A NOTE ON ED DEGREES OF GROUP-STABLE SUBVARIETIES IN POLAR REPRESENTATIONS

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Abstract. In a recent paper, Drusvyatskiy, Lee, Ottaviani, and Thomas establish a "transfer principle" by means of which the Euclidean distance degree of an orthogonally-stable matrix variety can be computed from the Euclidean distance degree of its intersection with a linear subspace. We generalise this principle.

1. Introduction

Fix a closed algebraic subvariety X of a finite-dimensional complex vector space V equipped with a non-degenerate symmetric bilinear form $\langle -|-\rangle\colon V \times V \to \mathbb{C}$. Denote by X^{reg} the smooth locus in X. Then for a sufficiently general *data point* $u \in V$ the number

$$\#\{x \in X^{\mathrm{reg}} | \, u - x \perp T_x X\}$$

of *ED critical points* for u on X is finite. Suppose that $V = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}$, the bilinear form is the complexification of a Euclidean inner product on $V_{\mathbb{R}}$ and X is the Zariski-closure of a real algebraic variety $X_{\mathbb{R}}$ that has real smooth points, then this number is, for $u \in V$ sufficiently general, positive and independent of u and is called the *Euclidean distance degree* (ED degree for short) of X in V. See [DHOST]. Here, the ED degree counts the number of critical points in the smooth locus of X of the distance function $d_u\colon X \to \mathbb{C}$ sending $x \mapsto \langle u - x | u - x\rangle$.

The goal of this note is to show that the ED degree of a variety X with a suitable group action can sometimes be computed from that of a simpler variety X_0 obtained by slicing X with a linear subspace of V.

For the simplest example of this phenomenon, let $C \subseteq \mathbb{C}^2$ be the unit circle with equation $x^2 + y^2 = 1$ where $\mathbb{C}^2$ is equipped with the standard form. The ED degree of C equals 2 and this is easily seen as follows. First, C is O_2-stable where O_2 is the orthogonal group preserving the bilinear form. For all $u \in \mathbb{C}^2$ and $g \in \mathrm{O}_2$, the map g restricts to a bijection between ED critical points on C for u and for gu. In particular, the number of ED critical points on C for a sufficiently general point $u \in \mathbb{C}^2$ equals that number for gu, for any choice of $g \in \mathrm{O}_2$. We may assume that u is not isotropic. Therefore, by choosing g suitably, we may assume that u lies on the horizontal axis. And then, since $u \not\perp T_p\,\mathrm{O}_2\,p = T_pC$ for any point $p \in C$ not on the horizontal axis, the search for ED critical points is reduced to the search for such points on the intersection of C with the horizontal axis. Clearly, both of the intersection points are critical.

Both authors were partially supported by the NWO Vici grant entitled *Stabilisation in Algebra and Geometry*.

In the paper [DLOT], a generalisation of this example is studied. They consider the vector space $V = \mathbb{C}^{n\times t}$ equipped with the trace bilinear form and with the group $G = \mathrm{O}_n \times \mathrm{O}_t$ acting by left and right multiplication. The variety X is chosen as the Zariski-closure of an $(\mathrm{O}_n(\mathbb{R})\times\mathrm{O}_t(\mathbb{R}))$-stable real algebraic variety $X_\mathbb{R}$ in $\mathbb{R}^{n\times t}$. This ensures that X is G-stable. The horizontal line is generalised to the $\min(n,t)$-dimensional space V_0 of diagonal matrices in V. They then prove that the ED degree of X in V equals the ED degree of $X_0 := X \cap V_0$ in V_0. In the paper, X_0 is defined in an *a priori* different manner, namely, as the Zariski-closure of the intersection of $X_\mathbb{R}$ with V_0. That this is the same thing as the intersection of X with V_0 is the content of [DLOT, Theorem 3.6], which is an application of the fact that the quotient map under a reductive (in fact, here finite) group sends closed, group-stable sets to closed sets.

Note that, like the unit circle and the horizontal line from the first example, the variety X and the subspace V_0 satisfy the following conditions:

(1) For $v_0 \in V_0$ sufficiently general, the vectorspace V is the orthogonal direct sum of V_0 and $T_{v_0}Gv_0$.
(2) The set GX_0 is dense in X.

The tangent space $T_{v_0}Gv_0$ is equal to $\mathfrak{g}\,v_0$ where $\mathfrak{g}$ is the Lie algebra of G, consists of all pairs (a,b) of skew-symmetric $n\times n$ and $t\times t$ matrices and acts by $(a,b)\cdot v = av - vb$ for all $v \in V$ and $(a,b) \in \mathfrak{g}$. From the fact that the bilinear form $\langle - | - \rangle$ is G-invariant, it follows that $\langle (a,b)v|w\rangle + \langle v|(a,b)w\rangle = 0$ for all $v,w \in V$ and $(a,b) \in \mathfrak{g}$. So condition (1) is equivalent to the statement that if $v_0 \in V_0$ is sufficiently general, then $w \in V$ satisfies $\mathrm{Tr}((aw)v_0^T) = \mathrm{Tr}((wb)v_0^T) = 0$ for all skew-symmetric $a \in \mathbb{C}^{n\times n}, b \in \mathbb{C}^{t\times t}$ if and only if w is a diagonal matrix. Using that symmetric matrices form the orthogonal complement, with respect to the trace form, of the space of skew-symmetric matrices, this is the content of [DLOT, Lemma 4.7]. Condition (2) follows from the fact that the Zariski-dense subset of X of *real* $n \times t$ matrices admit a singular value decomposition.

We will generalize the result of [DLOT] by showing that conditions (1) and (2) are sufficient for establishing that the ED degree of X in V equals that of X_0 in V_0, and we will describe the orthogonal representations that have such a subspace V_0—these turn out to be the *polar representations* of the title.

The remainder of the paper is organized as follows. In Section 2 we state our main results. Section 3 showcases a concrete optimization problem amenable to our techniques: given a real symmetric matrix, find a closest symmetric matrix with prescribed eigenvalues. In Section 4 we discuss the relation between complex varieties to which our theorem applies, acted upon by complex reductive groups; and their real counterparts acted upon by compact Lie groups. Section 5 contains the proof of our main theorem, and Section 6 discusses one possible approach for conclusively testing whether an orthogonal representation is polar. Finally, in Section 7 we discuss some of the most important polar representations coming from the irreducible real polar representation found in [Da].

2. Main results

Let V be a finite-dimensional complex vector space equipped with a non-degenerate symmetric bilinear form $\langle -|-\rangle \colon V \times V \to \mathbb{C}$. Let G a complex algebraic group and let $G \to O(V)$ be an orthogonal representation.

Main Theorem. *Suppose that V has a linear subspace V_0 such that, for sufficiently general $v_0 \in V_0$, the space V is the orthogonal direct sum of V_0 and the tangent space $T_{v_0}Gv_0$ of v_0 to its G-orbit. Let X be a G-stable closed subvariety of V. Set $X_0 := X \cap V_0$ and suppose that GX_0 is dense in X. Then the ED degree of X in V equals the ED degree of X_0 in V_0.*

Remark 1. The condition that for sufficiently general $v_0 \in V_0$ the space V is the orthogonal direct sum of V_0 and $T_{v_0}Gv_0$ implies that the restriction of the form $\langle -|-\rangle$ to V_0 is non-degenerate and that V_0 and $T_{v_0}Gv_0$ are perpendicular for all $v_0 \in V_0$.

Remark 2. When $T_xX \cap (T_xX)^\perp = \{0\}$ for some $x \in X^{\mathrm{reg}}$, then the ED degree of X in V is positive by [DHOST, Theorem 4.1]. Whenever X is the complexification of a real variety with smooth points, this condition is satisfied. Also note that this condition implies that $T_xX_0 \cap (T_xX_0)^\perp = \{0\}$ for some $x \in X_0^{\mathrm{reg}}$, so that the ED degree of X_0 in V_0 is positive as well.

The (proof of the) Main Theorem has the following real analogue.

Theorem 3. *Let $V_\mathbb{R}$ be a finite-dimensional real vector space equipped with a positive definite inner product. Let K be a Lie group and let $K \to \mathrm{O}(V_\mathbb{R})$ be an orthogonal representation. Suppose that $V_\mathbb{R}$ has a linear subspace $V_{\mathbb{R},0}$ such that, for sufficiently general $v_0 \in V_{\mathbb{R},0}$, the space $V_\mathbb{R}$ is the orthogonal direct sum of $V_{\mathbb{R},0}$ and $T_{v_0}Kv_0$. Then every K-orbit intersects $V_{\mathbb{R},0}$. Let X be a real K-stable closed subvariety of $V_\mathbb{R}$ and set $X_0 := X \cap V_{\mathbb{R},0}$. Then the number of real critical points of the distance function to a point is constant along orbits of K and the set of real critical points on X for a sufficiently general $v_0 \in V_{\mathbb{R},0}$ is contained in X_0.*

Remark 4. When we consider an arbitrary $v_0 \in V_{\mathbb{R},0}$, the space

$$N_{v_0} = \{v \in V_\mathbb{R} \mid v \perp T_{v_0}Kv_0\}$$

contains $V_{\mathbb{R},0}$, but may be bigger. So while it is still true that the critical points on X for v_0 are orthogonal to $T_{v_0}Kv_0$, this does not imply that they lie in $V_{\mathbb{R},0}$. However, in this case the stabilizer K_{v_0} acts on N_{v_0} and by [DK, Theorem 2.4] this representation again satisfies the conditions of Theorem 3 with the subspace $V_{\mathbb{R},0}$ of N_{v_0} again playing the same role. In particular, the K_{v_0}-orbit of any critical point on X for v_0 intersects $V_{\mathbb{R},0}$. This allows us to still restrict the search for critical points on X for v_0 to X_0. Since K_{v_0} preserves the distance to v_0, the same is true for closest points on X to v_0.

Apart from proving the Main Theorem, we also classify all orthogonal representations $G \to \mathrm{O}(V)$ for which a subspace V_0 as in the Main Theorem exists. Theorem 7 below relates this problem, in the case of reductive G, to the classification by Dadok and Kac of so-called *polar representations* [DK, Da].

Definition 5. A complex orthogonal representation V of a reductive algebraic group G is called stable polar when there exists a vector $v \in V$ such that the orbit Gv is closed and maximal-dimensional among all orbits of G and such that the codimension of the subspace $\{x \in V_{\mathbb{C}} | \mathfrak{g}\, x \subseteq \mathfrak{g}\, v\}$ equals the dimension of Gv where $\mathfrak{g}$ is the (complex) Lie algebra of G.

Definition 6. A real orthogonal representation $V_{\mathbb{R}}$ of a compact Lie group K is called polar when there exists a vector $v \in V_{\mathbb{R}}$ such that the orbit Kv is maximal-dimensional among all orbits of K and such that κu is perpendicular to $(\kappa v)^{\perp}$ for all $u \in (\kappa v)^{\perp}$ where κ is the (real) Lie algebra of K.

Theorem 7. *Let V be an orthogonal representation of a reductive group G. Then the following are equivalent:*

(i) *V satisfies the conditions of the Main Theorem;*
(ii) *V is a stable polar representation; and*
(iii) *V is the complexification of a polar representation of a maximal compact Lie group K contained in G.*

Remark 8. In (ii), we ask for the representation V to be stable, i.e. for there to exist a $v \in V$ whose orbit is closed and maximal-dimensional among all orbits. This is a notion coming from Geometric Invariant Theory and should not be confused with the notion of a subset X of V being G-stable, i.e. having $gX \subseteq X$ for all $g \in G$.

The only places in this paper where the word stable refers to the notion from GIT are in Definition 5 and Theorem 7.

Remark 9. Analogously to the equivalence (i)$\Leftrightarrow$(ii) of Theorem 7, the conditions on $V_{\mathbb{R}}$ in Theorem 3 are equivalent to $V_{\mathbb{R}}$ being a polar representation.

In the paper [Da], the irreducible real polar representations of compact Lie groups are completely classified, giving us a list of spaces on which our Main Theorem can be applied. We discuss some of these spaces in section 7.

3. Interlude: the closest symmetric matrix with prescribed eigenvalues

Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$ and given a sequance of real numbers $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n)$, how does one find the symmetric matrx $B \in \mathbb{R}^{n\times n}$ with spectrum λ that minimizes $d_A(B) := \sum_{i,j}(a_{ij} - b_{ij})^2$?

To cast this as an instance of Theorem 3, take for $V_{\mathbb{R}}$ the space of real symmetric matrices acted upon by the group $K = \mathrm{O}_n(\mathbb{R})$ of orthogonal $n \times n$-matrices via the action $\alpha : (g, A) \mapsto gAg^T$. The K-invariant inner product on $V_{\mathbb{R}}$ is given by

$$\langle C|D\rangle = \mathrm{Tr}\, C^T D = \sum_{i,j} c_{ij} d_{ij}.$$

We claim that the space $V_{\mathbb{R},0}$ of *diagonal* matrices has the properties of Theorem 3. Indeed, if D is any diagonal matrix with distinct eigenvalues, then differentiating α and using that the Lie algebra κ of K is the Lie algebra of skew-symmetric matrices, we find that

$$T_D KD = \{BD - DB \mid B^T = -B\}$$

is precisely the space of symmetric matrices with zeroes on the diagonal, i.e., the orthogonal complement of $V_{\mathbb{R},0}$.

Let X be the real-algebraic variety in $V_{\mathbb{R}}$ consisting of matrices with the prescribed spectrum λ. Then Theorem 3 says that, if A lies in V_0, so that $A = \mathrm{diag}(\mu_1, \ldots, \mu_n)$, then the critical points of d_A on X are the same as the critical points of the restriction of d_A to $X_0 := X \cap V_0$. If the λ_i are distinct, then this intersection consists of $n!$ diagonal matrices, one for each permutation of the λ_i.

Accordingly, the ED degree of the complexification of X (the subject of the Main Theorem) is then $n!$. If the λ_i are not distinct but come with multiplicities $n_1, \ldots, n_k$ adding up to n, then the ED degree is the multinomial coefficient $\frac{n!}{n_1!\cdots n_k!}$. The group S_n here is the Weyl group from Section 5. In Example 23 we will see a large class of examples where the ED degree equals the order of the Weyl group.

Still assuming that A is diagonal, we get a diagonal matrix $B \in X_0$ closest to A by arranging the λ_i in the same order as the μ_i. To see this, let $\pi \in S_n$ be a permutation. If $\mu_i < \mu_j$ and $\lambda_{\pi(i)} > \lambda_{\pi(j)}$ for some $i, j \in [n]$, then

$$(\mu_i - \lambda_{\pi(i)})^2 + (\mu_j - \lambda_{\pi(j)})^2 - (\mu_i - \lambda_{\pi(j)})^2 - (\mu_j - \lambda_{\pi(i)})^2 = 2(\mu_j - \mu_i)(\lambda_{\pi(i)} - \lambda_{\pi(j)}) > 0$$

and so π cannot minimize $\sum_i (\mu_i - \lambda_{\pi(i)})^2$.

Now when A is not diagonal to begin with, we first compute $g \in \mathrm{O}_n(\mathbb{R})$ such that $A_0 := gAg^T$ is diagonal, find a diagonal matrix B_0 closest to A_0 as above, and then $B := g^{-1}B_0g^{-T}$ is a solution to the original problem. In the same manner, one obtains all critical points of d_A from those of d_{A_0}.

4. Real compact versus complex reductive

We will use the correspondence between compact Lie groups and reductive complex linear algebraic groups.

Theorem 10.

(i) *Any reductive complex algebraic group G contains a maximal compact Lie group. All such subgroups are conjugate and Zariski dense in G.*

(ii) *Any compact Lie group is maximal in a reductive complex algebraic group, which is unique up to isomorphism.*

Proof. See for example [Pr, Subsection 8.7.2] and [OV, Section 5.2]. □

The following lemma is well known, but included for completeness.

Lemma 11. *The real orthogonal group $\mathrm{O}_n(\mathbb{R})$ is a maximal compact subgroup of the complex orthogonal group O_n.*

Proof. Any compact subgroup of O_n leaves invariant some Hermitian positive-definite form on $\mathbb{C}^n$. The only Hermitian positive definite forms that are $\mathrm{O}_n(\mathbb{R})$-invariant are multiples of the standard form. So any compact subgroup of O_n containing $\mathrm{O}_n(\mathbb{R})$ is contained in the unitary group $U(n)$. Since $\mathrm{O}_n(\mathbb{R}) = \mathrm{O}_n \cap U(n)$, we see that $\mathrm{O}_n(\mathbb{R})$ is maximal. □

Let G be a reductive linear algebraic group and let K be a maximal compact Lie group contained in G. Then the complexification of any real representation of K naturally has the structure of a representation of G.

Proposition 12. *A (complex) representation of G is orthogonal if and only if it is the complexification of a (real) representation of K that is orthogonal with respect to some positive definite inner product.*

Proof. Let V be an orthogonal real representation of K and let $V_\mathbb{C}$ be its complexification. Extend the inner product $\langle -|- \rangle$ on V to a non-degenerate symmetric bilinear form on $V_\mathbb{C}$. Then $\langle v|w \rangle = \langle gv|gw \rangle$ for all $v, w \in V_\mathbb{C}$ and $g \in K$. So since K is Zariski dense in G, we see that $V_\mathbb{C}$ is an orthogonal representation of G.

Let V be an orthogonal complex representation of G. Then the image of K in $O(V)$ is contained in some maximal compact subgroup H of $O(V)$. Let W be a real subspace of V with $W \otimes \mathbb{C} = V$ such that the bilinear form on V restricts to a $\mathbb{R}$-valued positive definite inner product on W. Since all maximal compact subgroups of $O(V)$ are conjugate, we see that

$$H = gO(W)g^{-1}$$

for some $g \in O(V)$. Let $V_{\mathbb{R}}$ be the real vector space gW with inner product $\langle v|w\rangle_{V_{\mathbb{R}}} = \langle g^{-1}v|g^{-1}w\rangle$ for all $v, w \in V_{\mathbb{R}}$. Then $V_{\mathbb{R}}$ is an orthogonal representation of K whose complexification is isomorphic to $V = W \otimes_{\mathbb{R}} \mathbb{C}$ by the map g^{-1}. □

Let $\mathfrak{g}$ be the (complex) Lie algebra of G and let κ be the (real) Lie algebra of K. The following theorem is a reformulation of Theorem 7.

Theorem 13. *Let $V_{\mathbb{R}}$ be an orthogonal representation of K and let $V_{\mathbb{C}}$ be its complexification. Then the following are equivalent:*

(i) *there exists a (complex) subspace $V_{\mathbb{C},0}$ of $V_{\mathbb{C}}$ such that, for $v_0 \in V_{\mathbb{C},0}$ sufficientely general, the space $V_{\mathbb{C}}$ is the orthogonal direct sum of $V_{\mathbb{C},0}$ and $\mathfrak{g}\, v_0$;*

(ii) *there exists a vector $v \in V_{\mathbb{C}}$ such that the orbit Gv is closed and maximal-dimensional among all orbits of G and such that the codimension of the subspace $\{x \in V_{\mathbb{C}} |\, \mathfrak{g}\, x \subseteq \mathfrak{g}\, v\}$ equals the dimension of Gv; and*

(iii) *there exists a vector $v \in V_{\mathbb{R}}$ such that the orbit Kv is maximal-dimensional among all orbits of K and such that κu is perpendicular to $(\kappa v)^{\perp}$ for all $u \in (\kappa v)^{\perp}$.*

Proof.

(ii)⇒(i) Let $v \in V_{\mathbb{C}}$ be a vector as in (ii) and take

$$V_{\mathbb{C},0} = \{x \in V_{\mathbb{C}} |\, \mathfrak{g}\, x \subseteq \mathfrak{g}\, v\}.$$

Then for $v_0 \in V_{\mathbb{C},0}$ sufficiently general, we have $\mathfrak{g}\, v_0 = \mathfrak{g}\, v$. So it suffices to prove that $V_{\mathbb{C}}$ is the orthogonal direct sum of $V_{\mathbb{C},0}$ and $\mathfrak{g}\, v$. By [DK, Corollary 2.5], we know that $V_{\mathbb{C}}$ is the direct sum of $V_{\mathbb{C},0}$ (there donoted c_v) and $\mathfrak{g}\, v$. We have

$$\langle V_{\mathbb{C},0} |\, \mathfrak{g}\, v\rangle = -\langle \mathfrak{g}\, V_{\mathbb{C},0} | v\rangle = -\langle \mathfrak{g}\, v | v\rangle = \{0\}$$

and therefore the direct sum is orthogonal.

(i)⇒(iii) Let $V_{\mathbb{C},0}$ be a subspace as in (i) and let U be a dense open subset of $V_{\mathbb{C},0}$ such that $V_{\mathbb{C}}$ is the orthogonal direct sum of $V_{\mathbb{C},0}$ and $\mathfrak{g}\, w$ for all $w \in U$. Then GU is a dense constructible subset of $V_{\mathbb{C}}$ and hence contains a dense open subset of $V_{\mathbb{C}}$. Note that the dimension of the orbit of any element of GU equals the codimension of $V_{\mathbb{C},0}$. So since GU is dense in $V_{\mathbb{C}}$, we see that these orbits must be maximal-dimensional among all orbits of G. Since $V_{\mathbb{R}}$ is dense in $V_{\mathbb{C}}$, the intersection of $V_{\mathbb{R}}$ with GU contains a vector $v = gw$ with $g \in G$ and $w \in U$. Since $v \in GU$, we see that

$$\dim_{\mathbb{R}}(Kv) = \dim_{\mathbb{R}}(\kappa v) = \dim_{\mathbb{C}}(\mathfrak{g}\, v) = \dim_{\mathbb{C}}(Gv)$$

is maximal among the dimensions of all orbits of K. The space $V_{\mathbb{C}}$ is the orthogonal direct sum of $gV_{\mathbb{C},0}$ and $\mathfrak{g}\, v$. Therefore we have

$$(\kappa v)^{\perp} = (\mathfrak{g}\, v)^{\perp} \cap V_{\mathbb{R}} \subseteq gV_{\mathbb{C},0}$$

and hence for all $u \in (\kappa v)^{\perp}$, we have

$$\langle \kappa u | (\kappa v)^{\perp}\rangle \subseteq \langle \mathfrak{g}\, u | gV_{\mathbb{C},0}\rangle = \langle g\, \mathfrak{g}\, g^{-1} u | gV_{\mathbb{C},0}\rangle = \langle \mathfrak{g}(g^{-1}u) | V_{\mathbb{C},0}\rangle = \{0\}.$$

(iii)⇒(ii) Let $v \in V_{\mathbb{R}}$ be a vector as in (iii). Since $\langle av|av\rangle = \langle v|v\rangle$ for all $a \in K$, we have $\langle bv|v\rangle + \langle v|bv\rangle = 0$ for all $b \in \kappa$. So $\langle \kappa v|v\rangle = \{0\}$ and v satisfies the condition of [DK, Theorem 1.1], because $\langle \mathfrak{g}\, v, v\rangle = \mathbb{C} \otimes \langle \kappa v|v\rangle = \{0\}$. Note that the Hermitian form $\langle -,-\rangle$ on $V_{\mathbb{C}}$ in that theorem is the extension of the inner product on $V_{\mathbb{R}}$ and that it is not equal to our bilinear form $\langle -|-\rangle$ on $V_{\mathbb{C}}$. By part (i) of Theorem 1.1, the orbit Gv is closed. Since K is dense in G and since the function $(u \mapsto \dim(Gu))$ is lower semicontinuous, we see that $\dim(Gv) = \dim(Kv)$ is maximal. As stated in the introduction of [DK], the dimension of $\{x \in V_{\mathbb{C}} |\, \mathfrak{g}\, x \subseteq \mathfrak{g}\, v\}$ is always at most the codimension of a maximal-dimensional orbit of G. Since

$$\mathbb{C} \otimes (\kappa v)^{\perp} \subseteq \mathbb{C} \otimes \{u \in V_{\mathbb{R}} | \kappa u \subseteq \kappa v\} \subseteq \{x \in V_{\mathbb{C}} |\, \mathfrak{g}\, x \subseteq \mathfrak{g}\, v\},$$

we must have equality. □

Example 14. Let G be the group $\mathrm{SL}_2(\mathbb{C})$ and let $V_{\mathbb{C}}$ be the irreducible 5-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. So $V_{\mathbb{C}}$ is the set of homogeneous polynomials in x and y of degree 4 and

$$\begin{array}{ccc} \mathfrak{sl}_2(\mathbb{C}) & \mapsto & \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}}) \\ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} & \mapsto & a\left(x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}\right) + bx\dfrac{\partial}{\partial y} + cy\dfrac{\partial}{\partial x} \end{array}$$

is the corresponding representation of $\mathfrak{sl}_2(\mathbb{C})$. Let the non-degenerate symmetric bilinear form $\langle -|-\rangle$ on $V_{\mathbb{C}}$ be given by the Gram matrix

$$\begin{pmatrix} &&&& 12 \\ &&& -3 & \\ && 2 && \\ & -3 &&& \\ 12 &&&& \end{pmatrix}$$

with respect to the basis $x^4, xy^3, x^2y^2, xy^3, y^4$ (obtained by setting $\langle x^4|y^4\rangle = 12$ and using $\langle gv|w\rangle = -\langle x|gw\rangle$ for all $v, w \in V_{\mathbb{C}}$ and $g \in \mathfrak{sl}_2(\mathbb{C})$). Then $\langle -|-\rangle$ is $\mathrm{SL}_2(\mathbb{C})$-invariant. A maximal compact subgroup of $\mathrm{SL}_2(\mathbb{C})$ is $K = \mathrm{SU}(2)$. The real subspace

$$V_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\left(x^4 + y^4, i(x^4 - y^4), x^2y^2, xy(x^2 - y^2), ixy(x^2 + y^2)\right).$$

of $V_{\mathbb{C}}$ is SU(2)-stable and has $V_{\mathbb{C}}$ as its complexification. See the proofs of [IRS, Propositions 3 and 5] for how $V_{\mathbb{R}}$ was obtained. We will now check that the three equivalent conditions of the theorem are satisfied.

(i) Take $V_{\mathbb{C},0} = \mathrm{span}_{\mathbb{C}}(x^4 + y^4, x^2 + y^2)$. Then $V_{\mathbb{C}}$ is the orthogonal direct sum of $V_{\mathbb{C},0}$ and

$$\mathfrak{sl}_2(\mathbb{C})v_0 = \mathrm{span}_{\mathbb{C}}\left(x^4 - y^4, x^3y, xy^3\right)$$

for all $v_0 = a(x^4 + y^4) + bx^2y^2$ with $4a^2 \neq b^2$.

(ii) Take $v = x^4 + y^4 + x^2y^2$. Then $\dim(\mathfrak{sl}_2(\mathbb{C})v) = 3 = \dim(\mathrm{SL}_2(\mathbb{C}))$. Hence the dimension of $\mathrm{SL}_2(\mathbb{C})v$ is maximal. As in the proof of the theorem, we see that the orbit $\mathrm{SL}_2(\mathbb{C})v$ is closed and

$$\{x \in V_{\mathbb{C}} | \mathfrak{sl}_2(\mathbb{C})x \subseteq \mathfrak{sl}_2(\mathbb{C})v\} = \mathrm{span}_{\mathbb{C}}(x^4 + y^4, x^2 + y^2)$$

has dimension $5 - 3 = 2$.

(iii) Again take $v = x^4 + y^4 + x^2y^2$. We have

$$\mathfrak{su}(2) = \mathrm{span}_{\mathbb{R}}\left(\begin{pmatrix} i & \\ & -i\end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & i \\ i & \end{pmatrix}\right)$$

and so we see that

$$\mathfrak{su}(2)v = \mathrm{span}_{\mathbb{R}}\left(i(x^4 - y^4), xy(x^2 - y^2), ixy(x^2 + y^2)\right)$$

has orthogonal complement

$$\mathrm{span}_{\mathbb{R}}\left(x^4 + y^4, x^2y^2\right)$$

and we have $\mathfrak{su}(2)u \subseteq \mathfrak{su}(2)v$ for all u in this complement.

5. Proof of the Main Theorem

Let $G \to O(V)$ be an orthogonal representation as in Section 2. Let X be a G-stable closed subvariety of V. We assume the conditions of the Main Theorem. Note that if we replace G by its unique irreducible component G° that contains the identity element, the conditions of the Main Theorem are still satisfied, because G° has finite index in G. So we may assume that G is irreducible. This implies that all irreducible components of X are also G-stable.

Lemma 15. *The set GV_0 is dense in V.*

Proof. The derivative of the multiplication map $G \times V_0 \to V$ at a (smooth) point $(1, v_0)$ equals the map

$$\begin{array}{rcl} \mathfrak{g} \oplus V_0 & \to & V \\ (A, u_0) & \mapsto & Av_0 + u_0 \end{array}$$

and has image $\mathfrak{g}\, v_0 + V_0$, which by assumption equals V for sufficiently general $v_0 \in V_0$. Hence the derivative is surjective at $(1, v_0)$ for some $v_0 \in V_0$. Therefore the multiplication map is dominant and its image GV_0 is dense in V. □

Lemma 16. *For elements $g \in G$ and $u \in V$, the ED critical points for gu are obtained from those of u by applying g.*

Proof. Let x be a point on X. The element $g \in G$ acts linearly and preserves X and X^{reg}. The derivative of the isomorphism $X \to X, y \mapsto gy$ at x is the isomorphism $T_xX \to T_{gx}X, w \mapsto gw$. So since g also preserves the billinear form, we have $u - x \perp T_xX$ if and only if $gu - gx \perp T_{gx}X$. □

Lemma 17. *A sufficiently general $x_0 \in X_0$ lies both in X_0^{reg} and in X^{reg}.*

Proof. A sufficiently general point on X_0 lies in X_0^{reg}. Since GX_0 is constructible and dense in X, it contains a G-stable dense open subset U of X^{reg}. The intersection of U with X_0 is dense in X_0. Hence a sufficiently general point on X_0 lies in X^{reg}. □

Define the Weyl group W by

$$W = N_G(V_0)/Z_g(V_0) = \{g \in G | gV_0 = V_0\}/\{g \in G | gw = w \forall w \in V_0\}.$$

Then the finite group W acts naturally on V_0. Consider the set S of G-stable closed subvarieties Y of V such that $G(Y \cap V_0)$ is dense in Y and the set R of W-stable closed subvarieties of V_0. Consider the maps

$$\begin{array}{rccc} \varphi\colon & S & \to & R \\ & Y & \mapsto & Y \cap V_0 \end{array} \quad \text{and} \quad \begin{array}{rccc} \psi\colon & R & \to & S \\ & Z & \mapsto & \overline{GZ} \end{array}$$

between these two sets.

Lemma 18. *The bijective maps φ and ψ are mutual inverses.*

Proof. Since S consists of the G-stable closed subvarieties Y of V such that Y equals the closure of $G(Y \cap V_0)$ in V, we see that $\psi \circ \varphi$ is the identity map on S. Let Z be a W-stable closed subvariety of V_0. It is clear that $Z \subseteq \varphi(\psi(Z))$ and we will show that in fact $\varphi(\psi(Z)) = Z$ holds. Since Z is W-stable and W is finite, the variety Z is defined by W-invariant polynomials $f_1, \dots, f_n \in \mathbb{C}[V_0]^W$. By [DK, Theorem 2.9], there exists G-invariant polynomials $g_1, \dots, g_n \in \mathbb{C}[V]^G$ such that f_i is the restriction of g_i to V_0 for all $i \in \{1, \dots, n\}$. Since $g_1, \dots, g_n$ are G-invariant and $g_1(z) = \cdots = g_n(z) = 0$ for all $z \in Z$, we see that (the closure of) GZ is contained in the zero set of the ideal generated by $g_1, \dots, g_n$. Hence

$$\varphi(\psi(Z)) = \overline{GZ} \cap V_0$$

is contained in the zero set of the ideal generated by the restrictions of $g_1, \dots, g_n$ to V_0. This zero set is Z and hence $\varphi(\psi(Z)) \subseteq Z$. So we see that $\varphi \circ \psi$ is the identity map on R. □

Lemma 19. *A sufficiently general $x_0 \in X_0$ satisfies $T_{x_0}X = T_{x_0}X_0 + T_{x_0}Gx_0$.*

Proof. By Lemma 17, we see that sufficiently general points of X_0 are contained in at most one irreducible component of X. Therefore each irreducible component of X_0 is contained in precisely one irreducible component of X. Let Y be an irreducible component of X and let $Z_1, \dots, Z_k$ be the irreducible components of X_0 contained in Y. Then the Weyl group W acts on the set $\{Z_1, \dots, Z_k\}$. Since GX_0 is dense in X, we see that $G(Z_1 \cup \cdots \cup Z_k)$ must be dense in Y. So GZ_i must be dense in Y for some $i \in \{1, \dots, k\}$. By the previous lemma, for this i we have

$$Z_1 \cup \cdots \cup Z_k = Y \cap V_0 = \bigcup_{g \in W} gZ_i$$

and hence W must act transitively on $\{Z_1, \dots, Z_k\}$. In particular, we see that GZ_j is in fact dense in Y for all $j \in \{1, \dots, k\}$.

Take $Z = Z_j$ for any $j \in \{1, \dots, k\}$. Then the multiplication map $G \times Z \to Y$ is dominant and G-equivariant when we let G act on itself by left multiplication. Therefore its derivative at $(1, z)$ is surjective for $z \in Z$ sufficiently general. This means that $T_zY = T_zZ + T_zGz$ for $z \in Z$ suffciently general. Since this holds for all components Z of X_0, we see that $T_{x_0}X = T_{x_0}X_0 + T_{x_0}Gx_0$ for $x_0 \in X_0$ suffciently general. □

Lemma 20. *Let Y be a closed subvariety in a complex affine space V. Let U be a dense open subset of Y and let Z be its complement in Y. Then for $v \in V$ sufficiently general, all ED critical points $y \in Y$ for v lie in U.*

Proof. See the proof of [DLOT, Lemma 4.2]. □

Lemma 21. *Let $v_0 \in V_0$ be sufficiently general. Then any ED critical point on X_0 for v_0 is an ED critical point on X for v_0.*

Proof. By combining the previous three lemmas, we may assume that all ED critical points $x_0 \in X_0$ for v_0 are not only elements of X_0^{reg} but also of X^{reg} and that they satisfy $T_{x_0}X = T_{x_0}X_0 + T_{x_0}Gx_0$. Let x_0 be an ED critical points of v_0. Then $v_0 - x_0 \perp T_{x_0}X_0$ by criticality and $v_0 - x_0 \in V_0 \perp T_{x_0}Gx_0$ by the conditions of the Main Theorem (here we do not need that $T_{x_0}Gx_0$ is the orthogonal complement of V_0—this may not be true—but only that it is contained in that complement). We see that

$$v_0 - x_0 \perp T_{x_0}X_0 + T_{x_0}Gx_0 = T_{x_0}X$$

and hence x_0 is an ED critical point on X for v_0. □

Lemma 22. *Let $v_0 \in V_0$ be sufficiently general. Then any ED critical point on X for v_0 is an ED critical point on X_0 for v_0.*

Proof. Let $x \in X$ be an ED critical point for v_0. Then in particular $v_0 - x \perp T_xGx = \mathfrak{g}\,x$. Together with $x \perp \mathfrak{g}\,x$, which holds by orthogonality of the representation, this implies that $v_0 \perp \mathfrak{g}\,x$. Using once more the orthogonality of the representation, we see that $\langle x|\,\mathfrak{g}\,v_0\rangle = -\langle \mathfrak{g}\,x|v_0\rangle = \{0\}$. So $x \perp T_{v_0}Gv_0$. Since v_0 is sufficiently general in V_0, the vector space V is the orthogonal direct sum of V_0 and $T_{v_0}Gv_0$ and therefore x is an element of V_0. So since also $x \in X$, we have $x \in X_0$. Since $v_0 - x \perp T_xX \supseteq T_xX_0$, we find that $x \in X_0$ is an ED critical point for v_0. □

Proof of the Main Theorem. By Lemmas 15 and 16 we may assume that the sufficiently general point on V is in fact a sufficiently general point v_0 on V_0. The previous two lemmas now tell us that the ED critical points for v_0 on X and on X_0 are the same. Hence the ED degrees of X in V and X_0 in V_0 are equal. □

Example 23. Let G be a complex semisimple algebraic group acting on its Lie algebra $V = \mathfrak{g}$ by conjugation, let V_0 be a Cartan subalgebra of $\mathfrak{g}$ and let W be the Weyl group. In Section 7, we show that V satisfies the conditions of the Main Theorem. Suppose X is the closed G-orbit of a sufficiently general point $v \in V_0$. Then the intersection $X_0 = X \cap V_0$ is a single W-orbit by [DK, Theorem 2.8]. Since X_0 is the W-orbit of a sufficiently general point of V_0, it is a set of size $\#W$. So the ED degree of X equals $\#W$.

Since v is sufficienlty general, the codimension of X in V equals the dimension of V_0. So the degree of the variety X, i.e. the cardinality of $X \cap V'$ for a sufficiently general subspace V' of V with $\dim(V') = \dim(V_0)$, is at least the cardinality of $X \cap V_0$, which is the ED degree of X. Let $f_1, \ldots, f_n$ be a set of invariant polynomial generating the algebra $\mathbb{C}[V]^G$. Then, since X is a closed G-orbit, we see that X is defined by the equations $f_1 = f_1(v), \ldots, f_n = f_n(v)$. Therefore the degree of X is at most the product of the degrees of $f_1, \ldots, f_n$. By Theorem [Hu, Theorem 3.19], this product equals the size of the Weyl group W. So the degree and ED degree of X are equal.

6. Testing for the conditions of the Main Theorem

In the paper [Da], the irreducible polar representations of compact Lie groups are completely classified. Now suppose we have a not-necessarily irreducible orthogonal representation V of a reductive algebraic group G. We would like to be able to test whether V satisfies the conditions of the Main Theorem. In [DK, Section 2], some methods are given. In this section, we describe one more such method.

Lemma 24. *For sufficiently general $v \in V$, the tangent space $\mathfrak{g}\,v$ of v to its orbit is maximal-dimensional and non-degenerate with respect to the bilinear form.*

Proof. By Proposition 12, we know that V is the complexification of a real subspace $V_{\mathbb{R}}$ and that the G-invariant bilinear form $\langle -|-\rangle$ is the extension of a positive definite inner product on $V_{\mathbb{R}}$. Since $V_{\mathbb{R}}$ is dense in V and the set of $v \in V$ such that the dimension of $\mathfrak{g}\,v$ is maximal is open and dense, there is an element w in the intersection. Note that $\mathfrak{g}\,w = \kappa w \otimes_{\mathbb{R}} \mathbb{C}$. Since $\langle -|-\rangle$ is an inner product on $V_{\mathbb{R}}$, its restriction to κw is non-degenerate. Therefore the restriction of $\langle -|-\rangle$ to $\mathfrak{g}\,w$ is non-degenerate as well.

Pick $\varphi_1, \dots, \varphi_n \in \mathfrak{g}$ such that $\varphi_1 w, \dots, \varphi_n w$ form a basis of $\mathfrak{g}\,w$. Then the set of $v \in V$ such that $\varphi_1 v, \dots, \varphi_n v$ are linearly independent and the restriction of $\langle -|-\rangle$ to their span is non-degenerate is an non-empty open subset of V. Since the dimension of $\mathfrak{g}\,w$ is maximal, we see that for every element v in this set, the tangent space $\mathfrak{g}\,v$ of v to its orbit is spanned by $\varphi_1 v, \dots, \varphi_n v$. So for sufficiently general $v \in V$, the vector space $\mathfrak{g}\,v$ is non-degenerate with respect to the bilinear form. □

Lemma 25. *A subspace V_0 as in the Main Theorem exists if and only if for sufficiently general $v \in V$ and for all $u_1, u_2 \perp \mathfrak{g}\,v$ we have $u_1 \perp \mathfrak{g}\,u_2$.*

Proof. Suppose such a subspace V_0 exists. Let $v \in V$ be sufficiently general and let $u_1, u_2 \perp \mathfrak{g}\,v$. For $v_0 \in V_0$ sufficiently general and for all $g \in G$, the vector space V is the orthogonal direct sum of gV_0 and $g\,\mathfrak{g}\,v_0 = \mathfrak{g}(gv_0)$. Since GV_0 contains an open dense subset of V, we may assume that $v = gv_0$ for such v_0 and g. So we see that $u_1, u_2 \in gV_0$. We have $V_0 \perp \mathfrak{g}\,u$ for all $u \in V_0$. Therefore we have $gV_0 \perp \mathfrak{g}\,u$ for all $u \in gV_0$ and hence $u_1 \perp \mathfrak{g}\,u_2$.

Let $v \in V$ be such that the $\mathfrak{g}\,v$ is maximal-dimensional, the restriction of the G-invariant bilinear form $\langle -|-\rangle$ to $\mathfrak{g}\,v$ is non-degenerate and $u_1 \perp \mathfrak{g}\,u_2$ for all $u_1, u_2 \perp \mathfrak{g}\,v$. Let V_0 be the orthogonal complement of $\mathfrak{g}\,v$. Then we see that V_0 is perpendicular to $\mathfrak{g}\,v_0$ for all $v_0 \in V_0$. Let $v_0 \in V_0$ be sufficiently general. Then we have $\mathfrak{g}\,v_0 = \mathfrak{g}\,v$ and hence V is the orthogonal direct sum of V_0 and $\mathfrak{g}\,v_0$. □

Lemma 26. *Let W be a finite-dimensional complex vector space, let*

$$f_1, \dots, f_m \colon V \to W$$

be linear maps and let $w \in W$ be an element. Then the following are equivalent:

(i) *For $v \in V$ sufficiently general, we have $w \in span_{\mathbb{C}}(f_1(v), \dots, f_m(v))$.*
(ii) *We have $1 \otimes w \in span_{\mathbb{C}(V^*)}(f_1, \dots, f_m) \subset \mathbb{C}(V^*) \otimes_{\mathbb{C}} W$.*

Proof. Suppose that

$$1 \otimes w = c_1 f_1 + \cdots + c_m f_m$$

for some $c_1, \dots, c_m \in \mathbb{C}(V^*)$. Then

$$w = c_1(v) f_1(v) + \cdots + c_m(v) f_m(v)$$

for the dense open subset of V consisting of all v where $c_1, \dots, c_m$ can be evaluated.

For the converse, suppose that for $v \in V$ sufficiently general we know that w is contained in the span of $f_1(v), \dots, f_m(v)$. We may assume that $f_1(v), \dots, f_m(v)$ are linearly independent for $v \in V$ sufficiently general. Let $v \in V$ be such that

$f_1(v), \dots, f_m(v)$ are linearly independent and w is contained in their span. Choose $w_1, \dots, w_k \in W$ such that $f_1(v), \dots, f_m(v), w_1, \dots, w_k$ form a basis of W. Now note that $f_1(v), \dots, f_m(v), w_1, \dots, w_k$ form a basis of W for $v \in V$ sufficiently general. By choosing a basis, we may assume that $W = \mathbb{C}^{n+k}$. This gives us a morphism

$$\begin{aligned} \varphi\colon V &\to \mathbb{C}^{(m+k)\times(m+k)} \\ v &\mapsto \begin{pmatrix} f_1(v) & \dots & f_m(v) & w_1 & \dots & w_k \end{pmatrix} \end{aligned}$$

such that $\varphi(v)$ is invertible for $v \in V$ sufficiently general. Consider the coefficients of $\varphi(v)$ as elements of the field $\mathbb{C}(V^*)$ of rational functions on V. Then the matrix $\varphi(v)$ is invertible and $c(v) = \varphi(v)^{-1}w$ is a vector with coefficients in $\mathbb{C}(V^*)$. We have

$$w = \varphi(v)c(v) = c_1(v)f_1(v) + \dots + c_m(v)f_m(v) + c_{n+1}(v)w_1 + \dots + c_{n+k}(v)w_k$$

for $v \in V$ sufficiently. Since we also know that $f_1(v), \dots, f_m(v), w_1, \dots, w_k$ form a basis and that w is contained in the span of $f_1(v), \dots, f_m(v)$ for $v \in V$ sufficiently general, we see that $c_{n+1}, \dots, c_{n+k}$ all must be equal to the zero function. Hence

$$1 \otimes w = c_1 f_1 + \dots + c_m f_m$$

is contained in the span of $f_1, \dots, f_m$ inside $\mathbb{C}(V^*) \otimes W$. □

Now we combine the previous two lemmas to reduce checking the existence of V_0 to a linear algebra problem over $\mathbb{C}(V^*)$. Take $U = W = V$ and consider U and W as affine spaces. Let $\varphi_1, \dots, \varphi_n$ form a basis of $\mathfrak{g}$. By Lemma 25, we know that the representation V satisfies the conditions of the Main Theorem if and only if, for $v \in V$ sufficiently general, the variety in $U \times W$ given by the linear equations

$$\langle u|\varphi_i v\rangle = \langle w|\varphi_i v\rangle = 0, \quad i = 1, \dots, n$$

is contained in the variety given by the equations $\langle u|\varphi_j w\rangle = 0$ for $j = 1, \dots, n$. The latter holds if and only if the polynomials $\langle u|\varphi_j w\rangle$ are contained in the ideal I of the coordinate ring $\mathbb{C}[U \times W]$ generated by $\langle u|\varphi_i v\rangle$ and $\langle w|\varphi_i v\rangle$ for $i = 1, \dots, n$. The polynomial $\langle u|\varphi_j w\rangle$ is homogeneous of degree 2. So for a fixed $v \in V$, it is contained in I if and only if

$$\langle u|\varphi_j w\rangle \in \operatorname{span}_{\mathbb{C}}\left(\mathbb{C}[U \times W]_{(1)} \cdot \left\{\langle u|\varphi_i v\rangle, \langle w|\varphi_i v\rangle \,\middle|\, i = 1, \dots, n\right\}\right).$$

So by Lemma 26, we see that V satisfies the conditions of the Main Theorem if and only if

$$\langle u|\varphi_j w\rangle \in \operatorname{span}_{\mathbb{C}(V^*)}\left(\mathbb{C}[U \times W]_{(1)} \cdot \left\{\langle u|\varphi_i v\rangle, \langle w|\varphi_i v\rangle \,\middle|\, i = 1, \dots, n\right\}\right)$$

for all $j \in \{1, \dots, n\}$. The latter condition can be checked efficiently on a computer, requiring as input the bilinear form $\langle -|-\rangle$ and the images in $\mathrm{End}(V)$ of a basis of $\mathfrak{g}$.

7. EXAMPLES

In this section we highlight some of the families of polar representations coming from [Da]. We also point out how some of these families are related by means of slice representations as defined in [DK]. Our Main Theorem can be applied to each of these families, thus generalizing [DLOT, Theorem 4.11].

Remark 27. A representation V of a group G satisfies the conditions of the Main Theorem if and only if the direct sum of V with the trivial representation does.

Remark 28. Let V be the orthogonal direct sum of two representations V_1 and V_2 of G. Then if V satisfies the conditions of the Main Theorem, so do V_1 and V_2.

7.1. **Adjoint representations.** Let G be a complex semisimple algebraic group acting on its Lie algebra $\mathfrak{g}$ by conjugation. This representation is orthogonal with respect to the Killing form B on $\mathfrak{g}$ defined by

$$B(v, w) = \mathrm{Tr}(\mathrm{ad}\, v\, \mathrm{ad}\, w)$$

for $v, w \in \mathfrak{g}$. Since G is semisimple, we know that B is non-degenerate. Since G acts by conjugation, the tangent space of a point $v \in \mathfrak{g}$ to its orbit equals $[\mathfrak{g}, v]$. We have

$$B(w, [\mathfrak{g}, v]) = -B([w, v], \mathfrak{g})$$

for all $w, v \in \mathfrak{g}$. So $w \perp \mathfrak{g}\, v$ if and only if $[v, w] = 0$. Let $h \in \mathfrak{g}$ and suppose $[h, v] \perp \mathfrak{g}\, v$. Then $h \in \ker(\mathrm{ad}\, v)^2 = \ker(\mathrm{ad}\, v)$ and hence $[h, v] = 0$. Hence $\mathfrak{g}\, v$ is non-degenerate. Let V_0 be a Cartan subalgebra of $\mathfrak{g}$. Let $v \in V_0$ be sufficiently general and let $u_1, u_2 \perp \mathfrak{g}\, v$. Then $V_0 = C_{\mathfrak{g}}(V_0) = C_{\mathfrak{g}}(v)$. So we have $u_1, u_2 \in V_0$ and hence $u_1 \perp \mathfrak{g}\, u_2$.

7.2. **Standard representations of groups of type B and D.** Let n be a positive integer and let G be the orthogonal group $\mathrm{O}(n)$ acting on $\mathbb{C}^n$ with the standard form. Let V_0 be the subspace of V spanned by the first basis vector e_1. For all $v \in V_0$ non-zero, we have $\mathfrak{g}\, v = \{Ae_1 | A \in \mathfrak{gl}_n \text{ skew-symmetric}\} = \mathrm{span}(e_2, \dots, e_n) = V_0^{\perp}$.

7.3. **Representations of groups of type B and D of highest weight $2\lambda_1$.** Let n be a positive integer and let G be the orthogonal group $\mathrm{SO}(n)$ acting on the vector space V of symmetric $n \times n$ matrices with trace zero by conjugation. The bilinear form given by

$$\langle A | B \rangle = \mathrm{Tr}(A^T B)$$

for $A, B \in V$ is non-degenerate and $\mathrm{SO}(n)$-invariant. Let V_0 be the subspace of V consisting of all diagonal matrices. For all $D \in V_0$ with pairwise distinct entries on the diagonal, we have $\mathfrak{g} \cdot D = \{AD - DA | A \in \mathfrak{gl}_n \text{ skew-symmetric}\} = V_0^{\perp}$.

7.4. **Tensor products of two standard representations of groups of type B and D.** Let $n \leq m$ be positive integers and let G be the group $\mathrm{O}(n) \times \mathrm{O}(m)$ acting on $n \times m$ matrices by left and right multiplication. The bilinear form given by

$$\langle A | B \rangle = \mathrm{Tr}(A^T B)$$

for $A, B \in \mathbb{C}^{n \times m}$ is non-degenerate and G-invariant. The subspace V_0 of $\mathbb{C}^{n \times m}$ consisting of diagonal matrices satisfies the conditions of the Main Theorem.

Remark 29. Consider the matrix $v = (I_n\ 0) \in \mathbb{C}^{n \times m}$. The stabilizer of v equals

$$G_v = \left\{ \left(g, \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \right) \middle| g \in \mathrm{O}(n), h \in \mathrm{O}(m - n) \right\}$$

and the orthogonal complement of $\mathfrak{g}\, v$ equals the set of matrices of the form $(A\ 0)$ where A is a symmetric $n \times n$ matrix. Ignoring the trivial action from $\mathrm{O}(m-n)$, we see that the slice representation of the element v is the direct sum of the representation from the previous subsection and the trivial representation.

7.5. **Second alternating powers of standard representations of groups of type** C**.** Let n be a positive integer and let G be the symplectic group

$$\mathrm{Sp}(n) = \left\{ A \in \mathrm{GL}_{2n} \,\middle|\, A \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A^T = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$$

acting on the second alternating power $\Lambda^2 \mathbb{C}^{2n}$ of the standard representation. The Lie algebra of $\mathrm{Sp}(n)$ equals

$$\begin{aligned} \mathfrak{sp}(n) &= \left\{ A \in \mathfrak{gl}_{2n} \,\middle|\, A \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A^T \right\} \\ &= \left\{ \begin{pmatrix} X & Y \\ Z & \Theta \end{pmatrix} \in \mathfrak{gl}_{2n} \,\middle|\, \begin{matrix} Y = Y^T,\ Z = Z^T \\ X + \Theta^T = 0 \end{matrix} \right\}. \end{aligned}$$

The $\mathrm{Sp}(n)$-invariant skew-symmetric form on $\mathbb{C}^{2n}$ induces the bilinear form on $\Lambda^2 \mathbb{C}^{2n}$ given by

$$\langle v \wedge w | x \wedge y \rangle = v^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} x \cdot w^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} y - v^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} y \cdot w^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} x$$

for $v, w, x, y \in \mathbb{C}^{2n}$. This form is symmetric, non-degenerate and $\mathrm{Sp}(2n)$-invariant. Let V_0 be the subspace of $\Lambda^2 \mathbb{C}^{2n}$ spanned by $e_i \wedge e_{n+i}$ for $i = 1, \dots, n$. Then for any linear combination v of $e_1 \wedge e_{n+1}, \dots, e_n \wedge e_{2n}$ with only non-zero coefficients, the vector space $\Lambda^2 \mathbb{C}^{2n}$ is the orthogonal direct sum of V_0 and $\mathfrak{g}\, v$.

Remark 30. The paper [Da] tells us that $\Lambda^2 \mathbb{C}^{2n}$ is isomorphic to $\mathfrak{gl}_{2n} / \mathfrak{sp}(n)$ acted on by $\mathrm{Sp}(n)$ by conjugation. In this case, the subspace V_0 of consists of matrices of the form

$$\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$$

with $D \in \mathfrak{gl}_n$ diagonal.

7.6. **Tensor products of two standard representations of groups of type** C**.** Let $n \le m$ be positive integers and let G be the group $\mathrm{Sp}(n) \times \mathrm{Sp}(m)$ acting on $2n \times 2m$ matrices by left and right multiplication. The bilinear form is defined by

$$\langle A | B \rangle = \mathrm{Tr}\left(\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} B^T \right)$$

for all $A, B \in \mathbb{C}^{2n \times 2m}$. This form is symmetric, non-degenerate and G-invariant. Let V_0 be the subspace of V consisting of matrices of the form

$$\begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & D & 0 \end{pmatrix}$$

where D is a diagonal $n \times n$ matrix. Then for every invertible diagonal $n \times n$ matrix D whose squares of diagonal entries are pairwise distinct, the vector space $\mathbb{C}^{2n \times 2m}$ is the orthogonal direct sum of V_0 and

$$\mathfrak{g} \cdot \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & D & 0 \end{pmatrix} = \mathfrak{sp}(n) \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & D & 0 \end{pmatrix} + \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & D & 0 \end{pmatrix} \mathfrak{sp}(m).$$

Remark 31. The slice representation of

$$\begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \end{pmatrix}$$

is a representation of $\mathrm{Sp}(n) \times \mathrm{Sp}(m-n)$ where the second factor acts trivially. Ignoring this factor, the slice representation is isomorphic to the representation $\mathfrak{gl}_{2n}/\mathfrak{sp}(n)$ from the previous remark.

7.7. **Direct sums of standard representations of groups of type A and their duals.** Let n be a positive integer and let G be the group SL_n acting on $\mathbb{C}^n \oplus \mathbb{C}^n$ by

$$g \cdot (v, w) = (gv, g^{-T} w)$$

for all $g \in \mathrm{SL}_n$ and $v, w \in \mathbb{C}^n$. Let the bilinear form be given by

$$\langle (v, w) | (x, y) \rangle = v^T y + x^T w.$$

for all $v, w, x, y \in \mathbb{C}^n$. This form is symmetric, non-degenerate and SL_n-invariant. Let V_0 be the subspace of $\mathbb{C}^n \oplus \mathbb{C}^n$ spanned by (e_1, e_1). Then $\mathbb{C}^n \oplus \mathbb{C}^n$ is the orthogonal direct sum of V_0 and $\mathfrak{g} \cdot v$ for all non-zero $v \in V_0$.

7.8. **Direct sums of representations of groups of type A of highest weight $2\lambda_1$ and their duals.** Let n be a positive integer and let G be the group GL_n acting on the vector space V of pairs of symmetric $n \times n$ matrices by $g \cdot (A, B) = (gAg^T, g^{-T} B g^{-1})$ for all $g \in \mathrm{GL}_n$ and $(A, B) \in V$. Let the bilinear form on V be given by

$$\langle (A, B) | (C, D) \rangle = \mathrm{Tr}(AD + BC)$$

for all symmetric matrices $A, B, C, D \in \mathfrak{gl}_n$. Let V_0 be the subspace

$$\{(D, D) | D \in \mathfrak{gl}_n \text{ diagonal}\}$$

of V. Then for every invertible diagonal $n \times n$ matrix D whose squares of diagonal entries are pairwise distinct, the vector space V is the orthogonal direct sum of V_0 and

$$\mathfrak{g} \cdot (D, D) = \{(AD + DA^T, -A^T D - DA) | A \in \mathfrak{gl}_n\}.$$

Remark 32. The slice representation of (I_n, I_n) is isomorphic to the set of symmetric $n \times n$ matrices acted on by O_n with conjugation.

7.9. **Direct sums of representations of groups of type A of highest weight λ_2 and their duals.** Let n be a positive integer and let G be the group GL_n acting on the vector space V of pairs of skew-symmetric $n \times n$ matrices by $g \cdot (A, B) = (gAg^T, g^{-T} B g^{-1})$ for all $g \in \mathrm{GL}_n$ and $(A, B) \in V$. Let the bilinear form on V be given by

$$\langle (A, B) | (C, D) \rangle = \mathrm{Tr}(AD + BC)$$

for all skew-symmetric matrices $A, B, C, D \in \mathfrak{gl}_n$. Let V_0 be the subspace

$$\left\{ \left(\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \right) \middle| E \in \mathfrak{gl}_{n/2} \text{ diagonal} \right\}$$

of V if n is even and the subspace

$$\left\{ \left(\begin{pmatrix} 0 & 0 & E \\ 0 & 0 & 0 \\ -E & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & E \\ 0 & 0 & 0 \\ -E & 0 & 0 \end{pmatrix} \right) \middle| E \in \mathfrak{gl}_{(n-1)/2} \text{ diagonal} \right\}$$

of V if n is odd. Then for every invertible diagonal $\lfloor n/2\rfloor \times \lfloor n/2\rfloor$ matrix E whose squares of diagonal entries are pairwise distinct, the vector space V is the orthogonal direct sum of V_0 and the tangent space at the corresponding element of V_0 to its orbit.

Remark 33. Suppose n is even. Then the slice representation of

$$\begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}$$

is isomorphic to the representation $\Lambda^2 \mathbb{C}^n$ of $\mathrm{Sp}(n/2)$.

7.10. **Direct sums of standard representations of groups of type C and their duals.** Let n be a positive integer and let G be the group $\mathrm{Sp}(n)$ acting on the vectorspace $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ with the form given by

$$\langle (v,w)|(x,y)\rangle = v^T \begin{pmatrix} o & I_n \\ -I_n & 0 \end{pmatrix} y + x^T \begin{pmatrix} o & I_n \\ -I_n & 0 \end{pmatrix} w$$

for all $v,w,x,y \in \mathbb{C}^{2n}$. Let V_0 be the subspace of $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ spanned by some (v,w) with $v_i, w_i \neq 0$ for all i and $v_i w_j - v_j w_i \neq 0$ for all $i \neq j$. Then it follows from the following lemma that $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ is the orthogonal direct sum of V_0 and the tangent space at any non-zero element V_0 of its orbit.

Lemma 34. *Let $v,w,x,y \in \mathbb{C}^m$ be such that $v_i, w_i \neq 0$ for all i and $v_i w_j \neq v_j w_i$ for all $i \neq j$. Then $v^T S y = x^T S w$ for all symmetric $m \times m$ matrices S if and only if $(x,y) = \lambda(v,w)$ for some $\lambda \in \mathbb{C}$.*

7.11. **Tensor products of two direct sums of standard representations of groups of type A and their duals.** Let $n \leq m$ be positive integers and let $\mathrm{GL}_n \times \mathrm{GL}_m$ act on $\mathbb{C}^{n\times m} \oplus \mathbb{C}^{n\times m}$ by

$$(g,h)(A,B) = \left(gAh^T, g^{-T}Bh^{-1}\right)$$

for all $g \in \mathrm{GL}_n$, $h \in \mathrm{GL}_m$ and $A, B \in \mathbb{C}^{n\times m}$. Let the bilinear form be given by

$$\langle (A,B)|(C,D)\rangle = \mathrm{Tr}\left(A^T D + C^T B\right)$$

for all $A,B,C,D \in \mathbb{C}^{n\times m}$. Let V_0 be the subspace

$$\left\{((D\ 0),(D\ 0))\,\middle|\, D \in \mathfrak{gl}_n \text{ diagonal}\right\}$$

of V. Then for all invertible diagonal $n \times n$ matrices D whose squares of diagonal entries are pairwise distinct, the vector space $\mathbb{C}^{n\times m} \oplus \mathbb{C}^{n\times m}$ is the orthogonal direct sum of V_0 and

$$\mathfrak{g}\,((D\ 0),(D\ 0)) = \left\{ \left((AD\ 0) + (D\ 0)B^T, (-A^T D\ 0) - (D\ 0)B\right) \,\middle|\, A \in \mathfrak{gl}_n, B \in \mathfrak{gl}_m \right\}.$$

Remark 35. The slice representation of the pair $((I_n\ 0),(I_n\ 0))$ is a representation of $\mathrm{GL}_n \times \mathrm{GL}_m$ where the second factor acts trivially. Ignoring this factor, we get the adjoint representation of GL_n.

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