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ON SUBELLIPTIC MANIFOLDS

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Dedicated to Mikhail Zaidenberg on the occasion of his 70-th birthday

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ABSTRACT

A smooth complex quasi-affine algebraic variety Y is flexible if its special group $\mathrm{SAut}(Y)$ of automorphisms (generated by the elements of one-dimensional unipotent subgroups of $\mathrm{Aut}(Y)$) acts transitively on Y, and an algebraic variety is stably flexible if its product with some affine space is flexible. An irreducible algebraic variety X is locally stably flexible if it is a union of a finite number of Zariski open sets each of which is stably flexible. The main result of this paper states that the blowup of a locally stably flexible variety along a smooth algebraic subvariety (not necessarily equidimensional or connected) is subelliptic, and, therefore, it is an Oka manifold.

Introduction

The notion of a subelliptic manifold (i.e., a complex manifold which admits a dominating family of sprays) was introduced by Forstnerič in [7], inspired by hints from Gromov in [10]. It is a natural generalization of the stronger condition of admitting a single dominating spray, called elliptic. The importance of the notion of subellipticity is that, as in the case of elliptic manifolds, it implies all Oka properties. In other words, a subelliptic manifold X is an Oka manifold as proven by Forstnerič in [7]. In particular, being an Oka manifold implies that every holomorphic map from a convex domain K in $\mathbb{C}^n$ into X can be approximated (in the compact-open topology) by a holomorphic map from $\mathbb{C}^n$ to X. Needless to say that this leads to many remarkable consequences (e.g., see [6]). On the other hand, having the same consequences, subellipticity is easier to establish than ellipticity, which is exemplified by the main results of the present paper.

The simplest example of an elliptic manifold is, of course, the affine space $\mathbb{C}^n$ itself. Furthermore, Gromov proved ellipticity in the case of the complement of an algebraic subvariety of codimension at least 2 in $\mathbb{C}^n$. Any complex manifold which is locally isomorphic to such complements (resp. $\mathbb{C}^n$) is called a manifold of class $\mathcal{A}$ (resp. $\mathcal{A}_0$) ([6, Definition 6.4.5.], [15, Remark 3]). Since in the algebraic case subellipticity turns out to be a local property, we see that a manifold of class $\mathcal{A}$ is always subelliptic. Gromov observed also the following.

Proposition 0.1: *Let X be a complex manifold of class $\mathcal{A}_0$ and Y be the result of blowing X up at a finite number of points. Then Y is also a manifold of class $\mathcal{A}_0$ and, therefore, subelliptic.*

This fact yields, in particular, subellipticity of compact rational surfaces (see [6, Corollary 6.4.8]). In a more general setting it is natural to ask whether subellipticity survives under blowups of submanifolds with positive dimensions. The first result in this direction is due to Lárusson and the third author [15] who proved the following.

Theorem 0.2: *Let X be a smooth complex algebraic variety*[1] *of class $\mathcal{A}$ and $\pi : \tilde{X} \to X$ be the blowup of X along a smooth algebraic (not necessarily connected) subvariety of codimension at least 2. Then $\tilde{X}$ is subelliptic.*

The proof in [15] made good use of the fact that the automorphism group of $\mathbb{C}^n$ is rather big. This last property is shared by the so-called flexible manifolds, extensively studied in [1]. Recall that one of equivalent definitions states that a smooth complex quasi-affine algebraic variety X of dimension at least 2 is flexible if its special group $\mathrm{SAut}(X)$ of automorphisms (generated by the elements of one-dimensional unipotent subgroups of $\mathrm{Aut}(X)$) acts transitively on X. It is easy to establish that flexible varieties are algebraically subelliptic (and even algebraically elliptic). Furthermore, there is no need to discuss complements of subvarieties of codimension at least 2 in flexible varieties because such complements are again flexible ([5]). This observation was a strong indication to us that the above construction can survive replacement of affine spaces by flexible manifolds. This is indeed true, and, furthermore, we can actually prove it for a more general class of locally stably flexible manifolds which we are going to define next.

Definition 0.3: (a) A smooth quasi-affine algebraic variety Y is **stably flexible** if $Y \times \mathbb{C}^n$ is flexible for some $n \geq 0$.

(b) An irreducible algebraic variety X is **locally stably flexible** if it is a union

$$\bigcup X_i$$

of a finite number of Zariski open sets, each of which is stably flexible.

Example 0.4: It is worth mentioning that a stably flexible manifold is not necessarily flexible. Indeed, one can consider Danielewski's surfaces

$$D_n = \{(x, y, z) \in \mathbb{C}^3 : \ x^n y = p(z)\}$$

[1] All algebraic varieties that appear in this paper are considered over the field $\mathbb{C}$ of complex numbers, i.e., every smooth variety is automatically a complex manifold.

where p is a polynomial of degree at least 2 with simple roots. The fact that D_1 is flexible can be extracted from the explicit description of its automorphism group in [16]. However, for $n \geq 2$ the surface D_n is the complement of a simple normal crossing divisor in a smooth projective surface and the dual graph of this divisor is not contractible to a linear one. Hence it is not a Gizatullin surface (i.e., a surface which becomes flexible after removing at most a finite number of points [9]).[2] On the other hand, $D_n \times \mathbb{C}$ is isomorphic to $D_1 \times \mathbb{C}$ for any n (see [2], [4]) which shows that every D_n is stably flexible.

Our main result is the following.

Theorem 0.5: *Let X be a locally stably flexible variety. Suppose that $\pi : \tilde{X} \to X$ is the blowup of X along a smooth subvariety Z, not necessarily equi-dimensional or connected. Then $\tilde{X}$ is algebraically subelliptic and, hence, an Oka manifold.*

The paper is organized as follows. Section 1 contains the definitions of sprays and subellipticity and some simple facts which are immediate consequences of the results presented in [6]. In Section 2 we describe the technique developed for flexible varieties in [5], [1], and [13]. In Section 3 we prove certain facts which, in particular, include ellipticity of flexible varieties. With all preparations done we obtain our main theorems in Section 4.

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[2] Another way to prove it is to note that by [14, Theorem 9.1] the Makar-Limanov invariant of D_n is $\mathbb{C}[x]$, which implies that every fiber of the function x is invariant under the action of the special automorphism group.

1. Sprays and subellipticity

Let us introduce some definitions which can be found in [6, Definition 5.5.11].

Definition 1.1: (i) A holomorphic vector bundle $p : E \to X$ over a complex manifold X is called a **spray** if there exists a holomorphic map $s : E \to X$ such that for every point y in the zero section S of E one has $s(y) = p(y)$. That is, a spray is a triple (E, p, s).

(ii) A spray is called **dominating** if for every $y \in S$ one has

$$\mathrm{d}s(T_y p^{-1}(x)) = T_x X$$

where $x = p(y)$.

(iii) A family of sprays $\{E_i, p_i, s_i\}_{i=1}^m$ on X is called **dominating** if for every $x \in X$

$$\mathrm{d}s_1(T_{y_1} p_1^{-1}(x)) + \mathrm{d}s_2(T_{y_2} p_2^{-1}(x)) + \cdots + \mathrm{d}s_m(T_{y_m} p_m^{-1}(x)) = T_x X \tag{1}$$

where y_i, $i = 1, \ldots, m$ is a point in the zero section of $s_i : E_i \to X$ for which $p_i(y_i) = x$.

(iv) A complex manifold X is called **elliptic** (resp. **subelliptic**) if it admits a dominating holomorphic spray (resp. a dominating family of holomorphic sprays).

(v) We say that a spray (E, p, s) is of **rank** k if for a general point $y \in S$ the dimension of the vector space $\mathrm{d}s(T_y p^{-1}(x))$ is k (precaution: in general, for such a spray the rank of the vector bundle $p : E \to X$ may be greater than k).

(vi) A spray (E, p, s) is called **simple** if the vector bundle $p : E \to X$ is trivial.

Convention 1.2: Except for Remark 1.5 we consider below only algebraic sprays (E, p, s) on algebraic varieties, which means that the vector bundle $p : E \to X$ is algebraic and the map $s : E \to X$ is a morphism. Hence from now on we omit this adjective "algebraic".

Under this convention the following definition makes sense.

Definition 1.3: (a) Let X_0 be a nonempty Zariski open subset of a smooth (not necessarily quasi-projective) algebraic variety X. An algebraic vector bundle $p : E \to X_0$ is called a spray on X_0 **with values** in X if there exists a morphism $s : E \to X$ such that for every point y in the zero section S_0 of E one has $s(y) = p(y)$.

(b) Let $s' : \tilde{E} \to X$ be another spray on X_0 with values in X where $\tilde{p} : \tilde{E} \to X_0$ is a vector bundle. We say that it is **equivalent** to the spray s from (a) if there exist a nonempty Zariski open subset U of X_0 and a bundle isomorphism $\lambda : \tilde{p}^{-1}(U) \to p^{-1}(U)$ (over U) such that for every $y \in S_0$ with $x = p(y) \in U$ one has

$$s \circ \lambda|_{E_x} = s'|_{E_x}. \tag{2}$$

(c) The notion of a **dominating** spray on X_0 with values in X is described exactly as in Definition 1.1 with y running over a section S_0 of $p : E \to X_0$. In the same fashion we deal with a **dominating family** of sprays on X_0 with values in X. Similarly, a spray on X_0 with values in X is called **simple** if the bundle $p : E \to X_0$ is trivial.

Convention 1.2 enables us to use the following fact (in the proof of which we essentially copy the argument from [6, Proposition 6.4.2]).

Proposition 1.4: *Let $p : E \to X$ be a vector bundle, $p_0 : E_0 \to X_0$ be its restriction over X_0, and $s : E_0 \to X$ be a spray on X_0 with values X.*

(i) *Then there exists an equivalent spray which extends to a spray $s' : E \to X$ on X.*

(ii) *If $H = X \setminus X_0$ is a principal divisor in X, then the spray $s' : E \to X$ can be chosen so that Formula (2) holds for every $x \in X_0$.*

Proof. Consider first the case when H is a principal divisor, i.e., it is the zero locus of a regular function h on X. Let λ from Formula (2) be such that on each fiber E_x it is a homothety given by multiplication of every vector by $h(x)^n$ (where n is natural) and let s' be the spray from Formula (2). Consider an affine Zariski dense open subset X' of X and $X'_0 = X' \cap X_0$. Making X' smaller we can suppose that the restriction $E' \to X'$ (resp. $E'_0 \to X'_0$) of $p : E \to X$ is a trivial vector bundle.

Claim: *For a sufficiently large n the spray $s'|_{X'_0}$ can be extended to a spray on X' (with values in X).*

Consider a closed embedding $X' \hookrightarrow \mathbb{C}^N$ where the affine space is equipped with a coordinate system $z = (z_1, \ldots, z_N)$. Increasing the ambient dimension N, if necessary, we can suppose that $h|_{X'}$ is the restriction of z_1, i.e., X'_0 is closed in $\mathbb{C}^*_{z_1} \times \mathbb{C}^{N-1}$. Then the variety E' (resp. E'_0) is contained in $\mathbb{C}^N \times \mathbb{C}^M$

(resp. $\mathbb{C}^*_{z_1} \times \mathbb{C}^{N-1} \times \mathbb{C}^M$) where $\mathbb{C}^M \simeq E_x$ is equipped with a linear coordinate system u. That is, the zero section S' of $E' \to X'$ is contained in

$$F = \{u = \bar{0}\} \subset \mathbb{C}^N \times \mathbb{C}^M.$$

Then $s|_{E'_0}$ is the restriction of a rational map $\mathbb{C}^{N+M} \dashrightarrow \mathbb{C}^N$ whose coordinates are of the form $\frac{q_i(z,u)}{z_1^{m_i} r_i(z,u)}$, where $m_i \in \mathbb{Z}$ and $q_i(z,u)$ and $r_i(z,u)$ are relatively prime polynomials on $\mathbb{C}^{N+M}$not divisible by z_1. Since the restriction of $s|_{E'_0}$ to $S'_0 = S' \setminus \{z_1 = 0\}$ is the identity map, we see that every m_i is non-negative and we can suppose that q_i is of the form $z_i z_1^{m_i} r_i(z,\bar{0}) + \hat{q}_i(z,u)$ where $\hat{q}_i(z,\bar{0}) = 0$. Furthermore, $s|_{E'_0}$ is regular in a neighborhood of S'_0 in X'_0, which implies that $R = \bigcup_{i=1}^N R_i$ where R_i, the zero locus of $r_i|_{X'_0}$, does not meet S'_0. In fact, we can suppose that each r_i does not vanish on $F_0 = F \setminus \{z_1 = 0\}$. Indeed, because R_i and F_0 are closed in $\mathbb{C}^*_{z_1} \times \mathbb{C}^{N-1}$ and disjoint one can choose a regular function $\tilde{r}_i$ on $\mathbb{C}^*_{z_1} \times \mathbb{C}^{N-1}$ which vanishes on R_i with the same multiplicities as r_i and equal to 1 on F_0. Multiplying $\tilde{r}_i$ by a power of z_1 we can make it regular on $\mathbb{C}^{N+M}$ and use it as r_i. This implies that r_i does not vanish on F since otherwise it has zeros on F_0 (i.e., $r_i(z,\bar{0}) = r_i|_F$ is a nonzero constant).

Note that $s'|_{X'_0}$ is the restriction of the rational map $\mathbb{C}^{N+M} \dashrightarrow \mathbb{C}^N$ whose coordinates are of the form $\frac{q_i(z,z_1^n u)}{z_1^{m_i} r_i(z,z_1^n u)}$. Choosing $n > \max_i(m_i)$ we see that such a function coincides with

$$\frac{z_i r_i(z,\bar{0}) + z_1^{n-m_i} \breve{q}_i(z,u)}{r_i(z,z_1^n u)}$$

where the polynomial $\breve{q}_i(z,u)$ is equal to $z_1^{-n} \hat{q}_i(z,z_1^n u)$. Since $r_i(z,\bar{0})$ does not vanish, this implies that the rational map is regular in a neighborhood of the hyperplane $z_1 = 0$ and its restriction to this hyperplane is given by $(z,u) \to z$. Hence $s'|_{X'}$ is regular. Since X can be covered by a finite number of sets like X' we obtain the Claim.

Cover X by a finite number of open sets as X' before and choose n such that the Claim is true for each of these sets. Then we get a spray $s' : E \to X$ on X with $s'|_{X_0}$ equivalent to s and Formula (2) valid for every $x \in X_0$, i.e., we have the second statement.

For the first statement we consider a Cartier divisor F containing $X \setminus X_0$. Let $\{U_i\}$ be a cover of X by Zariski open sets such that in each U_i the divisor $F \cap U_i$ is given by the zero locus of a regular function f_i with

$$g_{i,j} = f_j f_i^{-1}$$

being invertible on $U_i \cap U_j$. Let $\tilde{p} : \tilde{E} \to X$ be the trivial vector bundle extending $p : E \to X_0$ and let s_i be the restriction of our spray to $\tilde{p}^{-1}(U_i \setminus F)$. Then by the second statement the homothety with coefficient f_i^n generates a spray s_i' with values in X extendable to $\tilde{p}^{-1}(U_i)$. Replacing the bundle $\tilde{p} : \tilde{E} \to X$ by its tensor product $\hat{E} \to X$ with the line bundle with transition function $\{g_{i,j}\}$ we see that s_i' and s_j' agree over $U_i \cap U_j$. That is, we obtain a morphism $s' : \hat{E} \to X$ which yields the desired spray in (i). ∎

Remark 1.5: It is worth mentioning that Proposition 1.4 can be extended to a more general setting (which will not be used later). Recall that an étale covering of a smooth algebraic variety X is a family of étale morphisms $\varphi_i : U_i \to X$, $i = 1, \ldots, m$ such that $\bigcup_{i=1}^m \varphi_i(U_i) = X$. Suppose that $p : E \to X$ is a holomorphic vector bundle and for every $i = 1, \ldots, m$ there is a trivial algebraic bundle $p_i : E_i \to U_i$ with a holomorphic vector bundle map $\psi_i : E_i \to E$ over φ_i such that it maps each fiber of p_i isomorphically onto the corresponding fiber of p. We let $U_{ij} = U_i \times_X U_j$ and denote by $E_{ij}^i \to U_{ij}$ the lift of $p_i : E_i \to U_i$ to U_{ij}. Suppose that for every pair $1 \leq i \neq j \leq m$ the transition function between the trivial bundles E_{ij}^i and E_{ij}^j over U_{ij} is algebraic (i.e., $p : E \to X$ is an étale bundle).

Let X_0 be a Zariski dense open subset of X, $U_i^0 = \varphi_i^{-1}(X_0)$, $p_i^0 : E_i^0 \to U_i^0$ be the restriction of $p_i : E_i \to U_i$, and $\psi_i^0 : E_i^0 \to E$ be the restriction of ψ_i. Given a holomorphic spray $s : E|_{X_0} \to X$ with values in X we note that for every i it defines a holomorphic map from a neighborhood of the zero section of $E_i^0 \to U_i^0$ into U_i. Hence we call this spray étale if there are rational maps $\tilde{s}_i : E_i^0 \dashrightarrow U_i$ for which the following diagrams are commutative:

$$\begin{array}{ccc} E_i^0 & \overset{\tilde{s}_i}{\dashrightarrow} & U_i^0 \\ \Big\downarrow{\scriptstyle \psi_i^0} & & \Big\downarrow{\scriptstyle \varphi_i} \\ E & \xrightarrow{\ s\ } & X \end{array}$$

Consider, say, the case when $X \setminus X_0$ is the zero locus of a regular function h. Taking a composition of s with a homothety in the fibers of the bundle E given by multiplication by h^n, one obtains an equivalent spray s' on X_0 and rational maps $\tilde{s}' : E_i^0 \dashrightarrow U_i^0$ for which $\varphi_i \circ \tilde{s}_i' = s' \circ \psi_i^0$. Without loss of generality we can assume that each U_i is affine. Then the proof of the Claim in Proposition 1.4 implies that for sufficiently large n the map $\tilde{s}_i'$ extends to rational map $\breve{s}'_i : E_i \dashrightarrow U_i$ such that $\breve{s}'_i$ is regular in a neighborhood of the zero section of $p_i : E_i \to U_i$ and in a neighborhood of $E_i \setminus E_i^0$. Hence these maps $\breve{s}'_i$ can be pushed down to an étale spray $\breve{s}' : E \to X$ extending s' which, therefore, satisfies Formula (2) for every $x \in X_0$.

Corollary 1.6: *Let $\{U_i\}_i$ be a cover of a smooth algebraic variety X by Zariski open sets such that for every i there is a dominating family of simple sprays on U_i with values in X. Then there is a dominating family of sprays on X.*

Corollary 1.7: *Let $s : E \to X$ be a simple spray on X as in Definition 1.1 and let $\varphi : X \to Y$ be a birational morphism which yields an isomorphism between Zariski open subsets $X_0 \subset X$ and $Y_0 \subset Y$, i.e., one has the following commutative diagram:*

$$\begin{array}{ccc} E|_{X_0} & \xrightarrow{\psi} & F \\ \Big\downarrow{\scriptstyle p|_{X_0}} & & \Big\downarrow{\scriptstyle q} \\ X_0 & \xrightarrow{\varphi|_{X_0}} & Y_0 \end{array}$$

of isomorphic vector bundles. Then $r = \varphi \circ s \circ \psi^{-1} : F \to Y$ is a simple spray on Y_0 with values in Y. Furthermore, if $Y \setminus Y_0$ is a principal divisor in Y, then there is an equivalent spray $r' : F \to Y$ extendable to a spray on Y and such that $r(q^{-1}(y)) = r'(q^{-1}(y))$ for every $y \in Y_0$.

Corollary 1.8: *Let (E, p, s) be a spray of rank k on a smooth algebraic variety X and $\varphi : X \to Y$ be a birational morphism to a smooth algebraic variety Y. Then Y admits a spray of rank k.*

Proof. There are nonempty Zariski open affine subsets

$$X_0 \subset X \quad \text{and} \quad Y_0 = \varphi(X_0) \subset Y$$

such that the restriction $\varphi|_{X_0} : X_0 \to Y_0$ is an isomorphism. Taking a smaller X_0 we can suppose that the restriction $p_0 : E_0 \to X_0$ of the bundle $p : E \to X$ is trivial. Consider the bundle $F \to Y_0$ induced by $p_0 : E_0 \to X_0$. By Corollary 1.7 we get a spray $r' : F \to Y_0$ of rank k extendable to a spray on Y which yields the desired conclusion. ∎

The following descent property of subellipticity will be important below.[3]

[3] The authors are grateful to Finnur Lárusson for suggesting this argument.

Proposition 1.9: *Let $\hat{X}$ be a smooth algebraic variety and $X = \hat{X} \times \mathbb{C}^n$ for some $n \geq 0$. Suppose that X is subelliptic. Then so is $\hat{X}$.*

Proof. Let $\{E_i, p_i, s_i\}_{i=1}^m$ be a family of dominating sprays on X and let o be the origin of $\mathbb{C}^n$. Suppose that $\hat{p}_i : \hat{E}_i \to \hat{X}$ is the restriction of the bundle $p_i : E_i \to X$ to $\hat{X} \times o \simeq \hat{X}$. Then we have the morphism $s_i|_{\hat{E}_i} : \hat{E}_i \to X$. Consider the composition $\hat{s}_i : \hat{E}_i \to \hat{X}$ of this morphism with the natural projection $\tau : \hat{X} \times \mathbb{C}^n \to \hat{X}$. By construction $(\hat{E}_i, \hat{p}_i, \hat{s}_i)$ is a spray on $\hat{X}$. Furthermore, applying $\tau_* : TX \to T\hat{X}$ to Formula (1) we see that $\{\hat{E}_i, \hat{p}_i, \hat{s}_i\}_{i=1}^m$ is a dominating family of sprays which yields the desired conclusion. ∎

2. Flexible varieties

Recall the following facts which can be found in [1].

Definition 2.1: (1) A nontrivial derivation σ on the ring A of regular functions on a quasi-affine algebraic variety X is called **locally nilpotent** if for every $0 \neq a \in A$ there exists a natural n for which $\sigma^n(a) = 0$. For the smallest n with this property one defines the **degree** of a with respect to σ as $\deg_\sigma a = n - 1$. This derivation can be viewed as a vector field on X which we also call locally nilpotent. The phase flow of this vector field is an algebraic G_a-action on X, i.e., the action of the group $\mathbb{C}_+$ of complex numbers with respect to addition which can be viewed as a one-parameter unipotent group U in the group $\mathrm{Aut}(X)$ of all algebraic automorphisms of X. In fact, every G_a-action is generated by a locally nilpotent vector field (e.g, see [8]).

(2) A smooth quasi-affine variety X of dimension at least 2 is called **flexible** if for every $x \in X$ the tangent space T_xX is spanned by the tangent vectors to the orbits of one-parameter unipotent subgroups of $\mathrm{Aut}(X)$ through x.

(3) The subgroup $\mathrm{SAut}(X)$ of $\mathrm{Aut}\, X$ generated by all one-parameter unipotent subgroups is called special.

The next result provides equivalent definitions of flexibility [1], [5].

Theorem 2.2: *For every smooth irreducible quasi-affine algebraic variety X the following are equivalent:*

(i) *the special subgroup* $\mathrm{SAut}(X)$ *acts transitively on* X*;*
(ii) *the special subgroup* $\mathrm{SAut}(X)$ *acts infinitely transitively on* X *(i.e., for every natural* m *the action is* m*-transitive);*
(iii) X *is flexible.*

Remark 2.3: Suppose that G is a subgroup of $\mathrm{SAut}(X)$ generated by elements of the flows of locally nilpotent vector fields from a set $\mathcal{N}$ with the following property: for every σ in $\mathcal{N}$ and $f \in \mathrm{Ker}\,\sigma$ the field $f\sigma$ is also in $\mathcal{N}$. If one replaces $\mathrm{SAut}(X)$ in the formulation of Theorem 2.2 by such a group G, then the modified conditions (i)–(iii) remain equivalent [5, Theorem 2.12]. When any of these conditions holds we say that X is G-flexible.

THEOREM 2.4 ([1, Theorem 4.14 and Remark 4.16]): *Let $x_1, \ldots, x_m$ be distinct points in a G-flexible manifold X of $\dim X = n$ where G is as in Remark 2.3. Then there exists an automorphism $\alpha \in G$ such that it fixes points $x_1, \ldots, x_m$ and for every i the linear map $\mathrm{d}\alpha|_{T_{x_i}X}$ coincides with a prescribed element β_i of $\mathbf{SL}_n$.*

By the Rosenlicht Theorem (see [18, Theorem 2.3]) for X, A, and U as in Definition 2.1 one can find a finite set of U-invariant functions $a_1, \ldots, a_m \in A$ which separate general U-orbits in X. They generate a morphism $\varrho : X \to Q$ into an affine algebraic variety Q. Note that this set of invariant functions can be chosen so that Q is normal (since X is normal). The following notion borrowed from [5] will be used in the proof of Proposition 3.4 below.

Definition 2.5: Let X, A, U, and $\varrho : X \to Q$ be as before and let Q be normal. Then this morphism ϱ will be called a **partial quotient**. In the case when $a_1, \ldots, a_m$ generate the subring A^U of U invariant elements of A such a morphism is called the **categorical quotient**.[4]

Recall also the following notion introduced by Ramanujam [19].

Definition 2.6: Given irreducible algebraic varieties X and $\mathcal{A}$ and a map $\varphi : \mathcal{A} \to \mathrm{Aut}(X)$, we say that $(\mathcal{A}, \varphi)$ is an **algebraic family of automorphisms** on X if the induced map $\mathcal{A} \times X \to X$, $(\alpha, x) \mapsto \varphi(\alpha).x$, is a morphism.

The next fact is a special case of [13, Theorem 6.1 and Remark 6.8].

THEOREM 2.7: *Let $\varrho : X \to Q$ be a partial quotient morphism from a flexible algebraic variety X. Then there exists a connected family of algebraic automorphisms $\mathcal{A}$ of X such that for every locally closed reduced subvariety Z of X of codimension at least 2 and any smooth point z_0 of Z the following holds:*

[4] However, in general A^U is not finitely generated by the Nagata's example. That is why, following [5], we prefer to work with partial quotients.

for a general element $\alpha \in \mathcal{A}$ one can find a Zariski neighborhood V_0' of the point $\varrho(\alpha(z_0))$ in $\overline{\varrho \circ \alpha(Z)}$ (where the variety $\overline{\varrho \circ \alpha(Z)}$ is the Zariski closure of $\varrho \circ \alpha(Z)$ in Q) such that for $V_0 = \varrho^{-1}(V_0') \cap \alpha(Z)$ the map $\varrho|_{V_0} : V_0 \to V_0'$ is an isomorphism.

Corollary 2.8: *Let Y be a flexible variety and $X = Y \times \mathbb{C}^k_{u_1,\dots,u_k}$ (where $k \geq 2$ and by $\mathbb{C}^k_{u_1,\dots,u_k}$ we denote the affine space $\mathbb{C}^k$ equipped with a coordinate system $(u_1,\dots,u_k)$). Suppose that Z is a locally closed subvariety of X of pure dimension $\dim Y$ and $\varrho : X \to Y \times \mathbb{C}_{u_1}$ is the natural projection. Let z_0 be a smooth point of Z. Then there exists a connected family of algebraic automorphisms $\mathcal{A}$ of X such that for a general element $\alpha \in \mathcal{A}$ and a Zariski neighborhood $Z_0'' = Z_0''(\alpha)$ of $z_0'' = \varrho(\alpha(z_0))$ in $\overline{\varrho \circ \alpha(Z)}$ the restriction of ϱ to $\varrho^{-1}(Z_0'') \cap \alpha(Z)$ yields an isomorphism $\varrho^{-1}(Z_0'') \cap \alpha(Z) \to Z_0''$.*

Proof. Let us use induction by k. Suppose that $X' = Y \times \mathbb{C}^{k-1}_{u_1,\dots,u_{k-1}}$ and $\tau : X \to X'$ is the natural projection. Note that τ is the quotient morphism of the locally nilpotent vector field $\partial/\partial u_k$. By Theorem 2.7 there exists a connected family $\mathcal{A}$ of automorphisms of X such that for a general $\alpha \in \mathcal{A}$ and a neighborhood Z_0' of $z_0' = \tau(z_0)$ in $Z' = \overline{\tau(Z)}$ the restriction of τ to $\tau^{-1}(Z_0') \cap \alpha(Z)$ yields an isomorphism $\tau^{-1}(Z_0') \cap \alpha(Z) \to Z_0'$. Since $\varrho = \tau$ for $k = 2$ we get the first step of induction.

Assume now that the statement is true for $k-1$ and $\varrho' : X' \to Y \times \mathbb{C}_{u_1}$ is the natural projection. That is, there exists a connected family of algebraic automorphisms $\mathcal{A}'$ of X' such that for a general element $\alpha' \in \mathcal{A}'$ and a neighborhood Z_0'' of $z_0'' = \varrho'(z_0')$ in $Z'' = \overline{\varrho'(Z')}$ the restriction of ϱ' to $Z_0' = (\varrho')^{-1}(Z_0'') \cap \alpha'(Z')$ yields an isomorphism $Z_0' \to Z_0''$.

Note now that every automorphism α' of X' has a natural lift $\tilde{\alpha}'$ to X such that $\tau \circ \tilde{\alpha}' = \alpha' \circ \tau$. Consider the algebraic family $\tilde{\mathcal{A}}'$ of automorphisms of X that consists of such lifts for elements of $\mathcal{A}'$. Replace $\mathcal{A}$ by $\tilde{\mathcal{A}}' \cdot \mathcal{A}$. Since $\varrho = \varrho' \circ \tau$ this yields the desired family of automorphisms. ∎

Remark 2.9: The family $\mathcal{A}$ in Theorem 2.7 is a Zariski dense open subset in another family of automorphisms that contains the identity automorphism (see [13, Remark 6.8]). It follows from the proof that the same is true for the family of automorphisms in Corollary 2.8.

We need also the following technical fact.

Lemma 2.10: *Let the assumption of Corollary 2.8 hold and Z be irreducible. Suppose that $\tau : Y \times \mathbb{C} \to Y$ is the natural projection. There there exists an automorphism λ of X such that replacing Z by $\lambda(Z)$ one can suppose that for a general $\alpha \in \mathcal{A}$ the conclusion of Corollary 2.8 holds and the restriction of τ to $\varrho(\alpha(Z))$ is étale at $\varrho \circ \alpha(z_0)$.*

Proof. By [11, Chap. III, Corollary 10.7] one can choose a Zariski dense open subset Q_0 of $Y \times \mathbb{C}$ so that the morphism $\varrho|_{X_0} : X_0 \to Q_0$ is smooth where $X_0 = \tau^{-1}(Q_0)$. Consider a general point q_0 in Q_0, $y_0 = \tau(q_0)$, and a general point $x_0 \in X_0$ for which $\varrho(x_0) = q_0$. Let $(v_1, \ldots, v_n)$ be a local coordinate system at $y_0 \in Y$, i.e., $(v_1, \ldots, v_n, u_1)$ (resp. $(v_1, \ldots, v_n, u_1, \ldots, u_k)$) is a local coordinate system at $q_0 \in Q_0$ (resp. at $x_0 \in X$). By Theorem 2.2 one can choose an automorphism λ of X such that $x_0 = \lambda(z_0)$. Furthermore, by Theorem 2.4 one can suppose that $\lambda(Z)$ is tangent to the subvariety $u_1 = \cdots = u_k = 0$ in a neighborhood of x_0. Replace Z by $\lambda(Z)$. By construction, $\tau|_{\varrho(Z)}$ is étale at q_0.

By Remark 2.9 we can choose a general $\alpha \in \mathcal{A}$ from Theorem 2.7 as close to the identity automorphism as we wish. Then $\varrho \circ \alpha(z_0)$ remains a general point of Q_0 with the restriction of τ to $\varrho(\alpha(Z))$ being étale at $\varrho \circ \alpha(z_0)$ which yields the desired conclusion. ∎

3. First facts about sprays on flexible varieties

Theorem 3.1: *Every flexible variety X is elliptic and it admits a simple dominating spray.*

Proof. Let $x \in X$. By Definition 2.1 we can find locally nilpotent vector fields $\sigma_1, \ldots, \sigma_n$ for which $\sigma_{1,x}, \ldots, \sigma_{n,x}$ is a basis in T_xX. This implies that there is an open Zariski dense subset W of X so that for every point $y \in W$ the vectors $\sigma_{1,y}, \ldots, \sigma_{n,y}$ form a basis in T_yX. Note that $\dim(X \setminus W) = m \leq n-1$. Finding locally nilpotent vector fields that form a basis at general points of each of the components of $X \setminus W$, we can extend our sequence of vector fields to $\sigma_1, \ldots, \sigma_n, \sigma_{n+1}, \ldots, \sigma_l$ such that there is a Zariski open set $V \supset W$ for which $\dim(X \setminus V) < m$ and such that for every $y \in V$ these fields generate T_yX. Thus, using induction by dimension we can suppose that $\sigma_1, \ldots, \sigma_n, \sigma_{n+1}, \ldots, \sigma_l$ generate T_yX at every $y \in X$.

Let U^i be the one-parameter group of algebraic automorphisms associated with σ_i and let U^i_t be the element of this group for the value of the time parameter t. Consider the trivial vector bundle $\pi : E \to X$ of rank l, i.e., for every $x \in X$ the fiber $E_x = \pi^{-1}(x)$ is isomorphic to $\mathbb{C}^l$ with coordinates $(t_1, \dots, t_l)$.

Define the morphism $s : E \to X$ by the formula

$$(x, t_1, \dots, t_l) \to U^1_{t_1} \circ \cdots \circ U^l_{t_l}(x).$$

By construction, s is a simple dominating spray and we are done. ∎

By Corollary 1.6 we have the following.

Corollary 3.2: *Every locally flexible algebraic variety is subelliptic.*

Notation 3.3: Let $\pi : \tilde{X} \to X$ be the blowup of a quasi-affine variety X along a closed smooth algebraic subvariety Z of codimension $k + 1 \geq 2$. Suppose that F is the exceptional divisor of π, i.e.,

$$\pi|_F : F \to Z$$

is a locally trivial fibration with fiber $\mathbb{P}^k$. Note that every algebraic vector field σ on X, that vanishes on Z, can be lifted to an algebraic vector field $\tilde{\sigma}$ on $\tilde{X}$.[5] Furthermore, since

$$\mathbb{C}[\tilde{X}] = \mathbb{C}[X]$$

the vector field $\tilde{\sigma}$ is locally nilpotent provided σ is.

Proposition 3.4: *Let Notation 3.3 hold. For every $z \in Z$, $\tilde{x} \in \pi^{-1}(z)$, and a nonzero vector $w \in T_{\tilde{x}}\tilde{X}$ tangent to $\pi^{-1}(z)$ there exists a locally nilpotent vector field δ on $\tilde{X}$ for which its value $\delta_{\tilde{x}}$ at $\tilde{x}$ coincides with w. Furthermore, the subgroup of automorphisms of $\tilde{X}$ preserving $\pi^{-1}(z)$ acts transitively on $\pi^{-1}(z)$.*

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[5] Indeed, for every point of Z one can find its standard neigborhood U in X with an analytic coordinate system $(u_1, \dots, u_n)$ such that $Z \cap U$ is given by the equations $u_1 = \cdots = u_k = 0$. Then on an open dense subset V of $\pi^{-1}(U)$ we have the coordinate system $(v_1, \dots, v_n)$ such that $u_i = v_k v_i$ for $i \leq k-1$ and $u_j = v_j$ for $j \geq k$ (in particular, $V \cap F$ is given by $v_k = 0$). The Chain Rule implies that $\frac{\partial}{\partial u_i} = \frac{1}{v_k}\frac{\partial}{\partial v_i}$ for $i \leq k-1$, $\frac{\partial}{\partial u_j} = \frac{\partial}{\partial v_j}$ for $j \geq k+1$, and $\frac{\partial}{\partial u_k} = \frac{\partial}{\partial v_k} - \frac{v_i}{v_k}\frac{\partial}{\partial v_i}$. Note that $\sigma|_U = \sum_{i=1}^n b_i \frac{\partial}{\partial u_i}$ where b_i is a holomorphic function on U that vanishes on Z. Hence the function $\tilde{b}_i = b_i \circ \pi$ vanishes on F and thus its restriction to V is divisible by v_k. Since $\tilde{\sigma} = \sum_{i=1}^{k-1} \tilde{b}_i(\frac{1}{v_k}\frac{\partial}{\partial v_i}) + \sum_{i=k+1}^n \tilde{b}_i \frac{\partial}{\partial v_i} + \tilde{b}_k(\frac{\partial}{\partial v_k} - \frac{v_i}{v_k}\frac{\partial}{\partial v_i})$ we see that this field is regular on V which yields the desired conclusion.

Proof. Let $\varrho : X \to Q$ be a partial quotient associated with a nonzero locally nilpotent σ, x be a general point in X and $q = \varrho(x)$ a general point in Q. In particular, these points are smooth and one can choose local analytic coordinate systems at them. By Theorem 2.4 one can also choose an automorphism α which sends z to x such that in local analytic coordinate systems $(v_0, \dots, v_{n-1})$ (resp. $(u_0, \dots, u_n)$) at $q \in Q$ (resp. $x \in X$) one has $\alpha(Z)$ given by $u_{n-k} = \cdots = u_n = 0$ and $\varrho^*(v_j) = u_j$ for $j \leq n-1$. We replace Z by $\alpha(Z)$ and z by $\alpha(z)$ to make the argument local. In particular,

$$\pi^{-1}(z) \simeq \mathbb{P}^k$$

has homogeneous coordinates $U_{n-k} : U_{n-k+1} : \cdots : U_n$ such that $u_i U_j = u_j U_i$ for $n-k \leq i, j \leq n$. Without loss of generality consider the case when the vector w in $T_z \pi^{-1}(z)$ is tangent to the line $L \subset \mathbb{P}^k$ that consists of points with fixed ratios $U_{n-k+1} : \cdots : U_{n-1} : U_n$ where $U_n \neq 0$. Note that at the origin x_0 of the local coordinate system σ_{x_0} is proportional to the vector $\partial / \partial u_n$, i.e., we can assume that

$$\sigma_{x_0} = \partial / \partial u_n.$$

We can always suppose that v_{n-k} is the restriction of a regular function on Q, and, therefore, $u_{n-k} = \varrho^*(v_{n-k})$ can be treated as a regular function on X. Since $u_{n-k} \in \operatorname{Ker} \sigma$ we see that $u_{n-k}\sigma$ is also locally nilpotent. Denote by Φ the automorphism

$$\Phi = \exp(t u_{n-k} \sigma)$$

of X for some value of parameter $t \in \mathbb{C}$. By [1, Lemma 4.1] we have

$$\mathrm{d}_{x_0}\Phi(\nu) = \nu + t \mathrm{d} u_{n-k}(\nu) \partial / \partial u_n \tag{3}$$

for every $\nu \in T_{x_0} X$. Since $u_{n-k}\sigma$ vanishes on Z it can be lifted as a locally nilpotent derivation δ on $\tilde{X}$. Furthermore, Formula (3) shows that the elements of the flow of δ preserve $\pi^{-1}(z) \simeq \mathbb{P}^k$ and act on it as elementary transformations of form

$$(U_{n-k} : U_{n-k+1} : \cdots : U_n) \to ((U_{n-k} + tU_n) : U_{n-k+1} : \cdots : U_n).$$

That is, the action induced by δ is a translation along the affine line

$$\mathbb{C} \simeq L \setminus \{U_{n-k} = \infty\}$$

which yields the first statement. The fact that elementary transformations generate a special linear group implies the second statement and we are done. ∎

4. Main theorems

Now we are prepared for our main results.

Proposition 4.1: *Let Y be a flexible (and, therefore, smooth quasi-affine) variety and V be a smooth connected closed subvariety of Y of codimension at least 2. Let $k = \dim Y - \dim V$, $X = Y \times \mathbb{C}^k$, and $Z = V \times \mathbb{C}^k$. Suppose that $\pi : \tilde{X} \to X$ is the blowup of X along Z. Then $\tilde{X}$ is subelliptic.*

Proof. Choose any point $z_0 \in Z$ and a point $w_0 \in \pi^{-1}(z_0)$. We need to construct a family of sprays on $\tilde{X}$ of rank 1 that is dominating at w_0. Let $(u_1, \ldots, u_k)$ be a coordinate system on $\mathbb{C}^k$ and $Q = Y \times \mathbb{C}_{u_1}$. Consider the natural projection $\varrho : X \to Q$. By Corollary 2.8 we can find an automorphism α of X such that replacing Z by $\alpha(Z)$ we have the following: for a Zariski neighborhood Z_0' of $z_0' = \varrho(z_0)$ in the closure $Z' = \overline{\varrho(Z)}$ of $\varrho(Z)$ in Q, the restriction of ϱ to $Z_0 = \varrho^{-1}(Z_0') \cap Z$ yields an isomorphism $Z_0 \to Z_0'$. We can suppose that $Z_0' = Q_0 \cap Z'$ for a Zariski open affine subset Q_0 of Q and taking a smaller Q_0, if necessary, we can suppose that Z_0' is a smooth principal divisor in Q_0 given by the zero locus of a regular function $f \in \mathbb{C}[Q_0]$. Furthermore, by Lemma 2.10 the restriction of the natural projection $\theta : Q \to Y$ to Z_0' is a local embedding at z_0'.

Let $X_0 \simeq Q_0 \times \mathbb{C}^k$. Note that the isomorphism $Z_0 \to Z_0'$ implies that

$$Z_0 \subset Z_0' \times \mathbb{C}^k$$

is given by equations $u_2 = g_2, \ldots, u_k = g_k$ where each g_i is a regular function on Z_0'. Without loss of generality we suppose that Q_0 is affine and hence we can extend $g_2, \ldots, g_k$ to regular functions on Q_0 denoted by the same symbols. Consider the automorphism of X_0 over Q_0 given by

$$(u_2, \ldots, u_k) \to (u_2 - g_2, \ldots, u_k - g_k).$$

Observe that up to this automorphism the variety Z_0 can be treated as a strict complete intersection given by $f = u_2 = \cdots = u_k = 0$ and the locally nilpotent vector field $\delta = \partial/\partial u_1$ on X is not tangent to Z_0' at z_0' since $\theta|_{Z_0'}$ is étale at z_0'.

Let X_0' be the subvariety in $X_0 \times \mathbb{C}^{k-1}_{v_2,\ldots,v_k}$ given by the equations

$$v_2 f = u_2, \ldots, v_k f = u_k,$$

i.e., X_0' can be viewed as a Zariski open subset of $\tilde{X}$. By the Asanuma trick there is a natural isomorphism $\varphi : X_0' \to X_0$ over Q such that $\varphi^*(u_i) = v_i$

for $i \geq 2$ (e.g., see [12, Proposition 7.5]). Note that $f \circ \pi$ yields a regular function on X_0' whose zero locus may be viewed as a Zariski open subset W of $\pi^{-1}(Z)$ (and, moreover, this locus contains a Zariski open subset of $\pi^{-1}(z)$ for every $z \in Z_0$). By Proposition 3.4 the phase flow of a complete vector field on $\tilde{X}$ can move w_0 in a general position in $\pi^{-1}(z_0)$. Hence we can suppose that $w_0 \in W$. Since $X_0' \simeq X_0$ the field δ has a lift to a locally nilpotent vector field δ' on X_0'. The value δ'_{w_0} of δ' at w_0 is uniquely determined by the value $\delta_{z_0'}$ of δ at z_0. Since the latter is not tangent to Z_0 the field δ' is transversal to $\pi^{-1}(Z)$ at w_0.

By Proposition 1.4, δ' extends to a spray of rank 1 on $\tilde{X}$ and the only thing we have to show is that δ can be chosen so that δ'_{w_0} is a general vector.

This follows from Theorem 2.4 which provides us with an automorphism α of X such that $\alpha(z_0) = z_0$ and $\alpha_*(\delta_{z_0})$ is a general vector. This yields the desired conclusion in the case of an irreducible V. ∎

Theorem 4.2: *Let X be a stably flexible variety and Z be a smooth closed (not necessarily connected or pure-dimensional) subvariety of X of codimension at least 2. Suppose that $\pi : \tilde{X} \to X$ is the blowup of X along Z. Then $\tilde{X}$ is subelliptic.*

Proof. By Proposition 1.9 it suffices to consider the case when X is flexible. Let Z be a union of connected components $Z_1, \ldots, Z_m$. Choose

$$k = \max_i(\dim X - \dim Z_i)$$

and let $X' = X \times \mathbb{C}^k$, $Z' = Z \times \mathbb{C}^k$ and $Z_i' = Z_i \times \mathbb{C}^k$. Suppose that

$$X_i' = X' \setminus \bigcup_{j \neq i} Z_j'.$$

Note that X_i' is flexible by the main result of [5]. Let $\theta : \tilde{X}' \to X'$ be the blowing up along Z' and $\theta_i : \tilde{X}_i' \to X_i'$ be the blowing up along Z_i'. Then $\tilde{X}_i'$ is subelliptic by Proposition 4.1. Note also that $\tilde{X}_i'$ can be viewed as a Zariski open subset of $\tilde{X}'$ and

$$\bigcup_{i=1}^{m} \tilde{X}_i' = \tilde{X}'.$$

By Corollary 1.6 $\tilde{X}'$ is subelliptic. Since by construction $\tilde{X}' \simeq \tilde{X} \times \mathbb{C}^k$, the desired conclusion follows from Proposition 1.9. ∎

Applying Corollary 1.6 again we get the main result.

Corollary 4.3: *Let X be a locally stably flexible variety and $\pi : \tilde{X} \to X$ be the blowing up of X along a closed smooth algebraic (but not necessarily connected or pure-dimensional) submanifold of X with codimension at least 2. Then $\tilde{X}$ is subelliptic.*

It is interesting to ask whether the blowing up of $\tilde{X}$ in Corollary 4.3 preserves subellipticity. In a more general setting we can ask the following.

Question: Let $\tau : \hat{X} \to X$ be a proper birational morphism of smooth complex algebraic varieties such that X is subelliptic (say, locally stably flexible). Does it imply that $\hat{X}$ is subelliptic?

In the case of a locally stably flexible X the answer will be positive if the next question can be answered affirmatively.

Question: Let $\tau : \hat{X} \to X$ be a proper birational morphism of smooth complex algebraic varieties such that X is locally stably flexible. Does it imply that $\hat{X}$ is locally stably flexible?

In connection with this question it is worth mentioning that manifolds of class $\mathcal{A}$ from Theorem 0.2 are locally flexible but the converse statement is not true.

Example 4.4: (1) Note that $X = \mathbf{SL}_n$ is not contained in class $\mathcal{A}_0$ (or $\mathcal{A}$), i.e., it cannot be covered by open sets isomorphic to $\mathbb{C}^N$ (where $N = \dim X$). Indeed, $\mathbf{SL}_n$ is factorial since the ring of regular functions on every simply connected algebraic group is a factorial domain (e.g, see [17]). Thus, if one assumes existence of an open subset $U \simeq \mathbb{C}^N$ such that $U \neq X$, then $D = X \setminus U$ must be a divisor because of affineness. Factoriality implies that $D = f^{-1}(0)$ for a regular function on X. However, since this function does not vanish on $U \simeq \mathbb{C}^N$ it must be constant because every nonconstant complex polynomial has a root. That is, f is a nonzero constant on X. A contradiction.

(2) Let $H \subset \mathbb{C}^{n+2}_{u,v,\bar{x}}$ be a hypersurface given by $uv = p(\bar{x})$ in the case when the zero locus of p is smooth connected. Note that $H \setminus \{u = 0\} \simeq \mathbb{C}^*_u \times \mathbb{C}^n_{\bar{x}}$ is factorial and u is a prime element of the ring of regular functions on H. Hence H is also factorial by the Nagata lemma (e.g., [3]). Thus, unless H is isomorphic to $\mathbb{C}^{n+1}$ it does not belong to class $\mathcal{A}$ by the same argument as before.

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