

# A NOTE ON ED DEGREES OF GROUP-STABLE SUBVARIETIES IN POLAR REPRESENTATIONS

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ABSTRACT. In a recent paper, Drusvyatskiy, Lee, Ottaviani, and Thomas establish a “transfer principle” by means of which the Euclidean distance degree of an orthogonally-stable matrix variety can be computed from the Euclidean distance degree of its intersection with a linear subspace. We generalise this principle.

## 1. INTRODUCTION

Fix a closed algebraic subvariety  $X$  of a finite-dimensional complex vector space  $V$  equipped with a non-degenerate symmetric bilinear form  $\langle - | - \rangle: V \times V \rightarrow \mathbb{C}$ . Denote by  $X^{\text{reg}}$  the smooth locus in  $X$ . Then for a sufficiently general *data point*  $u \in V$  the number

$$\#\{x \in X^{\text{reg}} \mid u - x \perp T_x X\}$$

of ED critical points for  $u$  on  $X$  is finite. Suppose that  $V = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}$ , the bilinear form is the complexification of a Euclidean inner product on  $V_{\mathbb{R}}$  and  $X$  is the Zariski-closure of a real algebraic variety  $X_{\mathbb{R}}$  that has real smooth points, then this number is, for  $u \in V$  sufficiently general, positive and independent of  $u$  and is called the *Euclidean distance degree* (ED degree for short) of  $X$  in  $V$ . See [DHOST]. Here, the ED degree counts the number of critical points in the smooth locus of  $X$  of the distance function  $d_u: X \rightarrow \mathbb{C}$  sending  $x \mapsto \langle u - x | u - x \rangle$ .

The goal of this note is to show that the ED degree of a variety  $X$  with a suitable group action can sometimes be computed from that of a simpler variety  $X_0$  obtained by slicing  $X$  with a linear subspace of  $V$ .

For the simplest example of this phenomenon, let  $C \subseteq \mathbb{C}^2$  be the unit circle with equation  $x^2 + y^2 = 1$  where  $\mathbb{C}^2$  is equipped with the standard form. The ED degree of  $C$  equals 2 and this is easily seen as follows. First,  $C$  is  $O_2$ -stable where  $O_2$  is the orthogonal group preserving the bilinear form. For all  $u \in \mathbb{C}^2$  and  $g \in O_2$ , the map  $g$  restricts to a bijection between ED critical points on  $C$  for  $u$  and for  $gu$ . In particular, the number of ED critical points on  $C$  for a sufficiently general point  $u \in \mathbb{C}^2$  equals that number for  $gu$ , for any choice of  $g \in O_2$ . We may assume that  $u$  is not isotropic. Therefore, by choosing  $g$  suitably, we may assume that  $u$  lies on the horizontal axis. And then, since  $u \not\perp T_p O_2 p = T_p C$  for any point  $p \in C$  not on the horizontal axis, the search for ED critical points is reduced to the search for such points on the intersection of  $C$  with the horizontal axis. Clearly, both of the intersection points are critical.

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In the paper [DLOT], a generalisation of this example is studied. They consider the vector space  $V = \mathbb{C}^{n \times t}$  equipped with the trace bilinear form and with the group  $G = O_n \times O_t$  acting by left and right multiplication. The variety  $X$  is chosen as the Zariski-closure of an  $(O_n(\mathbb{R}) \times O_t(\mathbb{R}))$ -stable real algebraic variety  $X_{\mathbb{R}}$  in  $\mathbb{R}^{n \times t}$ . This ensures that  $X$  is  $G$ -stable. The horizontal line is generalised to the  $\min(n, t)$ -dimensional space  $V_0$  of diagonal matrices in  $V$ . They then prove that the ED degree of  $X$  in  $V$  equals the ED degree of  $X_0 := X \cap V_0$  in  $V_0$ . In the paper,  $X_0$  is defined in an *a priori* different manner, namely, as the Zariski-closure of the intersection of  $X_{\mathbb{R}}$  with  $V_0$ . That this is the same thing as the intersection of  $X$  with  $V_0$  is the content of [DLOT, Theorem 3.6], which is an application of the fact that the quotient map under a reductive (in fact, here finite) group sends closed, group-stable sets to closed sets.

Note that, like the unit circle and the horizontal line from the first example, the variety  $X$  and the subspace  $V_0$  satisfy the following conditions:

- (1) For  $v_0 \in V_0$  sufficiently general, the vectorspace  $V$  is the orthogonal direct sum of  $V_0$  and  $T_{v_0}Gv_0$ .
- (2) The set  $GX_0$  is dense in  $X$ .

The tangent space  $T_{v_0}Gv_0$  is equal to  $\mathfrak{g}v_0$  where  $\mathfrak{g}$  is the Lie algebra of  $G$ , consists of all pairs  $(a, b)$  of skew-symmetric  $n \times n$  and  $t \times t$  matrices and acts by  $(a, b) \cdot v = av - vb$  for all  $v \in V$  and  $(a, b) \in \mathfrak{g}$ . From the fact that the bilinear form  $\langle - | - \rangle$  is  $G$ -invariant, it follows that  $\langle (a, b)v | w \rangle + \langle v | (a, b)w \rangle = 0$  for all  $v, w \in V$  and  $(a, b) \in \mathfrak{g}$ . So condition (1) is equivalent to the statement that if  $v_0 \in V_0$  is sufficiently general, then  $w \in V$  satisfies  $\text{Tr}((aw)v_0^T) = \text{Tr}((wb)v_0^T) = 0$  for all skew-symmetric  $a \in \mathbb{C}^{n \times n}, b \in \mathbb{C}^{t \times t}$  if and only if  $w$  is a diagonal matrix. Using that symmetric matrices form the orthogonal complement, with respect to the trace form, of the space of skew-symmetric matrices, this is the content of [DLOT, Lemma 4.7]. Condition (2) follows from the fact that the Zariski-dense subset of  $X$  of *real*  $n \times t$  matrices admit a singular value decomposition.

We will generalize the result of [DLOT] by showing that conditions (1) and (2) are sufficient for establishing that the ED degree of  $X$  in  $V$  equals that of  $X_0$  in  $V_0$ , and we will describe the orthogonal representations that have such a subspace  $V_0$ —these turn out to be the *polar representations* of the title.

The remainder of the paper is organized as follows. In Section 2 we state our main results. Section 3 showcases a concrete optimization problem amenable to our techniques: given a real symmetric matrix, find a closest symmetric matrix with prescribed eigenvalues. In Section 4 we discuss the relation between complex varieties to which our theorem applies, acted upon by complex reductive groups; and their real counterparts acted upon by compact Lie groups. Section 5 contains the proof of our main theorem, and Section 6 discusses one possible approach for conclusively testing whether an orthogonal representation is polar. Finally, in Section 7 we discuss some of the most important polar representations coming from the irreducible real polar representation found in [Da].

2. MAIN RESULTS

Let  $V$  be a finite-dimensional complex vector space equipped with a non-degenerate symmetric bilinear form  $\langle - | - \rangle: V \times V \rightarrow \mathbb{C}$ . Let  $G$  a complex algebraic group and let  $G \rightarrow O(V)$  be an orthogonal representation.

**Main Theorem.** *Suppose that  $V$  has a linear subspace  $V_0$  such that, for sufficiently general  $v_0 \in V_0$ , the space  $V$  is the orthogonal direct sum of  $V_0$  and the tangent space  $T_{v_0}Gv_0$  of  $v_0$  to its  $G$ -orbit. Let  $X$  be a  $G$ -stable closed subvariety of  $V$ . Set  $X_0 := X \cap V_0$  and suppose that  $GX_0$  is dense in  $X$ . Then the ED degree of  $X$  in  $V$  equals the ED degree of  $X_0$  in  $V_0$ .*

**Remark 1.** The condition that for sufficiently general  $v_0 \in V_0$  the space  $V$  is the orthogonal direct sum of  $V_0$  and  $T_{v_0}Gv_0$  implies that the restriction of the form  $\langle - | - \rangle$  to  $V_0$  is non-degenerate and that  $V_0$  and  $T_{v_0}Gv_0$  are perpendicular for all  $v_0 \in V_0$ .

**Remark 2.** When  $T_xX \cap (T_xX)^\perp = \{0\}$  for some  $x \in X^{\text{reg}}$ , then the ED degree of  $X$  in  $V$  is positive by [DHOST, Theorem 4.1]. Whenever  $X$  is the complexification of a real variety with smooth points, this condition is satisfied. Also note that this condition implies that  $T_xX_0 \cap (T_xX_0)^\perp = \{0\}$  for some  $x \in X_0^{\text{reg}}$ , so that the ED degree of  $X_0$  in  $V_0$  is positive as well.

The (proof of the) Main Theorem has the following real analogue.

**Theorem 3.** *Let  $V_{\mathbb{R}}$  be a finite-dimensional real vector space equipped with a positive definite inner product. Let  $K$  be a Lie group and let  $K \rightarrow O(V_{\mathbb{R}})$  be an orthogonal representation. Suppose that  $V_{\mathbb{R}}$  has a linear subspace  $V_{\mathbb{R},0}$  such that, for sufficiently general  $v_0 \in V_{\mathbb{R},0}$ , the space  $V_{\mathbb{R}}$  is the orthogonal direct sum of  $V_{\mathbb{R},0}$  and  $T_{v_0}Kv_0$ . Then every  $K$ -orbit intersects  $V_{\mathbb{R},0}$ . Let  $X$  be a real  $K$ -stable closed subvariety of  $V_{\mathbb{R}}$  and set  $X_0 := X \cap V_{\mathbb{R},0}$ . Then the number of real critical points of the distance function to a point is constant along orbits of  $K$  and the set of real critical points on  $X$  for a sufficiently general  $v_0 \in V_{\mathbb{R},0}$  is contained in  $X_0$ .*

**Remark 4.** When we consider an arbitrary  $v_0 \in V_{\mathbb{R},0}$ , the space

$$N_{v_0} = \{v \in V_{\mathbb{R}} \mid v \perp T_{v_0}Kv_0\}$$

contains  $V_{\mathbb{R},0}$ , but may be bigger. So while it is still true that the critical points on  $X$  for  $v_0$  are orthogonal to  $T_{v_0}Kv_0$ , this does not imply that they lie in  $V_{\mathbb{R},0}$ . However, in this case the stabilizer  $K_{v_0}$  acts on  $N_{v_0}$  and by [DK, Theorem 2.4] this representation again satisfies the conditions of Theorem 3 with the subspace  $V_{\mathbb{R},0}$  of  $N_{v_0}$  again playing the same role. In particular, the  $K_{v_0}$ -orbit of any critical point on  $X$  for  $v_0$  intersects  $V_{\mathbb{R},0}$ . This allows us to still restrict the search for critical points on  $X$  for  $v_0$  to  $X_0$ . Since  $K_{v_0}$  preserves the distance to  $v_0$ , the same is true for closest points on  $X$  to  $v_0$ .

Apart from proving the Main Theorem, we also classify all orthogonal representations  $G \rightarrow O(V)$  for which a subspace  $V_0$  as in the Main Theorem exists. Theorem 7 below relates this problem, in the case of reductive  $G$ , to the classification by Dadok and Kac of so-called *polar representations* [DK, Da].

**Definition 5.** A complex orthogonal representation  $V$  of a reductive algebraic group  $G$  is called stable polar when there exists a vector  $v \in V$  such that the orbit  $Gv$  is closed and maximal-dimensional among all orbits of  $G$  and such that the

codimension of the subspace  $\{x \in V_{\mathbb{C}} \mid gx \subseteq gv\}$  equals the dimension of  $Gv$  where  $\mathfrak{g}$  is the (complex) Lie algebra of  $G$ .

**Definition 6.** A real orthogonal representation  $V_{\mathbb{R}}$  of a compact Lie group  $K$  is called polar when there exists a vector  $v \in V_{\mathbb{R}}$  such that the orbit  $Kv$  is maximal-dimensional among all orbits of  $K$  and such that  $\kappa u$  is perpendicular to  $(\kappa v)^{\perp}$  for all  $u \in (\kappa v)^{\perp}$  where  $\kappa$  is the (real) Lie algebra of  $K$ .

**Theorem 7.** *Let  $V$  be an orthogonal representation of a reductive group  $G$ . Then the following are equivalent:*

- (i)  $V$  satisfies the conditions of the Main Theorem;
- (ii)  $V$  is a stable polar representation; and
- (iii)  $V$  is the complexification of a polar representation of a maximal compact Lie group  $K$  contained in  $G$ .

**Remark 8.** In (ii), we ask for the representation  $V$  to be stable, i.e. for there to exist a  $v \in V$  whose orbit is closed and maximal-dimensional among all orbits. This is a notion coming from Geometric Invariant Theory and should not be confused with the notion of a subset  $X$  of  $V$  being  $G$ -stable, i.e. having  $gX \subseteq X$  for all  $g \in G$ .

The only places in this paper where the word stable refers to the notion from GIT are in Definition 5 and Theorem 7.

**Remark 9.** Analogously to the equivalence (i) $\Leftrightarrow$ (ii) of Theorem 7, the conditions on  $V_{\mathbb{R}}$  in Theorem 3 are equivalent to  $V_{\mathbb{R}}$  being a polar representation.

In the paper [Da], the irreducible real polar representations of compact Lie groups are completely classified, giving us a list of spaces on which our Main Theorem can be applied. We discuss some of these spaces in section 7.

### 3. INTERLUDE: THE CLOSEST SYMMETRIC MATRIX WITH PRESCRIBED EIGENVALUES

Given a symmetric matrix  $A \in \mathbb{R}^{n \times n}$  and given a sequence of real numbers  $\lambda = (\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n)$ , how does one find the symmetric matrix  $B \in \mathbb{R}^{n \times n}$  with spectrum  $\lambda$  that minimizes  $d_A(B) := \sum_{i,j} (a_{ij} - b_{ij})^2$ ?

To cast this as an instance of Theorem 3, take for  $V_{\mathbb{R}}$  the space of real symmetric matrices acted upon by the group  $K = O_n(\mathbb{R})$  of orthogonal  $n \times n$ -matrices via the action  $\alpha : (g, A) \mapsto gAg^T$ . The  $K$ -invariant inner product on  $V_{\mathbb{R}}$  is given by

$$\langle C|D \rangle = \text{Tr } C^T D = \sum_{i,j} c_{ij} d_{ij}.$$

We claim that the space  $V_{\mathbb{R},0}$  of *diagonal* matrices has the properties of Theorem 3. Indeed, if  $D$  is any diagonal matrix with distinct eigenvalues, then differentiating  $\alpha$  and using that the Lie algebra  $\kappa$  of  $K$  is the Lie algebra of skew-symmetric matrices, we find that

$$T_D K D = \{BD - DB \mid B^T = -B\}$$

is precisely the space of symmetric matrices with zeroes on the diagonal, i.e., the orthogonal complement of  $V_{\mathbb{R},0}$ .

Let  $X$  be the real-algebraic variety in  $V_{\mathbb{R}}$  consisting of matrices with the prescribed spectrum  $\lambda$ . Then Theorem 3 says that, if  $A$  lies in  $V_0$ , so that  $A = \text{diag}(\mu_1, \dots, \mu_n)$ , then the critical points of  $d_A$  on  $X$  are the same as the critical points of the restriction of  $d_A$  to  $X_0 := X \cap V_0$ . If the  $\lambda_i$  are distinct, then this intersection consists of  $n!$  diagonal matrices, one for each permutation of the  $\lambda_i$ .

Accordingly, the ED degree of the complexification of  $X$  (the subject of the Main Theorem) is then  $n!$ . If the  $\lambda_i$  are not distinct but come with multiplicities  $n_1, \dots, n_k$  adding up to  $n$ , then the ED degree is the multinomial coefficient  $\frac{n!}{n_1! \cdots n_k!}$ . The group  $S_n$  here is the Weyl group from Section 5. In Example 23 we will see a large class of examples where the ED degree equals the order of the Weyl group.

Still assuming that  $A$  is diagonal, we get a diagonal matrix  $B \in X_0$  closest to  $A$  by arranging the  $\lambda_i$  in the same order as the  $\mu_i$ . To see this, let  $\pi \in S_n$  be a permutation. If  $\mu_i < \mu_j$  and  $\lambda_{\pi(i)} > \lambda_{\pi(j)}$  for some  $i, j \in [n]$ , then

$$(\mu_i - \lambda_{\pi(i)})^2 + (\mu_j - \lambda_{\pi(j)})^2 - (\mu_i - \lambda_{\pi(j)})^2 - (\mu_j - \lambda_{\pi(i)})^2 = 2(\mu_j - \mu_i)(\lambda_{\pi(i)} - \lambda_{\pi(j)}) > 0$$

and so  $\pi$  cannot minimize  $\sum_i (\mu_i - \lambda_{\pi(i)})^2$ .

Now when  $A$  is not diagonal to begin with, we first compute  $g \in O_n(\mathbb{R})$  such that  $A_0 := gAg^T$  is diagonal, find a diagonal matrix  $B_0$  closest to  $A_0$  as above, and then  $B := g^{-1}B_0g^{-T}$  is a solution to the original problem. In the same manner, one obtains all critical points of  $d_A$  from those of  $d_{A_0}$ .

#### 4. REAL COMPACT VERSUS COMPLEX REDUCTIVE

We will use the correspondence between compact Lie groups and reductive complex linear algebraic groups.

**Theorem 10.**

- (i) *Any reductive complex algebraic group  $G$  contains a maximal compact Lie group. All such subgroups are conjugate and Zariski dense in  $G$ .*
- (ii) *Any compact Lie group is maximal in a reductive complex algebraic group, which is unique up to isomorphism.*

*Proof.* See for example [Pr, Subsection 8.7.2] and [OV, Section 5.2]. □

The following lemma is well known, but included for completeness.

**Lemma 11.** *The real orthogonal group  $O_n(\mathbb{R})$  is a maximal compact subgroup of the complex orthogonal group  $O_n$ .*

*Proof.* Any compact subgroup of  $O_n$  leaves invariant some Hermitian positive-definite form on  $\mathbb{C}^n$ . The only Hermitian positive definite forms that are  $O_n(\mathbb{R})$ -invariant are multiples of the standard form. So any compact subgroup of  $O_n$  containing  $O_n(\mathbb{R})$  is contained in the unitary group  $U(n)$ . Since  $O_n(\mathbb{R}) = O_n \cap U(n)$ , we see that  $O_n(\mathbb{R})$  is maximal. □

Let  $G$  be a reductive linear algebraic group and let  $K$  be a maximal compact Lie group contained in  $G$ . Then the complexification of any real representation of  $K$  naturally has the structure of a representation of  $G$ .

**Proposition 12.** *A (complex) representation of  $G$  is orthogonal if and only if it is the complexification of a (real) representation of  $K$  that is orthogonal with respect to some positive definite inner product.*

*Proof.* Let  $V$  be an orthogonal real representation of  $K$  and let  $V_{\mathbb{C}}$  be its complexification. Extend the inner product  $\langle - | - \rangle$  on  $V$  to a non-degenerate symmetric bilinear form on  $V_{\mathbb{C}}$ . Then  $\langle v | w \rangle = \langle gv | gw \rangle$  for all  $v, w \in V_{\mathbb{C}}$  and  $g \in K$ . So since  $K$  is Zariski dense in  $G$ , we see that  $V_{\mathbb{C}}$  is an orthogonal representation of  $G$ .

Let  $V$  be an orthogonal complex representation of  $G$ . Then the image of  $K$  in  $O(V)$  is contained in some maximal compact subgroup  $H$  of  $O(V)$ . Let  $W$  be a

real subspace of  $V$  with  $W \otimes \mathbb{C} = V$  such that the bilinear form on  $V$  restricts to a  $\mathbb{R}$ -valued positive definite inner product on  $W$ . Since all maximal compact subgroups of  $O(V)$  are conjugate, we see that

$$H = gO(W)g^{-1}$$

for some  $g \in O(V)$ . Let  $V_{\mathbb{R}}$  be the real vector space  $gW$  with inner product  $\langle v|w \rangle_{V_{\mathbb{R}}} = \langle g^{-1}v|g^{-1}w \rangle$  for all  $v, w \in V_{\mathbb{R}}$ . Then  $V_{\mathbb{R}}$  is an orthogonal representation of  $K$  whose complexification is isomorphic to  $V = W \otimes_{\mathbb{R}} \mathbb{C}$  by the map  $g^{-1}$ .  $\square$

Let  $\mathfrak{g}$  be the (complex) Lie algebra of  $G$  and let  $\kappa$  be the (real) Lie algebra of  $K$ . The following theorem is a reformulation of Theorem 7.

**Theorem 13.** *Let  $V_{\mathbb{R}}$  be an orthogonal representation of  $K$  and let  $V_{\mathbb{C}}$  be its complexification. Then the following are equivalent:*

- (i) *there exists a (complex) subspace  $V_{\mathbb{C},0}$  of  $V_{\mathbb{C}}$  such that, for  $v_0 \in V_{\mathbb{C},0}$  sufficiently general, the space  $V_{\mathbb{C}}$  is the orthogonal direct sum of  $V_{\mathbb{C},0}$  and  $\mathfrak{g}v_0$ ;*
- (ii) *there exists a vector  $v \in V_{\mathbb{C}}$  such that the orbit  $Gv$  is closed and maximal-dimensional among all orbits of  $G$  and such that the codimension of the subspace  $\{x \in V_{\mathbb{C}} | \mathfrak{g}x \subseteq \mathfrak{g}v\}$  equals the dimension of  $Gv$ ; and*
- (iii) *there exists a vector  $v \in V_{\mathbb{R}}$  such that the orbit  $Kv$  is maximal-dimensional among all orbits of  $K$  and such that  $\kappa u$  is perpendicular to  $(\kappa v)^{\perp}$  for all  $u \in (\kappa v)^{\perp}$ .*

*Proof.*

(ii) $\Rightarrow$ (i) Let  $v \in V_{\mathbb{C}}$  be a vector as in (ii) and take

$$V_{\mathbb{C},0} = \{x \in V_{\mathbb{C}} | \mathfrak{g}x \subseteq \mathfrak{g}v\}.$$

Then for  $v_0 \in V_{\mathbb{C},0}$  sufficiently general, we have  $\mathfrak{g}v_0 = \mathfrak{g}v$ . So it suffices to prove that  $V_{\mathbb{C}}$  is the orthogonal direct sum of  $V_{\mathbb{C},0}$  and  $\mathfrak{g}v$ . By [DK, Corollary 2.5], we know that  $V_{\mathbb{C}}$  is the direct sum of  $V_{\mathbb{C},0}$  (there denoted  $c_v$ ) and  $\mathfrak{g}v$ . We have

$$\langle V_{\mathbb{C},0} | \mathfrak{g}v \rangle = -\langle \mathfrak{g}V_{\mathbb{C},0} | v \rangle = -\langle \mathfrak{g}v | v \rangle = \{0\}$$

and therefore the direct sum is orthogonal.

(i) $\Rightarrow$ (iii) Let  $V_{\mathbb{C},0}$  be a subspace as in (i) and let  $U$  be a dense open subset of  $V_{\mathbb{C},0}$  such that  $V_{\mathbb{C}}$  is the orthogonal direct sum of  $V_{\mathbb{C},0}$  and  $\mathfrak{g}w$  for all  $w \in U$ . Then  $GU$  is a dense constructible subset of  $V_{\mathbb{C}}$  and hence contains a dense open subset of  $V_{\mathbb{C}}$ . Note that the dimension of the orbit of any element of  $GU$  equals the codimension of  $V_{\mathbb{C},0}$ . So since  $GU$  is dense in  $V_{\mathbb{C}}$ , we see that these orbits must be maximal-dimensional among all orbits of  $G$ . Since  $V_{\mathbb{R}}$  is dense in  $V_{\mathbb{C}}$ , the intersection of  $V_{\mathbb{R}}$  with  $GU$  contains a vector  $v = gw$  with  $g \in G$  and  $w \in U$ . Since  $v \in GU$ , we see that

$$\dim_{\mathbb{R}}(Kv) = \dim_{\mathbb{R}}(\kappa v) = \dim_{\mathbb{C}}(\mathfrak{g}v) = \dim_{\mathbb{C}}(Gv)$$

is maximal among the dimensions of all orbits of  $K$ . The space  $V_{\mathbb{C}}$  is the orthogonal direct sum of  $\mathfrak{g}V_{\mathbb{C},0}$  and  $\mathfrak{g}v$ . Therefore we have

$$(\kappa v)^{\perp} = (\mathfrak{g}v)^{\perp} \cap V_{\mathbb{R}} \subseteq \mathfrak{g}V_{\mathbb{C},0}$$

and hence for all  $u \in (\kappa v)^{\perp}$ , we have

$$\langle \kappa u | (\kappa v)^{\perp} \rangle \subseteq \langle \mathfrak{g}u | \mathfrak{g}V_{\mathbb{C},0} \rangle = \langle \mathfrak{g}g^{-1}u | \mathfrak{g}V_{\mathbb{C},0} \rangle = \langle \mathfrak{g}(g^{-1}u) | V_{\mathbb{C},0} \rangle = \{0\}.$$

(iii) $\Rightarrow$ (ii) Let  $v \in V_{\mathbb{R}}$  be a vector as in (iii). Since  $\langle av|av \rangle = \langle v|v \rangle$  for all  $a \in K$ , we have  $\langle bv|v \rangle + \langle v|bv \rangle = 0$  for all  $b \in \mathfrak{k}$ . So  $\langle \kappa v|v \rangle = \{0\}$  and  $v$  satisfies the condition of [DK, Theorem 1.1], because  $\langle \mathfrak{g}v, v \rangle = \mathbb{C} \otimes \langle \kappa v|v \rangle = \{0\}$ . Note that the Hermitian form  $\langle -, - \rangle$  on  $V_{\mathbb{C}}$  in that theorem is the extension of the inner product on  $V_{\mathbb{R}}$  and that it is not equal to our bilinear form  $\langle -|- \rangle$  on  $V_{\mathbb{C}}$ . By part (i) of Theorem 1.1, the orbit  $Gv$  is closed. Since  $K$  is dense in  $G$  and since the function  $(u \mapsto \dim(Gu))$  is lower semicontinuous, we see that  $\dim(Gv) = \dim(Kv)$  is maximal. As stated in the introduction of [DK], the dimension of  $\{x \in V_{\mathbb{C}} | \mathfrak{g}x \subseteq \mathfrak{g}v\}$  is always at most the codimension of a maximal-dimensional orbit of  $G$ . Since

$$\mathbb{C} \otimes (\kappa v)^{\perp} \subseteq \mathbb{C} \otimes \{u \in V_{\mathbb{R}} | \kappa u \subseteq \kappa v\} \subseteq \{x \in V_{\mathbb{C}} | \mathfrak{g}x \subseteq \mathfrak{g}v\},$$

we must have equality.  $\square$

**Example 14.** Let  $G$  be the group  $\mathrm{SL}_2(\mathbb{C})$  and let  $V_{\mathbb{C}}$  be the irreducible 5-dimensional representation of  $\mathrm{SL}_2(\mathbb{C})$ . So  $V_{\mathbb{C}}$  is the set of homogeneous polynomials in  $x$  and  $y$  of degree 4 and

$$\begin{aligned} \mathfrak{sl}_2(\mathbb{C}) &\mapsto \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}}) \\ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} &\mapsto a \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) + bx \frac{\partial}{\partial y} + cy \frac{\partial}{\partial x} \end{aligned}$$

is the corresponding representation of  $\mathfrak{sl}_2(\mathbb{C})$ . Let the non-degenerate symmetric bilinear form  $\langle -|- \rangle$  on  $V_{\mathbb{C}}$  be given by the Gram matrix

$$\begin{pmatrix} & & & & 12 \\ & & & -3 & \\ & & 2 & & \\ & -3 & & & \\ 12 & & & & \end{pmatrix}$$

with respect to the basis  $x^4, xy^3, x^2y^2, xy^3, y^4$  (obtained by setting  $\langle x^4|y^4 \rangle = 12$  and using  $\langle gv|w \rangle = -\langle x|gw \rangle$  for all  $v, w \in V_{\mathbb{C}}$  and  $g \in \mathfrak{sl}_2(\mathbb{C})$ ). Then  $\langle -|- \rangle$  is  $\mathrm{SL}_2(\mathbb{C})$ -invariant. A maximal compact subgroup of  $\mathrm{SL}_2(\mathbb{C})$  is  $K = \mathrm{SU}(2)$ . The real subspace

$$V_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}(x^4 + y^4, i(x^4 - y^4), x^2y^2, xy(x^2 - y^2), ixy(x^2 + y^2)).$$

of  $V_{\mathbb{C}}$  is  $\mathrm{SU}(2)$ -stable and has  $V_{\mathbb{C}}$  as its complexification. See the proofs of [IRS, Propositions 3 and 5] for how  $V_{\mathbb{R}}$  was obtained. We will now check that the three equivalent conditions of the theorem are satisfied.

(i) Take  $V_{\mathbb{C},0} = \mathrm{span}_{\mathbb{C}}(x^4 + y^4, x^2 + y^2)$ . Then  $V_{\mathbb{C}}$  is the orthogonal direct sum of  $V_{\mathbb{C},0}$  and

$$\mathfrak{sl}_2(\mathbb{C})v_0 = \mathrm{span}_{\mathbb{C}}(x^4 - y^4, x^3y, xy^3)$$

for all  $v_0 = a(x^4 + y^4) + bx^2y^2$  with  $4a^2 \neq b^2$ .

(ii) Take  $v = x^4 + y^4 + x^2y^2$ . Then  $\dim(\mathfrak{sl}_2(\mathbb{C})v) = 3 = \dim(\mathrm{SL}_2(\mathbb{C}))$ . Hence the dimension of  $\mathrm{SL}_2(\mathbb{C})v$  is maximal. As in the proof of the theorem, we see that the orbit  $\mathrm{SL}_2(\mathbb{C})v$  is closed and

$$\{x \in V_{\mathbb{C}} | \mathfrak{sl}_2(\mathbb{C})x \subseteq \mathfrak{sl}_2(\mathbb{C})v\} = \mathrm{span}_{\mathbb{C}}(x^4 + y^4, x^2 + y^2)$$

has dimension  $5 - 3 = 2$ .

(iii) Again take  $v = x^4 + y^4 + x^2y^2$ . We have

$$\mathfrak{su}(2) = \text{span}_{\mathbb{R}} \left( \begin{pmatrix} i & \\ & -i \end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & i \\ i & \end{pmatrix} \right)$$

and so we see that

$$\mathfrak{su}(2)v = \text{span}_{\mathbb{R}} (i(x^4 - y^4), xy(x^2 - y^2), ixy(x^2 + y^2))$$

has orthogonal complement

$$\text{span}_{\mathbb{R}} (x^4 + y^4, x^2y^2)$$

and we have  $\mathfrak{su}(2)u \subseteq \mathfrak{su}(2)v$  for all  $u$  in this complement.

## 5. PROOF OF THE MAIN THEOREM

Let  $G \rightarrow O(V)$  be an orthogonal representation as in Section 2. Let  $X$  be a  $G$ -stable closed subvariety of  $V$ . We assume the conditions of the Main Theorem. Note that if we replace  $G$  by its unique irreducible component  $G^\circ$  that contains the identity element, the conditions of the Main Theorem are still satisfied, because  $G^\circ$  has finite index in  $G$ . So we may assume that  $G$  is irreducible. This implies that all irreducible components of  $X$  are also  $G$ -stable.

**Lemma 15.** *The set  $GV_0$  is dense in  $V$ .*

*Proof.* The derivative of the multiplication map  $G \times V_0 \rightarrow V$  at a (smooth) point  $(1, v_0)$  equals the map

$$\begin{aligned} \mathfrak{g} \oplus V_0 &\rightarrow V \\ (A, u_0) &\mapsto Av_0 + u_0 \end{aligned}$$

and has image  $\mathfrak{g}v_0 + V_0$ , which by assumption equals  $V$  for sufficiently general  $v_0 \in V_0$ . Hence the derivative is surjective at  $(1, v_0)$  for some  $v_0 \in V_0$ . Therefore the multiplication map is dominant and its image  $GV_0$  is dense in  $V$ .  $\square$

**Lemma 16.** *For elements  $g \in G$  and  $u \in V$ , the ED critical points for  $gu$  are obtained from those of  $u$  by applying  $g$ .*

*Proof.* Let  $x$  be a point on  $X$ . The element  $g \in G$  acts linearly and preserves  $X$  and  $X^{\text{reg}}$ . The derivative of the isomorphism  $X \rightarrow X, y \mapsto gy$  at  $x$  is the isomorphism  $T_x X \rightarrow T_{gx} X, w \mapsto gw$ . So since  $g$  also preserves the bilinear form, we have  $u - x \perp T_x X$  if and only if  $gu - gx \perp T_{gx} X$ .  $\square$

**Lemma 17.** *A sufficiently general  $x_0 \in X_0$  lies both in  $X_0^{\text{reg}}$  and in  $X^{\text{reg}}$ .*

*Proof.* A sufficiently general point on  $X_0$  lies in  $X_0^{\text{reg}}$ . Since  $GX_0$  is constructible and dense in  $X$ , it contains a  $G$ -stable dense open subset  $U$  of  $X^{\text{reg}}$ . The intersection of  $U$  with  $X_0$  is dense in  $X_0$ . Hence a sufficiently general point on  $X_0$  lies in  $X^{\text{reg}}$ .  $\square$

Define the Weyl group  $W$  by

$$W = N_G(V_0)/Z_G(V_0) = \{g \in G \mid gV_0 = V_0\} / \{g \in G \mid gw = w \forall w \in V_0\}.$$



Then the finite group  $W$  acts naturally on  $V_0$ . Consider the set  $S$  of  $G$ -stable closed subvarieties  $Y$  of  $V$  such that  $G(Y \cap V_0)$  is dense in  $Y$  and the set  $R$  of  $W$ -stable closed subvarieties of  $V_0$ . Consider the maps

$$\begin{array}{ccc} \varphi: S & \rightarrow & R \\ Y & \mapsto & Y \cap V_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} \psi: R & \rightarrow & S \\ Z & \mapsto & \overline{GZ} \end{array}$$

between these two sets.

**Lemma 18.** *The bijective maps  $\varphi$  and  $\psi$  are mutual inverses.*

*Proof.* Since  $S$  consists of the  $G$ -stable closed subvarieties  $Y$  of  $V$  such that  $Y$  equals the closure of  $G(Y \cap V_0)$  in  $V$ , we see that  $\psi \circ \varphi$  is the identity map on  $S$ . Let  $Z$  be a  $W$ -stable closed subvariety of  $V_0$ . It is clear that  $Z \subseteq \varphi(\psi(Z))$  and we will show that in fact  $\varphi(\psi(Z)) = Z$  holds. Since  $Z$  is  $W$ -stable and  $W$  is finite, the variety  $Z$  is defined by  $W$ -invariant polynomials  $f_1, \dots, f_n \in \mathbb{C}[V_0]^W$ . By [DK, Theorem 2.9], there exists  $G$ -invariant polynomials  $g_1, \dots, g_n \in \mathbb{C}[V]^G$  such that  $f_i$  is the restriction of  $g_i$  to  $V_0$  for all  $i \in \{1, \dots, n\}$ . Since  $g_1, \dots, g_n$  are  $G$ -invariant and  $g_1(z) = \dots = g_n(z) = 0$  for all  $z \in Z$ , we see that (the closure of)  $GZ$  is contained in the zero set of the ideal generated by  $g_1, \dots, g_n$ . Hence

$$\varphi(\psi(Z)) = \overline{GZ} \cap V_0$$

is contained in the zero set of the ideal generated by the restrictions of  $g_1, \dots, g_n$  to  $V_0$ . This zero set is  $Z$  and hence  $\varphi(\psi(Z)) \subseteq Z$ . So we see that  $\varphi \circ \psi$  is the identity map on  $R$ .  $\square$

**Lemma 19.** *A sufficiently general  $x_0 \in X_0$  satisfies  $T_{x_0}X = T_{x_0}X_0 + T_{x_0}Gx_0$ .*

*Proof.* By Lemma 17, we see that sufficiently general points of  $X_0$  are contained in at most one irreducible component of  $X$ . Therefore each irreducible component of  $X_0$  is contained in precisely one irreducible component of  $X$ . Let  $Y$  be an irreducible component of  $X$  and let  $Z_1, \dots, Z_k$  be the irreducible components of  $X_0$  contained in  $Y$ . Then the Weyl group  $W$  acts on the set  $\{Z_1, \dots, Z_k\}$ . Since  $GX_0$  is dense in  $X$ , we see that  $G(Z_1 \cup \dots \cup Z_k)$  must be dense in  $Y$ . So  $GZ_i$  must be dense in  $Y$  for some  $i \in \{1, \dots, k\}$ . By the previous lemma, for this  $i$  we have

$$Z_1 \cup \dots \cup Z_k = Y \cap V_0 = \bigcup_{g \in W} gZ_i$$

and hence  $W$  must act transitively on  $\{Z_1, \dots, Z_k\}$ . In particular, we see that  $GZ_j$  is in fact dense in  $Y$  for all  $j \in \{1, \dots, k\}$ .

Take  $Z = Z_j$  for any  $j \in \{1, \dots, k\}$ . Then the multiplication map  $G \times Z \rightarrow Y$  is dominant and  $G$ -equivariant when we let  $G$  act on itself by left multiplication. Therefore its derivative at  $(1, z)$  is surjective for  $z \in Z$  sufficiently general. This means that  $T_z Y = T_z Z + T_z Gz$  for  $z \in Z$  sufficiently general. Since this holds for all components  $Z$  of  $X_0$ , we see that  $T_{x_0}X = T_{x_0}X_0 + T_{x_0}Gx_0$  for  $x_0 \in X_0$  sufficiently general.  $\square$

**Lemma 20.** *Let  $Y$  be a closed subvariety in a complex affine space  $V$ . Let  $U$  be a dense open subset of  $Y$  and let  $Z$  be its complement in  $Y$ . Then for  $v \in V$  sufficiently general, all ED critical points  $y \in Y$  for  $v$  lie in  $U$ .*

*Proof.* See the proof of [DLOT, Lemma 4.2].  $\square$

**Lemma 21.** *Let  $v_0 \in V_0$  be sufficiently general. Then any ED critical point on  $X_0$  for  $v_0$  is an ED critical point on  $X$  for  $v_0$ .*

*Proof.* By combining the previous three lemmas, we may assume that all ED critical points  $x_0 \in X_0$  for  $v_0$  are not only elements of  $X_0^{\text{reg}}$  but also of  $X^{\text{reg}}$  and that they satisfy  $T_{x_0}X = T_{x_0}X_0 + T_{x_0}Gx_0$ . Let  $x_0$  be an ED critical point of  $v_0$ . Then  $v_0 - x_0 \perp T_{x_0}X_0$  by criticality and  $v_0 - x_0 \in V_0 \perp T_{x_0}Gx_0$  by the conditions of the Main Theorem (here we do not need that  $T_{x_0}Gx_0$  is the orthogonal complement of  $V_0$ —this may not be true—but only that it is contained in that complement). We see that

$$v_0 - x_0 \perp T_{x_0}X_0 + T_{x_0}Gx_0 = T_{x_0}X$$

and hence  $x_0$  is an ED critical point on  $X$  for  $v_0$ .  $\square$

**Lemma 22.** *Let  $v_0 \in V_0$  be sufficiently general. Then any ED critical point on  $X$  for  $v_0$  is an ED critical point on  $X_0$  for  $v_0$ .*

*Proof.* Let  $x \in X$  be an ED critical point for  $v_0$ . Then in particular  $v_0 - x \perp T_xGx = \mathfrak{g}x$ . Together with  $x \perp \mathfrak{g}x$ , which holds by orthogonality of the representation, this implies that  $v_0 \perp \mathfrak{g}x$ . Using once more the orthogonality of the representation, we see that  $\langle x | \mathfrak{g}v_0 \rangle = -\langle \mathfrak{g}x | v_0 \rangle = \{0\}$ . So  $x \perp T_{v_0}Gv_0$ . Since  $v_0$  is sufficiently general in  $V_0$ , the vector space  $V$  is the orthogonal direct sum of  $V_0$  and  $T_{v_0}Gv_0$  and therefore  $x$  is an element of  $V_0$ . So since also  $x \in X$ , we have  $x \in X_0$ . Since  $v_0 - x \perp T_xX \supseteq T_xX_0$ , we find that  $x \in X_0$  is an ED critical point for  $v_0$ .  $\square$

*Proof of the Main Theorem.* By Lemmas 15 and 16 we may assume that the sufficiently general point on  $V$  is in fact a sufficiently general point  $v_0$  on  $V_0$ . The previous two lemmas now tell us that the ED critical points for  $v_0$  on  $X$  and on  $X_0$  are the same. Hence the ED degrees of  $X$  in  $V$  and  $X_0$  in  $V_0$  are equal.  $\square$

**Example 23.** Let  $G$  be a complex semisimple algebraic group acting on its Lie algebra  $V = \mathfrak{g}$  by conjugation, let  $V_0$  be a Cartan subalgebra of  $\mathfrak{g}$  and let  $W$  be the Weyl group. In Section 7, we show that  $V$  satisfies the conditions of the Main Theorem. Suppose  $X$  is the closed  $G$ -orbit of a sufficiently general point  $v \in V_0$ . Then the intersection  $X_0 = X \cap V_0$  is a single  $W$ -orbit by [DK, Theorem 2.8]. Since  $X_0$  is the  $W$ -orbit of a sufficiently general point of  $V_0$ , it is a set of size  $\#W$ . So the ED degree of  $X$  equals  $\#W$ .

Since  $v$  is sufficiently general, the codimension of  $X$  in  $V$  equals the dimension of  $V_0$ . So the degree of the variety  $X$ , i.e. the cardinality of  $X \cap V'$  for a sufficiently general subspace  $V'$  of  $V$  with  $\dim(V') = \dim(V_0)$ , is at least the cardinality of  $X \cap V_0$ , which is the ED degree of  $X$ . Let  $f_1, \dots, f_n$  be a set of invariant polynomial generating the algebra  $\mathbb{C}[V]^G$ . Then, since  $X$  is a closed  $G$ -orbit, we see that  $X$  is defined by the equations  $f_1 = f_1(v), \dots, f_n = f_n(v)$ . Therefore the degree of  $X$  is at most the product of the degrees of  $f_1, \dots, f_n$ . By Theorem [Hu, Theorem 3.19], this product equals the size of the Weyl group  $W$ . So the degree and ED degree of  $X$  are equal.

## 6. TESTING FOR THE CONDITIONS OF THE MAIN THEOREM

In the paper [Da], the irreducible polar representations of compact Lie groups are completely classified. Now suppose we have a not-necessarily irreducible orthogonal representation  $V$  of a reductive algebraic group  $G$ . We would like to

be able to test whether  $V$  satisfies the conditions of the Main Theorem. In [DK, Section 2], some methods are given. In this section, we describe one more such method.

**Lemma 24.** *For sufficiently general  $v \in V$ , the tangent space  $\mathfrak{g}v$  of  $v$  to its orbit is maximal-dimensional and non-degenerate with respect to the bilinear form.*

*Proof.* By Proposition 12, we know that  $V$  is the complexification of a real subspace  $V_{\mathbb{R}}$  and that the  $G$ -invariant bilinear form  $\langle - | - \rangle$  is the extension of a positive definite inner product on  $V_{\mathbb{R}}$ . Since  $V_{\mathbb{R}}$  is dense in  $V$  and the set of  $v \in V$  such that the dimension of  $\mathfrak{g}v$  is maximal is open and dense, there is an element  $w$  in the intersection. Note that  $\mathfrak{g}w = \kappa w \otimes_{\mathbb{R}} \mathbb{C}$ . Since  $\langle - | - \rangle$  is an inner product on  $V_{\mathbb{R}}$ , its restriction to  $\kappa w$  is non-degenerate. Therefore the restriction of  $\langle - | - \rangle$  to  $\mathfrak{g}w$  is non-degenerate as well.

Pick  $\varphi_1, \dots, \varphi_n \in \mathfrak{g}$  such that  $\varphi_1 w, \dots, \varphi_n w$  form a basis of  $\mathfrak{g}w$ . Then the set of  $v \in V$  such that  $\varphi_1 v, \dots, \varphi_n v$  are linearly independent and the restriction of  $\langle - | - \rangle$  to their span is non-degenerate is an non-empty open subset of  $V$ . Since the dimension of  $\mathfrak{g}w$  is maximal, we see that for every element  $v$  in this set, the tangent space  $\mathfrak{g}v$  of  $v$  to its orbit is spanned by  $\varphi_1 v, \dots, \varphi_n v$ . So for sufficiently general  $v \in V$ , the vector space  $\mathfrak{g}v$  is non-degenerate with respect to the bilinear form.  $\square$

**Lemma 25.** *A subspace  $V_0$  as in the Main Theorem exists if and only if for sufficiently general  $v \in V$  and for all  $u_1, u_2 \perp \mathfrak{g}v$  we have  $u_1 \perp \mathfrak{g}u_2$ .*

*Proof.* Suppose such a subspace  $V_0$  exists. Let  $v \in V$  be sufficiently general and let  $u_1, u_2 \perp \mathfrak{g}v$ . For  $v_0 \in V_0$  sufficiently general and for all  $g \in G$ , the vector space  $V$  is the orthogonal direct sum of  $gV_0$  and  $g\mathfrak{g}v_0 = \mathfrak{g}(gv_0)$ . Since  $GV_0$  contains an open dense subset of  $V$ , we may assume that  $v = gv_0$  for such  $v_0$  and  $g$ . So we see that  $u_1, u_2 \in gV_0$ . We have  $V_0 \perp \mathfrak{g}u$  for all  $u \in V_0$ . Therefore we have  $gV_0 \perp \mathfrak{g}u$  for all  $u \in gV_0$  and hence  $u_1 \perp \mathfrak{g}u_2$ .

Let  $v \in V$  be such that the  $\mathfrak{g}v$  is maximal-dimensional, the restriction of the  $G$ -invariant bilinear form  $\langle - | - \rangle$  to  $\mathfrak{g}v$  is non-degenerate and  $u_1 \perp \mathfrak{g}u_2$  for all  $u_1, u_2 \perp \mathfrak{g}v$ . Let  $V_0$  be the orthogonal complement of  $\mathfrak{g}v$ . Then we see that  $V_0$  is perpendicular to  $\mathfrak{g}v_0$  for all  $v_0 \in V_0$ . Let  $v_0 \in V_0$  be sufficiently general. Then we have  $\mathfrak{g}v_0 = \mathfrak{g}v$  and hence  $V$  is the orthogonal direct sum of  $V_0$  and  $\mathfrak{g}v_0$ .  $\square$

**Lemma 26.** *Let  $W$  be a finite-dimensional complex vector space, let*

$$f_1, \dots, f_m: V \rightarrow W$$

*be linear maps and let  $w \in W$  be an element. Then the following are equivalent:*

- (i) *For  $v \in V$  sufficiently general, we have  $w \in \text{span}_{\mathbb{C}}(f_1(v), \dots, f_m(v))$ .*
- (ii) *We have  $1 \otimes w \in \text{span}_{\mathbb{C}(V^*)}(f_1, \dots, f_m) \subset \mathbb{C}(V^*) \otimes_{\mathbb{C}} W$ .*

*Proof.* Suppose that

$$1 \otimes w = c_1 f_1 + \dots + c_m f_m$$

for some  $c_1, \dots, c_m \in \mathbb{C}(V^*)$ . Then

$$w = c_1(v)f_1(v) + \dots + c_m(v)f_m(v)$$

for the dense open subset of  $V$  consisting of all  $v$  where  $c_1, \dots, c_m$  can be evaluated.

For the converse, suppose that for  $v \in V$  sufficiently general we know that  $w$  is contained in the span of  $f_1(v), \dots, f_m(v)$ . We may assume that  $f_1(v), \dots, f_m(v)$  are linearly independent for  $v \in V$  sufficiently general. Let  $v \in V$  be such that

$f_1(v), \dots, f_m(v)$  are linearly independent and  $w$  is contained in their span. Choose  $w_1, \dots, w_k \in W$  such that  $f_1(v), \dots, f_m(v), w_1, \dots, w_k$  form a basis of  $W$ . Now note that  $f_1(v), \dots, f_m(v), w_1, \dots, w_k$  form a basis of  $W$  for  $v \in V$  sufficiently general. By choosing a basis, we may assume that  $W = \mathbb{C}^{n+k}$ . This gives us a morphism

$$\begin{aligned} \varphi: V &\rightarrow \mathbb{C}^{(m+k) \times (m+k)} \\ v &\mapsto \begin{pmatrix} f_1(v) & \dots & f_m(v) & w_1 & \dots & w_k \end{pmatrix} \end{aligned}$$

such that  $\varphi(v)$  is invertible for  $v \in V$  sufficiently general. Consider the coefficients of  $\varphi(v)$  as elements of the field  $\mathbb{C}(V^*)$  of rational functions on  $V$ . Then the matrix  $\varphi(v)$  is invertible and  $c(v) = \varphi(v)^{-1}w$  is a vector with coefficients in  $\mathbb{C}(V^*)$ . We have

$$w = \varphi(v)c(v) = c_1(v)f_1(v) + \dots + c_m(v)f_m(v) + c_{n+1}(v)w_1 + \dots + c_{n+k}(v)w_k$$

for  $v \in V$  sufficiently. Since we also know that  $f_1(v), \dots, f_m(v), w_1, \dots, w_k$  form a basis and that  $w$  is contained in the span of  $f_1(v), \dots, f_m(v)$  for  $v \in V$  sufficiently general, we see that  $c_{n+1}, \dots, c_{n+k}$  all must be equal to the zero function. Hence

$$1 \otimes w = c_1 f_1 + \dots + c_m f_m$$

is contained in the span of  $f_1, \dots, f_m$  inside  $\mathbb{C}(V^*) \otimes W$ .  $\square$

Now we combine the previous two lemmas to reduce checking the existence of  $V_0$  to a linear algebra problem over  $\mathbb{C}(V^*)$ . Take  $U = W = V$  and consider  $U$  and  $W$  as affine spaces. Let  $\varphi_1, \dots, \varphi_n$  form a basis of  $\mathfrak{g}$ . By Lemma 25, we know that the representation  $V$  satisfies the conditions of the Main Theorem if and only if, for  $v \in V$  sufficiently general, the variety in  $U \times W$  given by the linear equations

$$\langle u | \varphi_i v \rangle = \langle w | \varphi_i v \rangle = 0, \quad i = 1, \dots, n$$

is contained in the variety given by the equations  $\langle u | \varphi_j w \rangle = 0$  for  $j = 1, \dots, n$ . The latter holds if and only if the polynomials  $\langle u | \varphi_j w \rangle$  are contained in the ideal  $I$  of the coordinate ring  $\mathbb{C}[U \times W]$  generated by  $\langle u | \varphi_i v \rangle$  and  $\langle w | \varphi_i v \rangle$  for  $i = 1, \dots, n$ . The polynomial  $\langle u | \varphi_j w \rangle$  is homogeneous of degree 2. So for a fixed  $v \in V$ , it is contained in  $I$  if and only if

$$\langle u | \varphi_j w \rangle \in \text{span}_{\mathbb{C}} \left( \mathbb{C}[U \times W]_{(1)} \cdot \left\{ \langle u | \varphi_i v \rangle, \langle w | \varphi_i v \rangle \mid i = 1, \dots, n \right\} \right).$$

So by Lemma 26, we see that  $V$  satisfies the conditions of the Main Theorem if and only if

$$\langle u | \varphi_j w \rangle \in \text{span}_{\mathbb{C}(V^*)} \left( \mathbb{C}[U \times W]_{(1)} \cdot \left\{ \langle u | \varphi_i v \rangle, \langle w | \varphi_i v \rangle \mid i = 1, \dots, n \right\} \right)$$

for all  $j \in \{1, \dots, n\}$ . The latter condition can be checked efficiently on a computer, requiring as input the bilinear form  $\langle - | - \rangle$  and the images in  $\text{End}(V)$  of a basis of  $\mathfrak{g}$ .

## 7. EXAMPLES

In this section we highlight some of the families of polar representations coming from [Da]. We also point out how some of these families are related by means of slice representations as defined in [DK]. Our Main Theorem can be applied to each of these families, thus generalizing [DLOT, Theorem 4.11].

**Remark 27.** A representation  $V$  of a group  $G$  satisfies the conditions of the Main Theorem if and only if the direct sum of  $V$  with the trivial representation does.

**Remark 28.** Let  $V$  be the orthogonal direct sum of two representations  $V_1$  and  $V_2$  of  $G$ . Then if  $V$  satisfies the conditions of the Main Theorem, so do  $V_1$  and  $V_2$ .

**7.1. Adjoint representations.** Let  $G$  be a complex semisimple algebraic group acting on its Lie algebra  $\mathfrak{g}$  by conjugation. This representation is orthogonal with respect to the Killing form  $B$  on  $\mathfrak{g}$  defined by

$$B(v, w) = \text{Tr}(\text{ad } v \text{ ad } w)$$

for  $v, w \in \mathfrak{g}$ . Since  $G$  is semisimple, we know that  $B$  is non-degenerate. Since  $G$  acts by conjugation, the tangent space of a point  $v \in \mathfrak{g}$  to its orbit equals  $[\mathfrak{g}, v]$ . We have

$$B(w, [\mathfrak{g}, v]) = -B([w, v], \mathfrak{g})$$

for all  $w, v \in \mathfrak{g}$ . So  $w \perp \mathfrak{g}v$  if and only if  $[v, w] = 0$ . Let  $h \in \mathfrak{g}$  and suppose  $[h, v] \perp \mathfrak{g}v$ . Then  $h \in \ker(\text{ad } v)^2 = \ker(\text{ad } v)$  and hence  $[h, v] = 0$ . Hence  $\mathfrak{g}v$  is non-degenerate. Let  $V_0$  be a Cartan subalgebra of  $\mathfrak{g}$ . Let  $v \in V_0$  be sufficiently general and let  $u_1, u_2 \perp \mathfrak{g}v$ . Then  $V_0 = C_{\mathfrak{g}}(V_0) = C_{\mathfrak{g}}(v)$ . So we have  $u_1, u_2 \in V_0$  and hence  $u_1 \perp \mathfrak{g}u_2$ .

**7.2. Standard representations of groups of type  $B$  and  $D$ .** Let  $n$  be a positive integer and let  $G$  be the orthogonal group  $O(n)$  acting on  $\mathbb{C}^n$  with the standard form. Let  $V_0$  be the subspace of  $V$  spanned by the first basis vector  $e_1$ . For all  $v \in V_0$  non-zero, we have  $\mathfrak{g}v = \{Ae_1 | A \in \mathfrak{gl}_n \text{ skew-symmetric}\} = \text{span}(e_2, \dots, e_n) = V_0^\perp$ .

**7.3. Representations of groups of type  $B$  and  $D$  of highest weight  $2\lambda_1$ .** Let  $n$  be a positive integer and let  $G$  be the orthogonal group  $SO(n)$  acting on the vector space  $V$  of symmetric  $n \times n$  matrices with trace zero by conjugation. The bilinear form given by

$$\langle A | B \rangle = \text{Tr}(A^T B)$$

for  $A, B \in V$  is non-degenerate and  $SO(n)$ -invariant. Let  $V_0$  be the subspace of  $V$  consisting of all diagonal matrices. For all  $D \in V_0$  with pairwise distinct entries on the diagonal, we have  $\mathfrak{g} \cdot D = \{AD - DA | A \in \mathfrak{gl}_n \text{ skew-symmetric}\} = V_0^\perp$ .

**7.4. Tensor products of two standard representations of groups of type  $B$  and  $D$ .** Let  $n \leq m$  be positive integers and let  $G$  be the group  $O(n) \times O(m)$  acting on  $n \times m$  matrices by left and right multiplication. The bilinear form given by

$$\langle A | B \rangle = \text{Tr}(A^T B)$$

for  $A, B \in \mathbb{C}^{n \times m}$  is non-degenerate and  $G$ -invariant. The subspace  $V_0$  of  $\mathbb{C}^{n \times m}$  consisting of diagonal matrices satisfies the conditions of the Main Theorem.

**Remark 29.** Consider the matrix  $v = (I_n \ 0) \in \mathbb{C}^{n \times m}$ . The stabilizer of  $v$  equals

$$G_v = \left\{ \left( g, \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \right) \mid g \in O(n), h \in O(m-n) \right\}$$

and the orthogonal complement of  $\mathfrak{g}v$  equals the set of matrices of the form  $(A \ 0)$  where  $A$  is a symmetric  $n \times n$  matrix. Ignoring the trivial action from  $O(m-n)$ , we see that the slice representation of the element  $v$  is the direct sum of the representation from the previous subsection and the trivial representation.

**7.5. Second alternating powers of standard representations of groups of type C.** Let  $n$  be a positive integer and let  $G$  be the symplectic group

$$\mathrm{Sp}(n) = \left\{ A \in \mathrm{GL}_{2n} \mid A \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A^T = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$$

acting on the second alternating power  $\Lambda^2 \mathbb{C}^{2n}$  of the standard representation. The Lie algebra of  $\mathrm{Sp}(n)$  equals

$$\begin{aligned} \mathfrak{sp}(n) &= \left\{ A \in \mathfrak{gl}_{2n} \mid A \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A^T \right\} \\ &= \left\{ \begin{pmatrix} X & Y \\ Z & \Theta \end{pmatrix} \in \mathfrak{gl}_{2n} \mid \begin{array}{l} Y = Y^T, Z = Z^T \\ X + \Theta^T = 0 \end{array} \right\}. \end{aligned}$$

The  $\mathrm{Sp}(n)$ -invariant skew-symmetric form on  $\mathbb{C}^{2n}$  induces the bilinear form on  $\Lambda^2 \mathbb{C}^{2n}$  given by

$$\langle v \wedge w \mid x \wedge y \rangle = v^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} x \cdot w^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} y - v^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} y \cdot w^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} x$$

for  $v, w, x, y \in \mathbb{C}^{2n}$ . This form is symmetric, non-degenerate and  $\mathrm{Sp}(2n)$ -invariant. Let  $V_0$  be the subspace of  $\Lambda^2 \mathbb{C}^{2n}$  spanned by  $e_i \wedge e_{n+i}$  for  $i = 1, \dots, n$ . Then for any linear combination  $v$  of  $e_1 \wedge e_{n+1}, \dots, e_n \wedge e_{2n}$  with only non-zero coefficients, the vector space  $\Lambda^2 \mathbb{C}^{2n}$  is the orthogonal direct sum of  $V_0$  and  $\mathfrak{g}v$ .

**Remark 30.** The paper [Da] tells us that  $\Lambda^2 \mathbb{C}^{2n}$  is isomorphic to  $\mathfrak{gl}_{2n} / \mathfrak{sp}(n)$  acted on by  $\mathrm{Sp}(n)$  by conjugation. In this case, the subspace  $V_0$  consists of matrices of the form

$$\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$$

with  $D \in \mathfrak{gl}_n$  diagonal.

**7.6. Tensor products of two standard representations of groups of type C.** Let  $n \leq m$  be positive integers and let  $G$  be the group  $\mathrm{Sp}(n) \times \mathrm{Sp}(m)$  acting on  $2n \times 2m$  matrices by left and right multiplication. The bilinear form is defined by

$$\langle A \mid B \rangle = \mathrm{Tr} \left( \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} B^T \right)$$

for all  $A, B \in \mathbb{C}^{2n \times 2m}$ . This form is symmetric, non-degenerate and  $G$ -invariant. Let  $V_0$  be the subspace of  $V$  consisting of matrices of the form

$$\begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & D & 0 \end{pmatrix}$$

where  $D$  is a diagonal  $n \times n$  matrix. Then for every invertible diagonal  $n \times n$  matrix  $D$  whose squares of diagonal entries are pairwise distinct, the vector space  $\mathbb{C}^{2n \times 2m}$  is the orthogonal direct sum of  $V_0$  and

$$\mathfrak{g} \cdot \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & D & 0 \end{pmatrix} = \mathfrak{sp}(n) \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & D & 0 \end{pmatrix} + \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & D & 0 \end{pmatrix} \mathfrak{sp}(m).$$

**Remark 31.** The slice representation of

$$\begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \end{pmatrix}$$

is a representation of  $\mathrm{Sp}(n) \times \mathrm{Sp}(m-n)$  where the second factor acts trivially. Ignoring this factor, the slice representation is isomorphic to the representation  $\mathfrak{gl}_{2n}/\mathfrak{sp}(n)$  from the previous remark.

**7.7. Direct sums of standard representations of groups of type A and their duals.**

Let  $n$  be a positive integer and let  $G$  be the group  $\mathrm{SL}_n$  acting on  $\mathbb{C}^n \oplus \mathbb{C}^n$  by

$$g \cdot (v, w) = (gv, g^{-T}w)$$

for all  $g \in \mathrm{SL}_n$  and  $v, w \in \mathbb{C}^n$ . Let the bilinear form be given by

$$\langle (v, w)|(x, y) \rangle = v^T y + x^T w.$$

for all  $v, w, x, y \in \mathbb{C}^n$ . This form is symmetric, non-degenerate and  $\mathrm{SL}_n$ -invariant. Let  $V_0$  be the subspace of  $\mathbb{C}^n \oplus \mathbb{C}^n$  spanned by  $(e_1, e_1)$ . Then  $\mathbb{C}^n \oplus \mathbb{C}^n$  is the orthogonal direct sum of  $V_0$  and  $\mathfrak{g} \cdot v$  for all non-zero  $v \in V_0$ .

**7.8. Direct sums of representations of groups of type A of highest weight  $2\lambda_1$  and their duals.**

Let  $n$  be a positive integer and let  $G$  be the group  $\mathrm{GL}_n$  acting on the vector space  $V$  of pairs of symmetric  $n \times n$  matrices by  $g \cdot (A, B) = (gAg^T, g^{-T}Bg^{-1})$  for all  $g \in \mathrm{GL}_n$  and  $(A, B) \in V$ . Let the bilinear form on  $V$  be given by

$$\langle (A, B)|(C, D) \rangle = \mathrm{Tr}(AD + BC)$$

for all symmetric matrices  $A, B, C, D \in \mathfrak{gl}_n$ . Let  $V_0$  be the subspace

$$\{(D, D)|D \in \mathfrak{gl}_n \text{ diagonal}\}$$

of  $V$ . Then for every invertible diagonal  $n \times n$  matrix  $D$  whose squares of diagonal entries are pairwise distinct, the vector space  $V$  is the orthogonal direct sum of  $V_0$  and

$$\mathfrak{g} \cdot (D, D) = \{(AD + DA^T, -A^T D - DA)|A \in \mathfrak{gl}_n\}.$$

**Remark 32.** The slice representation of  $(I_n, I_n)$  is isomorphic to the set of symmetric  $n \times n$  matrices acted on by  $O_n$  with conjugation.

**7.9. Direct sums of representations of groups of type A of highest weight  $\lambda_2$  and their duals.**

Let  $n$  be a positive integer and let  $G$  be the group  $\mathrm{GL}_n$  acting on the vector space  $V$  of pairs of skew-symmetric  $n \times n$  matrices by  $g \cdot (A, B) = (gAg^T, g^{-T}Bg^{-1})$  for all  $g \in \mathrm{GL}_n$  and  $(A, B) \in V$ . Let the bilinear form on  $V$  be given by

$$\langle (A, B)|(C, D) \rangle = \mathrm{Tr}(AD + BC)$$

for all skew-symmetric matrices  $A, B, C, D \in \mathfrak{gl}_n$ . Let  $V_0$  be the subspace

$$\left\{ \left( \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \right) \middle| E \in \mathfrak{gl}_{n/2} \text{ diagonal} \right\}$$

of  $V$  if  $n$  is even and the subspace

$$\left\{ \left( \begin{pmatrix} 0 & 0 & E \\ 0 & 0 & 0 \\ -E & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & E \\ 0 & 0 & 0 \\ -E & 0 & 0 \end{pmatrix} \right) \middle| E \in \mathfrak{gl}_{(n-1)/2} \text{ diagonal} \right\}$$

of  $V$  if  $n$  is odd. Then for every invertible diagonal  $\lfloor n/2 \rfloor \times \lfloor n/2 \rfloor$  matrix  $E$  whose squares of diagonal entries are pairwise distinct, the vector space  $V$  is the orthogonal direct sum of  $V_0$  and the tangent space at the corresponding element of  $V_0$  to its orbit.

**Remark 33.** Suppose  $n$  is even. Then the slice representation of

$$\begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}$$

is isomorphic to the representation  $\Lambda^2 \mathbb{C}^n$  of  $\mathrm{Sp}(n/2)$ .

### 7.10. Direct sums of standard representations of groups of type C and their duals.

Let  $n$  be a positive integer and let  $G$  be the group  $\mathrm{Sp}(n)$  acting on the vectorspace  $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$  with the form given by

$$\langle (v, w)|(x, y) \rangle = v^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} y + x^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} w$$

for all  $v, w, x, y \in \mathbb{C}^{2n}$ . Let  $V_0$  be the subspace of  $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$  spanned by some  $(v, w)$  with  $v_i, w_i \neq 0$  for all  $i$  and  $v_i w_j - v_j w_i \neq 0$  for all  $i \neq j$ . Then it follows from the following lemma that  $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$  is the orthogonal direct sum of  $V_0$  and the tangent space at any non-zero element  $V_0$  of its orbit.

**Lemma 34.** Let  $v, w, x, y \in \mathbb{C}^m$  be such that  $v_i, w_i \neq 0$  for all  $i$  and  $v_i w_j \neq v_j w_i$  for all  $i \neq j$ . Then  $v^T S y = x^T S w$  for all symmetric  $m \times m$  matrices  $S$  if and only if  $(x, y) = \lambda(v, w)$  for some  $\lambda \in \mathbb{C}$ .

**7.11. Tensor products of two direct sums of standard representations of groups of type A and their duals.** Let  $n \leq m$  be positive integers and let  $\mathrm{GL}_n \times \mathrm{GL}_m$  act on  $\mathbb{C}^{n \times m} \oplus \mathbb{C}^{n \times m}$  by

$$(g, h)(A, B) = (gAh^T, g^{-T}Bh^{-1})$$

for all  $g \in \mathrm{GL}_n, h \in \mathrm{GL}_m$  and  $A, B \in \mathbb{C}^{n \times m}$ . Let the bilinear form be given by

$$\langle (A, B)|(C, D) \rangle = \mathrm{Tr}(A^T D + C^T B)$$

for all  $A, B, C, D \in \mathbb{C}^{n \times m}$ . Let  $V_0$  be the subspace

$$\{((D \ 0), (D \ 0)) \mid D \in \mathfrak{gl}_n \text{ diagonal}\}$$

of  $V$ . Then for all invertible diagonal  $n \times n$  matrices  $D$  whose squares of diagonal entries are pairwise distinct, the vector space  $\mathbb{C}^{n \times m} \oplus \mathbb{C}^{n \times m}$  is the orthogonal direct sum of  $V_0$  and

$$\mathfrak{g}((D \ 0), (D \ 0)) = \left\{ ((AD \ 0) + (D \ 0)B^T, (-A^T D \ 0) - (D \ 0)B) \mid A \in \mathfrak{gl}_n, B \in \mathfrak{gl}_m \right\}.$$

**Remark 35.** The slice representation of the pair  $((I_n \ 0), (I_n \ 0))$  is a representation of  $\mathrm{GL}_n \times \mathrm{GL}_m$  where the second factor acts trivially. Ignoring this factor, we get the adjoint representation of  $\mathrm{GL}_n$ .



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