Acta Math. Hungar., **156** (1) (2018), 56–81
https://doi.org/10.1007/s10474-018-0806-y
First published online March 17, 2018

# LIMIT THEOREMS FOR MULTIDIMENSIONAL RENEWAL SETS

A. ILIENKO[1] and I. MOLCHANOV[2,*]

[1]Department of Mathematical Analysis and Probability Theory,
Igor Sikorsky Kyiv Polytechnic Institute, Peremogy avenue 37, 03056 Kyiv, Ukraine
e-mail: ilienko@matan.kpi.ua

[2]Institute of Mathematical Statistics and Actuarial Science, University of Bern,
Alpeneggstrasse 22, CH-3012 Bern, Switzerland
e-mail: ilya.molchanov@stat.unibe.ch

*(Received January 4, 2018; accepted January 9, 2018)*

**Abstract.** Consider the multiple sums $S_n$ of i.i.d. random variables with a positive expectation on the $d$-dimensional integer grid. We prove the strong law of large numbers, the law of the iterated logarithm and the distributional limit theorem for random sets which appear as upper excursion sets of the interpolated multiple sums, that is, as the set of all arguments $x \in \mathbb{R}_+^d$ such that the interpolated multiple sums $S_x$ exceed $t$. The results are expressed in terms of set inclusions and using distances between sets.

## 1. Introduction

Classical renewal theorems can be viewed as inverse results to limit theorems for sums of i.i.d. random variables. In this paper we consider similar results for the multiple sums

$$S_n = \sum_{m \le n} \xi_m, \quad n \in \mathbb{N}^d,$$

of i.i.d. random variables $\xi_m$, $m \in \mathbb{N}^d$, on the $d$-dimensional grid, all being copies of an integrable random variable $\xi$ with a finite positive mean $\mu = \mathbf{E}\xi > 0$. Note that the discrete parameter random field $S_n$, $n \in \mathbb{N}^d$, has a positive drift.

* Corresponding author.

Supported by the Swiss National Science Foundation in the framework of the SCOPES programme, Grant No. IZ73Z0_152292.

*Key words and phrases:* excursion set, law of the iterated logarithm, multiple sum, random field, random set, renewal set, strong law of large numbers.

*Mathematics Subject Classification:* 60F15, 60K05, 60G60, 60D05, 60G50.

0236-5294/$20.00 

It is convenient to let $S_n = 0$ for $n$ with at least one vanishing component and to extend these multiple sums to all indices $x \in \mathbb{R}_+^d = [0, \infty)^d$ by the *piecewise multi-linear interpolation*, see, e.g., [19]. Let

$$S_x = \sum_{k \in C_x} v_k(x)\, S_{k^*}, \quad x \in \mathbb{R}_+^d, \tag{1.1}$$

where $C_x$ denotes the set of all vertices of the unit cube which contains $x$, $v_k(x)$ is the volume of the parallelepiped with $k$ and $x$ being diagonally opposite vertices and with faces parallel to the coordinate planes, and $k^*$ means the vertex opposite to $k$ in the cube that contains $x$. It is easily seen that (1.1) determines $S_x$ uniquely even if $x$ lies on the boundaries of several adjacent cubes. This interpolation technique, expressed in another way, was used by Bass and Pyke [3]. A special feature of this choice of interpolation is that $|x| = x^1 \cdots x^d$ (being a multilinear function in all coordinates) admits the exact interpolation.

Consider the *renewal set*

$$\mathcal{M}_t = \left\{ x \in \mathbb{R}_+^d : S_x \geq t \right\}, \quad t > 0,$$

which is also the *upper excursion set* at level $t$ for the random field $S_x$, $x \in \mathbb{R}_+^d$. Since the multi-linear interpolation (1.1) produces a continuous function, $\mathcal{M}_t$ is a random closed set in $\mathbb{R}_+^d$, see [15].

The paper is devoted to almost sure and distributional limit theorems for the scaled sets $\mathcal{M}_t$. As in the univariate case, these results rely on the inversion applied to the limit theorems for the sums. The strong laws of large numbers (SLLNs) for multiple sums were established in [17] and [10]. Unlike the conventional case of $d = 1$, they hold if and only if the generic summand has a logarithmic moment whose order depends on the dimension, see (2.4). By inverting this and other SLLNs for multiple sums, we show that the rescaled random sets $t^{-1/d}\mathcal{M}_t$ converge as $t \to \infty$ (in a sense to be specified) to the set

$$\mathcal{H} = \left\{ x \in \mathbb{R}_+^d : |x| \geq \mu^{-1} \right\}. \tag{1.2}$$

While the strong law of large numbers reduces to a direct inversion of the random field $S_x$, $x \in \mathbb{R}_+^d$, the techniques involved in the law of the iterated logarithm (LIL) and the central limit theorem are considerably more complicated. For instance, the LIL relies on perturbing $\mu^{-1}$ with an iterated logarithm term multiplied by a constant. We examine the values of the constant that ensure the validity of the LIL and show that the boundary values violate it. We also derive the LILs for distances between the scaled $\mathcal{M}_t$ and $\mathcal{H}$. Such distances involve the uniform behaviour of the perturbed sets

— the lack of a well-defined subtraction makes it necessary to carefully analyze the behaviour of multiple sums at various directions. While the upper limits are shown to be non-trivial, the lower limits vanish. The latter is rather surprising meaning that, inside any cone, the boundary of $t^{-1/d}\mathcal{M}_t$ infinitely often lies within a small envelope around the boundary of $\mathcal{H}$. The proof relies on considering the LILs for multiple sums inside a cone, outside it and further subtle results concerning the LIL for subsequences, see [5]. Finally, we establish the central limit theorem for radial functions that represent $\mathcal{M}_t$ in the spherical coordinates. The developed techniques may be extended to prove limit theorems for more general random fields with drift, provided they satisfy appropriate versions of the SLLN and the LIL.

The letters $m$, $n$, $k$, and $u$, $x$, $y$, $z$ stand for vectors from $\mathbb{N}^d$ or $\mathbb{R}_+^d$, or of spaces of other dimensions. Their components are denoted by the respective superscripted letters, e.g., $m = (m^1, \dots, m^d)$. Denote $\bar{1} = (1, \dots, 1)$. We will also make use of the standard componentwise partial order with $m \le n$ meaning that $m^i \le n^i$ for all $i = 1, \dots, d$, denote

$$|m| = m^1 \cdots m^d,$$

and write $m \to \infty$ if $\max\{m^1, \dots, m^d\} \to \infty$. For $m \in \mathbb{N}^d$, this is the case if and only if $|m| \to \infty$, while the condition $y \to \infty$ does not imply $|y| \to \infty$ for non-integer $y \in \mathbb{R}_+^d$.

Throughout the paper, $\log c$ and $\log\log c$ for $c \ge 0$ have the usual meanings except near zero; we set $\log c$ (respectively, $\log\log c$) to be 1 for $c \in [0, e)$ (respectively, $c \in [0, e^e)$). The extended logarithmic functions are positive and monotone on $\mathbb{R}_+$.

Sections 2 and 3 formulate the main results concerning the SLLN and the LIL, their longer proofs are postponed to Sections 5 and 6, respectively. The central limit theorem for radial functions is derived in Section 4. Special features of the one-dimensional case are considered in Section 7. In Appendix, we derive the SLLN and the LIL for multi-dimensional sums $S_n$ as $n \to \infty$ within a sector. These results differ from those available in the literature so far and complement the sectorial laws proved in [11].

## 2. Strong law of large numbers

We start with a rather general multidimensional inversion theorem which allows converting a.s. limit theorems for $S_n$ to their counterparts for $\mathcal{M}_t$ in terms of set inclusions. We will need the following generalisation of the regular variation property, which is due to Avacumović [2], see also [1,8] and references therein.

DEFINITION 2.1. A function $p\colon [0,\infty) \mapsto [0,\infty)$ which is positive for all sufficiently large arguments is said to be $\mathcal{O}$-regularly varying if, for all $c > 0$,

$$\limsup_{t\to\infty} \frac{p(ct)}{p(t)} < \infty.$$

The class of $\mathcal{O}$-regularly varying functions includes all regularly varying functions and many oscillating ones. The substitution $c \to c^{-1}$ leads to an equivalent characterisation:

$$\liminf_{t\to\infty} \frac{p(ct)}{p(t)} > 0. \tag{2.1}$$

For $c \in \mathbb{R}$, denote

$$\mathcal{H}(c) = \left\{ x \in \mathbb{R}_+^d : |x| \ge \mu^{-1} + c \right\}.$$

Then $\mathcal{H}(c)$ decreases in $c$, and $\mathcal{H}(0)$ becomes $\mathcal{H}$ from (1.2).

THEOREM 2.2 (multidimensional inversion). *Let $p$ be an $\mathcal{O}$-regularly varying function such that $p(t)$ is non-decreasing and $t^{-1}p(t)$ is non-increasing for all sufficiently large $t$. If*

$$S_n - \mu|n| = o(p(|n|)) \quad \textit{a.s. as } n \to \infty, \tag{2.2}$$

*then, for all $\varepsilon > 0$ and sufficiently large $t$,*

$$\mathcal{H}(\varepsilon p(t)t^{-1}) \subset t^{-1/d}\mathcal{M}_t \subset \mathcal{H}(-\varepsilon p(t)t^{-1}) \quad \textit{a.s.} \tag{2.3}$$

Theorem 2.2 yields the following Marcinkiewicz–Zygmund type SLLN for $\mathcal{M}_t$ in terms of set inclusions.

COROLLARY 2.3 (SLLN for renewal sets, set-inclusion version). *If*

$$\mathbf{E}\left(|\xi|^\beta \log^{d-1}|\xi|\right) < \infty \tag{2.4}$$

*for some $\beta \in [1,2)$, then, for each $\varepsilon > 0$ and all sufficiently large $t$,*

$$\mathcal{H}(\varepsilon t^{-1+1/\beta}) \subset t^{-1/d}\mathcal{M}_t \subset \mathcal{H}(-\varepsilon t^{-1+1/\beta}) \quad \textit{a.s.}$$

PROOF. According to the Marcinkiewicz–Zygmund type SLLN for multi-indexed sums due to Gut [10, Theorem 3.2] (see also [13, Corollary 9.3]), (2.4) implies (2.2) with the required function $p(t) = t^{1/\beta}$, $t > 0$, which satisfies the conditions of Theorem 2.2. To be more precise, in Gut's paper $n \to \infty$ means $\min\{n_1,\dots,n_d\} \to \infty$ instead of $\max\{n_1,\dots,n_d\} \to \infty$. However, the necessary refinement can be easily obtained. $\square$

Theorem 2.2 yields further strong laws of large numbers under other normalisations that still ensure the validity of the SLLNs for multiple sums as described in [13, Chapter 9].

In the following, $\mathbb{T}$ denotes a closed convex cone such that

$$\mathbb{T} \setminus \{0\} \subset \mathbb{R}^d_{++} = (0, \infty)^d. \tag{2.5}$$

If (2.2) is weakened to

$$S_n - \mu|n| = o(p(|n|)) \quad \text{a.s. as } \mathbb{T} \ni n \to \infty$$

for all such cones $\mathbb{T}$ (where $\mathbb{T} \ni n \to \infty$ means that $n \to \infty$ within $\mathbb{T}$), then (2.3) is replaced by

$$\mathbb{T} \cap \mathcal{H}(\varepsilon p(t)t^{-1}) \subset \mathbb{T} \cap t^{-1/d}\mathcal{M}_t \subset \mathbb{T} \cap \mathcal{H}(-\varepsilon p(t)t^{-1}). \tag{2.6}$$

The proof of (2.6) follows the lines of the proof of Theorem 2.2, see Section 5. These conical (or sectorial) versions of the a.s. limit theorems usually hold under weaker moment assumptions. The next result follows from the sectorial SLLN proved in Theorem 8.1.

Corollary 2.4. *If* $\mathbb{T} \setminus \{0\} \subset \mathbb{R}^d_{++}$ *and* $\mathbf{E}|\xi|^\beta < \infty$ *for some* $\beta \in [1, 2)$, *then* (2.6) *holds with* $p(t) = t^{1/\beta}$.

Note that (2.3) implies that $t^{-1/d}\mathcal{M}_t \to \mathcal{H}$ almost surely in the Fell topology on the family of closed sets, see, e.g., [15, Appendix C]. The convergence of sets can be quantified in various ways. The *Hausdorff* distance between two subsets $X$ and $Y$ of $\mathbb{R}^d$ is defined by

$$\rho_{\mathrm{H}}(X, Y) = \max\Big\{ \sup_{x\in X} \inf_{y\in Y} \rho(x, y), \sup_{y\in Y} \inf_{x\in X} \rho(x, y) \Big\},$$

with $\rho$ denoting the Euclidean distance in $\mathbb{R}^d$.

The *localised symmetric difference* distance (also called the Fréchet–Nikodym distance) between two Borel subsets $X$ and $Y$ of $\mathbb{R}^d$ is defined by

$$\rho^K_\triangle(X, Y) = \lambda_d(K \cap (X \triangle Y)),$$

where $\lambda_d$ is the Lebesgue measure on $\mathbb{R}^d$ and $K$ is a Borel set in $\mathbb{R}^d$ that determines the localisation.

Theorem 2.5 (SLLN for renewal sets, metric version). *If* (2.4) *holds for some* $\beta \in [1, 2)$, *then*

$$\rho_{\mathrm{H}}(t^{-1/d}\mathcal{M}_t, \mathcal{H}) = o(t^{-1+1/\beta}) \quad \textit{a.s. as } t \to \infty, \tag{2.7}$$

*and, for any compact set* $K \subset \mathbb{R}^d$,

$$\rho_{\triangle}^{K}(t^{-1/d}\mathcal{M}_t, \mathcal{H}) = \mathcal{O}(t^{-1+1/\beta}) \quad \text{a.s. as } t \to \infty. \tag{2.8}$$

*If, additionally,* $K \subset \mathbb{R}^d_{++}$, *then* (2.8) *holds provided only that* $\mathbf{E}|\xi|^\beta < \infty$. *Under this condition,* (2.7) *holds for* $\rho_{\mathrm{H}}((t^{-1/d}\mathcal{M}_t) \cap K, \mathcal{H} \cap K)$.

We now briefly consider discrete renewal sets $\mathcal{M}_t \cap \mathbb{N}^d$ constructed by non-interpolated partial sums. Strong limit theorems for the cardinality $N_t$ of the finite set $\mathbb{N}^d \setminus \mathcal{M}_t$ may be found in [13, Chapter 11]. In particular, the following SLLN holds.

THEOREM 2.6 [13, Theorem 11.7]. *Let* $\xi \geq 0$ *a.s. If* $\mathbf{E}(\xi \log^{d-1} \xi) < \infty$, *then*

$$\frac{N_t}{t \log^{d-1} t} \to \frac{1}{\mu (d-1)!} \quad \text{a.s. as } t \to \infty.$$

A similar result holds for $\mathbf{E} N_t$, see [13, Theorem 11.5]. Set-inclusion results for $t^{-1/d}(\mathcal{M}_t \cap \mathbb{N}^d)$ immediately follow from those for the continuous renewal sets, e.g., (2.3) holds with all sides intersected with $t^{-1/d}\mathbb{N}^d$. The situation with metric results is more complicated. In the most natural form, these results would look like a.s. limit theorems for the number of lattice points in $(t^{-1/d}\mathcal{M}_t) \triangle \mathcal{H}$. Such theorems might be derived from discretised set inclusions (2.3) by using bounds on the number of integer points between the sets $\partial\mathcal{H}(c)$ for different $c$'s. The latter, in turn, are closely related to the so-called generalised Dirichlet divisor problem in number theory.

For completeness, we now give some facts on this topic, following [13, Appendix 10]. For $k \geq 1$, let

$$T_k = \operatorname{card}\{n \in \mathbb{N}^d : |n| \leq k\}.$$

In order to bound the number of integer points between the sets $\partial\mathcal{H}(c)$, we need some results on the asymptotic behaviour of $T_k - T_j$ as $j, k \to \infty$. It can be proved that there exists a polynomial $\mathcal{P}_d$ of degree $d - 1$ such that

$$T_k = k\mathcal{P}_d(\log k) + \mathcal{O}(k^\alpha) \quad \text{as } k \to \infty,$$

for all $\alpha > \alpha_d$ with some $\alpha_d < 1$. Although there is a number of results concerning $\alpha_d$, their exact values are not yet known. According to the Hardy–Titchmarsh conjecture (that would follow from the Riemann hypothesis), $\alpha_d = (d-1)/(2d)$, and this bound would be sufficient in order not to dominate the stochastic factors. Without involving this and related number-theoretic conjectures, the necessary bounds can be obtained only in the case $d = 2$.

## 3. Laws of the iterated logarithm

Now we turn to the law of the iterated logarithm (LIL) for $\mathcal{M}_t$ in terms of set inclusions. Recall that $\mathbb{T}$ always denotes a closed convex cone such that (2.5) holds. Let

$$\mathcal{H}_{\mathbb{T}}(c) = \big(\mathbb{T} \cap \mathcal{H}(c)\big) \cup \big((\mathbb{R}^d_{++} \setminus \mathbb{T}) \cap \mathcal{H}(c\sqrt{d})\big). \tag{3.1}$$

In other words, $\mathcal{H}_{\mathbb{T}}(c)$ consists of all points $x \in \mathbb{R}^d_{++}$ such that $|x| \geq \mu^{-1} + c$ in case $x \in \mathbb{T}$ and $|x| \geq \mu^{-1} + c\sqrt{d}$ if $x \notin \mathbb{T}$.

Assume that $\xi$ has a finite variance denoted by $\sigma^2$ and denote

$$\varkappa(t) = \sqrt{2t^{-1} \log\log t}, \quad t > 0.$$

THEOREM 3.1 (LIL for renewal sets, set-inclusion version). *Let*

$$\mathbf{E}\Big[\xi^2 \frac{\log^{d-1}|\xi|}{\log\log|\xi|}\Big] < \infty. \tag{3.2}$$

(i) *If* $\gamma < -\mu^{-3/2}$, *then*

$$t^{-1/d}\mathcal{M}_t \subset \mathcal{H}_{\mathbb{T}}(\gamma\sigma\varkappa(t)) \quad a.s.$$

*for all sufficiently large* $t$.

(ii) *If* $-\mu^{-3/2} \leq \gamma \leq \mu^{-3/2}$, *then there are sequences* $\{t_i', i \geq 1\}$ *and* $\{t_i'', i \geq 1\}$ *depending on* $\omega$, $\mathbb{T}$, *and* $\gamma$ *such that* $t_i' \to \infty$ *and* $t_i'' \to \infty$ *almost surely, and*

$$(t_i')^{-1/d}\mathcal{M}_{t_i'} \not\subset \mathcal{H}_{\mathbb{T}}(\gamma\sigma\varkappa(t_i')), \tag{3.3}$$

$$(t_i'')^{-1/d}\mathcal{M}_{t_i''} \not\supset \mathcal{H}_{\mathbb{T}}(\gamma\sigma\varkappa(t_i'')), \tag{3.4}$$

*almost surely for all* $i$.

(iii) *If* $\gamma > \mu^{-3/2}$, *then*

$$t^{-1/d}\mathcal{M}_t \supset \mathcal{H}_{\mathbb{T}}(\gamma\sigma\varkappa(t)) \quad a.s.$$

*for all sufficiently large* $t$.

The idea of the proof of this theorem is to apply two laws of the iterated logarithm for multiple sums. First, a modification of the sectorial law from [11] with the limiting constant 1 (proved in Appendix) is applicable inside $\mathbb{T}$, while the law of the iterated logarithm from [21] in the full $\mathbb{R}^d_+$ with the limiting constant $\sqrt{d}$ is applicable in the complement of $\mathbb{T}$.

REMARK 3.2. Theorem 3.1 may be reformulated as

$$\sup\left\{\gamma : t^{-1/d}\mathcal{M}_t \subset \mathcal{H}_{\mathbb{T}}(\gamma\sigma\varkappa(t)) \text{ a.s. for large } t\right\} = -\mu^{-3/2},$$
$$\inf\left\{\gamma : t^{-1/d}\mathcal{M}_t \supset \mathcal{H}_{\mathbb{T}}(\gamma\sigma\varkappa(t)) \text{ a.s. for large } t\right\} = \mu^{-3/2},$$

and the supremum and infimum are not attained in the sense that the above inclusions do not hold for the critical values $\pm\mu^{-3/2}$.

As previously, we now quantify the results of Theorem 3.1 by means of the Hausdorff distance $\rho_{\mathrm{H}}$ and the localised symmetric difference metric $\rho_{\triangle}^K$. For any cone $\mathbb{T}$, define

$$L_{\mathbb{T}} = \frac{1}{d}\int_{\mathbb{T}\cap\mathbb{S}^{d-1}} |u|^{-1}\,\mathrm{d}u, \tag{3.5}$$

where $\mathbb{S}^{d-1}$ is the unit Euclidean sphere in $\mathbb{R}^d$. For compact set $K \subset \mathbb{R}^d_+$, let $\mathbb{T}_K$ denote the cone generated by $K \cap \partial\mathcal{H}$, that is, the smallest cone containing $K \cap \partial\mathcal{H}$. Note that $\mathbb{T}_K$ satisfies (2.5).

THEOREM 3.3 (LIL for renewal sets, metric version). *Under the assumptions of Theorem* 3.1,

$$\limsup_{t\to\infty} \frac{\rho_{\mathrm{H}}(t^{-1/d}\mathcal{M}_t, \mathcal{H})}{\varkappa(t)} = d^{-\frac{1}{2}}\sigma\mu^{-\frac{1}{2}-\frac{1}{d}} \quad \textit{a.s.}, \tag{3.6}$$

*and, for any compact set $K$ in* $\mathbb{R}^d$,

$$\limsup_{t\to\infty} \frac{\rho_{\triangle}^K(t^{-1/d}\mathcal{M}_t, \mathcal{H})}{\varkappa(t)} \le 2\sigma\mu^{-\frac{3}{2}} L_{\mathbb{T}_K} \quad \textit{a.s.} \tag{3.7}$$

*If $\xi$ is a.s. non-negative,* (3.7) *holds with the factor* 2 *on the right-hand side replaced by* 1. *If, additionally, $K \subset \mathbb{R}^d_{++}$, then* (3.7) *holds provided only that* $\mathbf{E}\xi^2 < \infty$.

Note that (3.6) gives the exact value of the upper limit unlike (3.7). This is achieved due to the high sensitivity of the Hausdorff metric to outlying points.

Assume that $\xi$ is a.s. non-negative. In the one-dimensional case, the corresponding *lower* limits in Theorem 3.3 equal zero. Indeed, it follows from the ordinary LIL and continuity of $S_x$ that $S_{t_i/\mu} = t_i$ along some sequence $t_i \to \infty$. Since $\xi \ge 0$ a.s., this implies

$$t_i^{-1}\mathcal{M}_{t_i} = [\mu^{-1}, \infty) = \mathcal{H} \quad \text{for all } i \ge 1,$$

and the claim follows. It is quite remarkable that this, even in a stronger form, holds in any dimension.

THEOREM 3.4. *Let $\xi$ be a.s. non-negative. If* (3.2) *holds, then*

$$\liminf_{t\to\infty} \sqrt{t}\,\rho_{\mathrm{H}}(t^{-1/d}\mathcal{M}_t,\mathcal{H}) = 0 \quad a.s., \tag{3.8}$$

$$\liminf_{t\to\infty} \sqrt{t}\,\rho_{\triangle}^{K}(t^{-1/d}\mathcal{M}_t,\mathcal{H}) = 0 \quad a.s. \tag{3.9}$$

*If, additionally, $K \subset \mathbb{R}^d_{++}$, then* (3.9) *holds provided only that* $\mathbf{E}\,\xi^2 < \infty$.

## 4. Convergence in distribution

Assume that $\sigma^2 = \mathbf{E}\,(\xi-\mu)^2 < \infty$. The limit theorem for multiple sums by Wichura [20, Corollary 1] yields that

$$\bar{S}_{t,x} = \frac{S_{[tx]} - |[tx]|\mu}{\sigma t^{d/2}}, \quad x \in [0,1]^d,$$

converges in distribution as $t \to \infty$ to the Chentsov field $Z_x$, $x \in [0,1]^d$, which is a centred Gaussian field with the covariance

$$\mathbf{E}\,(Z_x Z_y) = |x \wedge y|, \quad x, y \in \mathbb{R}^d_+.$$

Here the integer part $[\cdot]$ and the minimum $\wedge$ of vectors are defined componentwise. The convergence of $\bar{S}_{t,x}$ means that the value of each measurable functional continuous in the uniform metric converges in distribution to its value on the limiting Chentsov random field, see [20, Definition 1].

Bickel and Wichura [6] formalised this convergence as the weak convergence in the Skorokhod topology for random fields. The settings in [20] and [6] concerned the non-interpolated fields. The same convergence holds also for the interpolated fields

$$\tilde{S}_{t,x} = \frac{S_{tx} - |tx|\mu}{\sigma t^{d/2}}, \quad x \in [0,1]^d. \tag{4.1}$$

By [6, Theorem 2] or [18, Theorem 5.6], this follows from the weak convergence of finite-dimensional distributions and the tightness criterion

$$\lim_{\delta\downarrow 0}\limsup_{t\to\infty} \mathbf{P}\left\{ w_\delta(\tilde{S}_{t,\cdot}) > \varepsilon \right\} = 0 \tag{4.2}$$

for any $\varepsilon > 0$. Here $w_\delta$ stands for the $\delta$-modulus of continuity. The finite-dimensional convergence follows from the central limit theorem, whereas (4.2) holds by the inequality $w_\delta(\tilde{S}_{t,\cdot}) \le w_{2\delta}(\bar{S}_{t,\cdot})$, which is valid for large $t$, and the counterpart of (4.2) for $\bar{S}_{t,\cdot}$, which is derived in [20, Theorem 3] and [6, Theorem 5].

Notice that Bass and Pyke [3] considered random signed measures generated by the interpolated fields and established the convergence in the analogue of the uniform metric for set-indexed functions. The convergence of $\tilde{S}_{t,x}$ might be also directly derived from [3, Theorem 7.1] under a slightly stronger moment assumption $\mathbf{E}\,|\xi|^{2+\delta} < \infty$ for some $\delta > 0$, see Remark 8.5 ibid. We also note that the above convergence holds if $[0,1]^d$ is replaced by any compact set $K \subset \mathbb{R}^d_+$. Finally, we remark that both the pre-limiting and limiting fields are a.s. continuous, and so the convergence can also be regarded as the weak convergence in the uniform metric, see [7, p. 151].

The lack of a well-defined centring (and subtraction) for random sets makes it necessary to express limit theorems for the random sets $t^{-1/d}\mathcal{M}_t$ in terms of some real-valued functions of them. For this, choose the radial function

$$r_t(u) = \inf\left\{ a > 0 : au \in t^{-1/d}\mathcal{M}_t \right\}, \quad u \in \mathbb{R}^d_{++}.$$

In this section we will assume that the generic summand $\xi$ defining the multiple sums is almost surely non-negative. Hence, $S_{au} \le S_{bu}$ for $a \le b$, and so the radial function uniquely identifies the set $\mathcal{M}_t$.

By Corollary 2.3,

$$r_t(u) \to (\mu|u|)^{-1/d} \quad \text{as } t \to \infty$$

for all $u \in \mathbb{R}^d_{++}$. We may assume that the Euclidean norm of $u$ equals one.

THEOREM 4.1. *Assume that $\xi \ge 0$ a.s. Let $K$ be a compact subset of $\mathbb{S}^{d-1} \cap \mathbb{R}^d_{++}$ and let $f^-, f^+ \colon K \mapsto \mathbb{R}$ be continuous functions. Then*

$$\mathbf{P}\Big\{ f^-(u) < \sqrt{t}\Big((r_t(u))^d - \frac{1}{\mu|u|}\Big) \le f^+(u),\ u \in K \Big\}$$
$$\to \mathbf{P}\Big\{ f^-(u) < \frac{\sigma}{\mu^{3/2}}|u|^{-1} Z_{u/|u|^{1/d}} \le f^+(u),\ u \in K \Big\}$$

*as $t \to \infty$, where $Z_x$, $x \in \mathbb{R}^d_{++}$, is the Chentsov random field.*

PROOF. By the definition of the radial function,

$$\Big\{ \sqrt{t}\Big((r_t(u))^d - \frac{1}{\mu|u|}\Big) \le f^+(u),\ u \in K \Big\} \tag{4.3}$$
$$= \big\{ r_t(u) \le y_t^+(u),\ u \in K \big\} = \big\{ S_{t^{1/d} y_t^+(u) u} \ge t,\ u \in K \big\},$$

where

$$y_t^+(u) = \Big( \frac{f^+(u)}{\sqrt{t}} + \frac{1}{\mu|u|} \Big)^{1/d} = \Big( \frac{\mu f^+(u)|u|}{\sqrt{t}} + 1 \Big)^{1/d} (\mu|u|)^{-1/d}.$$

Let $M^+$ be the supremum of $\mu|f^+(u)||u|$ over $u \in K$, and so $y_t^+(u) = (\alpha^+)^{1/d}(\mu|u|)^{-1/d}$ with an $\alpha^+ = \alpha^+(t,u) \in [1 - M^+t^{-1/2}, 1 + M^+t^{-1/2}]$.

Thus, continuing (4.3),

$$\begin{aligned}
&\Big\{\sqrt{t}\Big((r_t(u))^d - \frac{1}{\mu|u|}\Big) \le f^+(u),\ u \in K\Big\} \\
&= \Big\{\frac{S_{t^{1/d}y_t^+(u)u} - |t^{1/d}y_t^+(u)u|\mu}{\sigma\sqrt{t}} \ge -\frac{\mu}{\sigma}f^+(u)|u|,\ u \in K\Big\} \\
&= \Big\{\frac{S_{(t\alpha^+/\mu|u|)^{1/d}u} - |(t\alpha^+/\mu|u|)^{1/d}u|\mu}{\sigma\sqrt{t}} \ge -\frac{\mu}{\sigma}f^+(u)|u|,\ u \in K\Big\} \\
&= \Big\{\tilde{S}_{(t/\mu)^{1/d},(\alpha^+)^{1/d}u/|u|^{1/d}} \ge -\frac{\mu^{3/2}}{\sigma}f^+(u)|u|,\ u \in K\Big\}
\end{aligned}$$

with $\tilde{S}$ defined by (4.1). It follows from the above equality and its counterpart for $f^-$ that

$$\begin{aligned}
&\mathbf{P}\Big\{f^-(u) < \sqrt{t}\Big((r_t(u))^d - \frac{1}{\mu|u|}\Big) \le f^+(u),\ u \in K\Big\} \\
&= \mathbf{P}\Big\{\tilde{S}_{(t/\mu)^{1/d},(\alpha^+)^{1/d}u/|u|^{1/d}} \ge -\frac{\mu^{3/2}}{\sigma}f^+(u)|u|, \\
&\qquad \tilde{S}_{(t/\mu)^{1/d},(\alpha^-)^{1/d}u/|u|^{1/d}} < -\frac{\mu^{3/2}}{\sigma}f^-(u)|u|,\ u \in K\Big\}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\big(\tilde{S}_{(t/\mu)^{1/d},(\alpha^+)^{1/d}u/|u|^{1/d}},\ \tilde{S}_{(t/\mu)^{1/d},(\alpha^-)^{1/d}u/|u|^{1/d}},\ u \in K\big) \\
&\qquad \to \big(Z_{u/|u|^{1/d}},\ Z_{u/|u|^{1/d}},\ u \in K\big)
\end{aligned}$$

weakly in the uniform metric as $t \to \infty$, since $\alpha^\pm(t,u) \to 1$ uniformly over $u \in K$. It remains to use the symmetry property of the Chentsov random field. □

REMARK 4.2. The random field

$$\zeta_u = |u|^{-1}Z_{u/|u|^{1/d}}, \quad u \in \mathbb{R}^d_{++},$$

which coincides with the limiting field up to a constant, has the covariance

$$\mathbf{E}(\zeta_u\zeta_v) = \frac{\big||u|^{1/d}v \wedge |v|^{1/d}u\big|}{(|u||v|)^2},$$

which becomes $|u \wedge v|$ if $|u| = |v| = 1$. Since

$$\zeta_{cu} = c^{-d}\zeta_u \quad \text{for any } c > 0, \tag{4.4}$$

and $\zeta_u = Z_u$ if $|u| = 1$, $\zeta_u$ can be obtained by extrapolation of $Z_u$ from $\{u \in \mathbb{R}^d_{++} : |u| = 1\}$ to $\mathbb{R}^d_{++}$ by means of (4.4).

## 5. Proofs for results in Section 2

Since we have to prove inclusions (2.3) only for large $t$, the function $p$ may be arbitrarily redefined in a neighbourhood of the origin. Particularly, we may assume that $p(t)$ is positive and non-decreasing for all $t \geq 0$, and $t^{-1}p(t)$ is non-increasing for all $t > 0$.

First, list some immediate properties of the function $p$ needed in the sequel.

LEMMA 5.1. *Under the assumptions of Theorem* 2.2,
(a) $p(t) = \mathcal{O}(t)$ *as* $t \to \infty$;
(b) $\liminf_{t\to\infty} p(t)/p(ct + \delta p(t)) > 0$ *for any* $c, \delta > 0$;
(c) $p(t) - \delta t$ *is non-increasing in* $t$ *for large* $\delta$ *and* $t$.

PROOF. (a) follows from the fact that $p(t)/t$ is non-increasing due to the positivity of $p$. By (a), taking into account that $p(t)$ is non-decreasing,

$$\liminf_{t\to\infty} \frac{p(t)}{p(ct + \delta p(t))} \geq \liminf_{t\to\infty} \frac{p(t)}{p(ct + \delta M t)}$$

with some $M > 0$, the right-hand side being positive due to (2.1).

Since $p(t)/t$ is non-increasing, $\delta - t^{-1}p(t)$ is positive and non-decreasing in $t$ for large $\delta$ and $t$. Hence,

$$p(t) - \delta t = -t(\delta - t^{-1}p(t))$$

does not increase. $\square$

Next, we show that the asymptotic behaviour of $S_n$ given by (2.2) is inherited by the interpolated sums.

LEMMA 5.2. *If* (2.2) *holds, then*

$$S_x - \mu|x| = o(p(|x|)) \quad \text{a.s. as } x \to \infty. \tag{5.1}$$

PROOF. Being multi-linear itself, $|x|$ can be exactly recovered by

$$|x| = \sum_{k\in C_x} v_k(x)\,|k^*|, \quad x \in \mathbb{R}^d_+. \tag{5.2}$$

Let $\widetilde{C}_x = \{k \in C_x : |k^*| \neq 0\}$. By (1.1), (5.2), and monotonicity of $p$, we have for all $x \in \mathbb{R}^d_+$

$$\begin{aligned}\frac{|S_x - \mu|x||}{p(|x|)} &\le \frac{\sum_{k\in\widetilde{C}_x} v_k(x)|S_{k^*} - \mu|k^*||}{p(\sum_{k\in\widetilde{C}_x} v_k(x)\,|k^*|)} \\ &= \sum_{k\in\widetilde{C}_x} \frac{v_k(x)|S_{k^*} - \mu|k^*||}{p(v_k(x)\,|k^*|)}\,\frac{p(v_k(x)\,|k^*|)}{p(\sum_{k\in\widetilde{C}_x} v_k(x)\,|k^*|)} \\ \le \sum_{k\in\widetilde{C}_x} \frac{v_k(x)|S_{k^*} - \mu|k^*||}{p(v_k(x)\,|k^*|)} &= \sum_{k\in\widetilde{C}_x} \frac{|S_{k^*} - \mu|k^*||}{p(|k^*|)}\,\frac{p(|k^*|)}{|k^*|}\,\frac{v_k(x)\,|k^*|}{p(v_k(x)\,|k^*|)}.\end{aligned}$$

Since $p(t)/t$ is non-increasing,

$$\frac{|S_x - \mu|x||}{p(|x|)} \le \sum_{k\in\widetilde{C}_x} \frac{|S_{k^*} - \mu|k^*||}{p(|k^*|)}\,,$$

and so (2.2) implies (5.1). $\square$

Proof of Theorem 2.2. Assume that the left-hand inclusion in (2.3) does not hold, that is, there are sequences $\{x_i, i \ge 1\}$ and $\{t_i, i \ge 1\}$ with $t_i \to \infty$, such that $x_i \in \mathcal{H}(\varepsilon p(t_i)t_i^{-1})$ and $x_i \notin t_i^{-1/d}\mathcal{M}_{t_i}$ for all $i$. Denoting $y_i = t_i^{1/d} x_i$, we may write the former inclusion as $|y_i| \ge \mu^{-1}t_i + \varepsilon p(t_i)$ and the latter one as $S_{y_i} < t_i$. The first inequality implies $y_i \to \infty$. Hence,

$$\begin{aligned}\alpha_i = \frac{S_{y_i} - \mu|y_i|}{p(|y_i|)} &= \frac{|y_i|}{p(|y_i|)}\Big(\frac{S_{y_i}}{|y_i|} - \mu\Big) \\ < \frac{|y_i|}{p(|y_i|)}\Big(\frac{t_i}{\mu^{-1}t_i + \varepsilon p(t_i)} - \mu\Big) &= -\frac{|y_i|}{p(|y_i|)}\frac{\varepsilon\mu p(t_i)}{\mu^{-1}t_i + \varepsilon p(t_i)}.\end{aligned}$$

Since $p(t)/t$ is non-increasing,

$$\alpha_i < -\frac{\mu^{-1}t_i + \varepsilon p(t_i)}{p(\mu^{-1}t_i + \varepsilon p(t_i))}\,\frac{\varepsilon\mu p(t_i)}{\mu^{-1}t_i + \varepsilon p(t_i)} = -\frac{\varepsilon\mu p(t_i)}{p(\mu^{-1}t_i + \varepsilon p(t_i))}\,. \tag{5.3}$$

Note that $\alpha_i \to 0$ as $i \to \infty$ by (5.1), whereas the negative right-hand side of (5.3) is bounded away from zero by Lemma 5.1(b). This contradiction proves the left-hand inclusion in (2.3).

Assume that the right-hand inclusion in (2.3) does not hold, so that there exist sequences $\{x_i, i \ge 1\}$ and $\{t_i, i \ge 1\}$ with $t_i \to \infty$ such that

$|y_i| < \mu^{-1}t_i - \varepsilon p(t_i)$ and $S_{y_i} \ge t_i$ for all $i$, where $y_i = t_i^{1/d} x_i$. Therefore, $S_{y_i} \to \infty$, which easily leads to $y_i \to \infty$ by (1.1). By Lemma 5.1(c),

$$|y_i| < \mu^{-1}S_{y_i} - \varepsilon p(S_{y_i}) \tag{5.4}$$

for large $i$ and sufficiently small $\varepsilon > 0$ (that may be smaller than the first chosen $\varepsilon$). Using the above definition of $\alpha_i$, we get

$$\alpha_i = \frac{|y_i|}{p(|y_i|)}\Big(\frac{S_{y_i}}{|y_i|} - \mu\Big) > \frac{|y_i|}{p(|y_i|)}\Big(\frac{S_{y_i}}{\mu^{-1}S_{y_i} - \varepsilon p(S_{y_i})} - \mu\Big)$$
$$= \frac{|y_i|}{\mu^{-1}S_{y_i} - \varepsilon p(S_{y_i})}\,\frac{\varepsilon\mu p(S_{y_i})}{p(|y_i|)}\,.$$

By (5.4) and taking into account the monotonicity of $p$, we have

$$\alpha_i > \frac{\mu|y_i|}{S_{y_i}}\,\frac{\varepsilon\mu p(S_{y_i})}{p(\mu^{-1}S_{y_i} - \varepsilon p(S_{y_i}))} \ge \frac{\mu|y_i|}{S_{y_i}}\,\frac{\varepsilon\mu p(S_{y_i})}{p(\mu^{-1}S_{y_i})}. \tag{5.5}$$

Note that

$$S_y - \mu|y| = \mathcal{O}(|y|) \quad \text{a.s. as } y \to \infty. \tag{5.6}$$

This is not a straightforward consequence of (5.1) and Lemma 5.1(a), since $y \to \infty$ need not imply $|y| \to \infty$ (which is possible if $y \to \infty$ while getting simultaneously closer to one of the coordinate planes). However, (5.6) may be proved in an alternative way: (2.2) and (a) lead to $S_n - \mu|n| = \mathcal{O}(|n|)$ a.s. as $n \to \infty$ in $\mathbb{N}^d$ (which is now equivalent to $|n| \to \infty$), and the latter in turn implies (5.6) in the same manner as (2.2) implies (5.1).

So, by (5.6)

$$\frac{\mu|y_i|}{S_{y_i}} \to 1 \quad \text{a.s. as } i \to \infty. \tag{5.7}$$

At the same time, the second factor on the right-hand side of (5.5) is bounded away from zero as $i \to \infty$ due to (2.1). This contradicts $\alpha_i \to 0$ and so proves the right-hand inclusion in (2.3). $\square$

The following results give bounds on the Hausdorff and the symmetric difference distances between the sets $\mathcal{H}(c)$ for different $c$'s.

LEMMA 5.3. *If* $-\mu^{-1} < c_1 \le c_2$, *then*

$$\rho_{\mathrm{H}}(\mathcal{H}(c_1), \mathcal{H}(c_2)) = \sqrt{d}\big((\mu^{-1} + c_2)^{1/d} - (\mu^{-1} + c_1)^{1/d}\big)\,. \tag{5.8}$$

*If* $c_1, c_2 \to 0$, *then*

$$\rho_{\mathrm{H}}(\mathcal{H}(c_1), \mathcal{H}(c_2)) = d^{-1/2}\mu^{1-1/d}(c_2 - c_1) + \mathcal{O}(c_2 - c_1). \tag{5.9}$$

PROOF. An elementary minimisation argument yields that

$$\inf\{\langle u,x\rangle : x\in\mathcal{H}(c)\} = d(c+\mu^{-1})^{1/d}|u|^{1/d}$$

for all $u\in\mathbb{S}^{d-1}\cap\mathbb{R}^d_+$. The above expression yields the negative of the support function of $\mathcal{H}(c)$ in direction $(-u)$. Since the Hausdorff distance between convex sets $\mathcal{H}(c_1)$ and $\mathcal{H}(c_2)$ equals the uniform distance between their support functions and the maximal value of $|u|$ is $d^{-d/2}$, (5.8) holds and easily yields (5.9). □

LEMMA 5.4. *Let $\mathbb{T}$ be a cone in $\mathbb{R}^d_{++}$. If $-\mu^{-1}<c_1\le c_2$, then*

$$\rho^{\mathbb{T}}_{\triangle}(\mathcal{H}(c_1),\mathcal{H}(c_2)) = L_{\mathbb{T}}(c_2-c_1), \tag{5.10}$$

*where $L_{\mathbb{T}}$ is given by* (3.5).

PROOF. Put $b_i(u)=(\mu^{-1}+c_i)^{1/d}|u|^{-1/d}$, $i=1,2$. Equation (5.10) easily follows by representing $\mathbb{T}\cap(\mathcal{H}(c_1)\setminus\mathcal{H}(c_2))$ in the spherical coordinates:

$$\rho^{\mathbb{T}}_{\triangle}(\mathcal{H}(c_1),\mathcal{H}(c_2)) = \int\limits_{\mathbb{T}\cap\mathbb{S}^{d-1}}\left(\int_{b_1(u)}^{b_2(u)} r^{d-1}\,\mathrm{d}r\right)\mathrm{d}u = \frac{c_2-c_1}{d}\int\limits_{\mathbb{T}\cap\mathbb{S}^{d-1}}|u|^{-1}\,\mathrm{d}u. \quad \square$$

PROOF OF THEOREM 2.5. By Corollary 2.3, (5.8) and (5.9),

$$\begin{aligned}&\rho_{\mathrm{H}}(t^{-1/d}\mathcal{M}_t,\mathcal{H})\\ &\le\sqrt{d}\big((\mu^{-1}+\varepsilon t^{-1+1/\beta})^{1/d}-(\mu^{-1}-\varepsilon t^{-1+1/\beta})^{1/d}\big) = \varepsilon\mathcal{O}(t^{-1+1/\beta}).\end{aligned}$$

A similar bound for the symmetric difference metric follows from (5.10). Since $\varepsilon$ can be chosen arbitrary small, (2.7) and (2.8) follow. If $K\subset\mathbb{R}^d_{++}$, then $K$ is a subset of a cone $\mathbb{T}$ with $\mathbb{T}\setminus\{0\}\subset\mathbb{R}^d_{++}$, so that Corollary 2.4 applies. □

## 6. Proofs for results in Section 3

It suffices to assume that $\sigma=1$. To simplify the notation, let

$$\chi(t)=\sqrt{2t\log\log t}=t\varkappa(t) \tag{6.1}$$

for $t\ge e^e$, and extend both $\chi$ and $\varkappa$ to $[0,\infty)$ and $(0,\infty)$, respectively, so that $\chi$ becomes positive and concave.

It follows from the law of the iterated logarithm for multi-indexed sums due to Wichura [21, Theorem 5], see also [13, Theorem 10.9], that, under (3.2),

$$\limsup_{n\to\infty} \frac{|S_n - \mu|n||}{\chi(|n|)} = \sqrt{d} \quad \text{a.s.}$$

Hence,

$$\limsup_{x\to\infty} \frac{|S_x - \mu|x||}{\chi(|x|)} = \sqrt{d} \quad \text{a.s.} \tag{6.2}$$

Indeed,

$$\frac{|S_x - \mu|x||}{\chi(|x|)} \le \frac{\sum_{k\in C_x} v_k(x)|S_{k^*} - \mu|k^*||}{\chi\big(\sum_{k\in C_x} v_k(x)\,|k^*|\big)}$$
$$\le \frac{\sum_{k\in C_x} v_k(x)|S_{k^*} - \mu|k^*||}{\sum_{k\in C_x} v_k(x)\,\chi(|k^*|)} \le \max_{k\in C_x} \frac{|S_{k^*} - \mu|k^*||}{\chi(|k^*|)},$$

where the second inequality relies on the concavity of $\chi$. The same argument applied to the sectorial version of the LIL proved in Theorem 8.1 leads to

$$\limsup_{\mathbb{T}\ni x\to\infty} \frac{|S_x - \mu|x||}{\chi(|x|)} = 1 \quad \text{a.s.} \tag{6.3}$$

Proof of Theorem 3.1. First, we prove the inclusion in (i). Taking (3.1) into account, we actually need to show that

$$t^{-1/d}\mathcal{M}_t \subset \mathcal{H}(\gamma\sqrt{d}\varkappa(t)) \tag{6.4}$$

and

$$t^{-1/d}(\mathbb{T}\cap\mathcal{M}_t) \subset (\mathbb{T}\cap\mathcal{H}(\gamma\varkappa(t)) \tag{6.5}$$

almost surely for all sufficiently large $t$.

In order to derive (6.4), we assume the contrary and consider the sequences $\{y_i, i\ge 1\}$ and $\{t_i, i\ge 1\}$ with $y_i, t_i \to \infty$ such that

$$|y_i| < \mu^{-1}t_i + \gamma\sqrt{d}\chi(t_i)$$

and $S_{y_i} \ge t_i$ for all $i$. Along the lines of the proof of Theorem 2.2 (with $-\gamma$ instead of $\varepsilon$ and $\sqrt{d}\chi(\cdot)$ instead of $p(\cdot)$), we arrive at an analogue of inequality (5.5):

$$\alpha_i = \frac{S_{y_i} - \mu|y_i|}{\sqrt{d}\chi(|y_i|)} > -\frac{\mu|y_i|}{S_{y_i}}\,\frac{\gamma\mu\chi(S_{y_i})}{\chi(\mu^{-1}S_{y_i})}\,.$$

Passing to the upper limit, by (6.2), (5.7), and (6.1) we arrive at the contradiction

$$1 \geq \limsup_{i\to\infty} \alpha_i \geq -\gamma\mu^{3/2} > 1.$$

The same argument with $y_i \in \mathbb{T}$ and a reference to (6.3) leads to (6.5).

The inclusion in (iii) may be deduced in a similar manner by means of (5.3) instead of (5.5) and lim inf instead of lim sup.

Let us now turn to the proof of (ii). Since $\mathcal{H}_{\mathbb{T}}(c)$ decreases in $c$, it suffices to prove that (3.3) and (3.4) hold with $\gamma = -\mu^{-3/2}$ and $\gamma = \mu^{-3/2}$, respectively. It will be shown that "exceptional" points which violate these inclusions may be found on the diagonal

$$\mathcal{D} = \{x \in \mathbb{R}^d_{++} : x^1 = \cdots = x^d\}.$$

This, however, requires a more delicate analysis. Introduce the sequence of diagonal integer points

$$\mathcal{D} \ni z_i = i \cdot \bar{1} = (i, \ldots, i), \quad i \geq 1,$$

and a (one-dimensional) sequence $\{\eta_j, j \geq 1\}$ of independent copies of $\xi$. For $i \geq 1$, denote $\tilde{S}_i = \sum_{j=1}^i \eta_j$. By [5, Theorem 1.1] (see also (1.14) ibid.), it may be easily checked that, under assumption

$$\mathbf{E}\,(\xi^2 \log\log|\xi|) < \infty, \tag{6.6}$$

which holds by (3.2),

$$q(t) = \sqrt{2t(\log\log t + 1)}, \quad t \geq 0, \tag{6.7}$$

is a lower function for $\{\tilde{S}_{i^d}, i \geq 1\}$. Hence, $q$ is a lower function for the sequence $\{S_{z_i}, i \geq 1\}$, which has the same distribution. In other words, each of the inequalities

$$S_{z_i} \leq \mu|z_i| - q(|z_i|), \quad S_{z_i} \geq \mu|z_i| + q(|z_i|) \tag{6.8}$$

holds infinitely often with probability one.

In order to prove the claim, it suffices to find (random) sequences $\{t'_i, i \geq 1\}$ and $\{t''_i, i \geq 1\}$ such that $t'_i, t''_i \to \infty$ a.s., and for large $i$ a.s.

$$(\mathcal{D} \cap \mathcal{M}_{t'_i}) \not\subset \mathcal{D} \cap \left\{ y \in \mathbb{R}^d_{++} : |y| \geq \mu^{-1}t'_i - \mu^{-3/2}\chi(t'_i) \right\},$$

$$(\mathcal{D} \cap \mathcal{M}_{t''_i}) \not\supset \mathcal{D} \cap \left\{ y \in \mathbb{R}^d_{++} : |y| \geq \mu^{-1}t''_i + \mu^{-3/2}\chi(t''_i) \right\}.$$

Following (6.8), we introduce (random) sequences of indices $\{z'_i, i \geq 1\}$ and $\{z''_i, i \geq 1\}$ such that $z'_i, z''_i \in \mathbb{N}^d \cap \mathcal{D}$, $z'_i, z''_i \to \infty$ a.s., and

$$S_{z'_i} \geq \mu|z'_i| + q(|z'_i|), \quad S_{z''_i} \leq \mu|z''_i| - q(|z''_i|)$$

almost surely for all sufficiently large $i$. Letting $t'_i = S_{z'_i}$ and $t''_i = S_{z''_i} + 1$ yields that $z'_i \in \mathcal{M}_{t'_i}$ and $z''_i \notin \mathcal{M}_{t''_i}$. Hence, we actually need to prove that the implications

$$S_{z'_i} \geq \mu|z'_i| + q(|z'_i|) \Rightarrow |z'_i| < \mu^{-1} S_{z'_i} - \mu^{-3/2}\chi(S_{z'_i}), \tag{6.9}$$

$$S_{z''_i} \leq \mu|z''_i| - q(|z''_i|) \Rightarrow |z''_i| \geq \mu^{-1}(S_{z''_i} + 1) + \mu^{-3/2}\chi(S_{z''_i} + 1) \tag{6.10}$$

hold a.s. for all sufficiently large $i$. Setting $\psi_-(u) = \mu u - q(u)$, $\psi_+(u) = \mu u + q(u)$, and denoting by $\psi_-^{\leftarrow}$ and $\psi_+^{\leftarrow}$ their inverses, we may write the left-hand inequalities in (6.9) and (6.10) as $|z'_i| \leq \psi_+^{\leftarrow}(S_{z'_i})$ and $|z''_i| \geq \psi_-^{\leftarrow}(S_{z''_i})$. Thus, it suffices to show that the inequalities

$$\psi_+^{\leftarrow}(u) < \mu^{-1}u - \mu^{-3/2}\chi(u), \quad \psi_-^{\leftarrow}(u) \geq \mu^{-1}(u+1) + \mu^{-3/2}\chi(u+1)$$

hold for large $u$. A straightforward calculation yields that these inequalities actually mean

$$q(\mu^{-1}u - \mu^{-3/2}\chi(u)) > \mu^{-1/2}\chi(u),$$
$$q(\mu^{-1}(u+1) + \mu^{-3/2}\chi(u+1)) \geq \mu^{-1/2}\chi(u+1) + 1.$$

Routine but rather tedious calculations (which we do not detail here) show that the above inequalities indeed hold for large $u$ with $\chi$ and $q$ defined by (6.1) and (6.7). This completes the proof of (ii) and of Theorem 3.1. □

Remark 6.1. The sectorial LIL proved in Theorem 8.1 does not require Wichura's condition (3.2). Hence, all parts of the foregoing proof based only on sectorial arguments remain true without (3.2). This particularly applies to (6.5) with $\gamma < -\mu^{-3/2}$ as well as to the reverse inclusion with $\gamma > \mu^{-3/2}$. For ease of reference, we reproduce them here in a slightly modified form

$$\mathbb{T} \cap \mathcal{H}(\gamma\varkappa(t)) \subset t^{-1/d}(\mathbb{T} \cap \mathcal{M}_t) \subset \mathbb{T} \cap \mathcal{H}(-\gamma\varkappa(t))$$

a.s. for $\gamma > \mu^{-3/2}$ and all sufficiently large $t$.

Proof of Theorem 3.3. Fix $\gamma > \mu^{-3/2}$ and a closed convex cone $\mathbb{T}$ with $\mathbb{T} \setminus \{0\} \subset \mathbb{R}^d_{++}$. Denote for brevity $\mathcal{H}^{\pm} = \mathcal{H}(\pm\gamma\varkappa(t))$ and $\mathcal{H}^{\pm}_{\mathbb{T}} = \mathcal{H}_{\mathbb{T}}(\pm\gamma\varkappa(t))$. By (i) and (iii) in Theorem 3.1,

$$\mathcal{H}^+_{\mathbb{T}} \subset t^{-1/d}\mathcal{M}_t \subset \mathcal{H}^-_{\mathbb{T}} \tag{6.11}$$

almost surely for all sufficiently large $t$. Therefore,

$$\rho_{\mathrm{H}}(t^{-1/d}\mathcal{M}_t, \mathcal{H}) \leq \max\{\rho_{\mathrm{H}}(\mathcal{H}, \mathcal{H}^-_{\mathbb{T}}), \rho_{\mathrm{H}}(\mathcal{H}, \mathcal{H}^+_{\mathbb{T}})\}$$

for all sufficiently large $t$.

Without loss of generality, assume that $\mathbb{T}$ is sufficiently large and contains the diagonal, so that $\rho_{\mathrm{H}}(\mathcal{H},\mathcal{H}^{\pm}_{\mathbb{T}})=\rho_{\mathrm{H}}(\mathcal{H},\mathcal{H}^{\pm})$. By (5.9),

$$\rho_{\mathrm{H}}(t^{-1/d}\mathcal{M}_t,\mathcal{H})\le d^{-1/2}\gamma\mu^{1-1/d}\varkappa(t)+\mathcal{O}(\varkappa(t))\quad\text{a.s. as } t\to\infty.$$

Dividing by $\varkappa(t)$ and letting $\gamma\downarrow\mu^{-3/2}$ yields the upper bound in (3.6):

$$\limsup_{t\to\infty}\frac{\rho_{\mathrm{H}}(t^{-1/d}\mathcal{M}_t,\mathcal{H})}{\varkappa(t)}\le d^{-1/2}\mu^{-1/2-1/d}\quad\text{a.s.}$$

In order to obtain the reverse inequality, we notice that the sequences $\{z_i', i\ge 1\}$ and $\{t_i', i\ge 1\}$ with $t_i'=S_{z_i'}$ constructed in the final part of the proof of Theorem 3.1 a.s. satisfy

$$(t_i')^{-1/d}z_i'\in(t_i')^{-1/d}\mathcal{M}_{t_i'},\quad (t_i')^{-1/d}z_i'\notin\mathcal{H}(-\mu^{-3/2}\varkappa(t_i'))$$

for large $i$. Since the supremum in the definition of $\rho_{\mathrm{H}}(\mathcal{H}(c_1),\mathcal{H}(c_2))$ is attained at a diagonal point, (5.9) implies

$$\frac{\rho_{\mathrm{H}}((t_i')^{-1/d}\mathcal{M}_{t_i'},\mathcal{H})}{\varkappa(t_i')}\ge\frac{\inf_{y\in\mathcal{H}}\rho((t_i')^{-1/d}z_i',y)}{\varkappa(t_i')}$$

$$>\frac{\rho_{\mathrm{H}}\left(\mathcal{H}(-\mu^{-3/2}\varkappa(t_i'),\mathcal{H}\right)}{\varkappa(t_i')}=d^{-1/2}\mu^{-1/2-1/d}+\mathcal{O}(1)\quad\text{as } i\to\infty.$$

Thus, we arrive at the lower bound in (3.6):

$$\limsup_{t\to\infty}\frac{\rho_{\mathrm{H}}(t^{-1/d}\mathcal{M}_t,\mathcal{H})}{\varkappa(t)}\ge d^{-1/2}\mu^{-1/2-1/d}\quad\text{a.s.}$$

Let us now turn to the proof of (3.7). Consider an enlarged closed convex cone $\mathbb{T}$ such that $\mathbb{T}\setminus\{0\}\subset\mathbb{R}^d_{++}$ and whose interior contains $\mathbb{T}_K\setminus\{0\}$. Notice that

$$\rho^{\mathbb{T}}_{\triangle}(\mathcal{H}^-_{\mathbb{T}},\mathcal{H}^+_{\mathbb{T}})=\rho^{\mathbb{T}}_{\triangle}(\mathcal{H}^-,\mathcal{H}^+),$$

since $\mathcal{H}^{\pm}_{\mathbb{T}}$ coincides with $\mathcal{H}^{\pm}$ within $\mathbb{T}$. Hence, by (6.11) and (5.10),

$$\rho^K_{\triangle}(t^{-1/d}\mathcal{M}_t,\mathcal{H})\le\rho^{\mathbb{T}}_{\triangle}(\mathcal{H}^-,\mathcal{H}^+)=2\gamma L_{\mathbb{T}}\varkappa(t)\tag{6.12}$$

almost surely for all sufficiently large $t$. Dividing by $\varkappa(t)$ and letting first $t\to\infty$ and then $\gamma\downarrow\mu^{-3/2}$ and $\mathbb{T}\downarrow\mathbb{T}_K$ yield (3.7).

Let now $K\subset\mathbb{R}^d_{++}$. Choose a cone $\mathbb{T}$, so that $K\subset\mathbb{T}\setminus\{0\}\subset\mathbb{R}^d_{++}$. By Remark 6.1,

$$K\cap\mathcal{H}^+\subset K\cap(t^{-1/d}\mathcal{M}_t)\subset K\cap\mathcal{H}^-\tag{6.13}$$

for all large $t$, provided only that $\mathbf{E}\xi^2 < \infty$. The rest of the proof follows the lines of the preceding proof, but with reference to (6.13) instead of (6.11).

Assume that $\xi$ is almost surely non-negative. Then, with each $x$, the set $\mathcal{M}_t$ contains also $ax$ for all $a \geq 1$. Hence, reflecting the set $t^{-1/d}\mathcal{M}_t \setminus \mathcal{H}$ symmetrically with respect to $\partial\mathcal{H}$ in the radial direction, we easily arrive at the counterpart of (6.12):

$$\rho^K_\triangle(t^{-1/d}\mathcal{M}_t, \mathcal{H}) \leq \rho^{\mathbb{T}}_\triangle(\mathcal{H}, \mathcal{H}^+) = \gamma L_{\mathbb{T}} \varkappa(t)$$

almost surely for all sufficiently large $t$, and then the proof proceeds as above. The case $K \subset \mathbb{R}^d_{++}$ is treated in the same way as before. □

Proof of Theorem 3.4. Fix a sufficiently large closed convex cone $\mathbb{T}$ such that $\mathbb{T} \setminus \{0\} \subset \mathbb{R}^d_{++}$, and put $F = \mathbb{T} \cap \mathbb{S}^{d-1}$. For $l \in \mathbb{N}$ and $c > 0$, let

$$A_{l,c} = \big\{\, \omega : \mathbb{T} \cap \mathcal{H}(cl^{-1/2}) \subset \mathbb{T} \cap (l^{-1/d}\mathcal{M}_l) \subset \mathbb{T} \cap \mathcal{H}(-cl^{-1/2}) \big\}.$$

The event $A_{l,c}$ means that, inside $\mathbb{T}$, the boundary of $l^{-1/d}\mathcal{M}_l$ lies within a relatively narrow strip $\mu^{-1} - cl^{-1/2} \leq |x| \leq \mu^{-1} + cl^{-1/2}$.

Let $R^{\pm}_{l,c}(u)$, $u \in F$, be the radial functions of $\mathcal{H}(\pm cl^{-1/2})$, that is,

$$R^{\pm}_{l,c}(u) = \inf\big\{\, a > 0 : au \in \mathcal{H}(\pm cl^{-1/2}) \big\} \;=\; \Big(\frac{\mu^{-1} \pm cl^{-1/2}}{|u|}\Big)^{1/d}.$$

Thus,

$$\begin{aligned} A_{l,c} &= \big\{\omega : R^-_{l,c}(u) \leq r_l(u) \leq R^+_{l,c}(u),\ u \in F\big\} \\ &\supset \big\{\omega\colon R^-_{l,c}(u) < r_l(u) \leq R^+_{l,c}(u),\ u \in F\big\} \end{aligned}$$

and the latter event is identical to

$$B_{l,c} = \Big\{\omega : -\frac{c}{|u|} < \sqrt{l}\Big((r_l(u))^d - \frac{1}{\mu|u|}\Big) \leq \frac{c}{|u|},\ u \in F\Big\}. \tag{6.14}$$

Since $\xi$ is a.s. non-negative, $B_{l,c}$ can be represented in terms of interpolated sums as

$$\begin{aligned} B_{l,c} = \Big\{\omega\colon & S_{l^{1/d}x} < l \text{ for all } x \in \mathbb{T} \text{ with } |x| = \mu^{-1} - cl^{-1/2}, \\ & S_{l^{1/d}x} \geq l \text{ for all } x \in \mathbb{T} \text{ with } |x| = \mu^{-1} + cl^{-1/2}\Big\}. \end{aligned} \tag{6.15}$$

By (6.14) and Theorem 4.1,

$$\lim_{l\to\infty} \mathbf{P}\,(B_{l,c}) = \mathbf{P}\Big\{ -c|u|^{-1} < \frac{\sigma}{\mu^{3/2}}|u|^{-1} Z_{u/|u|^{1/d}} \leq c|u|^{-1},\ u \in F\Big\}$$

$$\geq \mathbf{P}\Big\{-\frac{c\mu^{3/2}}{\sigma} < Z_{u/|u|^{1/d}} < \frac{c\mu^{3/2}}{\sigma},\ u \in F\Big\}.$$

It follows from general results on Gaussian measures in Banach spaces that the probability on the right-hand side is positive for any $c > 0$. For instance, this may be easily deduced from the infinite-dimensional Anderson inequality, see, e.g., [14, Corollary 7.1]. Hence, $\lim_{l\to\infty} \mathbf{P}(B_{l,c}) > 0$ for any $c > 0$, and

$$\mathbf{P}\{B_{l,c} \text{ i.o.}\} = \lim_{l\to\infty} \mathbf{P}\Big(\bigcup_{i\geq l} B_{i,c}\Big) \geq \lim_{l\to\infty} \mathbf{P}(B_{l,c}) > 0, \tag{6.16}$$

where i.o. stands for "infinitely often".

It follows from (6.15) that $B_{l,c}$ is measurable with respect to the $\sigma$-algebra generated by $S_{l^{1/d}x}$, $x \in \mathbb{T} \cap \mathcal{H}(-cl^{-1/2})$. So, the random event $\{B_{l,c} \text{ i.o.}\}$ is invariant under finite permutations of $\mathbb{N}^d$. Let $e : \mathbb{N}^d \mapsto \mathbb{N}$ be the usual zigzag enumeration of $\mathbb{N}^d$. Applying the Hewitt-Savage 0–1 law to the (one-dimensional) sequence $\{\xi_{e(m)}, m \in \mathbb{N}^d\}$ turns (6.16) into $\mathbf{P}\{B_{l,c} \text{ i.o.}\} = 1$. Hence, $\mathbf{P}\{A_{l,c} \text{ i.o.}\} = 1$.

So, Lemmas 5.3 and 5.4 imply

$$\liminf_{t\to\infty} \sqrt{t}\,\rho_{\mathrm{H}}(\mathbb{T} \cap t^{-1/d}\mathcal{M}_t, \mathbb{T} \cap \mathcal{H}) \leq 2cd^{-1/2}\mu^{1-1/d} \quad \text{a.s.}, \tag{6.17}$$

$$\liminf_{t\to\infty} \sqrt{t}\,\rho_{\triangle}^{K}(\mathbb{T} \cap t^{-1/d}\mathcal{M}_t, \mathbb{T} \cap \mathcal{H}) \leq 2cL_{\mathbb{T}} \quad \text{a.s.} \tag{6.18}$$

Under (3.2), it follows from (6.4) and the reverse inclusion that

$$\mathcal{H}(\gamma\sqrt{d}\varkappa(t)) \subset t^{-1/d}\mathcal{M}_t \subset \mathcal{H}(-\gamma\sqrt{d}\varkappa(t))$$

holds for any $\gamma > \mu^{-3/2}$ and all large $t$. By choosing a sufficiently large cone $\mathbb{T}$, we can make $\mathcal{H}(\gamma\sqrt{d}\varkappa(t))$ and $\mathcal{H}(-\gamma\sqrt{d}\varkappa(t))$ arbitrarily close to each other outside $\mathbb{T}$. Hence, $\liminf_{t\to\infty} \sqrt{t}\,\rho_{\mathrm{H}}(t^{-1/d}\mathcal{M}_t, \mathcal{H})$ is determined by the left-hand side of (6.17), and letting $c \to 0$ delivers (3.8).

The proof of (3.9) proceeds similarly to that of (3.7), but with reference to (6.11) replaced by that to

$$\widehat{\mathcal{H}}_{\mathbb{T}}^{+} \subset t^{-1/d}\mathcal{M}_t \subset \widehat{\mathcal{H}}_{\mathbb{T}}^{-}$$

with

$$\widehat{\mathcal{H}}_{\mathbb{T}}^{\pm} = \big(\mathbb{T} \cap \mathcal{H}(\pm ct^{-1/2})\big) \cup \big((\mathbb{R}_{++}^d \setminus \mathbb{T}) \cap \mathcal{H}(\pm\gamma\sqrt{d}\varkappa(t))\big)$$

and any $\gamma > \mu^{-3/2}$. Letting $c \to 0$ completes the proof of (3.9). Finally, if $K \subset \mathbb{R}_{++}^d$ then the claim immediately follows from (6.18) by choosing $\mathbb{T} \supset K$ and $c \to 0$. $\square$

## 7. The one-dimensional case

Let us now briefly discuss the case of $d = 1$. Then

$$\mathcal{H}(c) = [0, \infty) \cap [\mu^{-1} + c, \infty),$$

and there is no need to introduce the cone $\mathbb{T}$. The multidimensional inversion theorem (Theorem 2.2) and the set-inclusion SLLN (Corollary 2.3), together with their proofs, remain valid in this case, too. The set-inclusion LIL (Theorem 3.1) in the above form additionally requires that $\mathbf{E}\,(\xi^2 \log\log|\xi|) < \infty$ (see (6.6) above), which in the multidimensional case follows from Wichura's condition (3.2). Under this assumption, which goes back to Feller, we could apply a Kolmogorov–Petrovsky–Erdős–Feller type criterion in order to check whether a given function is upper or lower in the LIL for subsequences.

However, in the one-dimensional setting, this assumption actually affects only the behaviour at the critical values $\pm\mu^{-3/2}$. Indeed, if $|\gamma| > \mu^{-3/2}$ (parts (i) and (iii) in Theorem 3.1), the above proofs remain valid. In the case of $-\mu^{-3/2} < \gamma < \mu^{-3/2}$, the claim can be proved in the following alternative way which does not require (6.6).

According to the ordinary LIL, there is a (random) sequence of indices $\{n_k, k \geq 1\}$, such that $n_k \to \infty$ a.s. and

$$\lim_{k\to\infty} \frac{S_{n_k} - \mu n_k}{\sigma\chi(n_k)} = 1 \quad \text{a.s.} \tag{7.1}$$

Suppose (3.3) does not hold, and so $t^{-1}\mathcal{M}_t \subset \mathcal{H}(\gamma\sigma\varkappa(t))$ for all sufficiently large $t$. Therefore, $S_n \geq t$ implies that $n \geq \mu^{-1}t + \gamma\sigma\chi(t)$ for all sufficiently large $t$. Since $S_{n_k} \to \infty$ a.s., we may let $n = n_k$ and $t = S_{n_k}$, so that $n_k \geq \mu^{-1}S_{n_k} + \gamma\sigma\chi(S_{n_k})$. By (7.1), (6.1), and making use of the SLLN for $S_n$, we arrive at the contradiction

$$1 \leq -\gamma\mu \lim_{k\to\infty} \frac{\chi(S_{n_k})}{\chi(n_k)} = -\gamma\mu \lim_{k\to\infty} \sqrt{\frac{S_{n_k}}{n_k}} = -\gamma\mu^{3/2} < 1.$$

Statement (3.4) may be proved in a similar way, noticing that $S_n < t$ implies $n < \mu^{-1}t + \gamma\sigma\chi(t)$, and using

$$\lim_{k\to\infty} \frac{S_{n_k} - \mu n_k}{\sigma\chi(n_k)} = -1 \quad \text{a.s.}$$

instead of (7.1). So, Theorem 3.1 remains true in the one-dimensional case without condition (6.6) if $|\gamma| \neq \mu^{-3/2}$.

For the metric SLLN and LIL (Theorems 2.5 and 3.3) in case $d = 1$, one would rather define for $t > 0$ the first passage times

$$\nu(t) = \min\{n \geq 1 : S_n > t\}$$

and the last exit times

$$N(t) = \max\{n \ge 0 : S_n \le t\}.$$

The SLLN and LIL for $\nu(t)$ and $N(t)$ are given in [12, Theorems 3.4.4, 3.11.1]. Note that the right-hand sides in the cited results are actually identical to those in (2.7), (2.8) and (3.6) with $d = 1$.

Theorem 3.4 trivially holds in the one-dimensional case (see the argument above its statement). Theorem 4.1 actually reduces in this case to the classical central limit theorem for renewal processes (see, e.g., [12, Theorem 2.5.2]).

## 8. Appendix: strong limit theorems for the sectorial convergence

Fix a closed convex cone $\mathbb{T}$ with $\mathbb{T} \setminus \{0\} \subset \mathbb{R}^d_{++}$ and denote

$$S_n(\mathbb{T}) = \sum_{k \in \mathbb{T}, k \le n} \xi_k \quad \text{and} \quad R_n(\mathbb{T}) = \operatorname{card}\{k \in \mathbb{T} \cap \mathbb{N}^d : k \le n\}.$$

The a.s. limit theorems for $S_n(\mathbb{T})$ normalised by $R_n(\mathbb{T})$ were derived by Gut [11]. Then, lower moment assumptions on the summands suffice if $n$ converges to infinity inside the cone. Below we confirm that, with this mode of convergence, the strong limit theorems hold for $S_n(\mathbb{T})$ replaced by $S_n$ and $R_n(\mathbb{T})$ replaced by $|n|$.

Theorem 8.1. *If* $\mathbf{E}|\xi|^\beta < \infty$ *for some* $\beta \in [1, 2)$, *then*

$$S_n - \mu|n| = o(|n|^{1/\beta}) \quad \text{a.s. as } \mathbb{T} \ni n \to \infty. \tag{8.1}$$

*If* $\mathbf{E}\xi^2 < \infty$, *then*

$$\limsup_{\mathbb{T} \ni n \to \infty} \frac{|S_n - \mu|n||}{\sigma\chi(|n|)} = 1 \quad \text{a.s.} \tag{8.2}$$

Proof. We will partially apply the approach used in the proofs of [11, Theorems 3.1 and 5.1]. Fix $m_{\mathbb{T}} \in \mathbb{N}^d$ such that all $x \in \mathbb{T}$ with $|x| \le 1$ satisfy $x \le m_{\mathbb{T}}$, that is $m_{\mathbb{T}}$ dominates all points from $\{x \in \mathbb{T} : |x| \le 1\}$. The existence of such $m_{\mathbb{T}}$ is guaranteed by the fact that $\mathbb{T} \setminus \{0\}$ is a subset of $\mathbb{R}^d_{++}$.

We may clearly assume that $\mu = 0$ and, in the proof of (8.2), that $\sigma = 1$. Define

$$A(i) = \left\{ n \in \mathbb{N}^d \cap \mathbb{T} : 2^{d(i-1)} \le |n| < 2^{di} \right\}, \quad i \ge 1.$$

Then, for any $\varepsilon > 0$,

$$\sum_{i=1}^{\infty} \mathbf{P}\Big\{ \sup_{k \in A(i)} |S_k|/|k|^{1/\beta} > \varepsilon \Big\} \le \sum_{i=1}^{\infty} \mathbf{P}\Big\{ \sup_{k \in A(i)} |S_k| > \varepsilon 2^{d(i-1)/\beta} \Big\}. \tag{8.3}$$

By the multidimensional Lévy's inequality ([16, Theorem 1] or [13, Corollary 2.4]), assuming that $\xi$ is symmetric, we have

$$\mathbf{P}\Big\{\sup_{k\in A(i)}|S_k|>\varepsilon 2^{d(i-1)/\beta}\Big\}\le 2^d\mathbf{P}\big\{|Y_{l_i}|>\varepsilon 2^{d(i-1)/\beta}\big\}, \tag{8.4}$$

where $Y_{l_i}$ is the sum of $l_i=|m_{\mathbb{T}}|2^{di}$ i.i.d. copies of $\xi$. Next, by the one-dimensional Lévy's inequality,

$$\begin{aligned}\mathbf{P}\big\{|Y_{l_i}|>\varepsilon 2^{d(i-1)/\beta}\big\} &= \frac{1}{l_{i+1}-l_i}\sum_{j=l_i+1}^{l_{i+1}}\mathbf{P}\big\{|Y_{l_i}|>\varepsilon 2^{d(i-1)/\beta}\big\}\\ &\le \frac{2}{|m_{\mathbb{T}}|2^{di}(2^d-1)}\sum_{j=l_i+1}^{l_{i+1}}\mathbf{P}\big\{|Y_j|>\varepsilon 2^{d(i-1)/\beta}\big\}\\ &\le \frac{2^{d+1}}{2^d-1}\sum_{j=l_i+1}^{l_{i+1}}j^{-1}\mathbf{P}\big\{|Y_j|>\varepsilon_1 j^{1/\beta}\big\}\end{aligned} \tag{8.5}$$

with $\varepsilon_1=4^{-d/\beta}|m_{\mathbb{T}}|^{-1/\beta}\varepsilon$. Putting all the above inequalities together and noting that

$$\sum_{j=1}^{\infty}j^{-1}\mathbf{P}\big\{|Y_j|>\varepsilon_1 j^{1/\beta}\big\}<\infty$$

by [4, Theorem 1], we obtain that the series on the left-hand side of (8.3) converges for all $\varepsilon>0$, and so the Borel–Cantelli lemma applies. The desymmetrisation argument is standard (see, e.g., the proof of [10, Theorem 3.2]) and completes the proof of (8.1).

Let us now turn to the proof of (8.2). It is divided into two steps. First we show that

$$\limsup_{\mathbb{T}\ni n\to\infty}\frac{|S_n|}{\chi(|n|)}\le C\quad\text{a.s.} \tag{8.6}$$

for some $C>0$. Repeating the calculations from (8.3)–(8.5) with $\chi(|k|)$ instead of $|k|^{1/\beta}$ and $C$ instead of $\varepsilon$, we arrive at the inequality

$$\sum_{i=1}^{\infty}\mathbf{P}\Big\{\sup_{k\in A(i)}|S_k|/\chi(|k|)>C\Big\}\le\frac{2^{2d+1}}{2^d-1}\sum_{j=l_1+1}^{\infty}j^{-1}\mathbf{P}\big\{|Y_j|>C\chi(4^{-d}|m_{\mathbb{T}}|^{-1}j)\big\}.$$

It follows from [9, Theorem 4] that the series on the right-hand side converges for all $C>2^d|m_{\mathbb{T}}|^{1/2}$. An application of the Borel–Cantelli lemma and the desymmetrisation argument complete the proof of (8.6).

Next, we prove that

$$\limsup_{\mathbb{T}\ni n\to\infty} \frac{|S_n|}{\chi(|n|)} = 1 \quad \text{a.s.} \tag{8.7}$$

Fix a $\delta > 0$ and consider a further closed convex cone $\widehat{\mathbb{T}} \supset \mathbb{T}$ such that $\widehat{\mathbb{T}} \setminus \{0\} \subset \mathbb{R}^d_{++}$ and

$$(1-\delta')|n| > R_n(\widehat{\mathbb{T}}) > (1-\delta)|n| \quad \text{for all } n \in \mathbb{T} \tag{8.8}$$

with some $\delta' \in (0,\delta)$. Let $\widehat{\mathbb{T}}^c = \mathbb{R}^d_{++} \setminus \widehat{\mathbb{T}}$. Then

$$\frac{S_n}{\chi(|n|)} = \Big(\frac{S_n(\widehat{\mathbb{T}})}{\chi(R_n(\widehat{\mathbb{T}}))} + \frac{S_n(\widehat{\mathbb{T}}^c)}{\chi(R_n(\widehat{\mathbb{T}}^c))}\,\frac{\chi(R_n(\widehat{\mathbb{T}}^c))}{\chi(R_n(\widehat{\mathbb{T}}))}\Big)\frac{\chi(R_n(\widehat{\mathbb{T}}))}{\chi(|n|)}\,. \tag{8.9}$$

Note that

$$\limsup_{\mathbb{T}\ni n\to\infty} \frac{|S_n(\widehat{\mathbb{T}})|}{\chi(R_n(\widehat{\mathbb{T}}))} = 1 \tag{8.10}$$

by the sectorial law of the iterated logarithm from [11, Theorem 3.1]. Besides, (8.8) and (6.1) easily imply

$$\frac{\chi(R_n(\widehat{\mathbb{T}}))}{\chi(|n|)} > \sqrt{1-\delta}\,, \quad \frac{\chi(R_n(\widehat{\mathbb{T}}^c))}{\chi(R_n(\widehat{\mathbb{T}}))} < \frac{\sqrt{\delta}}{\sqrt{1-\delta}}, \tag{8.11}$$

$$\frac{\chi(|n|)}{\chi(R_n(\widehat{\mathbb{T}}^c))} < \frac{1}{\sqrt{\delta'}}\,, \quad \frac{\chi(R_n(\widehat{\mathbb{T}}))}{\chi(R_n(\widehat{\mathbb{T}}^c))} < \frac{\sqrt{1-\delta'}}{\sqrt{\delta'}}, \tag{8.12}$$

for all $n \in \mathbb{T}$ with sufficiently large $|n|$. Finally,

$$\frac{|S_n(\widehat{\mathbb{T}}^c)|}{\chi(R_n(\widehat{\mathbb{T}}^c))} \le \frac{|S_n|}{\chi(|n|)}\,\frac{\chi(|n|)}{\chi(R_n(\widehat{\mathbb{T}}^c))} + \frac{|S_n(\widehat{\mathbb{T}})|}{\chi(R_n(\widehat{\mathbb{T}}))}\,\frac{\chi(R_n(\widehat{\mathbb{T}}))}{\chi(R_n(\widehat{\mathbb{T}}^c))}. \tag{8.13}$$

As shown above,

$$\limsup_{\mathbb{T}\ni n\to\infty} \frac{|S_n|}{\chi(|n|)} < \infty.$$

So, (8.13), (8.10), and (8.12) lead to

$$\limsup_{\mathbb{T}\ni n\to\infty} \frac{|S_n(\widehat{\mathbb{T}}^c)|}{\chi(R_n(\widehat{\mathbb{T}}^c))} < \infty.$$

Due to (8.9), the latter along with (8.10) and (8.11) implies (8.7) since $\delta$ can be chosen arbitrarily small. $\square$

**Acknowledgement.** The authors are grateful to Oleg Klesov for bringing to them the problematic related to multiple sums.

## References

[1] S. Aljanvcić and D. Arandjelović, O-regularly varying functions, *Publ. Inst. Math.*, **22** (1977), 5–22.

[2] V. G. Avakumović, Über einen O-Inversionssatz, *Bull. Int. Acad. Youg. Sci*, **29** (1936), 107–117.

[3] R. F. Bass and R. Pyke, Functional law of the iterated logarithm and uniform central limit theorem for partial-sum processes indexed by sets, *Ann. Probab.*, **12** (1984), 13–34.

[4] L. E. Baum and M. Katz, Convergence rates in the law of large numbers, *Trans. Amer. Math. Soc.*, **120** (1965), 108–123.

[5] I. Berkes and M. Weber, Upper-lower class tests and frequency results along subsequences, *Stoch. Proc. Appl.*, **115** (2005), 679–700.

[6] P. J. Bickel and M. J. Wichura, Convergence criteria for multiparameter stochastic processes and some applications, *Ann. Math. Statist.*, **42** (1971), 1656–1670.

[7] P. Billingsley, *Convergence of Probability Measures*, Wiley (New York, 1968).

[8] V. V. Buldygin, K. H. Indlekofer, O. I. Klesov and J. G. Steinebach, *Pseudo Regularly Varying Functions and Generalized Renewal Processes*, TBiMC (Kyiv, 2012) (in Ukrainian).

[9] J. A. Davis, Convergence rates for the law of iterated logarithm, *Ann. Math. Statist.*, **39** (1968), 1479–1485.

[10] A. Gut, Marcinkiewicz laws and convergence rates in the law of large numbers for random variables with multidimensional indices, *Ann. Probab.*, **6** (1978), 469–482.

[11] A. Gut, Strong laws for independent identically distributed random variables indexed by a sector, *Ann. Probab.*, **11** (1983), 569–577.

[12] A. Gut, *Stopped Random Walks*, Springer (New York, 2009).

[13] O. Klesov, *Limit Theorems for Multi-Indexed Sums of Random Variables*, Springer (Berlin, 2014).

[14] M. Lifshits, *Lectures on Gaussian Processes*, Springer (Heidelberg, 2012).

[15] I. Molchanov, *Theory of Random Sets*, 2nd ed., Springer (London, 2017).

[16] S. R. Paranjape and C. Park, Laws of iterated logarithm of multiparameter Wiener processes, *J. Multivariate Anal.*, **3** (1973), 132–136.

[17] R. T. Smythe, Strong laws of large numbers for $r$-dimensional arrays of random variables, *Ann. Probab.*, **1** (1973), 164–170.

[18] M. L. Straf, Weak convergence of stochastic processes with several parameters, in: *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory, Univ. California Press (Berkeley, Calif., 1972), pp. 187–221.

[19] A. Weiser and S. E. Zarantonello, A note on piecewise linear and multilinear table interpolation in many dimensions, *Math. Comp.*, **50** (1988), 189–196.

[20] M. J. Wichura, Inequalities with applications to the weak convergence of random processes with multi-dimensional time parameters, *Ann. Math. Statist.*, **40** (1969), 681–687.

[21] M. J. Wichura, Some Strassen-type laws of the iterated logarithm for multiparameter stochastic processes with independent increments, *Ann. Probab.*, **1** (1973), 272–296.