the vicinity of the origin of the Coulomb branch. Thus the Coulomb and Higgs branches are quantum mechanically merged. This constraint also means that some global (and also gauge) symmetries are inevitably broken on the whole moduli space, and that the origin of the moduli space is not a vacuum.

We can test this phase by considering the Higgs branch. As in the previous case, let us introduce the rank $\langle M \rangle = 1$ vev. In this case the UV theory flows to a 3d $\mathcal{N} = 2$ SU(3) gauge theory with 2 flavors at the low-energy limit. The global non-abelian symmetry is enhanced since the $\mathbf{7}$ representations in G_2 yield $\mathbf{3} + \bar{\mathbf{3}}$. Therefore we have to carefully decompose the gauge invariant operators [13]. The symmetric meson is decomposed to $\hat{M}_f^{\bar{j}} + \hat{M}_g^{\bar{j}}$ and the cubic baryon becomes $ia(\hat{M}_2^3 - \hat{M}_3^2)$ where a is a vev of the fundamental squark. By inserting this expression we finally obtain

$$V_{\text{SU}(3)} \det \left(\hat{M}_i^{\bar{j}} \right) = 1, \tag{5.7}$$

where we rescaled the monopole operator and absorbed a^2 and the unimportant numerical factor. This result is precisely the 3d SU(3) result with 2 flavors.

$N_f = 4$: s-confinement. We present the dynamics of $N_f = 4$. The phase of the 3d $\mathcal{N} = 2$ G_2 gauge theory with 4 flavors is similar to a 4d $\mathcal{N} = 1$ G_2 theory with 5 flavors where one can see the s-confinement phase. The superpotential consistent with all the symmetries is

$$W = Z(-\det M + F^2 + B^i M_{ij} B^j), \tag{5.8}$$

where the relative coefficients are chosen as we reproduce the result of $N_f \leq 3$ when integrating out the massive flavors by introducing complex masses. The massless excitations are M_{ij} , B_i , F and the monopole operator Z . The interaction between these massless modes are described by the above potential. At the origin of the moduli space of vacua (in the present case, the origin belongs to the vacua as different from the $N_f = 3$ case.), none of the global symmetries is broken. So this phase is called s-confinement. We can see the consistency by calculating the parity anomaly. For the UV theory, each effective Chern-Simons level between the global U(1) and U(1)_R symmetries is computed as

$$k_{\text{U}(1)_R \text{U}(1)_R}^{UV} = \frac{1}{2} (7N_f \text{sign } M_Q + 14 \text{sign } M_\lambda) \in \begin{cases} \mathbb{Z} + \frac{1}{2} & (\text{odd } N_f) \\ \mathbb{Z} & (\text{even } N_f) \end{cases} \tag{5.9}$$

$$k_{\text{U}(1) \text{U}(1)_R}^{UV} = -\frac{7}{2} N_f \text{sign } M_Q \in \begin{cases} \mathbb{Z} + \frac{1}{2} & (\text{odd } N_f) \\ \mathbb{Z} & (\text{even } N_f) \end{cases} \tag{5.10}$$

$$k_{\text{U}(1) \text{U}(1)}^{UV} = \frac{1}{2} N_f \text{sign } M_Q \in \begin{cases} \mathbb{Z} + \frac{1}{2} & (\text{odd } N_f) \\ \mathbb{Z} & (\text{even } N_f) \end{cases}. \tag{5.11}$$

The similar calculation is performed for the IR description and we find the matching for $N_f = 4$:

$$k_{U(1)_R U(1)_R}^{IR} = \frac{1}{2} (10 \text{sign } M_M + 4 \text{sign } M_B + \text{sign } M_F + \text{sign } M_Z) \in \mathbb{Z} \quad (5.12)$$

$$k_{U(1)_L U(1)_R}^{IR} = \frac{1}{2} (-20 \text{sign } M_M - 12 \text{sign } M_B - 4 \text{sign } M_F - 8 \text{sign } M_Z) \in \mathbb{Z} \quad (5.13)$$

$$k_{U(1)_L U(1)_L}^{IR} = \frac{1}{2} (40 \text{sign } M_M + 36 \text{sign } M_B + 16 \text{sign } M_F + 64 \text{sign } M_Z) \in \mathbb{Z} \quad (5.14)$$

We can also test this phase by considering the Higgs branch as in [13]. By adding the rank $\langle M \rangle = 1$ vev to the theory, we flow to the 3d $\mathcal{N} = 2$ SU(3) gauge theory with $N_f = 3$ flavors. When flowing to the SU(3) gauge theory, the flavor symmetry is enhanced to $SU(3)_L \times SU(3)_R$. By introducing the vev to the 1st component of the flavor, the gauge invariant composites reduce to

$$M_{ij} = \hat{M}_i^{\bar{j}} + \hat{M}_j^{\bar{i}} \quad (5.15)$$

$$B^1 = i\sqrt{2}(b - \bar{b}) \quad (5.16)$$

$$B^i = \frac{ia}{2} \epsilon_{1ijk} (\hat{M}_j^{\bar{k}} - \hat{M}_k^{\bar{j}}), \quad (i, j, k = 2, 3, 4) \quad (5.17)$$

and the superpotential becomes

$$W = -8a^2 Z (\det \hat{M} - b\bar{b}) = -Y_{SU(3)} (\det \hat{M} - b\bar{b}), \quad (5.18)$$

where b and \bar{b} are the (anti-)baryonic operators for the SU(3) theory, \hat{M} is a meson with $SU(3)_L \times SU(3)_R$ indices and a is a vev for Q . We rescaled the Coulomb branch as $Y_{SU(3)} := 8a^2 Z$. This low-energy superpotential is consistent with the 3d $\mathcal{N} = 2$ SU(3) gauge theory with three flavors [45].

$N_f \geq 5$: interacting SCFT. Finally let us discuss the theory with $N_f \geq 5$. We expect that even for $N_f \geq 5$ the Coulomb branch is parametrized by one-dimensional coordinate Z and that the Higgs branch is described by M_{ij}, B_{ijk} and F_{ijkl} with classical constraints among them. One can still write down a consistent superpotential with all the symmetries although it contains a fractional power. For instance, the superpotential for $N_f = 5$ is given by

$$W_{N_f=5} = \left[Z (-\det M + B^{ij} B^{kl} M^{ik} M_{jl} + F^i M_{ij} F^j + \epsilon_{ijklm} B^{ij} B^{kl} F^m) \right]^{\frac{1}{2}}, \quad (5.19)$$

and for higher N_f we would have similar potentials. The presence of the fractional power leads to branch cut singularities on the moduli space of vacua, and so it signals that new massless degrees of freedom should be added to this effective description. Thus, we expect that an interacting SCFT appears at the origin of the moduli space, and probably we would have some Seiberg dual descriptions similar to the 4d dual [30].

6 G_2 SQCD on $S_1 \times \mathbb{R}_3$

We can connect the G_2 dynamics in 3d and 4d via compactification of the 4d theory on a circle and by taking into account non-perturbative effects from the twisted-monopole [37, 45] (it is known also as Kaluza-Klein monopole.). Generally speaking, if we compactify one direction of the space-time, the 4d BPST instanton is called a “caloron” (see for example [46].). This caloron configuration can be regarded as the bound state of the magnetic monopoles and the KK-monopole. Since the magnetic monopole has the same number of fundamental fermion zero-modes as the 4d instanton, the KK-monopole only has the adjoint zero-modes in our setup. Therefore the KK-monopole has only two gaugino zero-modes and it can contribute to the superpotential. In our present case we obtain

$$W = \eta Z = \eta V_\alpha V_\beta^2. \tag{6.1}$$

This is consistent with [16] since the Z direction is related with the lowest co-root.

Let us start with the analysis of the pure SYM on $S_1 \times \mathbb{R}_3$. We now have two contributions from the magnetic monopoles and the KK-monopole:

$$W_{N_f=0} = \frac{3}{V_\alpha} + \frac{1}{V_\beta} + \eta V_\alpha V_\beta^2 \tag{6.2}$$

Since the Coulomb moduli should be integrated out in a 4d limit, by solving the F-flatness conditions we find four discrete SUSY vacua and the superpotential

$$W_{4d \text{ limit}} = \pm 2^{3/2} \mathfrak{z}^{1/4} \eta^{1/4}, \quad \pm i 2^{3/2} \mathfrak{z}^{1/4} \eta^{1/4}, \tag{6.3}$$

which explains the gaugino condensation and is consistent with the fact that the 4d $\mathcal{N} = 1$ G_2 pure SYM has 4 discrete SUSY vacua. The coefficient is a fourth root of unity as it should be [5, 13, 15, 16].

For the theory with $N_f \leq 4$ fundamental matters, we again obtain the 4d superpotential by integrating out the monopole operator.

$$W_{N_f=1} = \frac{1}{(MZ)^{1/2}} + \eta Z \quad \rightarrow \quad W_{N_f=1}^{4d} = \left(\frac{\eta}{M}\right)^{\frac{1}{3}} \tag{6.4}$$

$$W_{N_f=2} = \frac{1}{Z \det M} + \eta Z \quad \rightarrow \quad W_{N_f=2}^{4d} = \left(\frac{\eta}{\det M}\right)^{\frac{1}{2}} \tag{6.5}$$

$$W_{N_f=3} = X(Z(\det M - B^2) - 1) + \eta Z \quad \rightarrow \quad W_{N_f=2}^{4d} = \frac{\eta}{\det M - B^2} \tag{6.6}$$

$$W_{N_f=4} = Z(-\det M + F^2 + B^i M_{ij} B^j) + \eta Z \quad \rightarrow \quad \det M - F^2 - B^i M_{ij} B^j = \eta \tag{6.7}$$

where for $N_f = 3$ we introduced a Lagrange multiplier field X to impose the constraint and X is also integrated out from the low-energy spectrum in the 4d limit. For $N_f = 4$, the monopole operator Z acts as a Lagrange multiplier.

For $N_f = 5$, we had the “effective” superpotential in 3d as

$$W_{N_f=5}^{3d} = \left[Z(-\det M + B^{ij} B^{kl} M_{ik} M_{jl} + F^i M_{ij} F^j + \epsilon_{ijklm} B^{ij} B^{kl} F^m) \right]^{\frac{1}{2}}. \tag{6.8}$$

This superpotential is singular at the origin of the moduli space and missing some massless degrees of freedom. However, we can use this “effective” description far away from the origin of the moduli space, and one can go back to the 4d theory as follows.

$$\begin{aligned}
 W_{N_f=5}^{\mathbb{S}^1 \times \mathbf{R}^3} &= W_{N_f=5}^{3d} + \eta Z \\
 &\rightarrow \frac{1}{\eta} (-\det M + B^{ij} B^{kl} M^{ik} M_{jl} + F^i M_{ij} F^j + \epsilon_{ijklm} B^{ij} B^{kl} F^m) \quad (6.9)
 \end{aligned}$$

7 Superconformal indices

In this section we calculate the superconformal indices [47–51] (see also [52–54] and [55]) for 3d $\mathcal{N} = 2$ G_2 gauge theories and confirm that the previous analysis is correct. Especially we will observe that the quantum Coulomb branch is indeed one-dimensional and described by the monopole operator Z .

The 3d superconformal indices (known as twisted partition functions on $\mathbb{S}_1 \times \mathbb{S}_2$) are given by a localization technique [56] and the result is

$$\begin{aligned}
 I(x, t) &= \sum_{s_1, s_2} \frac{1}{|\text{Sym}|} \oint \oint \prod_{i=1,2} \frac{dz_i}{2\pi i z_i} Z_{\text{vector}} Z_{\text{chiral}} \\
 Z_{\text{vector}} &= \prod_{\alpha \in \text{all the roots}} x^{-|\alpha(s)|} (1 - e^{i\alpha(h)} x^{2|\alpha(s)|}) \\
 Z_{\text{chiral}} &= \prod_{\Phi} \prod_{\rho_{\Phi} \in \text{all the weights}} (x^{1-\Delta_{\Phi}} e^{-i\rho(h)} t^{-1})^{|\rho(s)|} \frac{(e^{-i\rho(h)} t^{-1} x^{2|\rho(s)|+2-\Delta_{\Phi}}; x^2)_{\infty}}{(e^{i\rho(h)} t x^{2|\rho(s)|+\Delta_{\Phi}}; x^2)_{\infty}}, \quad (7.1)
 \end{aligned}$$

where $(a; x^2)_{\infty}$ is a q-Pochhammer symbol

$$(a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k), \quad (7.2)$$

and we introduced the fugacity t only for the global $U(1)$ symmetry for simplicity and it suffices for our purpose. Δ_{Φ} is a conformal weight of the chiral superfields and equal to the R-charge. Since we do not know a true value for it, we choose specific R-charge assignment in such a way that all the fields have positive conformal weights. In the following we will set $R = \frac{1}{8}$. The product $\prod_{\alpha \in \text{all the roots}}$ runs over all the roots of G_2 and $\prod_{\rho_{\Phi} \in \text{all the weights}}$ is including all the weights in a fundamental representation. The GNO charge s [57] is valued in a Cartan subalgebra and quantized as

$$s = s_1 H_1 + \sqrt{3} s_2 H_2, \quad s_1 \geq 3s_2 \geq 0, \quad s_1, s_2 \in \mathbb{Z}, \quad (7.3)$$

where we are restricting the summation of (s_1, s_2) by using the Weyl reflections of G_2 . $|\text{Sym}|$ is an order of the Weyl group for the unbroken gauge group after the introduction of a GNO charge (s_1, s_2) .

$N_f = 4$. Since the superconformal indices contain negative powers of x for $N_f \leq 3$, we start with the analysis of the superconformal indices from $N_f = 4$. In order to have the positive R-charges for the chiral operators including the monopole operators, R should be $0 < R < \frac{1}{4}$. Then the value $R = \frac{1}{8}$ is allowed. The theory with $N_f = 4$ flavors is s-confining, so we have the dual description without gauge group. We can compute the superconformal index on the electric theory and a magnetic one. We observe that these two indices exactly match. We first show the full conformal index for $N_f = 4$.

$$\begin{aligned}
 I(x, t)_{N_f=4} = & 1 + 10t^2x^{1/4} + 4t^3x^{3/8} + 56t^4x^{1/2} + 40t^5x^{5/8} + 240t^6x^{3/4} + 224t^7x^{7/8} \\
 & + \left(870t^8 + \frac{1}{t^8}\right)x + 940t^9x^{9/8} + \left(2782t^{10} + \frac{10}{t^6}\right)x^{5/4} + \left(3280t^{11} + \frac{4}{t^5}\right)x^{11/8} \\
 & + \left(8055t^{12} + \frac{55}{t^4}\right)x^{3/2} + \left(10008t^{13} + \frac{36}{t^3}\right)x^{13/8} + \left(21492t^{14} + \frac{220}{t^2}\right)x^{7/4} \\
 & + \left(27536t^{15} + \frac{180}{t}\right)x^{15/8} + \left(53495t^{16} + \frac{1}{t^{16}} + 698\right)x^2 + \dots \quad (7.4)
 \end{aligned}$$

Next, we list the index for each GNO charge. This is obtained from the indices of the electric theory.

- GNO charge: (0,0)

$$\begin{aligned}
 & 1 + 10t^2x^{1/4} + 4t^3x^{3/8} + 56t^4x^{1/2} + 40t^5x^{5/8} + 240t^6x^{3/4} + 224t^7x^{7/8} \\
 & + 870t^8x + 940t^9x^{9/8} + 2782t^{10}x^{5/4} + 3280t^{11}x^{11/8} + 8055t^{12}x^{3/2} \\
 & + 10008t^{13}x^{13/8} + 21492t^{14}x^{7/4} + 27536t^{15}x^{15/8} + (53495t^{16} - 16)x^2 + \dots \quad (7.5)
 \end{aligned}$$

- GNO charge: (1,0)

$$\frac{x}{t^8} + \frac{10x^{5/4}}{t^6} + \frac{4x^{11/8}}{t^5} + \frac{55x^{3/2}}{t^4} + \frac{36x^{13/8}}{t^3} + \frac{220x^{7/4}}{t^2} + \frac{180x^{15/8}}{t} + 714x^2 + \dots \quad (7.6)$$

- GNO charge: (2,0)

$$\frac{x^2}{t^{16}} + \frac{10x^{9/4}}{t^{14}} + \frac{4x^{19/8}}{t^{13}} + \frac{55x^{5/2}}{t^{12}} + \frac{36x^{21/8}}{t^{11}} + \frac{220x^{11/4}}{t^{10}} + \frac{180x^{23/8}}{t^9} + \frac{714x^3}{t^8} + \dots \quad (7.7)$$

- GNO charge: (3,0)

$$\frac{x^3}{t^{24}} + \frac{10x^{13/4}}{t^{22}} + \frac{4x^{27/8}}{t^{21}} + \frac{55x^{7/2}}{t^{20}} + \frac{36x^{29/8}}{t^{19}} + \frac{220x^{15/4}}{t^{18}} + \frac{180x^{31/8}}{t^{17}} + \frac{714x^4}{t^{16}} + \dots \quad (7.8)$$

- GNO charge: (3,1)

$$\begin{aligned}
 & \frac{x^8}{t^{32}} + \frac{4x^{65/8}}{t^{31}} + \frac{10x^{33/4}}{t^{30}} + \frac{20x^{67/8}}{t^{29}} + \frac{35x^{17/2}}{t^{28}} + \frac{56x^{69/8}}{t^{27}} \\
 & + \frac{84x^{35/4}}{t^{26}} + \frac{120x^{71/8}}{t^{25}} + \frac{165x^9}{t^{24}} + \dots \quad (7.9)
 \end{aligned}$$

- GNO charge: (4,0)

$$\frac{x^4}{t^{32}} + \frac{10x^{17/4}}{t^{30}} + \frac{4x^{35/8}}{t^{29}} + \frac{55x^{9/2}}{t^{28}} + \frac{36x^{37/8}}{t^{27}} + \frac{220x^{19/4}}{t^{26}} + \frac{180x^{39/8}}{t^{25}} + \frac{714x^5}{t^{24}} + \dots \quad (7.10)$$

- GNO charge: (4,1)

$$\frac{x^9}{t^{40}} + \frac{4x^{73/8}}{t^{39}} + \frac{10x^{37/4}}{t^{38}} + \frac{20x^{75/8}}{t^{37}} + \frac{35x^{19/2}}{t^{36}} + \frac{56x^{77/8}}{t^{35}} + \frac{84x^{39/4}}{t^{34}} + \frac{120x^{79/8}}{t^{33}} + \dots \quad (7.11)$$

We first explain low-lying operators in a sector with GNO charge (0,0). Since the G_2 gauge group is unbroken in this sector, it is simple enough to understand the BPS operators. The first contribution of unity is an identity operator with the GNO charges (0,0). From the state-operator mapping it is denoted as $|0,0\rangle$. The second contribution $10t^2x^{1/4}$ is a meson M_{ij} acting on $|0,0\rangle$. The third one $4t^3x^{3/8}$ is identified with $B^i|0,0\rangle$. The fourth term $56t^4x^{1/2}$ is from $F|0,0\rangle$ and $M_{ij} \otimes M_{kl}|0,0\rangle$, which are $\mathbf{1} + \mathbf{20} + \mathbf{35}$ in an SU(4) notation. The fifth term $40t^5x^{5/8}$ represents $M_{ij} \otimes B^k = \mathbf{4} + \mathbf{36}$. In this way we can find the chiral ring without monopole contributions.

Let us next consider a sector with the GNO charge (1,0). In this case the gauge group is broken to $SU(2) \times U(1)$, so the chiral ring constructed on the state $|1,0\rangle$ is modified from the previous case as in [52, 54]. From table 5, the monopole operator Z which has a minimal magnetic charge appear as $t^{-8}x^1$ and this is consistent with the index above. The second and third contributions $\frac{10x^{5/4}}{t^6} + \frac{4x^{11/8}}{t^5}$ are identified with $M_{ij}|1,0\rangle$ and $B^i|1,0\rangle$ respectively. The fourth term $\frac{55x^{3/2}}{t^4}$ only comes from $M_{ij} \otimes M_{kl}|1,0\rangle$ and the chiral ring does not have $F|1,0\rangle$. This is because we cannot construct the quartic baryons from the unbroken SU(2) sector. The fifth term $\frac{36x^{13/8}}{t^3}$ is also reduced because we cannot construct $\mathbf{4}$ which requires fourth order anti-symmetrization of the flavor indices and it is impossible. As the result, we only have a $\mathbf{36}$ representation. The sectors with GNO charges (2,0), (3,0) and (4,0) are consistent with (1,0) simply because the symmetry breaking pattern is the same.

For the sector with GNO charge (3,1), we have to first notice that the gauge group is broken to $SU(2) \times U(1)$, where this SU(2) is generated by the roots $\beta, -\beta$ and $\beta \cdot \mathbf{H}$. Under this breaking, the fundamental representation is decomposed as

$$\mathbf{7} \rightarrow \mathbf{2}_1 + \mathbf{2}_{-1} + \mathbf{1}_2 + \mathbf{1}_{-2} + \mathbf{1}_0. \quad (7.12)$$

Therefore we can construct gauge invariant operators by acting the last component $\mathbf{1}_0$ on the monopole background with a GNO charge (3,1). We do not have to combine two Q 's into M . The ground state $|3,1\rangle$ semi-classically corresponds to Z^3V_α and the first excited state $\frac{4x^{65/8}}{t^{31}}$ is $\mathbf{1}_0|3,1\rangle$. The remaining parts are just given by symmetrizing $(\mathbf{1}_0)^n$ about the flavor indices.

In a sector with a GNO charge (4,1), the gauge group is maximally broken to $U(1) \times U(1)$. In this broken phase the fundamental matters still supply the gauge singlet $\mathbf{1}_{(0,0)}$ so

that we can construct the states

$$|4, 1\rangle \Leftrightarrow \frac{x^9}{t^{40}}, \quad (7.13)$$

$$\mathbf{1}_{(0,0)} |4, 1\rangle \Leftrightarrow \frac{4x^{73/8}}{t^{39}}, \quad (7.14)$$

$$\mathbf{1}_{(0,0)} \mathbf{1}_{(0,0)} |4, 1\rangle \Leftrightarrow \frac{10x^{37/4}}{t^{38}}, \quad (7.15)$$

⋮

where the flavor indices of $\mathbf{1}_{(0,0)}$ are symmetrized.

$N_f = 5$. Firstly, the full index for $N_f = 5$ is presented. From table 5, we expect that the meson M_{ij} contributes as $15t^2x^{1/4}$ and the baryons B^{ij} , F^i should be represented as $10t^3x^{3/8}$ and $5t^4x^{1/2}$ respectively in the index. This can be easily checked from the index below. We again set $R = \frac{1}{8}$.

$$\begin{aligned} I(x, t)_{N_f=5} = & 1 + 15t^2x^{1/4} + 10t^3x^{3/8} + 125t^4\sqrt{x} + 150t^5x^{5/8} + 805t^6x^{3/4} + 1240t^7x^{7/8} \\ & + 4410t^8x + 7570t^9x^{9/8} + 21202t^{10}x^{5/4} + 37950t^{11}x^{11/8} + 91120t^{12}x^{3/2} \\ & + 164430t^{13}x^{13/8} + 355050t^{14}x^{7/4} + 634851t^{15}x^{15/8} + (1268710t^{16} - 25)x^2 \\ & + (2229135t^{17} - 50t)x^{17/8} + (4198290t^{18} - 400t^2)x^{9/4} + (7222165t^{19} - 950t^3)x^{19/8} \\ & + (12974178t^{20} - 3825t^4)x^{5/2} + (21827235t^{21} - 9225t^5)x^{21/8} \\ & + \left(37715930t^{22} + \frac{1}{t^{10}} - 27500t^6 \right) x^{11/4} + (62063820t^{23} - 63350t^7)x^{23/8} \\ & + \left(103778515t^{24} - 159750t^8 + \frac{15}{t^8} \right) x^3 + \dots \end{aligned} \quad (7.16)$$

We can also confirm that the first contribution with a negative U(1) charge appears as $\frac{1}{t^{10}}x^{11/4}$. This is precisely the Coulomb branch operator Z (see table 5), predicted as the correct moduli coordinate. Indeed, this does not prove that the Coulomb branch for $N_f \geq 5$ is described by a single operator Z , but the index computation corroborates our prediction. For completeness, let us list the index for each GNO charge.

- GNO charge: (0,0)

$$\begin{aligned} & 1 + 15t^2x^{1/4} + 10t^3x^{3/8} + 125t^4x^{1/2} + 150t^5x^{5/8} + 805t^6x^{3/4} + 1240t^7x^{7/8} + 8820t^8x \\ & + 7570t^9x^{9/8} + 21202t^{10}x^{5/4} + 37950t^{11}x^{11/8} + 91120t^{12}x^{3/2} + 164430t^{13}x^{13/8} \\ & + 355050t^{14}x^{7/4} + 634851t^{15}x^{15/8} + (-25 + 1268710t^{16})x^2 + (-50t + 2229135t^{17})x^{17/8} \\ & + (-400t^2 + 4198290t^{18})x^{9/4} + (-950t^3 + 7222165t^{19})x^{19/8} \\ & + (-3825t^4 + 12974178t^{20})x^{5/2} + (-9225t^5 + 21827235t^{21})x^{21/8} \\ & + (-27500t^6 + 37715930t^{22})x^{11/4} + (-63350t^7 + 62063820t^{23})x^{23/8} \\ & + (-159750t^8 + 103778515t^{24})x^3 + (-347425t^9 + 167175552t^{25})x^{25/8} + \dots \end{aligned} \quad (7.17)$$

- GNO charge: (1,0)

$$\begin{aligned} & \frac{x^{11/4}}{t^{10}} + \frac{15x^3}{t^8} + \frac{10x^{25/8}}{t^7} + \frac{120x^{13/4}}{t^6} + \frac{126x^{27/8}}{t^5} + \frac{680x^{7/2}}{t^4} \\ & + \frac{855x^{29/8}}{t^3} + \frac{3045x^{15/4}}{t^2} + \frac{4145x^{31/8}}{t} + 11427x^4 + \dots \end{aligned} \quad (7.18)$$

- GNO charge: (2,0)

$$\begin{aligned} & \frac{x^{11/2}}{t^{20}} + \frac{15x^{23/4}}{t^{18}} + \frac{10x^{47/8}}{t^{17}} + \frac{120x^6}{t^{16}} + \frac{126x^{49/8}}{t^{15}} + \frac{680x^{25/4}}{t^{14}} + \frac{855x^{51/8}}{t^{13}} \\ & + \frac{3045x^{13/2}}{t^{12}} + \frac{4145x^{53/8}}{t^{11}} + \frac{11427x^{27/4}}{t^{10}} + \frac{16080x^{55/8}}{t^9} + \frac{37310x^7}{t^8} + \dots \end{aligned} \quad (7.19)$$

- GNO charge: (3,0)

$$\frac{x^{33/4}}{t^{30}} + \frac{15x^{17/2}}{t^{28}} + \frac{10x^{69/8}}{t^{27}} + \frac{120x^{35/4}}{t^{26}} + \frac{126x^{71/8}}{t^{25}} + \frac{680x^9}{t^{24}} + \frac{855x^{73/8}}{t^{23}} + \frac{3045x^{37/4}}{t^{22}} + \dots \quad (7.20)$$

- GNO charge: (3,1)

$$\begin{aligned} & \frac{x^{15}}{t^{40}} + \frac{5x^{121/8}}{t^{39}} + \frac{15x^{61/4}}{t^{38}} + \frac{35x^{123/8}}{t^{37}} + \frac{70x^{31/2}}{t^{36}} \\ & + \frac{126x^{125/8}}{t^{35}} + \frac{210x^{63/4}}{t^{34}} + \frac{330x^{127/8}}{t^{33}} + \frac{495x^{16}}{t^{32}} + \dots \end{aligned} \quad (7.21)$$

- GNO charge: (4,0)

$$\begin{aligned} & \frac{x^{11}}{t^{40}} + \frac{15x^{45/4}}{t^{38}} + \frac{10x^{91/8}}{t^{37}} + \frac{120x^{23/2}}{t^{36}} + \frac{126x^{93/8}}{t^{35}} + \frac{680x^{47/4}}{t^{34}} \\ & + \frac{855x^{95/8}}{t^{33}} + \frac{3045x^{12}}{t^{32}} + \frac{4145x^{97/8}}{t^{31}} + \frac{11427x^{49/4}}{t^{30}} + \dots \end{aligned} \quad (7.22)$$

$N_f = 6$. Finally we will end up with the superconformal indices for $N_f = 6$. The value of R is set to be $\frac{1}{8}$, which is again allowed from table 5. We will list the index and give a simple explanation for each GNO sector. Let us start with a (0,0) sector.

- GNO charge: (0,0)

$$\begin{aligned} & 1 + 21t^2x^{1/4} + 20t^3x^{3/8} + 246t^4x^{1/2} + 420t^5x^{5/8} + 2261t^6x^{3/4} + 4830t^7x^{7/8} + 17766t^8x \\ & + 40740t^9x^{9/8} + 121569t^{10}x^{5/4} + 280140t^{11}x^{11/8} + 733194t^{12}x^{3/2} + 1651440t^{13}x^{13/8} \\ & + 3946974t^{14}x^{7/4} + 8597092t^{15}x^{15/8} + (-36 + 19195449t^{16})x^2 \\ & + (-90t + 40315392t^{17})x^{17/8} + (-876t^2 + 85267989t^{18})x^{9/4} \\ & + (-2610t^3 + 172772712t^{19})x^{19/8} + (-12636t^4 + 349323471t^{20})x^{5/2} \\ & + (-38100t^5 + 684175032t^{21})x^{21/8} + (-135576t^6 + 1330939701t^{22})x^{11/4} \\ & + (-386820t^7 + 2525733672t^{23})x^{23/8} + (-1160376t^8 + 4750153876t^{24})x^3 + \dots \end{aligned} \quad (7.23)$$

One can read off underlying BPS operators of the theory from a $(0, 0)$ sector which contains the chiral ring of the Higgs branch. The first term 1 can be identified with the identity $|0, 0\rangle$. The next four terms are regarded as

$$M_{ij} |0, 0\rangle, \quad B^{ijk} |0, 0\rangle, \quad (F^{ij} + M_{ij} \otimes M_{kl}) |0, 0\rangle, \quad \text{and} \quad M_{ij} \otimes B^{klm} |0, 0\rangle, \quad (7.24)$$

respectively. Note that the numerical coefficients are precisely the dimensions of the representations for the $SU(6)$ group.

Secondly, a sector with $(1, 0)$ is shown:

- GNO charge: $(1, 0)$

$$\begin{aligned} & \frac{x^{9/2}}{t^{12}} + \frac{21x^{19/4}}{t^{10}} + \frac{20x^{39/8}}{t^9} + \frac{231x^5}{t^8} + \frac{336x^{41/8}}{t^7} + \frac{1771x^{21/4}}{t^6} + \frac{2976x^{43/8}}{t^5} \\ & + \frac{10521x^{11/2}}{t^4} + \frac{18480x^{45/8}}{t^3} + \frac{51309x^{23/4}}{t^2} + \frac{90300x^{47/8}}{t} + 213479x^6 + \dots \end{aligned} \quad (7.25)$$

Recall that the symmetry breaking of G_2 to $SU(2) \times U(1)$ occurs in this sector. Here we can extract the monopole operator Z : it can be computed as $\frac{x^{9/2}}{t^{12}}$ from table 5, and indeed matches the first term above. Then we see the following states

$$M_{ij} |1, 0\rangle, \quad B^{ijk} |1, 0\rangle, \quad M_{ij} \otimes M_{kl} |1, 0\rangle, \quad \text{and} \quad M_{ij} \otimes B^{klm} |1, 0\rangle, \quad (7.26)$$

which corresponds to the preceding four terms, respectively. Still, for the last term, the restriction of the anti-symmetrization of flavor indices has to be taken into account. The above discussion applies likewise in $(2, 0)$, $(3, 0)$, $(4, 0)$, and $(5, 0)$ because of the same breaking pattern of the gauge group.

- GNO charge: $(2, 0)$

$$\begin{aligned} & \frac{x^9}{t^{24}} + \frac{21x^{37/4}}{t^{22}} + \frac{20x^{75/8}}{t^{21}} + \frac{231x^{19/2}}{t^{20}} + \frac{336x^{77/8}}{t^{19}} + \frac{1771x^{39/4}}{t^{18}} \\ & + \frac{2976x^{79/8}}{t^{17}} + \frac{10521x^{10}}{t^{16}} + \dots \end{aligned} \quad (7.27)$$

- GNO charge: $(3, 0)$

$$\begin{aligned} & \frac{x^{27/2}}{t^{36}} + \frac{21x^{55/4}}{t^{34}} + \frac{20x^{111/8}}{t^{33}} + \frac{231x^{14}}{t^{32}} + \frac{336x^{113/8}}{t^{31}} + \frac{1771x^{57/4}}{t^{30}} \\ & + \frac{2976x^{115/8}}{t^{29}} + \frac{10521x^{29/2}}{t^{28}} + \dots \end{aligned} \quad (7.28)$$

For another sector with the GNO charge $(3, 1)$, the same discussion goes as in the previous analysis. Due to the symmetry breaking (7.12), one can construct the gauge invariant states using the singlet $\mathbf{1}_0$, obtained from the decomposed fundamental representation.

- GNO charge: $(3, 1)$

$$\begin{aligned} & \frac{x^{22}}{t^{48}} + \frac{6x^{177/8}}{t^{47}} + \frac{21x^{89/4}}{t^{46}} + \frac{56x^{179/8}}{t^{45}} + \frac{126x^{45/2}}{t^{44}} \\ & + \frac{252x^{181/8}}{t^{43}} + \frac{462x^{91/4}}{t^{42}} + \frac{792x^{183/8}}{t^{41}} + \frac{1287x^{23}}{t^{40}} + \dots \end{aligned} \quad (7.29)$$

The first contribution can be semi-classically understood as $Z^3 V_\alpha$ but quantum mechanically this would be regarded as a highly composite product of Z and the Higgs branch coordinates.

Consequently, we can also apply the same interpretations to the cases $N_f \geq 5$ as for $N_f = 4$, and the results are still consistent with our finding that the Coulomb moduli space is labeled by the monopole operator Z .

8 Summary and discussion

We investigated the 3d $\mathcal{N} = 2$ supersymmetric G_2 gauge theory with (and without) fundamental matters. We found that the Coulomb branch of the moduli space of vacua is classically two-dimensional but the monopole-instantons generate the runaway-type superpotential and make the one-dimensional subspace massive. As a result, the quantum Coulomb moduli space is one-complex dimensional and we provided the proper monopole operator describing it. We introduced the chiral superfields in a fundamental representation and discussed that this one-dimensional direction remains after including the matters and their zero-modes. We also found that there are various phases depending on the number of the fundamentals. For $N_f \leq 2$, we have no supersymmetric vacuum. For $N_f = 3$, the Coulomb and Higgs moduli are quantum-mechanically merged and relating the weak- and strong-coupling regions. For $N_f = 4$, we found the s-confinement phase where the dual description is given by only gauge-singlet chiral superfields. As an independent check of our analysis, we calculated the superconformal indices and confirmed that the Coulomb branch is indeed parametrized by a Z field and observed the correct low-lying BPS operators.

In this paper we have shown the existence of the one-dimensional Coulomb branch, so it is possible to calculate a Hilbert series à la [58–62]. Hilbert series basically counts the holomorphic (gauge invariant) operators in a theory. Then we can study another aspect of the G_2 gauge theory and check the validity of our analysis. It is also interesting to consider the G_2 Chern-Simons theory.

A simple generalization of this work would be to study other exceptional groups in a framework of a 3d $\mathcal{N} = 2$ supersymmetry. In 4d, such theories do not have any s-confinement phases but have some quantum-deformed moduli spaces (see for instance [20–25]). Naively we expect that this is also the case in 3d. However, when connecting the physics between 3d and 4d, it is often the case where the s-confinement phase in 3d is de-compactified to the quantum deformed moduli space in 4d via the KK-monopole superpotential. So we can expect that some s-confinement phases might emerge in 3d for F_4, E_6, E_7 and E_8 being different from the 4d cases. It is also interesting to study the 3d Seiberg duality for those exceptional groups.

In this paper we only included the fundamental matters. So it is interesting to add some matter chiral superfields in various representations. In 4d if we include many matter fields, the theory is no longer asymptotically free. But in 3d the gauge coupling is a relevant interaction. Then it is interesting to study those cases. The possible matters would be adjoint. When studying those theories, the Coulomb branch becomes complicated to study because the Coulomb branch is no longer one-dimensional. Therefore it is a first attempt to

consider the (adjoint) matter with some superpotential. The presence of the superpotential caves the chiral ring and would simplify its analysis.

We could not find any Seiberg dual description for $N_f \geq 5$ where the “effective” superpotential had the singularities at the origin of the moduli space. This is implicitly telling us the presence of a magnetic gauge group and dual quarks. In 4d, the G_2 Seiberg dual is known in [30] and we reviewed it in section 2. Naively speaking we can derive the corresponding 3d Seiberg dual by dimensionally reducing the 4d electric and magnetic theories respectively. This method was studied in [28, 29], where those authors claimed that in reducing the 4d Seiberg dual pairs to 3d, it is important to take into account the non-perturbative effects from the twisted instantons (KK-monopoles) and carefully to take some low-energy limit on both sides. In our case of G_2 , we can easily find the electric theory on a circle. This is just including the superpotential $W = \eta Z$. On the magnetic side, however, it is difficult to study the full Coulomb branch structure and also difficult to derive the KK-monopole generated potential. More concretely we are not understanding dimensions of the Coulomb (quantum) moduli. For example, the Coulomb branch operator corresponding to

$$Y_{\text{SU}(N_f-3)} \leftrightarrow \begin{pmatrix} \phi & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -\phi \end{pmatrix} \tag{8.1}$$

is not gauge invariant because the magnetic theory is “chiral” in a four-dimensional sense, which includes only anti-fundamentals and not fundamentals. We can construct a dressed monopole operator by multiplying the chiral superfields a la [63, 64]. In the present case, we find that the following dressed monopole can be defined.

$$Y_{\text{dressed}} := Y_{\text{SU}(N_f-3)} s^{N_f-5} \tag{8.2}$$

This is quite plausible because the superpotential on the magnetic theory contains $W \ni \det s = s^{N_f-3}$ and such a dressed monopole would be generated by absorbing the fermion zero-modes from the symmetric matter. We also found that the non-perturbative superpotential

$$W = \tilde{\eta} Y_{\text{dressed}} \tag{8.3}$$

would be generated via the KK-monopole and dressing effects. This superpotential is consistent with all the (spurious) symmetries. But we do not understand whether any other Coulomb branch directions quantum mechanically remain massless and whether other types of KK-monopole superpotential might be generated or not. We have to also take a 3d limit in order to turn off the electric superpotential $W = \eta Z$. This can be achieved by introducing real masses by background gauging the flavor symmetry $\text{SU}(N_f)$. On the magnetic side, this deformation would lead to the higgsing of the dual gauge group. Under this higgsing the Coulomb branch operator Y_{dressed} is non-trivially transformed and additional Coulomb

branch operators would also emerge. We have to rewrite the magnetic superpotential in this new set of monopole operators. This is highly non-trivial and we could not find any natural dual description.

It remains important to study a 3d $\mathcal{N} = 2$ Spin(7) theory with spinorial representations $\mathbf{8}$ since the G_2 gauge theory comes from this by higgsing the Spin(7) gauge group via a vev of the spinorial scalar field. Although we could not find a 3d G_2 dual description from the 4d G_2 Seiberg duality, it is quite plausible that we can find the G_2 duals after constructing the 3d Seiberg duality for Spin(7). We will come back to this problem and near future we would like to address some progresses on this direction.

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A Notations for G_2

Here we summarize symbols and notations for the G_2 group and its Lie algebra, which we have used in this paper. For the details of the G_2 algebra and its representations, for example, see [65, 66].

A.1 Group invariants

The group G_2 is the smallest exceptional Lie group with a trivial center. It has rank 2 and dimension 14. Also, the G_2 has maximal subgroups SU(3) and SU(2) \times SU(2).

The Dynkin index $T_{\mathbf{r}}$ is defined as a constant appearing in $\text{Tr}(t^a t^b) = T_{\mathbf{r}} \delta^{ab}$. For fundamental and adjoint representations it is given by

$$T_{\mathbf{7}} = 1, \quad T_{\mathbf{Adj.}} = 4. \tag{A.1}$$

A one-loop beta function in a 4d $\mathcal{N} = 1$ SQCD is given by

$$\beta(g) = -\frac{g^3}{16\pi^2} b, \quad b = 3T_{\mathbf{Adj.}} - \sum_i T_{\mathbf{r}_i}, \tag{A.2}$$

where $3T_{\mathbf{Adj.}}$ is a contribution from the vector superfield and the other is from chiral superfields with representations \mathbf{r}_i .

It is helpful to enumerate the group invariant tensors for the group G_2 . We have two invariant tensors. The first one is a Kronecker delta symbol δ_{ab} where $a, b = 1, \dots, 7$. The second one is a totally anti-symmetric tensor f_{abc} . In addition to these tensors we can construct the fourth order totally antisymmetric tensor $\tilde{f}_{abcd} := f_{e[ab} f_{cd]e}$, which is also expressed by the dual of f_{abc} .

The G_2 gauge invariants can be constructed by contracting these invariant tensors with the fundamental quarks. The composite fields are thus given by

$$\begin{aligned}
 M_{ij} &= \delta_{ab} Q^a_i Q^b_j, \\
 B_{ijk} &= \frac{1}{3!} f_{abc} Q^a_i Q^b_j Q^c_k, \\
 F_{ijkl} &= \frac{1}{4!} \tilde{f}_{abcd} Q^a_i Q^b_j Q^c_k Q^d_l
 \end{aligned}
 \tag{A.3}$$

A.2 Representations of G_2

We follow the notation of the Lie algebra for G_2 used in [67] although we are relabeling the names. The adjoint representation is represented by 7×7 matrices with 14 generators, which are decomposed into two Cartan matrices and 12 raising and lowering operators. In this representation, the fundamental representation with 7 dimensions are taking a 7 dimensional column vector and the matrices below naturally act on the column vector. The explicit parametrization for the Cartan subalgebra and 12 roots are as follows.

$$X_\alpha = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{-\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{A.4}$$

$$X_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad X_{-\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.5}$$

$$X_{\alpha+\beta} = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{-\alpha-\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.6}$$

$$X_{2\alpha+\beta} = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{-2\alpha-\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.7}$$

$$X_{3\alpha+\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{-3\alpha-\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.8})$$

$$X_{3\alpha+2\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{-3\alpha-2\beta} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.9})$$

$$H_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad H_2 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \quad (\text{A.10})$$

Note that those generators in Cartan subalgebra are normalized such that

$$\text{Tr } H_1 H_1 = \text{Tr } H_2 H_2 = 1, \quad \text{Tr } H_1 H_2 = 0. \quad (\text{A.11})$$

The two simple roots are expressed in a (H_1, H_2) -plane as

$$\alpha(H) = \left(0, \frac{1}{\sqrt{3}}\right), \quad \beta(H) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \quad (\text{A.12})$$

and the other positive roots are $\alpha + \beta$, $2\alpha + \beta$, $3\alpha + \beta$ and $3\alpha + 2\beta$. For a fundamental representation, we chose a following set of weights:

$$\rho_i \cong {}^t(0, \dots, \overset{i\text{-th}}{1}, \dots, 0), \quad i = 1, \dots, 7, \quad (\text{A.13})$$

which can be parametrized on the (H_1, H_2) -plane as

$$\rho_i(H) = \left((H_1)_{ii}, (H_2)_{ii} \right), \quad i : \text{ not summed} \quad (\text{A.14})$$

The G_2 root system and a weight diagram of a fundamental representation are depicted in figure 1.

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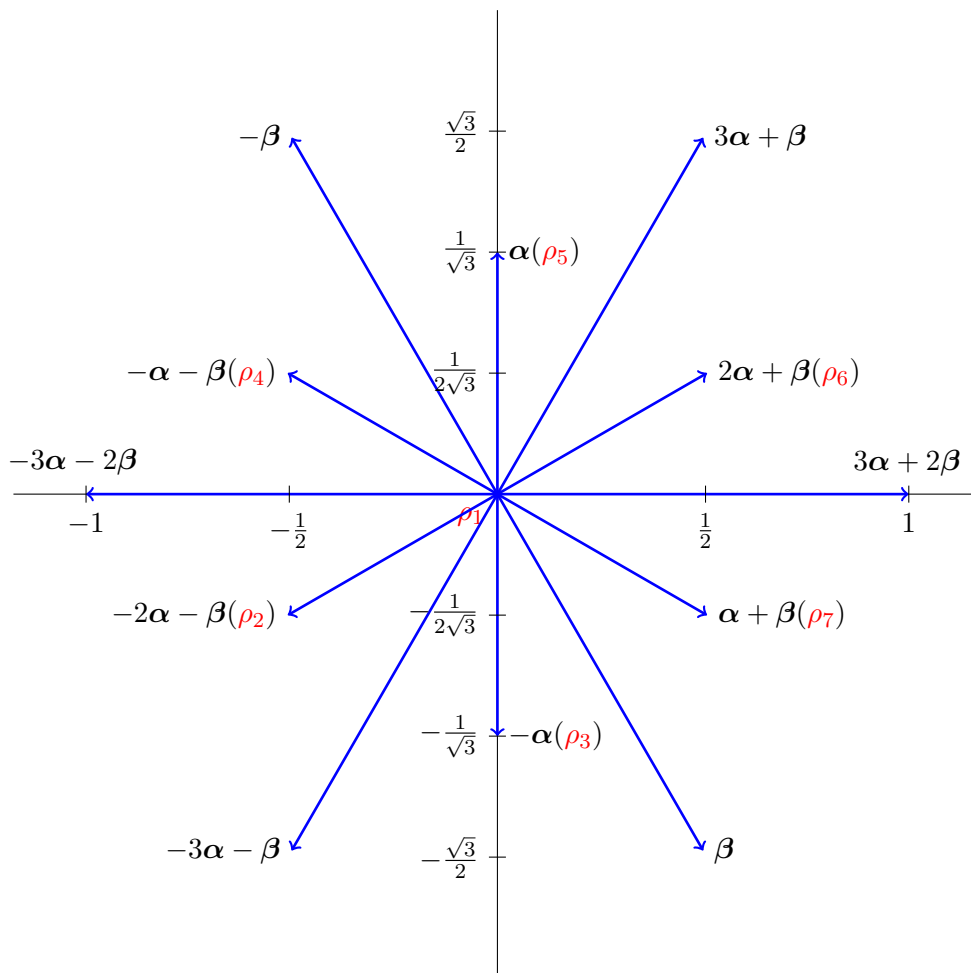


Figure 1. The G_2 roots and weights of fundamental representations.

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