

Article

THE STABILITY OF THE AGGREGATE LOSS DISTRIBUTION

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Abstract: In this article we introduce the stability analysis of a compound sum: it consists in computing the standardized variation of the survival function of the sum resulting from an infinitesimal perturbation of the common distribution of the summands. Stability analysis is complementary to the classical sensitivity analysis, which consists in computing the derivative of an important indicator of the model, with respect to a model parameter. We obtain a computational formula for this stability from the saddlepoint approximation. We apply the formula to the compound Poisson insurer loss with gamma individual claim amounts and to the compound geometric loss with Weibull individual claim amounts.

Keywords: Dirac distribution; gamma-Poisson, Weibull-geometric compound distributions; Gâteaux differential; saddlepoint approximation.

MSC: 41A60, 65C05, 60K10

1. Introduction

This article presents a computational formula for the stability of the survival function (s.f.) of the compound sum of independent and identically distributed (i.i.d.) random variables that are independent of their summation index. The compound sum typically represents the insurer total claim amount during a fixed period (e.g. a year): the i.i.d. random variables are the individual claim amounts and the number of claims within the period is a counting random variable or a counting stochastic process, if we let the period length vary. We define the stability of a sum as the standardized variation of the s.f. of the sum resulting from an infinitesimal perturbation at some point $x \in \mathbb{R}$ of the distribution of the summands.

More precisely, let Δ_x denote the Dirac distribution function (d.f.) over \mathbb{R} with mass one at x (thus jumping from level 0 to level 1 at point x). If F denotes the d.f. of the summands, then

$$F_{x\varepsilon} = (1 - \varepsilon)F + \varepsilon\Delta_x \quad (1)$$

is the ε -perturbation of F at x , for any choice of $\varepsilon \in [0, 1]$. The derivative of the s.f. of the sum under $F_{x\varepsilon}$ with respect (w.r.t.) ε evaluated at $\varepsilon = 0$ is the s.f. stability (s.f.s.) at the perturbation point x .

This concept differs from the one of sensitivity of queueing theory or risk theory, which is defined as the derivative of the s.f. of the sum w.r.t. a parameter of F ; cf. e.g. [Asmussen and Albrecher \(2010\)](#), Section IV.9. From an abstract point of view, a parametric model spans only a low-dimensional or narrow subset of the space of probability distributions. Such a narrow subset is indeed beneficial to statistical data reduction, but often does not contain all realistic perturbations of the assumed model.

28 In this sense, the sensitivity is a limited indicator of the model stability. Allowing for perturbations
 29 in all possible directions provides a more complete or realistic analysis of the model stability. In this
 30 sense, our concept of stability is preferable. This concept is in fact an important idea of robust statistics,
 31 see e.g. [Hampel et al. \(1986\)](#). Mathematically, the quantity of interest of a stochastic model is regarded
 32 as a functional and a functional derivative is computed. This approach is used for example in renewal
 33 theory by [Grübel \(1989\)](#), where the renewal function is a functional of the lifetime distribution, or by
 34 [Politis \(2006\)](#), for the probability of ruin of the risk process.

35 Practically, for a given actuarial aggregate loss model in form of a compound sum, if a stability
 36 of low magnitude results from the perturbation of a new large individual claim amount (viz. a large
 37 value of x), then the loss model is reliable under perturbations through extreme large claims. In the
 38 context of uncertainty (where for example catastrophic events are not incorporated in the model), this
 39 notion of stability appears practically relevant. The s.f.s. informs the risk manager about the variation
 40 of the upper tail probability of the aggregate loss when an uncertain large claim amount is considered.
 41 Still from the practical point of view, the sensitivity as described above has the alternative role of
 42 identifying important model parameters: the most significant ones have large sensitivity value. But
 43 this interpretation holds only when the model is really the correct one (which is often not simple to
 44 establish). Of course, both sensitivity and stability analyses can be carried out simultaneously.

45 [Field and Ronchetti \(1985\)](#) considered this type of stability for the sample mean and called it
 46 “tail area influence function”. Their applications concerned statistical testing. They computed the tail
 47 area influence function with the saddlepoint approximation of [Daniels \(1954\)](#). This article generalizes
 48 this approximation to the stability of the compound sum and suggests using this concept in risk
 49 management. The new formula is easy and fast to compute. A numerical illustration for the total claim
 50 amount with gamma individual claim amounts and Poisson number of claims is provided.

51 Most methods for computing sensitivities rely on Monte Carlo simulation; see e.g. [Asmussen](#)
 52 [and Rubinstein \(1999\)](#) and [Asmussen and Glynn \(2007\)](#), Section VII. One exception is [Gatto and](#)
 53 [Peeters \(2015\)](#), who proposes evaluating the sensitivity of the s.f. of the random sum w.r.t. the
 54 parameter of the summation index distribution (which is either Poisson or geometric) with the
 55 saddlepoint approximation. [Gatto and Peeters \(2015\)](#) shows numerically that the sensitivities
 56 obtained by the saddlepoint approximation and by simulation with importance sampling are very close,
 57 even though importance sampling is computationally intensive. The high accuracy of the saddlepoint
 58 approximation is well illustrated in the literature of statistics and applied probability; refer e.g. to
 59 [Jensen \(1995\)](#) or to [Gatto and Mosimann \(2012\)](#) in the context of risk theory.

60 The next parts of this article are the following. Section 2 provides the approximations to s.f.s.
 61 based on the saddlepoint approximation. Section 2.1 considers the deterministic sum and Section
 62 2.2 the compound sum, viz. the insurer aggregate claim amount. Section 3 provides numerical
 63 illustrations. Section 3.1 considers the aggregate claim amount with Poisson distributed number of
 64 claims and gamma distributed individual claim amounts. In Section 3.2, the number of claim follows
 65 the geometric distribution and the individual claim amounts follow the Weibull distribution. Some
 66 related long derivatives are provided in the Appendix.

67 2. Saddlepoint approximation to the stability

68 This section has two parts: in Section 2.1 an approximation to s.f.s. of the deterministic sum
 69 is derived. It corresponds to the formula of [Field and Ronchetti \(1985\)](#), although the derivation is
 70 different. Section 2.2 generalizes the formula to the compound sum, which is an essential quantity of
 71 risk theory.

72 2.1. The sum

Let X_1, \dots, X_n be independent random variables with d.f. F , (moment generating function) m.g.f. M and (cumulant generating function) c.g.f. $K = \log M$. Define the sample mean by $\bar{X}_n = \sum_{i=1}^n X_i/n$.

The Legendre-Fenchel transform (or convex conjugate or large deviations index) of the c.g.f. K and at point $t \in \mathbb{R}$ is given by

$$\hat{K}(t) = \sup_{v \in \text{dom } K} vt - K(v) \in \mathbb{R}_+ \cup \{\infty\}, \quad (2)$$

where $\text{dom } \varphi = \{x \in \mathbb{R} \mid |\varphi(x)| < \infty\}$ is the domain of definition of $\varphi : \mathbb{R} \rightarrow \mathbb{R}$. The transform \hat{K} is clearly nonnegative. One can show that it is convex and that it attains its minimum at $\mu = E[X_1]$, when the expectation exists. Assume that the supremum in (2) is attained at $v_t \in \text{int dom } K$. This condition is satisfied without restrictions on t when F is light-tailed, in the sense of having exponentially decaying tails. Under this assumption, v_t solves w.r.t. v the equation

$$K'(v) = t \quad (3)$$

and convexity tells that it is the unique solution. It is called the saddlepoint at t and $\hat{K}(t) = v_t t - K(v_t)$. Define the sample mean by $\bar{X}_n = \sum_{j=1}^n X_j/n$ and the s.f. $\bar{H}_n(t) = P_F[\bar{X}_n \geq t]$. Chernoff's large deviations theorem states that $\forall t \geq \mu$,

$$\frac{1}{n} \log \bar{H}_n(t) = -\hat{K}(t) + o(1), \text{ as } n \rightarrow \infty. \quad (4)$$

73 Although (4) is a large deviations approximation, the asymptotics is in the logarithmic scale of the
74 probability.

This article is based on the saddlepoint approximation of [Lugannani and Rice \(1980\)](#), because it is known that it provides a very accurate approximation to the s.f. $\bar{H}_n(t)$. It has bounded relative error on the probability scale, instead of the logarithmic scale. From now on, we assume that F is absolutely continuous. Under this additional assumption, Lugannani and Rice's approximation to $\bar{H}_n(t)$ at $t \neq \mu$ is given by

$$\bar{G}_n(t) = 1 - \Phi(n^{\frac{1}{2}}r) + n^{-\frac{1}{2}}\phi(n^{\frac{1}{2}}r) \left(\frac{1}{s} - \frac{1}{r} \right), \quad (5)$$

where

$$s = v_t \{K''(v_t)\}^{\frac{1}{2}}, \quad r = \text{sgn}(v_t) \{2\hat{K}(t)\}^{\frac{1}{2}}, \quad (6)$$

75 ϕ and Φ are the standard normal density and d.f., respectively. The relative error of approximation (5)
76 is $O(n^{-1})$, as $n \rightarrow \infty$. For comparison, (4) re-expressed in terms of the new variable r leads to the quite
77 dissimilar approximation to $\bar{H}_n(t)$ given by $\sqrt{2\pi}\phi(n^{1/2}r)$.

The s.f.s. of \bar{X}_n at tail value t and perturbation point $x \in \mathbb{R}$ is given by the Gâteaux differential

$$\tau_n(t; x, F) = \frac{\partial}{\partial \varepsilon} P_{x\varepsilon}[\bar{X}_n \geq t] \Big|_{\varepsilon=0}, \quad (7)$$

78 where $P_{x\varepsilon}$ is the probability measure obtained by the replacement of the summand d.f. F by its
79 ε -perturbation at x , that is $F_{x\varepsilon}$ defined in (1), for some $\varepsilon \in [0, 1]$. The following result gives an
80 approximation to the s.f.s. obtained from (5).

Theorem 1. *Under the previous assumptions, the s.f.s. of \bar{X}_n given in (7), at $t \neq \mu$ and at perturbation point $x \in \mathbb{R}$, can be approximated by*

$$\tilde{\tau}_n(t; x, F) = -n^{\frac{1}{2}}\phi(n^{\frac{1}{2}}r) \left\{ \frac{r\dot{r}_x}{s} + O(n^{-1}) \right\} \quad (8)$$

where s and r are given by (6), v_t is given by (3) and

$$\dot{r}_x = \frac{1}{r} \left(1 - \exp \left\{ v_t(x - t) + \frac{r^2}{2} \right\} \right). \quad (9)$$

The reminder term in (8) is given by

$$-n^{-1} \left(\frac{\dot{r}_x}{r^2} - \frac{\dot{s}_x}{s^2} \right),$$

where

$$\dot{s}_x = \frac{v_t}{2s} \exp \left\{ v_t(x-t) + \frac{r^2}{2} \right\} \left\{ (t-x) \left(2 + \frac{v_t^3}{s^2} K'''(v_t) \right) + v_t(t-x)^2 - \frac{s^2}{v_t} \right\}. \quad (10)$$

81 Proof. Let $x \in \mathbb{R}$ and $\varepsilon \in [0, 1]$. The approximate s.f.s. (8) is obtained by differentiating w.r.t. ε the
82 Lugannani and Rice saddlepoint approximation (5) at $F_{x\varepsilon}$ and by evaluating it at $\varepsilon = 0$.

Let $v \in \mathbb{R}$, denote $M_{x\varepsilon}(v) = \int_{\mathbb{R}} e^{vy} dF_{x\varepsilon}(y)$ and $K_{x\varepsilon} = \log M_{x\varepsilon}$. Then

$$M_{x\varepsilon}(v) = (1 - \varepsilon)M(v) + \varepsilon e^{vx} \quad (11)$$

(because for any Borel function $g: \mathbb{R} \rightarrow \mathbb{R}$, $\int_{\mathbb{R}} g(y) d\Delta_x(y) = g(x)$). The perturbed saddlepoint $v_{tx\varepsilon}$ at point $t \in \mathbb{R}$ is defined by $K'_{x\varepsilon}(v_{tx\varepsilon}) = t$. Thus from

$$K'_{x\varepsilon}(v_{tx\varepsilon}) \sim K'_{x\varepsilon}(v_t) + (v_{tx\varepsilon} - v_t)K''_{x\varepsilon}(v_t), \text{ as } \varepsilon \rightarrow 0,$$

we obtain

$$\frac{v_{tx\varepsilon} - v_t}{\varepsilon} \sim \frac{t - K'_{x\varepsilon}(v_t)}{\varepsilon K''_{x\varepsilon}(v_t)}, \text{ as } \varepsilon \rightarrow 0.$$

Consequently,

$$\dot{v}_{tx} = \frac{\partial}{\partial \varepsilon} v_{tx\varepsilon} \Big|_{\varepsilon=0} = \lim_{\varepsilon \rightarrow 0} \frac{t - K'_{x\varepsilon}(v_t)}{\varepsilon K''_{x\varepsilon}(v_t)} = \lim_{\varepsilon \rightarrow 0} \frac{-\frac{\partial}{\partial \varepsilon} K'_{x\varepsilon}(v_t)}{K''_{x\varepsilon}(v_t) + \varepsilon \frac{\partial}{\partial \varepsilon} K''_{x\varepsilon}(v_t)} = -\frac{\dot{K}'_x(v_t)}{K''(v_t)}, \quad (12)$$

where $\dot{K}'_x(v) = \partial/\partial \varepsilon K'_{x\varepsilon}(v) |_{\varepsilon=0}$, see (A2) of the Appendix. Thus we obtain

$$\dot{v}_{tx} = \frac{e^{vtx}(t-x)}{M(v_t)K''(v_t)}.$$

Denote $r = r(F)$ in (6), then

$$\dot{r}_x = \frac{\partial}{\partial \varepsilon} r(F_{x\varepsilon}) \Big|_{\varepsilon=0} = \text{sgn}(v_t) \frac{1}{2} \{2[v_t t - K(v_t)]\}^{-\frac{1}{2}} 2 \left[\dot{v}_{tx} t - \frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v_{tx\varepsilon}) \Big|_{\varepsilon=0} \right]. \quad (13)$$

Note that small perturbations do not affect the sign of v_t when tail probabilities are considered. Precisely, if $t \neq E[X_1]$, then $\text{sgn } v_{tx\varepsilon} = \text{sgn } v_t, \forall \varepsilon \in [0, \varepsilon_0]$, for some $\varepsilon_0 > 0$. Thus $\partial/\partial \varepsilon \text{sgn } v_{tx\varepsilon} = 0, \forall \varepsilon \in [0, \varepsilon_0]$. Define $g(\varepsilon, v) = K_{x\varepsilon}(v)$, $g'_1(\varepsilon, v) = \partial/\partial \varepsilon g(\varepsilon, v)$ and $g'_2(\varepsilon, v) = \partial/\partial v g(\varepsilon, v)$. Then, from the multivariate chain rule,

$$\frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v_{tx\varepsilon}) = \frac{\partial}{\partial \varepsilon} g(\varepsilon, v_{tx\varepsilon}) = g'_1(\varepsilon, v_{tx\varepsilon}) + g'_2(\varepsilon, v_{tx\varepsilon}) \frac{\partial}{\partial \varepsilon} v_{tx\varepsilon} = \left(\frac{\partial}{\partial \varepsilon} K_{x\varepsilon} \right) (v_{tx\varepsilon}) + K'_{x\varepsilon}(v_{tx\varepsilon}) \frac{\partial}{\partial \varepsilon} v_{tx\varepsilon}.$$

Hence we obtain

$$\frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v_{tx\varepsilon}) \Big|_{\varepsilon=0} = \dot{K}_x(v_t) + K'(v_t) \dot{v}_{tx},$$

83 where $\dot{K}_x(v) = \partial/\partial \varepsilon K_{x\varepsilon}(v) |_{\varepsilon=0}$, see (A1) of the Appendix. By inserting this result into (13) we obtain
84 (9).

Denote $s = s(F)$, then (6) leads to

$$\dot{s}_x = \left. \frac{\partial}{\partial \varepsilon} s(F_{x\varepsilon}) \right|_{\varepsilon=0} = \dot{v}_{tx} \{K''(v_t)\}^{\frac{1}{2}} + \frac{v_t}{2} \{K''(v_t)\}^{-\frac{1}{2}} \left. \frac{\partial}{\partial \varepsilon} K''_{x\varepsilon}(v_{tx\varepsilon}) \right|_{\varepsilon=0}.$$

From the multivariate chain rule we obtain

$$\left. \frac{\partial}{\partial \varepsilon} K''_{x\varepsilon}(v_{tx\varepsilon}) \right|_{\varepsilon=0} = \dot{K}'_x(v_t) + K'''(v_t) \dot{v}_{tx},$$

85 where $\dot{K}'_x(v) = \partial/\partial \varepsilon K''_{x\varepsilon}(v) |_{\varepsilon=0}$, see (A3) of the Appendix. These two last results yield (10). \square

86 The leading term of the approximation to the s.f.s. (8) is equal to the formula (3.1) in [Field and](#)
 87 [Ronchetti \(1985\)](#), which is however not derived from the saddlepoint approximation (5) but from the
 88 Laplace approximation to the integral of the saddlepoint approximation to the density of [Daniels](#)
 89 [\(1954\)](#). In order to control this equality, the following correspondences between the two notations can
 90 be useful: $C_F(t) = \exp\{\hat{K}(t)\}$, $\alpha_F(t) = v_t$, $\sigma_F(t) = \{K''(v_t)\}^{1/2}$ and $\psi(x;t) = x - t$. Thus Theorem 1
 91 provides an alternative derivation of the s.f.s. of [Field and Ronchetti \(1985\)](#) as well as the exact form
 92 of the error term. However, numerical studies suggest that it is preferable using the first order term
 93 alone.

94 Regarding the sum, let $S_n = \sum_{j=1}^n X_j$, then $P[S_n \geq t] = \bar{H}_n(t/n)$ is its s.f., its saddlepoint
 95 approximation is $\bar{G}_n(t/n)$, $\partial/\partial \varepsilon P_{x\varepsilon}[S_n \geq t] |_{\varepsilon=0} = \tau_n(t/n; x, F)$ is its s.f.s. and the saddlepoint
 96 approximation is $\tilde{\tau}_n(t/n; x, F)$.

97 2.2. The compound sum

Let the random variable X_1, X_2, \dots fulfill the assumptions given in Section 2.1 and let F denote their common d.f. Let N be an independent random variable taking values in $\{0, 1, \dots\}$ with probability function $p_n = P[N = n]$, for $n = 0, 1, \dots$. Consider the compound sum

$$Z = \sum_{j=0}^N X_j,$$

where $X_0 = 0$ by definition. Define the indicator $I\{A\}$ as the function equal to 1, if the statement A is true, or equal to 0, if A is false. The s.f. of Z at $t \in \mathbb{R}$ can be written as

$$\bar{H}_Z(t) = P_F[Z \geq t] = p_0 I\{t \leq 0\} + \sum_{n=1}^{\infty} \int_{\mathbb{R}^n} I\left\{\sum_{k=1}^n x_k \geq t\right\} \prod_{k=1}^n dF(x_k) p_n, \quad (14)$$

98 which is generally not a computational formula. This section provides the saddlepoint approximation
 99 to (14) and then the associated approximation to the s.f.s.

In (14) we see that the distribution of Z is a linear combination of a distribution with mass one at zero and an absolutely continuous distribution. The mass at zero must be eliminated in order to apply the saddlepoint approximation. Denote by M_N and K_N the m.g.f. and the c.g.f. of N and by K the c.g.f. of X_1 . Then the m.g.f. of Z is $M_Z = M_N \circ K$ and its c.g.f. is $K_Z = K_N \circ K$. Let Z_0 be a random variable with the conditional distribution of Z given $N > 0$. Then $\bar{H}_{Z_0}(t) = P[Z \geq t | N > 0]$ and $K_{Z_0}(v) = \log E[e^{vZ} | N > 0]$ are the s.f. and the c.g.f. of Z_0 , respectively. The Legendre-Fenchel transform of the c.g.f. K_{Z_0} at $t \in \mathbb{R}$ is given by

$$\hat{K}_{Z_0}(t) = \sup_{v \in \text{dom } K_{Z_0}} vt - K_{Z_0}(v). \quad (15)$$

We assume that the supremum in (15) is attained at $v_t \in \text{int dom } K_{Z_0}$. Under this assumption, v_t solves w.r.t. v the equation

$$K'_{Z_0}(v) = t. \quad (16)$$

The solution v_t is unique and called saddlepoint at t . We obtain the saddlepoint approximation to $\bar{H}_{Z_0}(t)$ at $t \neq E[Z_0]$, denoted $\bar{G}_{Z_0}(t)$, by the left side of (5) with $n = 1$ and with

$$s = v_t \{K''_{Z_0}(v_t)\}^{\frac{1}{2}} \quad \text{and} \quad r = \text{sgn}(v_t) \{2\hat{K}_{Z_0}(t)\}^{\frac{1}{2}}. \quad (17)$$

It follows from

$$K_{Z_0}(v) = \log \frac{M_Z(v) - p_0}{1 - p_0}$$

that (16) can be re-expressed as

$$\frac{M'_N(K(v))K'(v)}{M_N(K(v)) - p_0} = t. \quad (18)$$

More explicit expressions of s and r than those in (17) are obtained with

$$K''_{Z_0}(v_t) = t \left(\frac{M''_N(K(v_t))K'(v_t)}{M'_N(K(v_t))} + \frac{K''(v_t)}{K'(v_t)} - t \right), \quad (19)$$

see (A6) in the Appendix, and by

$$r = \text{sgn}(v_t) \{2[v_t t - \log\{M_N(K(v_t)) - p_0\} + \log\{1 - p_0\}]\}^{\frac{1}{2}}. \quad (20)$$

It follows from

$$\bar{H}_Z(t) = \bar{H}_{Z_0}(t)(1 - p_0) + \mathbf{I}\{t \leq 0\}p_0 \quad (21)$$

that the saddlepoint approximation to $\bar{H}_Z(t)$ is given by

$$\bar{G}_Z(t) = \bar{G}_{Z_0}(t)(1 - p_0) + \mathbf{I}\{t \leq 0\}p_0. \quad (22)$$

The s.f.s. of Z is the Gâteaux differential

$$\tau_Z(t; x, F) = \left. \frac{\partial}{\partial \varepsilon} \text{P}_{x\varepsilon}[Z \geq t] \right|_{\varepsilon=0'} \quad (23)$$

100 where $F_{x,\varepsilon}$ is given by (1), $\forall x \in \mathbb{R}$ and $\varepsilon \in [0, 1]$. The following result gives an approximation to the
 101 s.f.s. $\tau_Z(t; x, F)$ obtained from the first order approximation of the s.f.s. of the mean given in Theorem
 102 1.

Theorem 2. *Under the previous assumptions, the s.f.s. given in (23), for the s.f. of Z at $t \neq E[N]E[X_1]/(1 - p_0)$ and at perturbation $x \in \mathbb{R}$, can be approximated by*

$$\tilde{\tau}_Z(t; x, F) = \tilde{\tau}_{Z_0}(t; x, F)(1 - p_0), \quad (24)$$

where

$$\tilde{\tau}_{Z_0}(t; x, F) = -\phi(r) \frac{r \dot{r}_x}{s},$$

s and r are given by (17), (19) and (20), v_t is given in (18),

$$\dot{r}_x = -\frac{\dot{K}_{Z_0x}(v_t)}{r} \quad (25)$$

and

$$\dot{K}_{Z_0x}(v_t) = t \frac{\exp\{v_t x - K(v_t)\} - 1}{K'(v_t)}. \quad (26)$$

Proof. This proof is similar to the one of Theorem 1 and so only the main arguments are given. Let $x \in \mathbb{R}$ and $\varepsilon \in [0, 1]$. Let us define the perturbed m.g.f. $M_{x\varepsilon}$ as in (11), $K_{x\varepsilon} = \log M_{x\varepsilon}$ and for $v \in \mathbb{R}$,

$$K_{Z_0x\varepsilon}(v) = \log \frac{M_N(K_{x\varepsilon}(v)) - p_0}{1 - p_0}.$$

By following the reasoning that lead to (12) in the proof of Theorem 1, we obtain the perturbed saddlepoint at point $t \in \mathbb{R}$ as

$$\dot{v}_{tx} = -\frac{\dot{K}'_{Z_0x}(v_t)}{K''_{Z_0}(v_t)}, \quad (27)$$

where $\dot{K}'_{Z_0x}(v) = \partial/\partial\varepsilon K'_{Z_0x\varepsilon}(v) |_{\varepsilon=0}$ is given by (A5) in the Appendix. With (18) it simplifies to

$$\dot{K}'_{Z_0x}(v_t) = t \left(\frac{M''_N(K(v_t))\dot{K}_x(v_t)}{M'_N(K(v_t))} + \frac{\dot{K}'_x(v_t)}{K'(v_t)} - \frac{\dot{K}_x(v_t)}{K'(v_t)} t \right), \quad (28)$$

103 where $\dot{K}_x(v)$ and $\dot{K}'_x(v)$ are respectively given in (A1) and (A2) of the Appendix.

By denoting $r = r(F)$, we find for $t \neq E[Z_0]$,

$$\dot{r}_x = \frac{\partial}{\partial\varepsilon} r(F_{x\varepsilon}) \Big|_{\varepsilon=0} = \text{sgn}(v_t) \frac{1}{2} \{2[v_t t - K_{Z_0}(v_t)]\}^{-\frac{1}{2}} 2 \left[\dot{v}_{tx} t - \frac{\partial}{\partial\varepsilon} K_{Z_0x\varepsilon}(v_{tx\varepsilon}) \Big|_{\varepsilon=0} \right].$$

The multivariate chain rule yields

$$\frac{\partial}{\partial\varepsilon} K_{Z_0x\varepsilon}(v_{tx\varepsilon}) \Big|_{\varepsilon=0} = K'_{Z_0}(v_t) \dot{v}_{tx} + \dot{K}_{Z_0x}(v_t),$$

104 where $\dot{K}_{Z_0x} = \partial/\partial\varepsilon K'_{Z_0x\varepsilon}(v) |_{\varepsilon=0}$ is given in (A4) in the Appendix. By joining these two last expressions
105 one obtains (25) in the theorem. Then (26) is obtained from (A4) and (A1) in the Appendix and from
106 (18).

The s.f.s. of Z_0 is given by

$$\tau_{Z_0}(t; x, F) = \frac{\partial}{\partial\varepsilon} \text{P}_{x\varepsilon}[Z_0 \geq t] \Big|_{\varepsilon=0}.$$

Thus it follows from (21) that

$$\tau_Z(t; x, F) = \tau_{Z_0}(t; x, F)(1 - p_0).$$

107 This justifies (24) in the theorem. \square

Remark 1. Another approximation to the s.f.s. of Z can be obtained by generalizing the reminder term given in Theorem 1. This yields the approximation at $t \neq E[Z_0]$ given by

$$-\phi(r) \left\{ \frac{r\dot{r}_x}{s} - \left(\frac{\dot{r}_x}{r^2} - \frac{\dot{s}_x}{s^2} \right) \right\} (1 - p_0),$$

where

$$\dot{s}_x = \frac{v_t}{s} \left\{ -\dot{K}'_{Z_0x}(v_t) \left(1 + \frac{v_t}{2} \frac{K''_{Z_0}(v_t)}{K'_{Z_0}(v_t)} \right) + \frac{v_t}{2} K''_{Z_0x}(v_t) \right\} \quad (29)$$

and with other quantities given in Theorem 2. The derivatives appearing in (29) are given by (28), (19),

$$K''_{Z_0}(v_t) = t \left\{ \frac{M'''_N(K(v_t)) \{K'(v_t)\}^2 + 3M''_N(K(v_t))K''(v_t) + \frac{K'''(v_t)}{K'(v_t)}}{M'_N(K(v_t))} - 3 \left(\frac{M''_N(K(v_t))K'(v_t)}{M'_N(K(v_t))} + \frac{K''(v_t)}{K'(v_t)} \right) t + 2t^2 \right\} \quad (30)$$

and by

$$\begin{aligned} \dot{K}''_{Z_0x}(v_t) = t & \left\{ \frac{M'''_N(K(v_t))\dot{K}_x(v_t)K'(v_t) + 2M''_N(K(v_t))\dot{K}'_x(v_t) + \frac{M''_N(K(v_t))\dot{K}_x(v_t)K''(v_t)}{M'_N(K(v_t))K'(v_t)}}{M'_N(K(v_t))} \right. \\ & \left. + \frac{\dot{K}''_x(v_t)}{K'(v_t)} - \left(3 \frac{M''_N(K(v_t))\dot{K}_x(v_t)}{M'_N(K(v_t))} + 2 \frac{\dot{K}'_x(v_t)}{K'(v_t)} + \frac{K''(v_t)\dot{K}_x(v_t)}{\{K'(v_t)\}^2} \right) t + 2 \frac{\dot{K}_x(v_t)}{K'(v_t)} t^2 \right\}. \end{aligned} \quad (31)$$

$\dot{K}_x(v_t)$, $\dot{K}'_x(v_t)$ and $\dot{K}''_x(v_t)$ in (31) can be found respectively in (A1), (A2) and (A3) in the Appendix. The justification follows the proof of Theorem 1. By denoting $s = s(F)$, we have

$$\dot{s}_x = \frac{\partial}{\partial \varepsilon} s(F_{x\varepsilon}) \Big|_{\varepsilon=0} = \dot{v}_{tx} \{K''_{Z_0}(v_t)\}^{\frac{1}{2}} + \frac{v_t}{2} \{K''_{Z_0}(v_t)\}^{-\frac{1}{2}} \frac{\partial}{\partial \varepsilon} K''_{Z_0x\varepsilon}(v_{tx\varepsilon}) \Big|_{\varepsilon=0}.$$

From the multivariate chain rule we obtain

$$\frac{\partial}{\partial \varepsilon} K''_{Z_0x\varepsilon}(v_{tx\varepsilon}) \Big|_{\varepsilon=0} = K'''_{Z_0}(v_t) \dot{v}_{tx} + \dot{K}''_{Z_0x}(v_t).$$

These two last expressions and (27) give (29). Then (30) follows from (A8), (28) follows from (A5) and (31) follows from (A7).

3. Numerical illustrations

This section provides numerical illustrations of the results of Section 2.2 for two important aggregate loss models: the Poisson number of occurrences with gamma individual claim amounts, in Section 3.1, and the geometric number of occurrences with Weibull individual claim amounts, in Section 3.2.

This numerical study is performed with *Matlab* and the function `fminsearch` is used for computing the saddlepoint. *Matlab*'s programs used for these computations are available at <http://www.stat.unibe.ch>.

3.1. Poisson-gamma total claim amount

Assume that the total number of claims of an insurance company that occur during a fixed time horizon, denoted by N , is Poisson distributed with parameter $\lambda > 0$; viz. $p_n = \mathbb{P}[N = n] = e^{-\lambda} \lambda^n / n!$, for $n = 0, 1, \dots$. Let $v \in \mathbb{R}$. The m.g.f. of N and its derivatives are given by

$$M_N(v) = \exp\{\lambda(e^v - 1)\}, \quad M'_N(v) = \lambda e^v \exp\{\lambda(e^v - 1)\} \quad \text{and} \quad M''_N(v) = \lambda e^v (1 + \lambda e^v) \exp\{\lambda(e^v - 1)\}.$$

Assume that the individual claim amounts or losses X_1, X_2, \dots are gamma distributed, with density $f(y) = \beta^\alpha / \Gamma(\alpha) y^{\alpha-1} e^{-\beta y}$, $\forall y > 0$, for some parameters $\alpha, \beta > 0$. Let $v < \beta$. The c.g.f. of X_1 and its derivatives are given by

$$K(v) = \alpha \log \frac{\beta}{\beta - v}, \quad K'(v) = \frac{\alpha}{\beta - v} \quad \text{and} \quad K''(v) = \frac{\alpha}{(\beta - v)^2}.$$

The m.g.f. of the aggregate loss $Z = \sum_{j=0}^N X_j$ is given by

$$M_Z(v) = \exp \left\{ \lambda \left[\left(\frac{\beta}{\beta - v} \right)^\alpha - 1 \right] \right\}$$

and so the c.g.f. of Z_0 , viz. Z given $N > 0$, is given by

$$K_{Z_0}(v) = \log \frac{\exp \left\{ \lambda \left(\frac{\beta}{\beta - v} \right)^\alpha \right\} - 1}{e^\lambda - 1}.$$

121 With these formulae we can obtain the values of s , r and r_x required in Theorem 2. So we can compute
122 the s.f.s. $\tilde{\tau}(t; x, F)$ given in (24).

123 For the numerical illustration, we fix $\lambda = 2$, $\alpha = 2$ and $\beta = 3$. The results are shown in Figure
124 1. The dashed curve shows the saddlepoint approximation $\tilde{G}_Z(t)$ to the s.f., see (22), for all relevant
125 values of t . The four solid curves of Figure 1 show the approximation to the stability $\tilde{\tau}_Z(t; x, F)$, for
126 the perturbation points $x = 1, 2, 5, 10$ and for relevant values of t . The highest curves correspond to
127 the largest values of x . This is what we would have expected. A large perturbation point x yields a
128 large increase of the upper tail probability, so a large value of the stability. A vanishing perturbation
129 point x yields either a small increase or a decrease of the upper tail probability, so a small value
130 of the stability. We should note that the numerical computation of these curves is very fast. Thus
131 the proposed approximation to the s.f.s. inherits the well-known computational efficiency of the
132 saddlepoint approximation. Any purely numerical technique (like Monte Carlo simulation) would be
133 computational intensive and thus computationally slower.

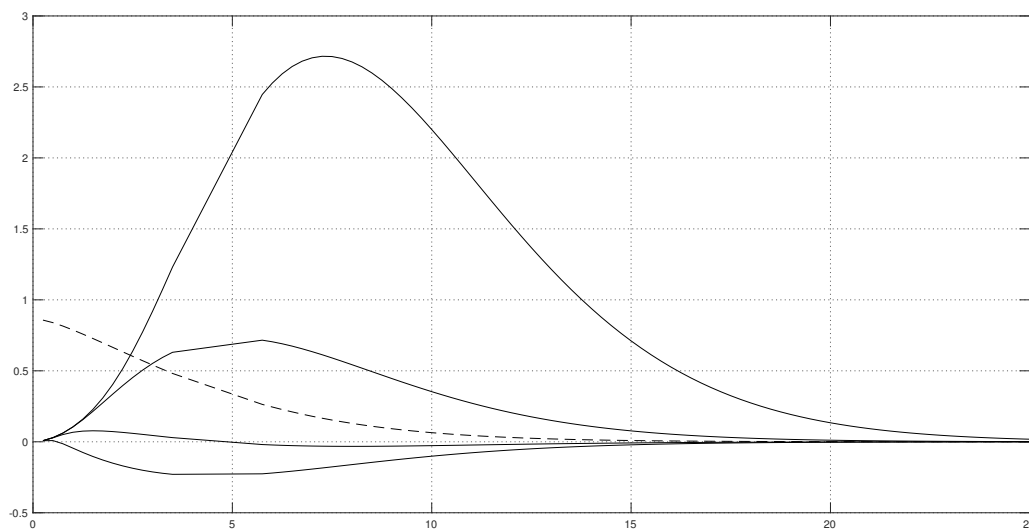


Figure 1. Poisson(2) compound sum of independent Gamma(2,3) random variables. Dashed curve: s.f. Continuous curve, from lowest to highest curve: approximate stabilities for perturbation points $x = 1, 2, 5, 10$, respectively.

134 For a practical illustration, consider the following values from the setting of Figure 1: $\bar{G}_Z(14.75) =$
 135 $0.0099 \simeq 1\%$ and $\tilde{\tau}_Z(14.75; 10, F) = 0.7639$. If the insurance believes that additional claim amounts
 136 of $x = 10$ with small frequency $\varepsilon = 1\%$ have to be considered, then the tail probability of the
 137 non-perturbed model would rise by 7%, because $0.0099 + 0.001 \cdot 0.7636 = 0.0107$.

138 3.2. Geometric-Weibull total claim amount

The suggested approximation is tested with a different aggregate loss model. Assume that the total number of claims N follows the geometric distribution with parameter $\rho \in (0, 1)$, precisely $p_n = P[N = n] = \rho(1 - \rho)^n$, for $n = 0, 1, \dots$. The m.g.f. of N and its derivatives at $v < -\log(1 - \rho)$ are given by

$$M_N(v) = \frac{\rho}{1 - (1 - \rho)e^v}, \quad M'_N(v) = \frac{\rho(1 - \rho)e^v}{\{1 - (1 - \rho)e^v\}^2} \quad \text{and} \quad M''_N(v) = \frac{\rho(1 - \rho)e^v \{1 + (1 - \rho)e^v\}}{\{1 - (1 - \rho)e^v\}^3}.$$

Assume the individual losses X_1, X_2, \dots follow the Weibull distribution with density $f(y) = \alpha y^{\alpha-1} \exp\{-y^\alpha\}$, $\forall y > 0$, for some $\alpha > 0$. We can easily compute its moments $\mu_k = E[X_1^k] = \Gamma(1 + k/\alpha)$, for $k = 1, 2, \dots$. The m.g.f. of the Weibull distribution $M(v) = \int_0^\infty \exp\{vx^{1/\alpha} - x\} dx$ exists for all v over a neighborhood of zero iff $\alpha \geq 1$. Thus, the Weibull distribution is light-tailed in this sense iff $\alpha \geq 1$. Therefore, the power series representation $M(v) = \sum_{k=0}^\infty \mu_k v^k / k!$ holds for any v within a neighborhood of zero. Moreover, for v in this neighborhood,

$$M^{(l)}(v) = \sum_{k=0}^\infty \Gamma\left(1 + \frac{l+k}{\alpha}\right) \frac{v^k}{k!}, \quad \text{for } l = 0, 1, \dots,$$

with $M^{(0)} = M$. With this, the m.g.f. of the aggregate loss can be expressed as $Z = \sum_{j=0}^N X_j$ is given by

$$M_Z(v) = \left\{ 1 - \rho \sum_{k=0}^\infty \Gamma\left(1 + \frac{k}{\alpha}\right) \frac{v^k}{k!} \right\}^{-1}$$

and the c.g.f. of Z_0 can be written as

$$K_{Z_0}(v) = \log \frac{\left\{ 1 - \rho \sum_{k=0}^\infty \Gamma\left(1 + \frac{k}{\alpha}\right) \frac{v^k}{k!} \right\}^{-1} - \rho}{1 - \rho}.$$

139 These formulae allow us to compute s , r and r_x of Theorem 2 and thus we can compute the s.f.s.
 140 $\tilde{\tau}(t; x, F)$ given in (24).

141 For the numerical example, we consider $\rho = 3/10$ and $\alpha = 3$. Figure 2 shows the numerical results.
 142 The dashed curve indicates the saddlepoint approximation $\bar{G}_Z(t)$ to the s.f., cf. (22), for all relevant
 143 values of t . The four solid curves of Figure 2 show the approximation to the s.f.s. $\tilde{\tau}_Z(t; x, F)$, for the
 144 perturbation points $x = 1/2, 3/2, 3, 7$ and for relevant values of t . The highest curves correspond to
 145 the largest values of x . The numerical evaluation of the above series representations of m.g.f. and c.g.f.
 146 does not give any particular problem: after few summands only, numerical convergence is obtained.
 147 We note that the numerical results are similar in nature to the ones of the Poisson-gamma aggregate
 148 loss of Section 3.1. Also, as with the Poisson-gamma model, the approximate s.f.s. can be computed
 149 very fast. Thus it can be conveniently applied to practical problems and it provides an additional
 150 indicator of reliability of the model under uncertainty.

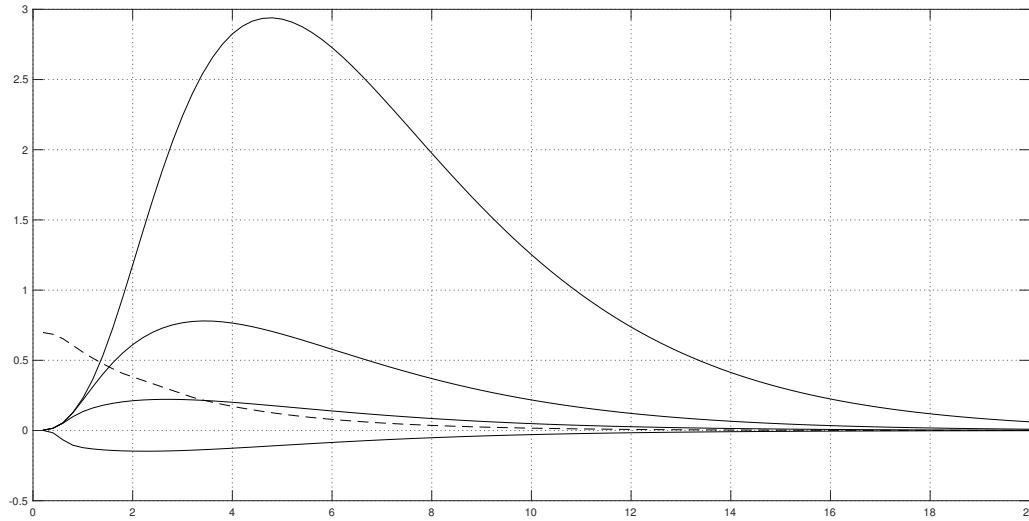


Figure 2. Geometric(3/10) compound sum of independent Weibull(3) random variables. Dashed curve: s.f. Continuous curve, from lowest to highest curve: approximate stabilities for perturbation points $x = 1/2, 3/2, 3, 7$, respectively.

151 Appendix

152 This appendix provides various elementary but long derivatives appearing in the previous
153 developments.

154 Appendix A.1 Derivatives of the cumulant generating function of the perturbed summand

Recall that M and K denote the m.g.f. and the c.g.f. of X_1 . This section gives some derivatives of $K(v)$ under the ε -perturbation, viz. of $K_{x\varepsilon}(v) = \log((1 - \varepsilon)M(v) + \varepsilon e^{vx})$, w.r.t. to v and ε . The results are the following:

$$\frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v) = \frac{e^{vx} - M(v)}{(1 - \varepsilon)M(v) + \varepsilon e^{vx}},$$

$$\dot{K}_x(v) = \frac{e^{vx}}{M(v)} - 1 = e^{vx-K(v)} - 1, \quad (\text{A1})$$

$$K'_{x\varepsilon}(v) = \frac{(1 - \varepsilon)M'(v) + \varepsilon x e^{vx}}{(1 - \varepsilon)M(v) + \varepsilon e^{vx}},$$

$$\frac{\partial}{\partial \varepsilon} K'_{x\varepsilon}(v) = \frac{-M'(v) + x e^{vx}}{(1 - \varepsilon)M(v) + \varepsilon e^{vx}} - \frac{((1 - \varepsilon)M'(v) + \varepsilon x e^{vx})(-M(v) + e^{vx})}{((1 - \varepsilon)M(v) + \varepsilon e^{vx})^2},$$

$$\dot{K}'_x(v) = \frac{e^{vx}(M(v)x - M'(v))}{M^2(v)} = e^{vx-K(v)}(x - K'(v)), \quad (\text{A2})$$

$$K''_{x\varepsilon}(v) = \frac{(1-\varepsilon)M''(v) + \varepsilon x^2 e^{vx}}{(1-\varepsilon)M(v) + \varepsilon e^{vx}} - \frac{((1-\varepsilon)M'(v) + \varepsilon x e^{vx})^2}{((1-\varepsilon)M(v) + \varepsilon e^{vx})^2},$$

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} K''_{x\varepsilon}(v) &= \frac{-M''(v) + x^2 e^{vx}}{(1-\varepsilon)M(v) + \varepsilon e^{vx}} - \frac{((1-\varepsilon)M''(v) + \varepsilon x^2 e^{vx})(-M(v) + e^{vx})}{((1-\varepsilon)M(v) + \varepsilon e^{vx})^2} \\ &\quad - 2 \frac{((1-\varepsilon)M'(v) + \varepsilon x e^{vx})(-M'(v) + x e^{vx})}{((1-\varepsilon)M(v) + \varepsilon e^{vx})^2} \\ &\quad + 2 \frac{((1-\varepsilon)M'(v) + \varepsilon x e^{vx})^2(-M(v) + e^{vx})}{((1-\varepsilon)M(v) + \varepsilon e^{vx})^3} \end{aligned}$$

and

$$\begin{aligned} \dot{K}''_x(v) &= \frac{M^2(v)x^2 e^{vx} - M''(v)M(v)e^{vx} - 2M(v)M'(v)xe^{vx} + 2(M'(v))^2 e^{vx}}{M^3(v)} \\ &= e^{vx-K(v)}((x-K'(v))^2 - K''(v)). \end{aligned} \quad (\text{A3})$$

155 Appendix A.2 Derivatives of the cumulant generating function of the perturbed compound sum

Recall that M_N , K and K_{Z_0} denote the m.g.f. of N and the c.g.f. of X_1 and of Z_0 . This Section gives some derivatives of $K_{Z_0}(v)$ under ε -perturbation of the distribution of X_1 , viz. of $K_{Z_0x\varepsilon}(v) = \log(M_N(K_{x\varepsilon}(v)) - p_0) - \log(1 - p_0)$, w.r.t. to v and ε . The following results are expressed in terms of the derivatives of Appendix A.1:

$$\frac{\partial}{\partial \varepsilon} K_{Z_0x\varepsilon}(v) = \frac{M'_N(K_{x\varepsilon}(v)) \frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v)}{M_N(K_{x\varepsilon}(v)) - p_0},$$

$$\dot{K}_{Z_0x}(v) = \frac{M'_N(K(v)) \dot{K}_x(v)}{M_N(K(v)) - p_0}, \quad (\text{A4})$$

$$K'_{Z_0x\varepsilon}(v) = \frac{M'_N(K_{x\varepsilon}(v)) K'_{x\varepsilon}(v)}{M_N(K_{x\varepsilon}(v)) - p_0},$$

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} K'_{Z_0x\varepsilon}(v) &= \frac{M''_N(K_{x\varepsilon}(v)) \frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v) K'_{x\varepsilon}(v)}{M_N(K_{x\varepsilon}(v)) - p_0} + \frac{M'_N(K_{x\varepsilon}(v)) \frac{\partial}{\partial \varepsilon} K'_{x\varepsilon}(v)}{M_N(K_{x\varepsilon}(v)) - p_0} \\ &\quad - \frac{(M'_N(K_{x\varepsilon}(v)))^2 K'_{x\varepsilon}(v) \frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v)}{(M_N(K_{x\varepsilon}(v)) - p_0)^2}, \end{aligned}$$

$$\dot{K}'_{Z_0x}(v) = \frac{M''_N(K(v)) \dot{K}_x(v) K'(v)}{M_N(K(v)) - p_0} + \frac{M'_N(K(v)) \dot{K}'_x(v)}{M_N(K(v)) - p_0} - \frac{(M'_N(K(v)))^2 K'(v) \dot{K}_x(v)}{(M_N(K(v)) - p_0)^2}, \quad (\text{A5})$$

$$K''_{Z_0x\varepsilon}(v) = \frac{M''_N(K_{x\varepsilon}(v)) (K'_{x\varepsilon}(v))^2}{M_N(K_{x\varepsilon}(v)) - p_0} + \frac{M'_N(K_{x\varepsilon}(v)) K''_{x\varepsilon}(v)}{M_N(K_{x\varepsilon}(v)) - p_0} - \frac{(M'_N(K_{x\varepsilon}(v)))^2 (K'_{x\varepsilon}(v))^2}{(M_N(K_{x\varepsilon}(v)) - p_0)^2}, \quad (\text{A6})$$

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon} K_{Z_0 x \varepsilon}''(v) &= \frac{M_N'''(K_{x\varepsilon}(v)) \frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v) (K'_{x\varepsilon}(v))^2}{M_N(K_{x\varepsilon}(v)) - p_0} + 2 \frac{M_N''(K_{x\varepsilon}(v)) K'_{x\varepsilon}(v) \frac{\partial}{\partial \varepsilon} K'_{x\varepsilon}(v)}{M_N(K_{x\varepsilon}(v)) - p_0} \\
&\quad - 3 \frac{M_N''(K_{x\varepsilon}(v)) (K'_{x\varepsilon}(v))^2 M_N'(K_{x\varepsilon}(v)) \frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v)}{(M_N(K_{x\varepsilon}(v)) - p_0)^2} \\
&\quad + \frac{M_N''(K_{x\varepsilon}(v)) \frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v) K_{x\varepsilon}''(v)}{M_N(K_{x\varepsilon}(v)) - p_0} + \frac{M_N'(K_{x\varepsilon}(v)) \frac{\partial}{\partial \varepsilon} K_{x\varepsilon}''(v)}{M_N(K_{x\varepsilon}(v)) - p_0} \\
&\quad - \frac{(M_N'(K_{x\varepsilon}(v)))^2 K_{x\varepsilon}''(v) \frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v)}{(M_N(K_{x\varepsilon}(v)) - p_0)^2} - 2 \frac{(M_N'(K_{x\varepsilon}(v)))^2 K'_{x\varepsilon}(v) \frac{\partial}{\partial \varepsilon} K'_{x\varepsilon}(v)}{(M_N(K_{x\varepsilon}(v)) - p_0)^2} \\
&\quad + 2 \frac{(M_N'(K_{x\varepsilon}(v)))^3 (K'_{x\varepsilon}(v))^2 \frac{\partial}{\partial \varepsilon} K_{x\varepsilon}(v)}{(M_N(K_{x\varepsilon}(v)) - p_0)^3}, \\
\dot{K}_{Z_0 x}''(v) &= \frac{M_N'''(K(v)) \dot{K}_x(v) (K'(v))^2}{M_N(K(v)) - p_0} + 2 \frac{M_N''(K(v)) K'(v) \dot{K}'_x(v)}{M_N(K(v)) - p_0} \\
&\quad - 3 \frac{M_N''(K(v)) (K'(v))^2 M_N'(K(v)) \dot{K}_x(v)}{(M_N(K(v)) - p_0)^2} + \frac{M_N''(K(v)) \dot{K}_x(v) K''(v)}{M_N(K(v)) - p_0} \\
&\quad + \frac{M_N'(K(v)) \dot{K}_x''(v)}{M_N(K(v)) - p_0} - \frac{(M_N'(K(v)))^2 K''(v) \dot{K}_x(v)}{(M_N(K(v)) - p_0)^2} \\
&\quad - 2 \frac{(M_N'(K(v)))^2 K'(v) \dot{K}'_x(v)}{(M_N(K(v)) - p_0)^2} + 2 \frac{(M_N'(K(v)))^3 (K'(v))^2 \dot{K}_x(v)}{(M_N(K(v)) - p_0)^3} \tag{A7}
\end{aligned}$$

and

$$\begin{aligned}
K_{Z_0}'''(v) &= \frac{M_N'''(K(v)) (K'(v))^3}{M_N(K(v)) - p_0} + 3 \frac{M_N''(K(v)) K'(v) K''(v)}{M_N(K(v)) - p_0} \\
&\quad - 3 \frac{M_N''(K(v)) (K'(v))^3 M_N'(K(v))}{(M_N(K(v)) - p_0)^2} + \frac{M_N'(K(v)) K'''(v)}{M_N(K(v)) - p_0} \\
&\quad - 3 \frac{(M_N'(K(v)))^2 K''(v) K'(v)}{(M_N(K(v)) - p_0)^2} + 2 \frac{(M_N'(K(v)))^3 (K'(v))^3}{(M_N(K(v)) - p_0)^3}. \tag{A8}
\end{aligned}$$

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