A NITSCHE FINITE ELEMENT APPROACH FOR ELLIPTIC PROBLEMS WITH DISCONTINUOUS DIRICHLET BOUNDARY CONDITIONS

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Abstract. We present a numerical approximation method for linear diffusion-reaction problems with possibly discontinuous Dirichlet boundary conditions. The solution of such problems can be represented as a linear combination of explicitly known singular functions as well as of an H^2-regular part. The latter part is expressed in terms of an elliptic problem with regularized Dirichlet boundary conditions, and can be approximated by means of a Nitsche finite element approach. The discrete solution of the original problem is then defined by adding the singular part of the exact solution to the Nitsche approximation. In this way, the discrete solution can be shown to converge of second order with respect to the mesh size.

1. Introduction

Given a bounded, open and convex polygonal domain $\Omega \subset \mathbb{R}^2$ with straight edges, we consider the linear diffusion-reaction problem

$$-\Delta u + \mu u = f \qquad \text{in } \Omega, \tag{1}$$

$$u = g \qquad \text{on } \Gamma, \tag{2}$$

where $\Gamma = \partial\Omega$ denotes the boundary of Ω, $\mu \in L^\infty(\Omega)$ is a nonnegative function, $f \in L^2(\Omega)$ is a source term, and $g \in L^2(\partial\Omega)$ is a possibly discontinuous function on Γ whose precise regularity will be specified later on.

Various formulations for (1)–(2), where the Dirichlet boundary data does not necessarily belong to $H^{1/2}(\Gamma)$, exist in the literature. For instance, the *very weak formulation* is based on twofold integration by parts of (1) and, thereby, incorporates the Dirichlet boundary conditions in a natural way. It seeks a solution $u \in L^2(\Omega)$ such that

$$-\int_\Omega u\Delta v \,\mathrm{d}\boldsymbol{x} + \int_\Omega \mu uv \,\mathrm{d}\boldsymbol{x} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x} - \int_\Gamma g\,\nabla v \cdot \mathbf{n} \,\mathrm{d}s$$

for any $v \in H^2(\Omega) \cap H^1_0(\Omega)$, where we write $\mathbf{n}$ for the unit outward normal vector to the boundary Γ. Alternatively, the following saddle point formulation, which traces back to the work [9], may be applied: provided that $g \in H^{1/2-\varepsilon}(\partial\Omega)$, for some $\varepsilon \in [0, 1/2)$, find $u \in H^{1-\varepsilon}(\Omega)$ with $u|_\Gamma = g$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} + \int_\Omega \mu uv \,\mathrm{d}\boldsymbol{x} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x} \tag{3}$$

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for all $v \in H^{1+\varepsilon}(\Omega) \cap H^1_0(\Omega)$; for results dealing with finite element approximations of (3), we refer to [4]. Another related approach is based on weighted Sobolev spaces (accounting for the local singularities of solutions with discontinuous boundary data), and has been analyzed in the context of *hp*-type discontinuous Galerkin methods in [7].

The main idea of this paper is to represent the (weak) solution of (1)–(2) in terms of a regular H^2 part as well as an explicitly known singular part (Section 2.4). The latter is expressed by means of suitable singular functions which account for the local discontinuities in the Dirichlet boundary data (Section 2.2). Here, it is crucial to ensure that the boundary data of the regular problem is sufficiently smooth as to provide an H^2 trace lifting (see Section 2.3). We shall employ a classical Nitsche technique in order to discretize the regular part of the solution, and define the numerical approximation of (1)–(2) by adding back the (exact) singular part (Section 3.2). A numerical experiment (Section 3.3) underlines that our approach provides optimally converging results.

Throughout the paper we shall use the following notation: For an open domain $\mathcal{D} \subset \mathbb{R}^n$, $n \in \{1, 2\}$, and $p \in [1, \infty]$, we denote by $L^p(\mathcal{D})$ the class of Lebesgue spaces on $\mathcal{D}$. For $p = 2$, we write $\|\cdot\|_{0,\mathcal{D}}$ to signify the L^2-norm on $\mathcal{D}$. Furthermore, for an integer $k \in \mathbb{N}_0$, we let $H^k(\mathcal{D})$ be the usual Sobolev space of order k on $\mathcal{D}$, with norm $\|\cdot\|_{k,D}$ and semi-norm $|\cdot|_{k,\mathcal{D}}$. The set $H^1_0(\mathcal{D})$ represents the subspace of $H^1(\mathcal{D})$ of all functions with zero trace along $\partial\mathcal{D}$. If $\mathcal{D}$ is represented as a (disjoint) finite union of open sets, that is, $\overline{\mathcal{D}} = \bigcup_i \overline{\mathcal{D}}_i$, and X is any class of function spaces, then we write $X_{\mathsf{pw}}(\mathcal{D}) = \Pi_i X(D_i)$ to mean the set of all functions which belong *piecewise* (with respect to the partition $\{\mathcal{D}_i\}_i$) to X.

2. Problem formulation

The aim of this section is to establish a suitable framework for the weak solution of (1)–(2).

2.1. **Notation.** Let $\mathcal{A} = \{\boldsymbol{A}_i\}_{i=1}^M \subset \partial\Omega$, with $\boldsymbol{A}_i \neq \boldsymbol{A}_j$, for $1 \leq i \neq j \leq M$, be a finite set of points on the boundary of the polygonal domain Ω, which are numbered in counter-clockwise direction along $\partial\Omega$; the points in $\mathcal{A}$ mark the locations where the Dirichlet boundary condition g from (2) exhibits discontinuities. Furthermore, we denote by $\Gamma_i \subset \Gamma$, $i = 1, 2, \ldots, M$, the open edge which connects the two points $\boldsymbol{A}_i$ and $\boldsymbol{A}_{i+1}$; in the sequel, we shall identify indices $0 \simeq M$, $1 \simeq M+1$, etc.; for instance, we have $\boldsymbol{A}_{M+1} = \boldsymbol{A}_1$ and $\boldsymbol{A}_0 = \boldsymbol{A}_M$, or $\Gamma_{M+1} = \Gamma_1$ and $\Gamma_0 = \Gamma_M$, etc. Moreover, let $\omega_i \in (0, \pi]$ signify the interior angle of Ω at $\boldsymbol{A}_i$ (in counter-clockwise direction). Finally, for $\phi \in C^0_{\mathsf{pw}}(\Gamma)$, i.e., $\phi|_{\Gamma_i} \in C^0(\Gamma_i)$, for $1 \leq i \leq M$, we set $\phi_i := \phi|_{\Gamma_i}$, and define the one-sided limits

$$\phi(\boldsymbol{A}_i^+) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{A}_i \\ \boldsymbol{x}\in\Gamma_i}} \phi_i(\boldsymbol{x}), \qquad \phi(\boldsymbol{A}_i^-) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{A}_i \\ \boldsymbol{x}\in\Gamma_{i-1}}} \phi_{i-1}(\boldsymbol{x}),$$

and the jumps $[\![\phi]\!]_i = \phi(\boldsymbol{A}_i^+) - \phi(\boldsymbol{A}_i^-)$, for $i = 1, \ldots, M$.

2.2. **Singular functions.** In the following, based on the partition $\overline{\Gamma} = \bigcup_{i=1}^M \overline{\Gamma}_i$, we assume that the boundary data g from (2) satisfies

$$g \in H^2_{\mathsf{pw}}(\Gamma), \tag{4}$$

i.e., with the notation above, we have $g_i \in H^2(\Gamma_i)$, for $1 \leq i \leq M$. We note the continuous Sobolev embedding $H^1(\Gamma_i) \hookrightarrow L^\infty(\Gamma_i)$, i.e.,

$$\sup_{\boldsymbol{x}\in\Gamma_i} |v(\boldsymbol{x})| \leq C\|v\|_{1,\Gamma_i}, \qquad \forall v \in H^1(\Gamma_i), \quad i = 1,\dots,M, \tag{5}$$

for a constant $C = C(\Gamma_i) > 0$. In particular, this implies that the values of $g(\boldsymbol{A}_i^{\pm})$ and $g'(\boldsymbol{A}_i^{\pm})$, with g' denoting the (edgewise) tangential derivative of g in counterclockwise direction along Γ, are well-defined. Hence, for $r_i \neq 0$, we may consider the singular functions (cf. [8, Lemma 6.1.1]), for $1 \leq i \leq M$,

$$\Theta_i(r_i,\theta_i) = \begin{cases} g(\boldsymbol{A}_i^+) - \dfrac{\theta_i}{\omega_i}[\![g]\!]_i & \text{if } \omega_i \in (0,\pi), \\ g(\boldsymbol{A}_i^+) - \dfrac{1}{\pi}\left(\theta_i[\![g]\!]_i + \sigma_i(r_i,\theta_i)[\![g']\!]_i\right) & \text{if } \omega_i = \pi, \end{cases} \tag{6}$$

with

$$\sigma_i(r_i,\theta_i) = r_i\left(\ln(r_i)\sin(\theta_i) + \theta_i\cos(\theta_i)\right).$$

Here, (r_i,θ_i) denote polar coordinates with respect to a local coordinate system centered at $\boldsymbol{A}_i$ such that $\theta_i = 0$ on Γ_i, and $\theta_i = \omega_i$ on Γ_{i-1}. We note that Θ_i is harmonic away from $\boldsymbol{A}_i$, i.e., $\Delta\Theta_i = 0$ in Ω. Since Θ_i is smooth away from $\boldsymbol{A}_i$, there holds

$$[\![\Theta_i]\!]_j = \delta_{ij}[\![g]\!]_i, \qquad 1 \leq i,j \leq M, \tag{7}$$

where δ_{ij} is Kronecker's delta. In addition, for $\omega_j = \pi$, we have

$$[\![\Theta_i']\!]_j = \delta_{ij}[\![g']\!]_j, \tag{8}$$

for $i = 1,\dots,M$.

2.3. **Trace lifting.** Defining the function

$$\widehat{g} : \Gamma \to \mathbb{R}, \qquad \widehat{g} := g - \sum_{i=1}^{M} \Theta_i|_\Gamma, \tag{9}$$

with Θ_i from (6), and recalling (7), we note that

$$[\![\widehat{g}]\!]_j = [\![g]\!]_j - \sum_{i=1}^{M}[\![\Theta_i]\!]_j = 0, \qquad 1 \leq j \leq M, \tag{10}$$

i.e., $\widehat{g}$ is continuous along the boundary Γ. Similarly, whenever $\omega_j = \pi$, using (8), we have

$$[\![\widehat{g}']\!]_j = [\![g']\!]_j - \sum_{i=1}^{M}[\![\Theta_i']\!]_j = 0. \tag{11}$$

Lemma 2.1. *There holds the estimate*

$$\sum_{i=1}^{M} \|\widehat{g}_i\|_{2,\Gamma_i} \leq C\sum_{i=1}^{M} \|g_i\|_{2,\Gamma_i},$$

where $C > 0$ is a constant independent of g.

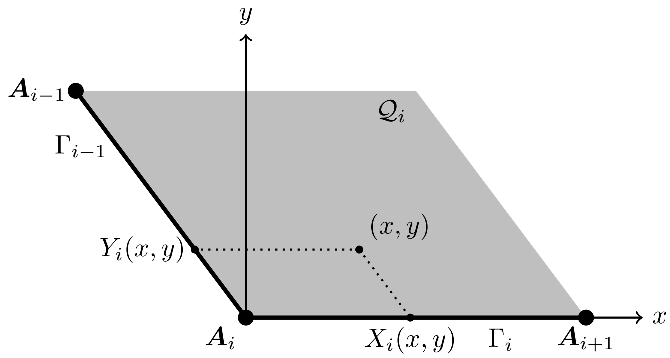

FIGURE 1. Graphical illustration of (local) trace lifting construction.

Proof. By definition of $\widehat{g}$, see (9), for any $1 \leq i \leq M$, there holds

$$\|\widehat{g}_i\|_{2,\Gamma_i} \leq \|g_i\|_{2,\Gamma_i} + \sum_{j=1}^{M} \|\Theta_j\|_{2,\Gamma_i}.$$

Since Θ_j is a linear function along both Γ_{j-1} and Γ_j and smooth on $\bigcup_{k\neq j-1,j} \overline{\Gamma}_k$, we deduce the bound

$$\|\Theta_j\|_{2,\Gamma_i} \leq C_{ij} \left(|g(\boldsymbol{A}_j^+)| + \omega_j^{-1}|[\![g]\!]_j| + |[\![g']\!]_j|\right),$$

where $C_{ij} > 0$ is a constant depending on $\boldsymbol{A}_j$ and Γ_i. Hence,

$$\begin{aligned}\sum_{i=1}^{M} \|\widehat{g}_i\|_{2,\Gamma_i} &\leq \sum_{i=1}^{M} \|g_i\|_{2,\Gamma_i} + \sum_{i,j=1}^{M} C_{ij} \left(|g(\boldsymbol{A}_j^+)| + \omega_j^{-1}|[\![g]\!]_j| + |[\![g']\!]_j|\right) \\ &\leq \sum_{i=1}^{M} \|g_i\|_{2,\Gamma_i} + C \sum_{i=1}^{M} \left(\|g_i\|_{\infty,\Gamma_i} + \|g'\|_{\infty,\Gamma_i}\right).\end{aligned}$$

Using (5), the proof is complete. □

The identities (10) and (11) together with the previous lemma imply the following result.

Lemma 2.2. *There exists a lifting $\widehat{U} \in H^2(\Omega)$ of the boundary data $\widehat{g}$, i.e., $\widehat{U}|_\Gamma = \widehat{g}$ in the sense of traces, with*

$$\|\widehat{U}\|_{2,\Omega} \leq C \sum_{i=1}^{M} \|g_i\|_{2,\Gamma_i}, \tag{12}$$

where $C > 0$ is a constant independent of g.

Proof. We use a partition of unity approach. Specifically, to each corner $\boldsymbol{A}_i$ of Ω, we associate a function $\phi_i \in C^\infty(\overline{\Omega})$ such that $\sum_{i=1}^{M} \phi_i(\boldsymbol{x}) = 1$ for any $\boldsymbol{x} \in \Gamma$, and $\operatorname{supp}(\phi_i) \cap \Gamma \subset \Gamma_{i-1} \cup \{\boldsymbol{A}_i\} \cup \Gamma_i$, for $1 \leq i \leq M$.

Fix $i \in \{1, \ldots, M\}$. If $0 < \omega_i < \pi$, we may assume, without loss of generality, that $\boldsymbol{A}_i$ coincides with the origin $(0,0)$, and the edge Γ_i can be placed on the first coordinate axis. Denoting the (Cartesian) coordinates in this system by (x, y), we let

$$X_i(x,y) = \left(x - \frac{y}{\tan(\omega_i)}, 0\right), \qquad Y_i(x,y) = \left(\frac{y}{\tan(\omega_i)}, y\right);$$

see Figure 1 for a graphical illustration. Observe that

$$X_i|_{\Gamma_{i-1}} = \mathbf{0}, \qquad X_i|_{\Gamma_i} = \mathsf{id},$$
$$Y_i|_{\Gamma_{i-1}} = \mathsf{id}, \qquad Y_i|_{\Gamma_i} = \mathbf{0},$$

where id is the identity function. Then, for $(x, y) \in \Omega$, we define the lifting

$$\widehat{U}_i = \begin{cases} \left(\widehat{g}|_{\Gamma_i} \circ X_i + \widehat{g}|_{\Gamma_{i-1}} \circ Y_i - \widehat{g}(\boldsymbol{A}_i)\right) \phi_i & \text{in } \mathcal{Q}_i \cap \Omega, \\ 0 & \text{on } \Omega \setminus \mathcal{Q}_i, \end{cases}$$

where

$$\mathcal{Q}_i = \{\boldsymbol{x} = \omega_1(\boldsymbol{A}_{i+1} - \boldsymbol{A}_i) + \omega_2(\boldsymbol{A}_{i-1} - \boldsymbol{A}_i) : \omega_1, \omega_2 \in (0,1)\};$$

cf. the gray area in Figure 1. The lifting $\widehat{U}_i$ satisfies the boundary condition

$$\widehat{U}_i|_\Gamma = \widehat{g}\phi_i|_\Gamma. \tag{13}$$

Furthermore, we note that

$$\|\widehat{U}_i\|_{2,\Omega} \leq C\left(\|\widehat{g}_{i-1}\|_{2,\Gamma_{i-1}} + \|\widehat{g}_i\|_{2,\Gamma_i} + |\widehat{g}(\boldsymbol{A}_i)|\right).$$

Using (5), we obtain

$$\|\widehat{U}_i\|_{2,\Omega} \leq C\left(\|\widehat{g}_{i-1}\|_{2,\Gamma_{i-1}} + \|\widehat{g}_i\|_{2,\Gamma_i}\right). \tag{14}$$

If $\omega_i = \pi$, then the function $\widehat{g}\phi_i|_\Gamma$ belongs to $H^{3/2}(\Gamma)$, and by the trace theorem, there exists $\widehat{U}_i \in H^2(\Omega)$ which again satisfies (13) as well as (14). Therefore, letting

$$\widehat{U} = \sum_{i=1}^{M} \widehat{U}_i,$$

we see that $\widehat{U}|_\Gamma = \widehat{g}$, and

$$\|\widehat{U}\|_{2,\Omega} \leq \sum_{i=1}^{M} \|\widehat{U}_i\|_{2,\Omega} \leq C \sum_{i=1}^{M} \|\widehat{g}_i\|_{2,\Gamma_i}.$$

Employing Lemma 2.1 completes the argument. □

2.4. **Weak solution.** Let

$$\widehat{f} := f - \mu \sum_{i=1}^{M} \Theta_i \in L^2(\Omega). \tag{15}$$

Then, proceeding analogously as in the proof of Lemma 2.1, we deduce that

$$\|\widehat{f}\|_{0,\Omega} \leq \|f\|_{0,\Omega} + \mu \sum_{i=1}^{M} \|\Theta_i\|_{0,\Omega} \leq \|f\|_{0,\Omega} + C \sum_{i=1}^{M} \|g_i\|_{2,\Gamma_i}, \tag{16}$$

with a constant independent of f and g. Consider the regularized problem

$$-\Delta\widehat{u} + \mu\widehat{u} = \widehat{f} \qquad \text{in } \Omega, \tag{17}$$
$$\widehat{u} = \widehat{g} \qquad \text{on } \Gamma, \tag{18}$$

where $\widehat{g}$ is the boundary function from (9).

Proposition 2.3. *Let Ω be a convex and bounded polygonal domain. Then, there exists a unique solution $\widehat{u} \in H^2(\Omega)$ to (17)–(18) that satisfies the stability bound*

$$\|\widehat{u}\|_{2,\Omega} \leq C\left(\|f\|_{0,\Omega} + \sum_{i=1}^{M} \|g_i\|_{2,\Gamma_i}\right), \tag{19}$$

with a constant $C > 0$ depending on Ω, and on μ.

Proof. Proposition 2.2 provides the existence of a function $\widehat{U} \in H^2(\Omega)$ with $\widehat{U}\big|_\Gamma = \widehat{g}$. Since $\widehat{f} + \Delta\widehat{U} - \mu\widehat{U}$ belongs to $L^2(\Omega)$, elliptic regularity theory in convex polygons (see, e.g., [2, 5, 6]) implies the existence of a unique remainder function $\widehat{\rho} \in H^2(\Omega)$ with

$$\begin{aligned} -\Delta\widehat{\rho} + \mu\widehat{\rho} &= \widehat{f} + \Delta\widehat{U} - \mu\widehat{U} && \text{in } \Omega, \\ \widehat{\rho} &= 0 && \text{on } \Gamma, \end{aligned}$$

and

$$\|\widehat{\rho}\|_{2,\Omega} \leq C\|\widehat{f} + \Delta\widehat{U} - \mu\widehat{U}\|_{0,\Omega} \leq C\left(\|\widehat{U}\|_{2,\Omega} + \|\widehat{f}\|_{0,\Omega}\right). \tag{20}$$

Thus, the function $\widehat{u} := \widehat{U} + \widehat{\rho}$ belongs to $H^2(\Omega)$. Furthermore, it holds that

$$-\Delta\widehat{u} + \mu\widehat{u} = -\Delta\widehat{U} + \mu\widehat{U} - \Delta\widehat{\rho} + \mu\widehat{\rho} = \widehat{f} \qquad \text{in } \Omega,$$

as well as

$$\widehat{u}\big|_\Gamma = \widehat{U}\big|_\Gamma + \widehat{\rho}\big|_\Gamma = \widehat{g}.$$

In addition, combining (12) and (20) yields

$$\begin{aligned} \|\widehat{u}\|_{2,\Omega} &\leq \|\widehat{U}\|_{2,\Omega} + \|\widehat{\rho}\|_{2,\Omega} \leq C\left(\|\widehat{U}\|_{2,\Omega} + \|\widehat{f}\|_{0,\Omega}\right) \\ &\leq C\left(\sum_{i=1}^{M} \|g_i\|_{2,\Gamma_i} + \|\widehat{f}\|_{0,\Omega}\right), \end{aligned}$$

which, by virtue of (16), results in the bound (19). □

Definition 2.4. *We call the function u defined by*

$$u := \widehat{u} + \sum_{i=1}^{M} \Theta_i, \tag{21}$$

with $\widehat{u}$ the unique H^2-solution of (17)–(18), the weak solution *of (1)–(2).*

Remark 2.5. It can be verified easily that the weak solution defined in (21) belongs to a class of weighted Sobolev spaces; cf., e.g., [2, 3]. The norms of these spaces contain local radial weights at the discontinuity points $\mathcal{A}$ of the Dirichlet boundary data, and, thereby, account for possible singularities in the solution of (1)–(2). Based on an inf-sup theory, the work [7] shows that (1)–(2) exhibits a unique solution within this framework.

3. Numerical approximation

The purpose of this section is to discretize (1)–(2) by a finite element approach. Specifically, we will employ a Nitsche method to obtain a numerical approximation of the elliptic problem (17), with the possibly non-homogeneous Dirichlet boundary condition (18). The discrete solution will then be defined similarly as in (21).

3.1. **Meshes and spaces.** We consider regular, quasi-uniform meshes $\mathcal{T}_h$ of mesh size $h > 0$, which partition $\Omega \subset \mathbb{R}^2$ into open disjoint triangles and/or parallelograms $\{K\}_{K\in\mathcal{T}_h}$, i.e., $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$. Each element $K \in \mathcal{T}_h$ is an affinely mapped image of the reference triangle $\widehat{T} = \{(\widehat{x}, \widehat{y}) : -1 < \widehat{x} < 1, -1 < \widehat{y} < -\widehat{x}\}$ or the reference square $\widehat{S} = (-1, 1)^2$, respectively. Moreover, we define the conforming finite element space

$$\mathbb{V}(\mathcal{T}_h) = \{v \in H^1(\Omega) : v|_K \in \mathbb{S}(K), K \in \mathcal{T}_h\},$$

where, for $K \in \mathcal{T}_h$, we write $\mathbb{S}(K)$ to mean either the space $\mathbb{P}_1(K)$ of all polynomials of total degree at most 1 on K or the space $\mathbb{Q}_1(K)$ of all polynomials of degree at most 1 in each coordinate direction on K.

3.2. **Nitsche discretization.** The classical Nitsche approach [10] for the numerical approximation of (17)–(18) is given by finding $\widehat{u}_h \in \mathbb{V}(\mathcal{T}_h)$ such that

$$a_h(\widehat{u}_h, v) = l_h(v) \quad \text{for all } v \in \mathbb{V}(\mathcal{T}_h). \tag{22}$$

Here, denoting by ∇_h the elementwise gradient operator, we define the bilinear form

$$\begin{aligned} a_h(w, v) = &\int_\Omega \{\nabla_h w \cdot \nabla_h v + \mu w v\} \,\mathrm{d}\boldsymbol{x} \\ &- \int_{\partial\Omega} v\,(\nabla_h w \cdot \mathbf{n}) \,\mathrm{d}s - \int_{\partial\Omega} w\,(\nabla_h v \cdot \mathbf{n}) \,\mathrm{d}s + \frac{\gamma}{h}\int_{\partial\Omega} w v \,\mathrm{d}s, \end{aligned}$$

as well as the linear functional

$$l_h(v) = \int_\Omega \widehat{f} v \,\mathrm{d}\boldsymbol{x} - \int_{\partial\Omega} \widehat{g}\,(\nabla_h v \cdot \mathbf{n}) \,\mathrm{d}s + \frac{\gamma}{h}\int_{\partial\Omega} \widehat{g} v \,\mathrm{d}s,$$

with $\widehat{g}$ and $\widehat{f}$ from (9) and (15), respectively. The penalty parameter $\gamma > 0$ appearing in both forms is chosen sufficiently large (but independent of the mesh size) as to guarantee the well-posedness of the weak formulation (22); this can be shown in a similar way as in the context of discontinuous Galerkin methods; see, e.g., [1]. In addition, referring to [10, Satz 2], cf. also [1, Section 5.1], there holds the *a priori* error estimate

$$\|\widehat{u} - \widehat{u}_h\|_{0,\Omega} \le Ch^2 |\widehat{u}|_{2,\Omega}, \tag{23}$$

with a constant $C = C(\mu, \widehat{f}, \widehat{g}) > 0$ independent of the mesh size h.

Definition 3.1. *Analogously to* (21)*, we define the discrete solution of* (1)*–*(2) *by*

$$u_h := \widehat{u}_h + \sum_{i=1}^{M} \Theta_i, \tag{24}$$

where $\widehat{u}_h \in \mathbb{V}(\mathcal{T}_h)$ is the Nitsche solution from (22)*, and $\{\Theta_i\}_{i=1}^M$ are the singular functions from* (6)*.*

Theorem 3.2. *Let u be the solution of* (1)*–*(2) *given by* (21)*, and u_h its discrete counterpart from* (24)*. Then, there holds the* a priori *error estimate*

$$\|u - u_h\|_{0,\Omega} \le Ch^2, \tag{25}$$

with a constant $C = C(\mu, f, g) > 0$ independent of h.

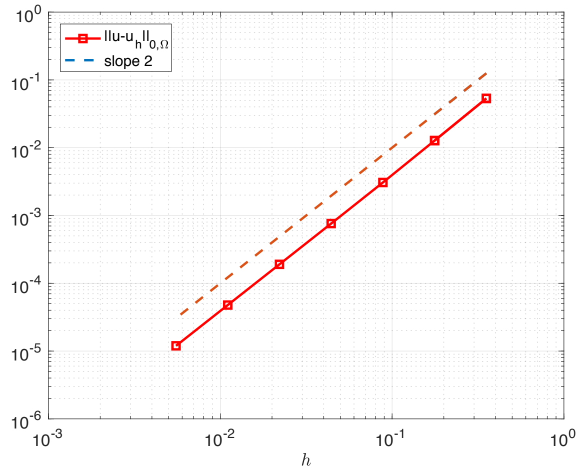

FIGURE 2. L^2 error $\|u-u_h\|_{0,\Omega}$ against mesh size h compared to a reference line with slope 2 (expected behaviour).

Proof. We recall the definitions (21) and (24) in order to notice

$$u - u_h = \widehat{u} + \sum_{i=1}^{M} \Theta_i - \left(\widehat{u}_h + \sum_{i=1}^{M} \Theta_i \right) = \widehat{u} - \widehat{u}_h.$$

Therefore, applying (23) yields

$$\|u - u_h\|_{0,\Omega} \leq Ch^2 |\widehat{u}|_{2,\Omega}.$$

Finally, recalling (19) completes the proof. □

3.3. **Numerical example.** On the rectangle $\Omega = (-1,1) \times (0,1)$ we consider the elliptic boundary value problem

$$\begin{aligned} -\Delta u + u &= e^{-r^2}(5 - 4r^2)\theta \qquad &&\text{in } \Omega \\ u &= g &&\text{on } \Gamma, \end{aligned}$$

with the Dirichlet boundary data g chosen such that the analytical solution is given by

$$u(r,\theta) = e^{-r^2}\theta.$$

Here, (r,θ) denote polar coordinates in $\mathbb{R}^2$. Note that the solution u is smooth along Γ except at the origin, where it exhibits a discontinuity jump. In particular, it follows that $u \notin H^1(\Omega)$.

Starting from a regular coarse mesh, we investigate the practical performance of the a priori error estimate derived in Theorem 3.2 within a sequence of uniformly refined $\mathbb{P}_1$ elements. In Figure 2 we present a comparison of the L^2 norm of the error versus the mesh size h on a log-log scale for each of the meshes. Our results are in line with the *a priori* error estimate (25), and show that the discrete solution u_h from (24) converges of second order with respect to the mesh size h. Moreover, in Figure 3 we show the Nitsche solution $\widehat{u}_h \in \mathbb{V}(\mathcal{T}_h)$ defined in (22), as well as the computed solution u_h for a mesh consisting of 1024 elements.

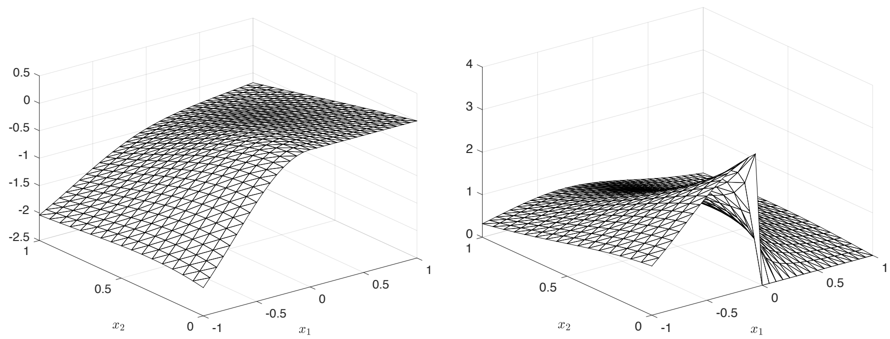

FIGURE 3. Nitsche solution (left) and discrete solution (right) based on a uniform mesh with 1024 elements.

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