Category Theory
in
Explicit Mathematics

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Introduction

The goal of this thesis is to develop a practical framework for category theory in Explicit Mathematics. While doing so we explore some candidates for a category of sets and examine their properties and differences. We also compare notions of universes coming from Explicit Mathematics and from category theory. While universes in Explicit Mathematics have a conceptually very simple description, their definition leads to some problems when used for category theory. For instance they are not closed under isomorphisms. In fact they are not even closed under extensional equality of classes. It turns out to be surprisingly complicated to construct a model for a categorical universe, respectively to prove that it has the required properties. But before giving a more in depth outline, we present a historical overview of foundations in and of category theory and some of Solomon Feferman's ideas which play into them.

Since categories were first introduced by Samuel Eilenberg and Saunders Mac Lane in 1942-1945, there has been lots of work done concerning the foundations of category theory. There have been different approaches to this. One is to try to give an explanation for categories themselves. Another is to use the language of category theory to describe foundations of mathematics. A third one is the usage of existence axioms for specific categories to use as a universe in a similar way as ZFC provides a universe of sets.

The goal of this thesis is not to try and give yet another foundational system, but to see if it is actually possible to reasonably work inside Explicit Mathematics if the questions one is interested in happen to be formulated in categorical language. The need for such a framework arose when trying to formalize a model of a version of cubical type theory in Explicit Mathematics, specifically the system described in [26]. But the fact that there was no intention to develop any kind of foundational system does not mean it was developed in a historical vacuum. Simply by working in Explicit Mathematics, this thesis inherits a lot of ideas of Solomon Feferman, who originally designed and described that framework.Feferman had some strong opinions about category theory and he worked on several systems to use as foundations of what he called 'naive' or 'unrestricted' category theory [19]. While Explicit Mathematics in the form used here is not exactly what he had in mind for category theory, it shares some of the main ideas. The most important being that operations and collections are supposed to be the fundamental concepts for any foundation of categories and indeed of all of mathematics. Since the author of this thesis has no strong opinion on this particular matter, he feels it is best to let some of the involved tell the story themselves:[1]

Feferman wrote in *Categorical foundations and foundations of category theory,* [19]

> The point is simply that *when explaining* the general notion of structure and of particular kinds of structures such as groups, rings, categories, etc. we implicitly *presume as understood* the ideas of *operation* and *collection* (p.151)

> The *logical* and *psychological priority* if not primacy of the notions of operation and collection is thus evident.
>
> It follows that a theory whose objects are supposed to be highly structured and which does not explicitly reveal assumptions about operations and collections cannot claim to constitute a foundation for mathematics, simply because those assumptions are unexamined.

So, his reasons for wanting to explain operations and collections were *not* mathematically motivated, and certainly not fixated on some particular system:

> There are at present two (more or less) coherent and comprehensive approaches to these, based respectively on the Platonist and the constructive viewpoints.[...] It is distinctive of [the first] approach that it is *extensional,* i.e. collections are considered independent of any means of definition. Further, operations are identified with their graphs.
>
> On the other hand, it is distinctive of the constructive point of view that the basic notions are conceived to be *intensional,* i.e.

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[1] Any typos and similar mistakes were introduced in the copying process and are entirely the fault of the author of this thesis.

> operations are supposed to be *given by rules* and collections are supposed to be given by *defining properties.*
>
> [. . .]
>
> Since neither the realist nor the constructivist point of view encompasses the other, there cannot be any present claim to a *universal foundation* for mathematics, unless one takes the line of rejecting all that lies outside the favored scheme. Indeed, *multiple foundations* in this sense may be necessary (p.152)

In particular he rejected the notion that categorical language could be used as a foundation in general, because

> My claim above is that the general concepts of operation and collection have logical priority with respect to structural notions (such as ’group’, ’category’, etc.)
>
> [. . .]
>
> I realize that workers in category theory are so at home in their subject that they find it more natural to think in categorical rather than set-theoretical terms, but I would liken this to not needing to hear, once one has learned to compose music. (p.153)

Feferman gave a specific example of what he considered ’logical priority’:

> My use of ’logical priority’ refers not to relative strength of formal theories but to the order of definition of concepts, in cases where certain of these *must* be defined before others. For example, the concept of vector space is logically prior to that of linear transformation (p.152)

Colin McLarty replied to this specific quote in *Learning from Questions on Categorical Foundations* [35]

> This brings us face to face with mathematical practice. The first mathematicians to work with linear transformations defined them as functions on lists of numbers. They did not define ’vector space’ at all, and at most defined a ’vector’ as either a ’directed line segment’ or an ’ordered triple of numbers’. Those definitions of ’vector’ are still used today by some engineers and even a few physicists. (p.46)

But actually, in the same article, McLarty agreed with a lot of Feferman's notions even if he reinterpreted some of them for his own purposes.

> Obviously I agree with Feferman that foundations of mathematics should lie in a general theory of operations and collections, only I say the currently best general theory of those calls them *arrows* and *objects*. It is category theory. And I think there is no question of whether to seek foundations for structural mathematics. Of course we should. The theoretical unity and practical power of modern structural methods make them, to my ear, actually finer music than proof theory or realism *versus* constructivism. The efficacy of Structuralism in practice makes it a *more* compelling topic for foundations, to me. But Feferman asks the right question - the question of whether we 'hear the music'. It is not a matter of merely technical logic. (p.49)

Even among category theorists not everyone agrees that category theory is a foundational framework, or that such a thing is even necessary. Steve Awodey for example wrote in response to Geoffrey Hellman's [25] in *An answer to Hellman's question: 'Does category theory provide a framework for mathematical Structuralism?'* [1] the following:

> Thus according to our view, there is neither a once-and-for-all universe of mathematical objects, not a once-and-for-all system of all mathematical inferences. Are there, then, various and changing universes and systems? How are they determined, and how are they related? Here I would rather say that there are *no* such universes or systems; or rather, that the question itself is still based on a “foundationalist” preconception about the nature of mathematical statements. (p.4)

He gives an example about the complex numbers:

> To understand (describe) a piece of mathematics (say, that in the complex numbers $i^5 = i$) the foundationalist must “construct” the terms involved (the complex numbers and their multiplication operation, and perhaps even the identity relation) and then prove that the specific entities so constructed do indeed have the stated property. The structuralist can simply observe that
>
> (i) in any ring, if $x^2 = -1$ then $x^5 = x$, and

> (ii) the complex numbers are by definition a ring with an element i such that $i^2 = -1$, and having a couple of other distinctive properties. (p.4)

Awodey described category theory as “schematic”, such that the defined structures apply to many different situations. However, he explicit distinguished the interpretation of such structures from the case of formulas with an implicit universal quantification.

> The “schematic” element in mathematical theorems, definitions, and even proofs is not captured by treating the indeterminate objects involved as universally quantified variables, as quantification requires a fixed domain over which the range of the variable is restricted. This schematic character is more akin, rather, to the phenomenon Russell’s “typical ambiguity” was intended to capture. (p.7)

In a footnote to this, he added that

> In [22], Feferman recognizes the similarity between Russell’s typical ambiguity and category theory’s relative use of the concept of “smallness”.

He went on to describe that this didn’t mean that consistency is irrelevant, rather

> The truth of the consequent statement doesn’t depend on some unknown or unknowable antecedent conditions; rather it *applies* only to those cases specified by the antecedent description. In cases where we are not sure whether the conditions at issue are ever satisfied, i.e. whether they are consistent, we have no recourse but to investigate their consequences in order to gain more information. (p.9)

Finally he had another example of the difference in the interpretation of schematic structures as top-down descriptions, as opposed to a more traditional bottom-up piece by piece construction of all the objects involved.

> Thus rather than saying, for example, “now suppose this particular solar system is an atom in some huge piece of matter in an enormous solar system”, one is instead saying “now suppose

> this particular configuration of bodies occurs not as a solar system, but as an atom in some piece of matter in a solar system". The former assumption indeed requires additional (outrageous) existence assumptions, while the latter requires none.

So far, we have left out the case where we take a specific category, rather than category theory, as our mathematical universe. McLarty commented in *Exploring Categorical Structuralism* [34] on a claim by Hellman [25]

> So Hellman's claim that 'category theory . . . *lacks substantive axioms for mathematical existence*' is a misunderstanding (p.138, Hellman's italics). Indeed category theory *per se* is a general theory applicable to many structures. Each specific categorical foundation offers various quite strong existence axioms. (p.42)

In fact categorical foundations such as CCAF (The Category of Categories as Foundation), have been proposed . McLarty pointed out in [35], that such foundations are exactly *not* set-theoretic foundations, in that

> When we axiomatize a metacategory of categories by the axioms CCAF, the categories are *not 'anything satisfying the algebraic axioms of category theory'* -i.e. the Eilenberg-Mac Lane axioms. They are *anything whose existence follows from the CCAF axioms.* They are precisely not *sets* satisfying the Eilenberg-Mac Lane axioms. They are *categories* as described by Lawvere's CCAF axioms. (p.52)

This thesis is explicitly not an example of this last described kind of foundation described by McLarty. In fact we take the, as Awodey calls it, "foundationalist" approach and describe conditions on classes, respectively on forumlas, to say what it means for something to be a category. It turns out that after we have the required definitions it is possible to do proofs in category theory in a fairly natural style as long a they consist of "diagram chasing". On the other hard, we have strong differences between classes, which are the fundamental tool to describe collections Explicit Mathematics, and sets in any usual set theory. This complicates the translation of proofs which involve the "category of sets" and they require a lot more setup to go through at all.

The thesis is structured in the following way: Chapter one gives an overview over our particular axiomatization **EM** of Explicit Mathematics. Chapter two introduces fundamental definitions of category theory such as categories themselves, functors, natural transformations and limits as well as some further, more specific definitions which are used in the subsequent chapters such as regular categories and cartesian closure. Chapter three sets out to give a category which recovers as many properties as possible of the usual category of sets. As mentioned in the beginning, the universes of Explicit Mathematics don't fit the categorical point of view very well. In chapter 4 we therefore construct an (weaker) instance of a universe of morphisms introduced in [38] which satisfies categorical closure conditions. At the same time it still maintains the closure in the sense of Explicit Mathematics in that any object represented by some element in a universe u of Explicit Mathematics is also present as a morphism from the terminal object to this element in the categorical universe. We also give two more examples of "sets", i.e. objects of one of the categories constructed in chapter 3, which describe cardinal numbers and terms of combinatory logic. Finally, in chapter 5 we prove a version of the Yoneda Lemma and give an outline for future work.

1. Explicit Mathematics

The system used throughout this thesis is a variant of a family of similar systems all going by the name of *Explicit Mathematics.* The first version was described by Solomon Feferman in a series of articles [18], [20] and [21]. The version used here is based on an extension of the usual predicate calculus called the *logic of partial terms* **LPT** due to Beeson [4]. The axiomatization used is due to [23]. When Feferman first proposed it, his system was meant as a formalized but usable version of Bishop's approach to constructive mathematics as it is described in [6]. Later it turned out to be a very useful for proof theory but only few people ever used it for it's initial purpose. One example is the dissertation [40] and articles [39],[41] and [42] by Thomas Studer which uses explicit mathematics to give among other things a semantics for a subset of the programming language Java. Another example are the articles [44],[45] by Sergei Tupailo which give realizability interpretations of subsystems of analysis and constructive set theory into explicit mathematics.

Definition 1.0.1 (Language of **EM**)**.** Unlike many other presentations of the systems of explicit mathematics, we will be very liberal with the inclusion of abbreviations and syntax which does not really belong to the system, but is built on top of it. The official language of $\mathcal{L}_{\mathbf{EM}}$ is built-up from two sorts of variables. The first sort is called *individual variables* and the second sort is called *class variables.* To this we add the following symbols.

(a) *Constants of the first sort*

$$\mathrm{k}, \mathrm{s}, \mathrm{p}, \mathrm{p}_0, \mathrm{p}_1, 0, \mathrm{s}_N, \mathrm{p}_N, \mathrm{d}_N$$

(denoting the usual applicative constants)

$$\mathrm{nat}, \mathrm{id}, \mathrm{co}, \mathrm{un}, \mathrm{dom}, \mathrm{inv}, \mathrm{i}, \sum, \wr$$

(denoting class constructors)

(b) *Constants of the second sort*

N (denoting the natural numbers)

(c) *relation symbols of the first sort*

$=$ (denoting equality on individual terms)

$\downarrow$ (denoting definedness for individual terms)

(d) *Further symbols*

$\cdot$ (denoting a binary function symbol for first sort term application)

$\in$ (denoting a binary relation symbol between individual terms and classes)

$\mathfrak{R}$ (denoting a naming/representation relation between individual terms and classes)

$*$ (denoting the element of "the" one-element class.[1])

◇

Definition 1.0.2 (Individual terms)**.** Individual terms s, t, r are defined inductively from individual variables and constants by use of the binary function symbol $\cdot$ as usual. ◇

Notation 1.0.3. The following notations and abbreviations will serve as basic building blocks.

(a) $1 :\equiv \mathrm{s}_N 0$ and $k = \mathrm{s}_N \cdots \mathrm{s}_N 0$ for natural numbers

(b) *Term application on n inputs* is defined recursively on $n \geq 0$

$$st_1 \dots t_n :\equiv s(t_1, \dots, t_n) :\equiv \begin{cases} s & n = 0 \\ (s \cdot t_1)t_2 \dots t_n & n > 0 \end{cases}$$

(c) *General n-tuple building* is defined recursively on $n \geq 0$

$$\langle s_0, \dots, s_{n-1} \rangle :\equiv \begin{cases} 0 & n = 0 \\ \mathrm{p} s_0 \langle s_1, \dots, s_{n-1} \rangle & n > 0 \end{cases}$$

---

[1] This is added to make the intention of talking about a non-specific one-element class $\{*\}$ clearer. We could have chosen 0 for this and in fact it doesn't matter which (defined) term we use. We will overload $*$ in some contexts to mean other things.

(d) *General projections for n-tuples* are defined recursively on $n \geq 0$

$$\pi_n(s) :\equiv \begin{cases} \mathrm{p}_0 s & n = 0 \\ \pi_{n-1}(\mathrm{p}_1 s) & n > 0 \end{cases}$$

(e) *Lambda abstraction of a variable* x on a term t is defined recursively on the build-up of t

$$\lambda x.t :\equiv \begin{cases} \mathrm{skk} & \text{if } t \text{ is } x \\ \mathrm{k}t & \text{if } t \text{ is a constant or} \\ & \text{a variable that is different from } x \\ \mathrm{s}(\lambda.t_1)(\lambda x.t_2) & \text{if } t \text{ is } t_1 t_2 \end{cases}$$

The term $\lambda x.t$ does *not* contain the variable x. In general, *lambda abstraction* of a list of variables $\overrightarrow{x} :\equiv x_1, \ldots, x_n$ over a term t is defined recursively on $n \geq 0$.

$$\lambda \overrightarrow{x}.t :\equiv \begin{cases} t & n = 0 \\ \lambda x_1.(\lambda x_2 \ldots x_n).t & n > 0 \end{cases}$$

(f) *Partial equality*

$$s \simeq t :\equiv (s{\downarrow} \vee t{\downarrow}) \to (s = t)$$
$$s \neq t :\equiv (s{\downarrow} \wedge t{\downarrow} \wedge \neg(s = t))$$

Note that this means that $(s \neq t)$ is *not* equivalent to $\neg(s = t)$. This convention is because of the strictness axioms which we will introduce below.

(g) *Abbreviations for N*

$$(\exists x \in N)A :\equiv \exists x(x \in N \wedge A)$$
$$(\forall x \in N)A :\equiv \forall x(x \in N \to A)$$
$$t \in (N \to N) :\equiv (\forall x \in N)(tx \in N)$$
$$t \in (N^{k+1} \to N) :\equiv (\forall x \in N)(tx \in (N^k \to N))$$

(h) *Abbreviations for classes*

$$\begin{aligned} U \subseteq V &:\equiv \forall x(x \in U \to x \in V) \\ s \mathrel{\dot\in} t &:\equiv \exists X(\mathfrak{R}(t, X) \wedge s \in X) \\ (\exists x \mathrel{\dot\in} s)A[x] &:\equiv \exists x(x \mathrel{\dot\in} s \wedge A[x]) \\ (\forall x \mathrel{\dot\in} s)A[x] &:\equiv \forall x(x \mathrel{\dot\in} s \to A[x]) \\ s \mathrel{\dot=} t &:\equiv \exists X(\mathfrak{R}(s, X) \wedge \mathfrak{R}(t, X)) \\ s \mathrel{\dot\subseteq} t &:\equiv \exists X \exists Y(\mathfrak{R}(s, X) \wedge \mathfrak{R}(t, Y) \wedge X \subseteq Y) \\ \mathfrak{R}(s) &:\equiv \exists X \mathfrak{R}(s, X) \end{aligned}$$

◇

Definition 1.0.4. The collection of Formulas A, B of **EM** is given by induction

-
  - $t \in N, \qquad t\downarrow, \qquad (s = t)$
  - $(U = V), \qquad t \in U, \qquad \mathfrak{R}(t, U)$
-
  - $\neg A, \qquad A \wedge B, \qquad A \vee B, \qquad A \to B$
  - $\exists x A, \qquad \forall x A$
  - $\exists X A, \qquad \forall X A$

◇

Definition 1.0.5. The theory of **EM** is based on the logic of partial terms with equality **LPT** due to Beeson [4].

(a) *Propositional Axioms and rules*

These comprise of the axioms and rules of inference of some sound Hilbert calculus for classical (or intuitionistic) propositional logic as spelled out in any of the standard texts.

(b) *Quantification*

For A being a formula and t an individual term we have

- $(\forall x A) \wedge t\downarrow \to A[t/x]$

- $A[t/x] \wedge t{\downarrow} \rightarrow \exists x A[t]$

And for A, B being formulas and x not a free variable of A we have rules

$$\frac{A \rightarrow B}{A \rightarrow \forall x B}$$

$$\frac{B \rightarrow A}{\exists x B \rightarrow A}$$

(c) *Equality axioms*

For all variables $u, u_1, \ldots, u_n, v_1, \ldots, v_n$ of the first sort and $U, U_1, \ldots, U_n, V_1, \ldots, V_n$ of the second sort

- $u = u$
- $(u_1 = v_1 \wedge \cdots \wedge u_n = v_n \wedge A[u_1, \ldots, u_n]) \rightarrow A[v_1, \ldots, v_n]$
- $U = U$
- $(U_1 = V_1 \wedge \cdots \wedge U_n = V_n \wedge A[U_1, \ldots, U_n]) \rightarrow A[V_1, \ldots, V_n]$

(d) *Definedness axioms*

For all constants r, all variables u, all terms $s, t, t_1, \ldots, t_n$, all n-ary function symbols F and all n-ary relation symbols R

- $r{\downarrow} \wedge u{\downarrow}$
- $F(t_1, \ldots, t_n){\downarrow} \rightarrow t_1{\downarrow} \wedge \cdots t_n{\downarrow}$
- $t \in N \rightarrow t{\downarrow} \wedge t \in U \rightarrow t{\downarrow}$
- $\mathfrak{R}\,(t, U) \rightarrow t{\downarrow}$
- $R(t_1, \ldots, t_n) \rightarrow t_1{\downarrow} \wedge \ldots \wedge t_n{\downarrow}$

(e) *Partial combinatory algebra*

- $\mathrm{k}ab = a$
- $\mathrm{s}ab{\downarrow} \wedge \mathrm{s}abc \simeq (ac)(bc)$

(f) *Pairing and projection*

- $\mathrm{p}_0(\mathrm{p}ab) = a \wedge \mathrm{p}_1(\mathrm{p}ab) = b$

(g) *Natural numbers*

- $0 \in N \wedge (a \in N \rightarrow \mathrm{s}_N a \in N)$

- $a \in N \to (\mathrm{s}_N a \neq 0 \wedge \mathrm{p}_N(\mathrm{s}_N a) = a)$
- $(a \in N \wedge a \neq 0) \to (\mathrm{p}_N a \in N \wedge \mathrm{s}_N(\mathrm{p}_N a) = a)$

(h) *Definition by numerical cases*

- $a \in N \wedge b \in N \wedge a = b \to \mathrm{d}_N uvab = u$
- $a \in N \wedge b \in N \wedge a \neq b \to \mathrm{d}_N uvab = v$

(i) *Primitive recursion on N*

- $f \in (N^2 \to N) \wedge a \in N \to \mathrm{r}_N fa \in (N \to N)$
- $f \in (N^2 \to N) \wedge a \in N \wedge b \in N \wedge h = \mathrm{r}_N fa$
  $\to h0 = a \wedge h(\mathrm{s}_N b) = fb(hb)$

(j) *Explicit representation and extensionality*

- $\exists x \mathfrak{R}(x, U)$
- $\mathfrak{R}(s, U) \wedge \mathfrak{R}(s, V) \to U = V$
- $\forall x(x \in U \leftrightarrow x \in V) \to U = V$

(k) *Class existence axioms*

- $\mathfrak{R}(\mathrm{nat}) \wedge \forall x(x \mathrel{\dot{\in}} \mathrm{nat} \leftrightarrow x \in N)$
- $\mathfrak{R}(\mathrm{id}) \wedge \forall x(x \mathrel{\dot{\in}} \mathrm{id} \leftrightarrow \exists y(x = \langle y, y\rangle))$
- $\mathfrak{R}(s) \to \mathfrak{R}(\mathrm{co}(s)) \wedge \forall x(x \mathrel{\dot{\in}} \mathrm{co}(s) \leftrightarrow \neg(x \mathrel{\dot{\in}} s))$
- $\mathfrak{R}(s) \wedge \mathfrak{R}(t) \to \mathfrak{R}(\mathrm{un}(s,t)) \wedge \forall x(x \mathrel{\dot{\in}} \mathrm{un}(s,t) \leftrightarrow x \mathrel{\dot{\in}} s \vee x \mathrel{\dot{\in}} t)$
- $\mathfrak{R}(s) \wedge \mathfrak{R}(t) \to \mathfrak{R}(\mathrm{int}(s,t)) \wedge \forall x(x \mathrel{\dot{\in}} \mathrm{int}(s,t) \leftrightarrow x \mathrel{\dot{\in}} s \wedge x \mathrel{\dot{\in}} t)$
- $\mathfrak{R}(s) \to \mathfrak{R}(\mathrm{dom}(s)) \wedge \forall x(x \mathrel{\dot{\in}} \mathrm{dom}(s) \leftrightarrow \exists y(\langle x, y\rangle \mathrel{\dot{\in}} s))$
- $\mathfrak{R}(s) \to \mathfrak{R}(\mathrm{all}(s)) \wedge \forall x(x \mathrel{\dot{\in}} \mathrm{all}(s) \leftrightarrow \forall y(\langle x, y\rangle \mathrel{\dot{\in}} s))$
- $\mathfrak{R}(s) \to \mathfrak{R}(\mathrm{inv}(s,f)) \wedge \forall x(x \mathrel{\dot{\in}} \mathrm{inv}(s,f) \leftrightarrow fx \mathrel{\dot{\in}} s)$

Note that this axiomatization is not minimal when using classical logic.

(l) *$\mathcal{L}$ induction on N* ($\mathcal{L}$-I_N)

For all $\mathcal{L}$ formulas $A[u]$ we have

- $A[0] \wedge (\forall x \in N)(A[x] \to A[\mathrm{s}_N x]) \to (\forall x \in N)(A[x]))$

(m) *Class induction on N* (C-I_N)

- $\forall X(0 \in X \land (\forall x \in N)(x \in X \to \mathrm{s}_N x \in X)$
  $\to (\forall x \in N)(x \in X))$

With Induction is restricted to (C-I_N), we will call the system **REM** while **EM** is reserved for the case where full $\mathcal{L}$ induction on N is available.

◇

Proposition 1.0.6 (Some Properties about pairing and lambda abstraction)**.** *The following all hold*

- *Let* $0 \leq k < n$ *be fixed natural numbers and* $a_0, \ldots, a_{n-1}$ *variables, then*
  - $\pi_k(\langle a_0, \ldots, a_{n-1}\rangle) = a_k$
- *For each term* t *and all variables* x :
  - $(\lambda x.t){\downarrow} \land (\lambda x.t)x \simeq t$
  - $s{\downarrow} \to (\lambda x.t)s \simeq t[s/x]$
- *For each term* t *and distinct variables* x
  - $((\lambda x.t)[s/y]x) \simeq (\lambda x.t[s/y])x.$
- *There exists a closed term fix such that*
  - $\mathit{fix} f{\downarrow} \land \mathit{fix} f x \simeq f(\mathit{fix} f)x.$

Proof. The first claim follows by induction, the rest is proved in [23]. ☐

Definition 1.0.7 (Join and inductive generation axioms)**.** The following two axioms will be pointed out when we use them and will not be assumed by default.

(n) *Join axiom*

- $\mathfrak{R}(a) \land (\forall x \mathrel{\dot\in} a)\mathfrak{R}(fx) \to \mathfrak{R}(\sum(a, f)) \land$
  $\forall x(x \mathrel{\dot\in} \sum(a, f) \leftrightarrow x = \langle \pi_0 x, \pi_1 x\rangle \land \pi_0 x \mathrel{\dot\in} a \land \pi_1 x \mathrel{\dot\in} f(\pi_0 x))$

(o) *Inductive generation*

For this let D be an arbitrary formula. We first define the formula

$$\mathrm{Closed}[a, r, D] :\equiv (\forall x \mathrel{\dot\in} a)(\forall y(\langle y, x\rangle \mathrel{\dot\in} r \to D[y]) \to D[x]).$$

Now the IG axioms can be written as

- $\mathfrak{R}\,(a) \wedge \mathfrak{R}\,(b) \to \mathfrak{R}\,(\mathrm{i}(a,b)) \wedge \mathrm{Closed}[a,b,i(a,b)]$
- $\mathfrak{R}\,(a) \wedge \mathfrak{R}\,(b) \wedge \mathrm{Closed}[a,b,D] \to (\forall x \in \mathrm{i}(a,b))D[x]$ ◇

Definition 1.0.8 (Universes and the limit axiom [27])**.** To define universes, we need some additional notation. Let $\mathcal{C}[S,a]$ be the closure condition given by the disjunction of the following formulas:

(a) $a = \mathrm{nat} \vee a = \mathrm{id}$,

(b) $\exists x(a = \mathrm{co}(x) \wedge x \in S)$,

(c) $\exists x \exists y(a = \mathrm{un}(x,y) \wedge x \in S \wedge y \in S)$,

(d) $\exists x \exists y(a = \mathrm{int}(x,y) \wedge x \in S \wedge y \in S)$,

(e) $\exists f(a = \mathrm{dom}(x) \wedge x \in S)$,

(f) $\exists f(a = \mathrm{all}(x) \wedge x \in S)$,

(g) $\exists f \exists x(a = \mathrm{inv}(f,x) \wedge x \in S)$,

(h) $\exists x \exists f(a = \sum(x,f) \wedge x \in S \wedge (\forall y \in x)(fy \in S))$.

The formula $\forall x(\mathcal{C}[S,x] \to x \in S)$ describes that S is a class which is closed under the class constructors mentioned. A universe is then a class which consists of names only and which satisfies this closure condition.

We write use the following formula to say that a class S is a universe:

$$\begin{aligned} U(S) &:\equiv \forall x(\mathcal{C}[S,x] \to x \in S) \wedge (\forall x \in S)\mathfrak{R}\,(x) \\ \mathcal{U}(t) &:\equiv \exists X(\mathfrak{R}\,(t,X) \wedge U(S)) \end{aligned}$$

The limit axiom is then given by

$$\forall x(\mathfrak{R}\,(x) \to \mathcal{U}(\mathfrak{l}x) \wedge x \mathrel{\dot{\in}} \mathfrak{l}x). \qquad \diamond$$

Definition 1.0.9. We call a formula of **EM** *elementary* if it contains no bound class variables and no instance of the symbol $\mathfrak{R}$. ◇

Theorem 1.0.10 (Elementary Comprehension [23])**.** *For every elementary formula $A[u,\vec{v},\vec{W}]$ with at most the indicated free variables there exists a closed term t_A such that* **REM** *proves:*

(a) $\mathfrak{R}(\vec{z}) \to \mathfrak{R}(t_A(\vec{y},\vec{z}))$

(b) $\mathfrak{R}\left(\vec{z},\vec{Z}\right) \to \forall x(x \mathrel{\dot{\in}} t_A(\vec{y},\vec{z}) \leftrightarrow A[x,\vec{y},\vec{Z}])$.

Proof. We prove this by induction on the structure of the elementary formula $A[u,\vec{v},\vec{W}]$.

(a)

$$t_A :\equiv \begin{cases} \lambda\vec{v}.\lambda\vec{w}.\mathrm{inv}(\mathrm{id},\lambda u.\langle r,s\rangle) & \text{if } A[u,\vec{v},\vec{W}] \text{ is } r=s, \\ \lambda\vec{v}.\lambda\vec{w}.\mathrm{inv}(\mathrm{id},\lambda u.\langle s,s\rangle) & \text{if } A[u,\vec{v},\vec{W}] \text{ is } s\downarrow, \\ \lambda\vec{v}.\lambda\vec{w}.\mathrm{inv}(\mathrm{nat},\lambda u.s) & \text{if } A[u,\vec{v},\vec{W}] \text{ is } s \in N, \\ \lambda\vec{v}.\lambda\vec{w}.\mathrm{inv}(w_i,\lambda u.s) & \text{if } A[u,\vec{v},\vec{W}] \text{ is } s = W_i. \end{cases}$$

(b) If $A[u,\vec{v},\vec{W}]$ is of the form $\neg B[u,\vec{v},\vec{W}]$ we set

$$t_A :\equiv \lambda\vec{v}.\lambda\vec{w}.\mathrm{co}(t_B(\vec{v},\vec{w})).$$

(c) If $A[u,\vec{v},\vec{W}]$ is of the form $B_1[u,\vec{v},\vec{W}] \vee B_2[u,\vec{v},\vec{W}]$ we set

$$t_A :\equiv \lambda\vec{v}.\lambda\vec{w}.\mathrm{un}(t_{B_1}(\vec{v},\vec{w}),t_{B_2}(\vec{v},\vec{w})).$$

(d) If $A[u,\vec{v},\vec{W}]$ is of the form $B_1[u,\vec{v},\vec{W}] \wedge B_2[u,\vec{v},\vec{W}]$ we set

$$t_A :\equiv \lambda\vec{v}.\lambda\vec{w}.\mathrm{int}(t_{B_1}(\vec{v},\vec{w}),t_{B_2}(\vec{v},\vec{w})).$$

(e) If $A[u,\vec{v},\vec{W}]$ is of the form $\exists a B[a,u,\vec{v},\vec{W}]$ we set

$$C[u,\vec{v},\vec{W}] :\equiv B[\pi_1 u,\pi_0 u,\vec{v},\vec{W}].$$

By induction hypothesis we get

a) $\mathfrak{R}(\vec{z}) \to \mathfrak{R}(t_C(\vec{y},\vec{z}))$
b) $\mathfrak{R}\left(\vec{z},\vec{Z}\right) \to \forall x(x \mathrel{\dot{\in}} t_C(\vec{y},\vec{z}) \leftrightarrow B[\pi_1 x,\pi_0 x,\vec{y},\vec{Z}])$.

This allows us to set

$$t_A :\equiv \lambda\vec{v}.\lambda\vec{w}.\mathrm{dom}(t_C(\vec{v},\vec{w})).$$

To verify this we use the axiom about dom :

$$\begin{aligned} x \mathrel{\dot\in} \mathrm{dom}(t_C(\vec{v},\vec{w})) &\leftrightarrow \exists y(\langle x,y\rangle \mathrel{\dot\in} t_C(\vec{v},\vec{w})) \\ &\leftrightarrow \exists y(B[\pi_1\langle x,y\rangle, \pi_0\langle x,y\rangle, \vec{v},\vec{w}]) \\ &\leftrightarrow \exists y(B[y,x,\vec{v},\vec{w}]) \end{aligned}$$

(f) If $A[u,\vec{v},\vec{W}]$ is of the form $\forall a B[a,u,\vec{v},\vec{W}]$ we set

$$C[u,\vec{v},\vec{W}] :\equiv B[\pi_1 u, \pi_0 u, \vec{v},\vec{W}].$$

By induction hypothesis we get

a) $\mathfrak{R}\,(\vec{z}) \to \mathfrak{R}\,(t_C(\vec{y},\vec{z}))$

b) $\mathfrak{R}\left(\vec{z},\vec{Z}\right) \to \forall x(x \mathrel{\dot\in} t_C(\vec{y},\vec{z}) \leftrightarrow B[\pi_1 x, \pi_0 x, \vec{y}, \vec{Z}])$.

This allows us to set

$$t_A :\equiv \lambda\vec{v}.\lambda\vec{w}.\mathrm{all}(t_C(\vec{v},\vec{w})).$$

To verify this we use the axiom about dom :

$$\begin{aligned} x \mathrel{\dot\in} \mathrm{all}(t_C(\vec{v},\vec{w})) &\leftrightarrow \forall y(\langle x,y\rangle \mathrel{\dot\in} t_C(\vec{v},\vec{w})) \\ &\leftrightarrow \forall y(B[\pi_1\langle x,y\rangle, \pi_0\langle x,y\rangle, \vec{v},\vec{w}]) \\ &\leftrightarrow \forall y(B[y,x,\vec{v},\vec{w}]) \end{aligned} \qquad \square$$

Now that we have our “official” language, we will add some general notes on abbreviations. Where it is clear from context, we will deviate from the language described above. Most importantly, we will freely use lower-case Greek letters for individual terms. If we introduce terms representing binary functions, we will generally use infix notation. In particular we will write $(f \circ g)$ for the application of the term $\circ$ to f and g, $(\circ \cdot f) \cdot g$. Similarly, if we introduce a symbol for a formula with two free variables we will also sometimes write this infix for example in the case of $a =_o b$ which stands for some formula with free variables u, v which get substituted with a and b resulting in $=_o[a/u, b/v]$.

Unlike most other presentations of Explicit Mathematics we will not use subscripts like $(\langle a,b,c\rangle)_2 = c$ but instead use subscripts generally as part of variable names like we would in informal proofs. So r_0, r_1 may well be the first two elements of some tuple $\langle r_0, r_1, \dots, t\rangle$ but if so this would be coincidental. Another example will be terms for an inverse of one kind or another. Those will be written as $f^{-1}\{x\}$ for the class of a preimage (restrictions of $\mathrm{inv}(s, f)$) and $(\cdot)^{-1}$ for an inverse morphism as used in a group. Keep in mind that when we desugar this it will just be a term $t_{(\cdot)^{-1}}$ with the intuition that $t_{(\cdot)^{-1}}(a)$ should represent $(a)^{-1}$.

Notation 1.0.11. We will sometimes use set-builder for elementary comprehension. $\{u \mid A[u, \vec{v}, \vec{W}]\}$ will stand for the term t_A which elementary comprehension provides for the formula $A[u, \vec{v}, \vec{W}]$. We will also write $\{k_0, \dots, k_n\}$ for the term t_B where $B :\equiv u = k_0 \vee \dots \vee u = k_n$. Furthermore we will often define a term the following way:

$$s(\vec{v}, \vec{w}) :\equiv \{u \mid A[u, \vec{v}, \vec{W}]\}.$$

This is again meant to be a meta-variable for a term t_A we get from elementary comprehension such that

$$\bigwedge \Re\,(w_i, W_i) \to (x \,\dot{\in}\, t_A(\vec{v}, \vec{w}) \leftrightarrow A[x, \vec{v}, \vec{W}]).$$

Note that these names are in principle not unique! however, our proof of elementary comprehension gives a recursive procedure (from the outside) to generate these names uniquely so for that purpose we can treat t_A for a fixed formula A as a unique name. This means as long as there are no other undetermined variables in $A[u]$ it makes sense to write

$$x \,\dot{\in}\, \{u \mid A[u]\}. \qquad \diamond$$

Notation 1.0.12. Syntax for definition by numerical cases: Consider the following function on the natural numbers

$$f(x) :\equiv \begin{cases} 1 & x = 0 \\ 2 & x = 1 \\ x - 1 & x > 1. \end{cases}$$

This is not valid syntax in explicit mathematics, but it can be approximated by recursively defining the right-hand side using the constant d_N for definition by numerical cases.

$$\text{"}\begin{cases} k_0 & x = i_0 \end{cases}\text{"} :\equiv \qquad \mathrm{d}_N k_0 0 x i_0$$

$$\text{"}\begin{cases} k_0 & x = i_0 \\ k_1 & x = i_1 \\ \vdots & \\ k_n & x = i_n \end{cases}\text{"} :\equiv \qquad \mathrm{d}_N k_0 \left(\begin{cases} k_1 & x = i_1 \\ \vdots & \\ k_n & x = i_n \end{cases} \right) x i_0$$

we will also allow an optional clause for “otherwise” which translates to “provably not equal”

$$\begin{cases} k_0 & x = i_0 \\ k_1 & \mathit{otherwise} \end{cases} :\equiv \qquad \mathrm{d}_N k_0 k_1 x i_0$$

Note that the usage of definition by cases in notation 1.0.3 is *from the outside* to define terms by induction, while it is used here as part of the syntax. ◇

Notation 1.0.13. Having mentioned binary infix operations above we add some “standard” names of classes

$$\begin{aligned} P_2[u, A, B] &:\equiv u = \langle \pi_0 u, \pi_1 u \rangle \wedge \pi_0 u \in A \wedge \pi_1 u \in B \\ P_\infty[u, A, B] &:\equiv (\forall x \in A)(\langle x, ux \rangle \in B) \\ F[u, A, B] &:\equiv (\forall x \in A)(ux \in B) \\ Z[u] &:\equiv u = * \\ E[u] &:\equiv 0 = 1 \\ C[u, A, B] &:\equiv u \in A \wedge u \in B \\ D[u, A] &:\equiv u = \langle \pi_0 u, \pi_1 u \rangle \wedge \pi_0 u = \pi_1 u \wedge \pi_0 u \in A \end{aligned}$$

The terms resulting from elementary comprehension will then be called

$$\begin{aligned}
a \times b &:\equiv t_{P_2}(a,b) \\
\prod(a,f) &:\equiv t_{P_\infty}(a,\sum(a,f)) \\
a \dot{\to} b &:\equiv t_F(a,b) \\
\mathbb{1} &:\equiv t_Z \\
\emptyset &:\equiv t_E \\
a \cap b &:\equiv t_C(a,b) \\
\Delta(a) &:\equiv t_D(a)
\end{aligned}$$

◇

Finally, when writing down classes built from the Join axiom, we will adopt a type theory-inspired style where it looks nicer:

$$\begin{aligned}
\sum_{x:a}(px) :\equiv \sum_{x \dot{\in} a}(px) :\equiv \sum(a, \lambda x.px) \\
\prod_{x:a}(px) :\equiv \prod_{x \dot{\in} a}(px) :\equiv \prod(a, \lambda x.px).
\end{aligned}$$

As an example consider the class of all operations between any two classes in a universe u :

$$\sum_{x:u}\sum_{y:u}(x \dot{\to} y)$$

2. Axiomatic Category Theory

2.1. Axiomatic Category Theory

When Explicit Mathematics was first introduced by Feferman, it was designed as a system to be actually used for formalizing mathematics. Here we are going demonstrate how to formalize (parts of) category theory.

Definition 2.1.1 (Language of Category Theory)**.** The language for category theory in our framework doesn't need any extension in the technical sense. We simply fix some standard variable names for four formulas and two terms. Mac Lane gives a formulation in CWM ([33]) which mentions only morphisms and morphism equality but the language used here has been chosen for convenience more than for minimalism.

$$\begin{aligned}
\mathit{Ob}(\cdot) &\text{ – Objects (formula with one parameter)} \\
\mathit{Mor}(\cdot) &\text{ – Morphisms (formula with one parameter)} \\
(\cdot =_o \cdot) &\text{ – equality on objects (formula with two parameters)} \\
(\cdot =_m \cdot) &\text{ – equality on morphisms (formula with two parameters)} \\
\mathit{id}(\cdot) &\text{ – identity morphism (term)} \\
(\cdot \circ \cdot) &\text{ – composition of morphisms (term)}
\end{aligned}$$

We will also fix terms $\mathit{dom}(\cdot)$ and $\mathit{cod}(\cdot)$ for domain and codomain right now as abbreviations and add another abbreviation called $\overline{(\cdot)}$ which while non-standard, is needed for technical reasons. Given a three-tuple f we set:

$$\mathit{dom}(f) :\equiv \pi_0 f \qquad \mathit{cod}(f) :\equiv \pi_1 f \qquad \overline{f} :\equiv \pi_2 f \qquad \diamond$$

Definition 2.1.2 (Weak category)**.** Given a tuple of formulas and terms $(Ob, Mor, =_o, =_m, id, \circ)$, we define the abbreviation

$$(f : a \overset{m}{\to} b) :\equiv Mor(f) \wedge dom(f) =_o a \wedge cod(f) =_o b.$$

Such a tuple is a weak category if the following axioms are satisfied.

$$\begin{aligned}
&(C1) && Mor(f) \to \exists g_0, g_1, g_2(f = \langle g_0, g_1, g_2\rangle \\
& && \wedge Ob(g_0) \wedge Ob(g_1)) \\
&(C2) && Ob(x) \to id(x) : x \overset{m}{\to} x \\
&(C3) && Mor(f) \wedge Mor(g) \wedge dom(f) =_o cod(g) \\
& && \to (f \circ g){\downarrow} \wedge (f \circ g) : dom(g) \overset{m}{\to} cod(f) \\
&(C4) && dom(f) =_o cod(g) \wedge dom(g) =_o cod(h) \\
& && \to f \circ (g \circ h) =_m (f \circ g) \circ h \\
&(C5) && dom(f) =_o c \to f \circ id(c) =_m f \\
&(C6) && cod(f) =_o c \to id(c) \circ f =_m f
\end{aligned}$$

The above are the axioms of what Mac Lane ([33]) calls a *metacategory* if we assume $=_o$ and $=_m$ to be equality. For the general case we have to add those axioms explicitly.

$$\begin{aligned}
&(EI1) && c =_o d \to id(c) =_m id(d) \\
&(EI2) && f =_m g \to dom(f) =_o dom(g) \\
&(EI3) && f =_m g \to cod(f) =_o cod(g) \\
&(CO1) && dom(f) =_o cod(h) \wedge f =_m g \to f \circ h =_m g \circ h \\
&(CO2) && dom(h) =_o cod(f) \wedge f =_m g \to h \circ f =_m h \circ g \\
&(EO1) && (x =_o y \to Ob(x) \wedge Ob(y)) \wedge (Ob(x) \to x =_o x) \\
&(EO2) && x =_o y \to y =_o x \\
&(EO3) && x =_o y \wedge y =_o z \to x =_o z \\
&(EM1) && (f =_m g \to Mor(f) \wedge Mor(g)) \wedge (Mor(f) \to f =_m f) \\
&(EM2) && f =_m g \to g =_m f \\
&(EM3) && f =_m g \wedge g =_m h \to f =_m h
\end{aligned}$$

If the context is clear, this can also be extended to several morphism-terms which stands for the conjunction of the above formula for all morphisms. Similarly, we will sometimes annotate whole equations with an arrow to specify the domain and codomain:

$$f =_m g : a \overset{m}{\to} b.$$

This will be taken to mean

$$(f, g : a \overset{m}{\to} b) \wedge f =_m g.$$

Similarly, we're going to write

$$\begin{aligned} Ob(x_1, \ldots, x_k) &:\equiv Ob(x_1) \wedge \cdots \wedge Ob(x_k) \\ Mor(f_1, \ldots, f_l) &:\equiv Mor(f_1 \wedge \cdots \wedge Mor(f_l) \end{aligned}$$

for the conjunctions.

Given a tuple $(Ob, Mor, =_o, =_m, id, \circ)$ We will use $CAT[\mathcal{C}]$ as a shorthand for the conjunction of the above formulas where we substitute specific formulas and terms for Ob, Mor etc. The variables $\mathcal{C}, \mathcal{D}, \ldots$ will run over weak categories and will be used to distinguish between the tuples of formulas and terms. Given $CAT[\mathcal{C}]$ and $CAT[\mathcal{D}]$ we will then write $Ob_{\mathcal{C}}$, $Mor_{\mathcal{D}}$ etc. when we have to talk about those weak categories. ◇

Remark 2.1.3. The axioms $(EO1)$ and $(EM1)$ require that $x =_o y \to Ob(x) \wedge Ob(y)$ and similarly for morphisms. In the examples given below we will often give equality-predicates which are defined on a much wider collection of terms e.g. equality $a = b$ on objects even if objects are only defined $Ob(x) :\equiv (x = 0)$. In these cases we will consistently omit any explicit restriction, but it's clear that we can always just add $Ob(x) \wedge Ob(y)$ to a formula to make these axioms true. ◇

Example 2.1.4 (Weak category of one object and one morphism)**.** The simplest possible non-empty category can be defined by

$$\begin{aligned} Ob(o) &:\equiv (o = 0) \\ Mor(f) &:\equiv (f = \langle 0, 0, 0\rangle) \\ a =_o b &:\equiv a = b \\ f =_m g &:\equiv f = g \\ id(o) &:\equiv \langle 0, 0, 0\rangle \\ \circ(f, g) &:\equiv f. \end{aligned}$$

Here we chose the arbitrary term $0 \in N$ as a representation for our only object, but any t with $t\downarrow$ would work. Note that if we define all relations to be $\top$ with constant operations we actually get an isomorphic[1] category. ◇

Example 2.1.5 (Weak empty category)**.** A weak empty category is given by

$$\begin{aligned} Ob(o) &:\equiv \bot \\ Mor(f) &:\equiv \bot \\ a =_o b &:\equiv \bot \\ f =_m g &:\equiv \bot \\ id(o) &:\equiv \langle 0, 0, 0\rangle \\ \circ(f, g) &:\equiv f. \end{aligned}$$

The terms for the identity morphisms and composition are completely arbitrary, since the axioms only require them to behave correctly on actual objects and morphisms. ◇

The definition of a weak category is useful because it can be applied to a lot of structures. But similar to categories formalized in set theories, we would like our categories to be small in relation to some universe(s). We will work with classes we get from elementary comprehension, the join-axiom and the limit axiom for some universe. Instead of restricting a weak category to a certain form, we will present the axioms again in a self-contained manner. Of course it will still hold that any category is a

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[1] For an appropriate notion of *isomorphism.*

weak category by letting class membership be the corresponding formula while keeping the operations the same.

Definition 2.1.6 (Category)**.** A *category* (which we will occasionally call *non-weak* for emphasis) is a six-tuple $\langle ob, mor, id, \circ, =_o, =_m \rangle$ which satisfies the following properties:

$$
\begin{array}{rl}
(CL) & \mathfrak{R}\,(ob) \wedge \mathfrak{R}\,(mor) \wedge \mathfrak{R}\,(=_o) \wedge \mathfrak{R}\,(=_m) \\
(MOR) & (\forall m \mathrel{\dot{\in}} mor)(\exists x, y \mathrel{\dot{\in}} ob)(m = \langle x, y, \pi_2 m \rangle) \\
(CMP1) & (\forall f, g \mathrel{\dot{\in}} mor)(cod(g) =_o dom(f) \\
 & \quad \to (f \circ g)\!\downarrow \wedge\, (f \circ g) : dom(g) \overset{m}{\to} cod(f)) \\
(CMP2) & (\forall f, g, h \mathrel{\dot{\in}} mor) \\
 & \quad (cod(g) =_o dom(f) \wedge cod(h) =_o dom(g) \\
 & \quad \to (f \circ g) \circ h =_m f \circ (g \circ h)) \\
(EQO1) & (=_o \dot{\subset}\ ob \times ob) \wedge (\forall x \mathrel{\dot{\in}} ob)(x =_o x) \\
(EQO2) & (\forall x, y \mathrel{\dot{\in}} ob)(x =_o y \to y =_o x) \\
(EQO3) & (\forall x, y, z \mathrel{\dot{\in}} ob)(x =_o y \wedge y =_o z \to x =_o z) \\
(EQM1) & (=_m \dot{\subset}\ mor \times mor) \wedge (\forall f \mathrel{\dot{\in}} mor)(f =_m f) \\
(EQM2) & (\forall f, g \mathrel{\dot{\in}} mor)(f =_m g \to g =_m f) \\
(EQM3) & (\forall f, g, h \mathrel{\dot{\in}} mor)(f =_m g \wedge g =_m h \to f =_m h) \\
(EQC1) & (\forall f, g, h \mathrel{\dot{\in}} mor)(f =_m g \wedge cod(h) =_o dom(f) \\
 & \quad \to (f \circ h =_m g \circ h)) \\
(EQC2) & (\forall f, g, h \mathrel{\dot{\in}} mor)(g =_m h \wedge cod(g) =_o dom(f) \\
 & \quad \to (f \circ g =_m f \circ h)) \\
(ID1) & (\forall x \mathrel{\dot{\in}} ob)(id(x) \mathrel{\dot{\in}} mor \\
 & \quad \wedge\, dom(id(x)) =_o x \wedge cod(id(x)) =_o x) \\
(ID2) & (\forall x \in ob)(\forall f \mathrel{\dot{\in}} mor)(dom(f) =_o x \to f \circ id(x) =_m f) \\
(ID3) & (\forall x \in ob)(\forall f \mathrel{\dot{\in}} mor)(cod(f) =_o x \to id(x) \circ f =_m f)
\end{array}
$$

Given a universe u (such that $\mathcal{U}(u)$ holds) and a category which additionally satisfies $(UCAT)$, we call it a *category relative to* u.

$$\begin{aligned}(UCAT)\quad & (ob \mathrel{\dot{\in}} u) \land (mor \mathrel{\dot{\in}} u) \land ({=_o} \mathrel{\dot{\in}} u) \land ({=_m} \mathrel{\dot{\in}} u) \\ & \land (\{ob, mor, =_o, =_m, cod, dom, id, \circ\} \mathrel{\dot{\in}} u)\end{aligned}$$

Here we have used the abbreviations:

$$\begin{aligned} x =_o y &:\equiv \langle x, y\rangle \mathrel{\dot{\in}} {=_o} \\ f =_m g &:\equiv \langle f, g\rangle \mathrel{\dot{\in}} {=_m} \\ f \circ g &:\equiv \circ(f, g) \\ f : a \overset{m}{\to} b &:\equiv f \mathrel{\dot{\in}} mor \land dom(f) =_o a \land cod(f) =_o b \end{aligned}$$

$\diamond$

Notation 2.1.7. Now that we have classes which describe our category, we can use them in elementary comprehension. Using this in a readable way requires another abbreviation. Suppose we are given a category $\langle ob, mor, id, \circ, =_o=_m\rangle$. If we write down an arbitrary formula which is elementary except for sub-formulas of the form $t \mathrel{\dot{\in}} ob$, $t \mathrel{\dot{\in}} mor$, $s =_o t$ and $s =_m t$ for any terms s, t. Then we can translate this into an elementary formula by substituting $t \in OB$ for $t \mathrel{\dot{\in}} ob$ etc. This new formula now has all free variables from before plus four new class variables. If it were of the form $A[u, ob, mor, id, \circ, =_o, =_m, \vec{x}, \vec{W}]$ then it would now have the form $A'[u, ob, mor, id, \circ, =_o, =_m, \vec{x}, \vec{W}, OB, MOR, EQO, EQM]$ and we can use elementary comprehension to get the term t_A and apply it to the following arguments:

$$t_A(ob, mor, id, \circ, =_o=_m, \vec{x}, \vec{w}, ob, mor, =_o, =_m).$$

Now writing this out for every formula gets tedious, so consider $\mathcal{C}$ standing for the category given by $\langle ob, mor, id, \circ, =_o, =_m\rangle$. The class of all morphisms with domain and codomain a is given by:

$$\begin{aligned} M[u, a, OB_{\mathcal{C}}, MOR_{\mathcal{C}}, EQO_{\mathcal{C}}] :\equiv\ & u \in MOR_{\mathcal{C}} \land a \in OB_{\mathcal{C}} \\ & \land \langle a, dom(u)\rangle \in EQO_{\mathcal{C}} \\ & \land \langle a, cod(u)\rangle \in EQO_{\mathcal{C}} \end{aligned}$$

For which elementary comprehension then gives a term

$$m_M(a, ob, mor, =_o).$$

This gets more readable if we shorten it instead to

$$\begin{aligned} M[u, a, \mathcal{C}] :\equiv u \in MOR_{\mathcal{C}} \wedge a \in OB_{\mathcal{C}} \\ \wedge \langle a, dom(u)\rangle \in EQO_{\mathcal{C}} \\ \wedge \langle a, cod(u)\rangle \in EQO_{\mathcal{C}} \end{aligned}$$

For which elementary comprehension then gives a term

$$m_M(a, \mathcal{C}).$$

What we mean is, that $\mathcal{C}$ as an argument in a formula A stands for the ten-tuple given above, and when used in a term provided by elementary comprehension it stands for the six tuple of $\mathcal{C}$ and the (repeated) class names of $\mathcal{C}$ used as arguments for OB, MOR, EQO, EQM.

$$\begin{aligned} &A[\vec{v}, \vec{W}, \mathcal{C}] \\ &t_A(\vec{v}, \vec{w}, \mathcal{C}) :\equiv t_A(\vec{v}, id, \circ, ob_{\mathcal{C}}, mor_{\mathcal{C}}, =^{\mathcal{C}}_o, =^{\mathcal{C}}_m, \vec{w}, ob_{\mathcal{C}}, mor_{\mathcal{C}}, =^{\mathcal{C}}_o, =^{\mathcal{C}}_m). \end{aligned}$$

As another example take the class of all pairs of a morphism from one of two categories $\mathcal{C}$ and $\mathcal{D}$ labeled by the their morphism classes $mor_{\mathcal{C}}$ and $mor_{\mathcal{D}}$. A completely useless class, but it helps to illustrate the notation:

$$\begin{aligned} M[u, \mathcal{C}, \mathcal{D}] :\equiv \exists a, b(u = \langle a, b\rangle \wedge \\ (a = mor_{\mathcal{C}} \wedge b \in MOR_{\mathcal{C}}) \\ \vee (a = mor_{\mathcal{D}} \wedge b \in MOR_{\mathcal{D}})). \end{aligned}$$

Elementary comprehension then provides a term

$$t_M(\mathcal{C}, \mathcal{D}).$$

If we expand this, we get

$$\begin{aligned}M[u, & mor_{\mathcal{C}}, mor_{\mathcal{D}}, MOR_{\mathcal{C}}, MOR_{\mathcal{D}}] :\equiv \\ & \exists a, b(u = \langle a, b\rangle \wedge \\ & (a = mor_{\mathcal{C}} \wedge b \in MOR_{\mathcal{C}}) \vee \\ & (a = mor_{\mathcal{D}} \wedge b \in MOR_{\mathcal{D}})) \\ & t_M(mor_{\mathcal{C}}, mor_{\mathcal{D}}, mor_{\mathcal{C}}, mor_{\mathcal{D}}).\end{aligned}$$

This gives

$$\begin{aligned}\forall x(x \mathrel{\dot\in} t_M(\mathcal{C}, \mathcal{D}) \leftrightarrow & \exists a(x = \langle mor_{\mathcal{C}}, a\rangle \wedge a \mathrel{\dot\in} mor_{\mathcal{C}} \\ & \vee x = \langle mor_{\mathcal{D}}, a\rangle \wedge a \mathrel{\dot\in} mor_{\mathcal{D}}))\end{aligned}$$

Yet another example is the following formula

$$A[u, ob, O, E] = \exists a, b(u = \langle ob, a, b\rangle \wedge a, b \in O \wedge \langle a, b\rangle \in E),$$

which by elementary comprehension gives us a name $t_A(ob, ob, =_o)$ of the (completely useless) class consisting of tuples of equal objects and the name of the object class. Here we have only used three of the ten variables and it already becomes unwieldy. So, shortened this would be

$$A[u, \mathcal{C}] = \exists a, b(u = \langle ob_{\mathcal{C}}, a, b\rangle \wedge a, b \in OB_{\mathcal{C}} \wedge \langle a, b\rangle \in EQO_{\mathcal{C}}).$$

with resulting abbreviated term $t_A(\mathcal{C})$.

In general, the variables $\mathcal{C}, \mathcal{D}, \mathcal{E}, \ldots$ will always be used as meta-variables for categories. When used as subscripts for operations or classes they only serve to disambiguate between several categories (e.g. in the formula $x \mathrel{\dot\in} ob_{\mathcal{C}} \leftrightarrow \langle x, f\rangle \mathrel{\dot\in} ob_{\mathcal{D}}$ where $ob_{\mathcal{C}}$ and $ob_{\mathcal{D}}$ are the names of object-classes for two different categories.) ◇

Remark 2.1.8. Note that we did *not* list the class $\{f \mid f : a \overset{m}{\to} b\}$ in any of the definitions above although it's clearly given by an elementary formula. The reason for this is that one might make the unfortunate connection to the *set* $hom_{\mathcal{C}}(a, b)$. While such a definition might seem reasonable at first sight, this is not what we want. First of all it fixes the category of class names and operations (as defined in section 3.1) as our category of sets which is not very well-behaved (see section 3.3 and in particular definition

3.3.8 for a better candidate). Furthermore, $hom_{\mathcal{C}}(a, b)$ defined in this way completely ignores equality on morphisms. If we try to prove something like Yoneda's lemma for this class (and category) it must fail. The problem is that any term t for which $t \mathrel{\dot\in} hom_{\mathcal{C}}(a, b)$ holds represents actually a quotient $[t]_{=_m}$ since Mor is modded out by morphism-equality. Of course in most cases there will be many representatives in this class which should not be distinguished. Note that this is one of the few cases where this problem actually show up. Strictly speaking we should check well-definedness of morphisms every time we write one down, but in most cases this is actually unnecessary as in the case of operations up to extensionality w.r.t. some equivalence relation. Since ◇

Proposition 2.1.9. *As noted before, any category is a weak category.*

Proof. Suppose we have some category $\mathcal{C}$. As it turns out we don't have much to verify. Where the axioms of a category differ from those of a weak category they do so only by substitution of the definitions below.

$$\begin{aligned}
Ob(x) &:\equiv x \mathrel{\dot\in} ob_{\mathcal{C}} \\
Mor(f) &:\equiv f \mathrel{\dot\in} mor_{\mathcal{C}} \\
x =_o y &:\equiv \langle x, y \rangle \mathrel{\dot\in} =^{\mathcal{C}}_o \\
f =_m g &:\equiv \langle f, g \rangle \mathrel{\dot\in} =^{\mathcal{C}}_m \\
id(x) &:\equiv id_{\mathcal{C}}(x) \\
f \circ g &:\equiv f \circ^{\mathcal{C}} g.
\end{aligned}$$

□

It is often convenient or even necessary to have categories with properties somewhere in-between weak categories (definition 2.1.2) and non-weak categories (definition 2.1.6.) It's tempting to call categories which have classes of morphisms for any two objects *locally small* but this carries a meaning of cardinality which does not apply here. (Remember that $(0 = 0)$ is elementary and can be used with elementary comprehension to get the class containing everything.) Instead we will adopt some non-standard terminology.

Definition 2.1.10. (proper) Local weak categories
A weak category $\mathcal{C}$ is called *local* if there is a term $lhomclass_{\mathcal{C}}$ which satisfies

$$\begin{aligned}(LM)\qquad & (Ob(c)\wedge Ob(d)\rightarrow \mathfrak{R}\,(lhomclass_{\mathcal{C}}(c,d))\\ & \wedge\forall f((Mor(f)\wedge dom(f)=c\wedge cod(f)=d)\\ & \quad\leftrightarrow f\mathrel{\dot\in} lhomclass_{\mathcal{C}}(c,d)).\end{aligned}$$

If there also exists a term $eqm_{\mathcal{C}}$, and $\mathcal{C}$ also satisfies $(LMEQ)$ we call it *proper local.*

$$\begin{aligned}(LMEQ)\qquad & (Ob(c)\wedge Ob(d)\rightarrow \mathfrak{R}\,(eqm_{\mathcal{C}}(c,d))\\ & \wedge\forall f,g(f,g:c\overset{m}{\rightarrow}d\\ & \rightarrow(f=_m g\leftrightarrow\langle f,g\rangle\mathrel{\dot\in} eqm_{\mathcal{C}}(c,d)))\end{aligned}$$

Note that if the join axiom is allowed, and there also exist classes for objects and object-equality, the property of being proper local, implies that the category is non-weak if the join axiom is allowed. This can be done by using the classes

$$\sum_{x,y\mathrel{\dot\in} ob} lhomclass_{(}x,y),$$

$$\left\{\langle f,g\rangle\mid\exists x,y\left(\langle x,y,f,g\rangle\mathrel{\dot\in}\sum_{x,y\mathrel{\dot\in} ob} eqm_{\mathcal{C}}(x,y)\right)\right\}.$$

◇

Definition 2.1.11 (u-proper local category)**.**
A proper local category is called u-proper local if it satisfies

$$\begin{aligned}(SU)\qquad & \mathcal{U}(u)\\ (SCONST)\qquad & \{lhomclass_{\mathcal{C}},dom,cod,id,\circ\}\mathrel{\dot\in} u\\ (SM)\qquad & \forall c,d(Ob(c)\wedge Ob(d)\rightarrow lhomclass_{\mathcal{C}}(c,d)\mathrel{\dot\in} u)\\ (SUMEQ)\qquad & \forall c,d(Ob(c)\wedge Ob(d)\rightarrow eqm_{\mathcal{C}}(c,d)\mathrel{\dot\in} u).\end{aligned}$$

◇

Definition 2.1.12. Opposite category

The opposite category $\mathcal{C}^{op}$ of some category $\mathcal{C}$ is defined as usual.

$$\begin{aligned} Ob_{\mathcal{C}^{op}} &:\equiv Ob_{\mathcal{C}} \\ Mor_{\mathcal{C}^{op}}(f) &:\equiv \exists a, b, c(f = \langle a, b, c\rangle \wedge Mor_{\mathcal{C}}(\langle b, a, c\rangle)) \\ id_{\mathcal{C}^{op}}(x) &:\equiv id_{\mathcal{C}}(x) \\ f \circ_{\mathcal{C}^{op}} g &:\equiv \langle cod(g), dom(g), \overline{g}\rangle \circ_{\mathcal{C}} \langle cod(f), dom(f), \overline{f}\rangle \qquad \Diamond \end{aligned}$$

Definition 2.1.13. A proper local category $\mathcal{C}$ is a *thin category* iff

$$\begin{aligned} (\mathit{THIN}) \qquad & \forall c, d(Ob_{\mathcal{C}}(c) \wedge Ob_{\mathcal{C}}(d) \\ & \to (\exists f \mathrel{\dot{\in}} lhomclass_{\mathcal{C}}(c, d)) \\ & \qquad (\forall g \mathrel{\dot{\in}} lhomclass_{\mathcal{C}}(c, d))(f =_m^{\mathcal{C}} g) \qquad \Diamond \end{aligned}$$

Definition 2.1.14. Monomorphism

We call a morphism $f : b \overset{m}{\to} c$ in category $\mathcal{C}$ *mono* or *monic* iff

$$\begin{aligned} (\mathit{MONO}) \qquad & \forall a(Ob_{\mathcal{C}}(a) \to \\ & \forall g, h(g, h : a \overset{m}{\to} b \wedge f \circ_{\mathcal{C}} g =_m f \circ_{\mathcal{C}} h \to g =_m^{\mathcal{C}} h)) \qquad \Diamond \end{aligned}$$

Notation 2.1.15. Occasionally, we will write monomorphisms $f : a \overset{m}{\to} b$ as $f : a \overset{m}{\rightarrowtail} b$ for emphasis. ◇

Definition 2.1.16. Epimorphism

We call a morphism $f : a \overset{m}{\to} b$ in category $\mathcal{C}$ *epi* or *epic* iff

$$\begin{aligned} (\mathit{EPI}) \qquad & \forall c(Ob_{\mathcal{C}}(c) \to \\ & \forall g, h(g, h : b \overset{m}{\to} c \wedge g \circ_{\mathcal{C}} f =_m h \circ_{\mathcal{C}} f \to g =_m^{\mathcal{C}} h)) \qquad \Diamond \end{aligned}$$

Definition 2.1.17. Given a category $\mathcal{C}$, and a morphism $Mor_{\mathcal{C}}(f)$, we say that f is an *isomorphism* iff

$$\begin{aligned} (\mathit{ISO}) \qquad & \exists g(Mor_{\mathcal{C}}(g) \wedge f \circ_{\mathcal{C}} g =_m^{\mathcal{C}} id_{\mathcal{C}}(dom(g)) \\ & \wedge g \circ_{\mathcal{C}} f =_m^{\mathcal{C}} id_{\mathcal{C}}(dom(f)) \end{aligned}$$

We add Notation for isomorphic objects:

$$a \cong b :\equiv \exists f (f : a \overset{m}{\to} b \wedge ISO[\mathcal{C}, f]) \qquad \diamond$$

Definition 2.1.18. A category $\mathcal{C}$ is a *groupoid* iff

$$(GROUPOID) \qquad \forall f (Mor_{\mathcal{C}}(f) \to ISO[\langle \mathcal{C}, f \rangle]) \qquad \diamond$$

Commutative Diagrams

Definition 2.1.19 (Commutative Diagrams used to specify categories and classes)**.** When a diagram is used instead as (part of) a specification for a new class or to create a new (weak) category we will instead implicitly use the translation to a set of equations. In case that the collections of objects, morphisms and equality on morphisms in $\mathcal{C}$ are classes, these equations will even be elementary.

The description of the translation of labeled graphs into equations inside Explicit Mathematics is outside of the scope this thesis, the translation of such a labeled graph into $\mathcal{L}$-formulas is clearly always possible, and in fact clearly trivial.

While we will also give a more general definition of (commutative) diagrams in the appendix (see definitions A.2.3 and A.2.5) which should be seen as the *official* one, we will use this one as our working definition. ◇

Example 2.1.20 (Composition)**.** If the composition of two morphisms has to be a specific third morphism this is written as

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "h"'] & b \arrow[d, "g"] \\
& c
\end{tikzcd}$$

which is short for the conjunction of

(i) $Ob(a, b, c)$

(ii) $f : a \overset{m}{\to} b,\ g : b \overset{m}{\to} c,\ h : a \overset{m}{\to} c$

(iii) $g \circ f =_m h$ ◇

Example 2.1.21 (Commutativ Square)**.** A commutative square (meaning one composition of two morphisms is the same as some other composition of two morphisms,) is given by

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, "h"] \\ c \arrow[r, "i"'] & d \end{tikzcd}$$

which is short for the conjunction of

(i) $Ob(a, b, c, d)$

(ii) $f : a \overset{m}{\to} b,\ g : a \overset{m}{\to} c,\ i : c \overset{m}{\to} d\ h : b \overset{m}{\to} d$

(iii) $h \circ f =_m i \circ g$ ◇

Example 2.1.22 (Commutativity)**.** As a special case, we have the usual commutativity of two maps. For example take f and g to be morphisms (linear maps) on $\mathbb{Q}^k, k \in N$ represented by diagonal matrices w.r.t. the standard basis in the category of finite dimensional $\mathbb{Q}$-vector spaces (which we could define in this setting.) Then we have:

$$\begin{tikzcd} \mathbb{Q}^k \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{Q}^k \arrow[d, "g"] \\ \mathbb{Q}^k \arrow[r, "f"'] & \mathbb{Q}^k \end{tikzcd}$$

which is short for the conjunction of

(i) $Ob(\mathbb{Q}^k)$

(ii) $f : \mathbb{Q}^k \overset{m}{\to} \mathbb{Q}^k,\ g : \mathbb{Q}^k \overset{m}{\to} \mathbb{Q}^k$

(iii) $g \circ f =_m f \circ g$ ◇

Functors and Natural Transformations

Definition 2.1.23 (Functor)**.** Let f be given by a pair $\langle f_o, f_m \rangle$.

We say a term f is a functor between two (weak) categories $\mathcal{C}$ and $\mathcal{D}$ (Notation: $Functor_{\mathcal{C},\mathcal{D}}[f]$ or $f : \mathcal{C} \to \mathcal{D}$, if no confusion arises) if the conjunction of the following properties holds:

$$\begin{array}{ll}
(F1) & Ob_{\mathcal{C}}(x,y) \wedge x =^{\mathcal{C}}_{o} y \to f_o(x) =^{\mathcal{D}}_{o} f_o(d) \\
(F2) & Mor_{\mathcal{C}}(g,h) \wedge g =^{\mathcal{C}}_{m} h \to f_m(g) =^{\mathcal{D}}_{m} f_m(h) \\
(F3) & Mor_{\mathcal{C}}(g) \to dom(f_m(g)) =^{\mathcal{D}}_{o} f_o(dom(g)) \\
(F4) & Mor_{\mathcal{C}}(g) \to cod(f_m(g)) =^{\mathcal{D}}_{o} f_o(cod(g)) \\
(F5) & Ob_{\mathcal{C}}(x) \to f_m(id(x)) =^{\mathcal{D}}_{m} id_{\mathcal{D}}(f_o(x)) \\
(F6) & dom(g) =^{\mathcal{C}}_{o} cod(h) \to f_m(g \circ_{\mathcal{C}} h) =^{\mathcal{D}}_{m} f_m(g) \circ_{\mathcal{D}} f_m(h)
\end{array} \qquad \diamond$$

Definition 2.1.24 (Natural Transformation)**.** Given two functors $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{C} \to \mathcal{D}$ between fixed (weak) categories, we call a tuple $\eta = \langle f, g, \overline{\eta} \rangle$ a natural transformation (Notation: $\eta : f \Rightarrow g$) if $(NAT1)$ and $(NAT2)$ are satisfied.

$$\begin{array}{ll}
(NAT1) & \forall x (Ob_{\mathcal{C}}(x) \to (\overline{\eta}(x) : f_o(x) \overset{m}{\to} g_o(x))) \\
(NAT2) & \forall h (Mor_{\mathcal{C}}(h) \\
& \quad \to \overline{\eta}(cod(h)) \circ_{\mathcal{D}} f_m(h) =^{\mathcal{D}}_{m} g_m(h) \circ_{\mathcal{D}} \overline{\eta}(dom(h)))
\end{array} \qquad \diamond$$

Remark 2.1.25. The notation $\eta : f \Rightarrow g$ is heavily shortened. It should be taken as the conjunction of the formulas $(NAT1)$, $(NAT2)$ and the formulas asserting that f and g are functors with the same categories as (co)domains, which is already a big formula consisting of $\bigwedge_i (Fi)$ and the (weak) category axioms for $\mathcal{C}$ and $\mathcal{D}$. $\diamond$

Definition 2.1.26. Functor Category
A (covariant) functor category from (fixed) (weak) categories $\mathcal{C}$ to $\mathcal{D}$ is a (weak) category $\mathcal{D}^{\mathcal{C}}$ for which the following axioms hold.

$$\begin{aligned}
&(FCT1) && Ob(f) \leftrightarrow Functor_{\mathcal{C},\mathcal{D}}[f]\\
&(FCT2) && Mor(\eta) \leftrightarrow \exists f, g(Ob(f,g) \wedge \eta : f \Rightarrow g)\\
&(FCT3) && Ob(f,g) \rightarrow (f =_o g\\
& && \quad \leftrightarrow (\forall x(Ob_{\mathcal{C}}(x) \rightarrow f_o(x) =_o^{\mathcal{D}} g_o(x))\\
& && \qquad \wedge \forall h(Mor_{\mathcal{C}}(h) \rightarrow f_m(h) =_o^{\mathcal{D}} g_m(h))))\\
&(FCT4) && Mor(\eta,\nu) \rightarrow (\eta =_m \nu\\
& && \quad \leftrightarrow (\forall x(Ob_{\mathcal{C}}(x) \rightarrow \overline{\eta}(x) =_m^{\mathcal{D}} \overline{\nu}(x)))\\
&(FCT5) && Mor(\eta,\nu) \wedge cod(\eta) =_o dom(\nu)\\
& && \quad \rightarrow ((\nu \circ \eta) =_m \langle dom(\eta), cod(\nu), \lambda x.\overline{\nu}x \circ_{\mathcal{D}} \overline{\eta}x\rangle)
\end{aligned}$$

All omitted subscripts in $(FCT1)$ – $(FCT5)$ would be $\mathcal{D}^{\mathcal{C}}$. Objects of $\mathcal{D}^{\mathcal{C}}$ are called covariant functors.

An object of $\mathcal{D}^{\mathcal{C}^{op}}$ is called a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$. Given a contravariant functor and a morphism in $\mathcal{C}$ we will often use the reversed axioms of functor categories instead of explicitly reversing domain and codomain of the morphism. ◇

Functors take commutative diagrams to commutative diagrams:

$$\begin{tikzcd}
a \arrow[loop left, "id(a)"] \arrow[r, "f"] \arrow[dr, "g\circ f"'] & b \arrow[d, "g"] & \overset{p}{\longmapsto} & p_o(a) \arrow[loop left, "id(p_o(a))"] \arrow[r, "p_m(f)"] \arrow[dr, "p_m(g\circ f)"'] & p_o(b) \arrow[d, "p_m(g)"] \\
& c & & & p_o(c)
\end{tikzcd}$$

Natural transformations are characterized by the naturality square

$$\begin{tikzcd}
f_o(dom(h)) \arrow[r, "\eta(dom(h))"] \arrow[d, "f_m(h)"] & g_o(dom(h)) \arrow[d, "g_m(h)"] \\
f_o(cod(h)) \arrow[r, "\eta(cod(h))"] & g_o(cod(h)).
\end{tikzcd}$$

Remark 2.1.27. For categories in a set-theoretic context we would have the usual size-issues of functor categories. (If $\mathcal{C}$ is locally small and $\mathcal{D}$ is small the $\mathcal{C}^{\mathcal{D}}$ is locally small, if $\mathcal{C}$ is large and $\mathcal{D}$ is small then $\mathcal{C}^{\mathcal{D}}$ is large [33].) But classes in the explicit mathematics sense are not sets and "size" doesn't have any real meaning.[2] What we can say is, that if either $\mathcal{C}$ or $\mathcal{D}$ are not categories in the sense of definition 2.1.6 the functor category will be weak. ◇

Definition 2.1.28. Cone/Cocone
Given a non-weak category $\mathcal{I}$, a weak category $\mathcal{C}$ and a functor $f : \mathcal{I} \to \mathcal{C}$, a (co)cone is a pair $\langle c, p \rangle$ such that

$$\begin{aligned}
&(CONE1) && Ob_{\mathcal{C}}(c) \wedge (\forall i \mathrel{\dot\in} ob_{\mathcal{I}})(p(i) : c \overset{m}{\to} f_o(i)) \\
&(CONE2) && (\forall h \mathrel{\dot\in} mor_{\mathcal{I}})(p(cod(h)) =^{\mathcal{C}}_{m} f_m(h) \circ p(dom(h)))
\end{aligned}$$

$$\begin{aligned}
&(COCONE1) && Ob_{\mathcal{C}}(c) \wedge (\forall i \mathrel{\dot\in} ob_{\mathcal{I}})(p(i) : f_o(i) \overset{m}{\to} c)) \\
&(COCONE2) && (\forall h \mathrel{\dot\in} mor_{\mathcal{I}})(p(dom(h)) =^{\mathcal{C}}_{m} p(cod(h)) \circ f_m(h)) \diamond
\end{aligned}$$

Proposition 2.1.29. *(co)cones in a non-weak category build a class when we allow the join axiom.*

Proof.

$$\begin{aligned}
C1[u, O, P] &:\equiv \exists c, p(u = \langle c, p \rangle \wedge c \in O \wedge p \in P) \\
cOp(f) &:\equiv \prod (ob_{\mathcal{I}}, \lambda i.lhomclass_{\mathcal{C}}(c, f_o(i))) \\
cone1 &:\equiv t_{C1}(ob_{\mathcal{C}}, cOp(f)) \\
C2[u, f, C, M, =_m] &:\equiv u \in C \wedge (\forall h \in M) \\
&\qquad ((u)_1(cod(h)) =_m f_m(h) \circ (u)_1(dom(h))) \\
cone &:\equiv t_{C2}(f, cone1, mor_{\mathcal{C}})
\end{aligned}$$

Note that $(CO)C2$ works only because $(\cdot =_m \cdot)$ is a class. □

Definition 2.1.30. Limit/Colimit
Given a non-weak category $\mathcal{I}$, a weak category $\mathcal{C}$ and a functor $f : \mathcal{I} \to \mathcal{C}$. Writing $C[c, p, \mathcal{I}, \mathcal{C}, f]$ for the conjunction of $(CONE1)$ and $(CONE2)$ with

---

[2] Remember that $ob :\equiv \{0 = 0\}$, $mor :\equiv \{\langle u, v, 0 \rangle \mid u\downarrow \wedge v\downarrow\}$ with equality and the obvious composition is a perfectly cromulent "small" (i.e. non-weak) category with the whole universe as objects.

the appropriate parameters, a Cone $\langle c, p\rangle$ with a term h is called a limit of f if the following holds

$$
\begin{aligned}
&(\mathit{LIMIT}1) && C[c, p, \mathcal{I}, \mathcal{C}, f]\\
&(\mathit{LIMIT}2) && \forall d, q(C[d, q, \mathcal{I}, \mathcal{C}, f] \rightarrow (h(d,q) : d \overset{m}{\rightarrow} c))\\
&&& \quad \wedge (\forall i \dot{\in} ob_{\mathcal{I}})(q(i) =_m p(i) \circ h(d,q))\\
&(\mathit{LIMIT}3) && \forall d, q(C[d, q, \mathcal{I}, \mathcal{C}, f] \rightarrow \forall g((g : d \overset{m}{\rightarrow} c\\
&&& \quad \wedge (\forall i \dot{\in} ob_{\mathcal{I}})(q(i) =_m p(i) \circ g)) \rightarrow h(d,q) =_m g))
\end{aligned}
$$

Note that $g, h(d,q) : d \overset{m}{\rightarrow} c$ refer to morphisms in $\mathcal{C}$. The axioms for colimits are the same except for the reversal of the morphism $h(d,q)$ and the equation $q(i) =_m h(d,q) \circ p(i)$ in C. ◇

Proposition 2.1.31. *(co)limits in a non-weak category build a class when we allow the join axiom.*

Proof. Specifically, the class $lim(f)$:

$$
\begin{aligned}
cn &:\equiv cone(f, ob_{\mathcal{C}}, ob_{\mathcal{I}}, lhomclass_{\mathcal{C}}, mor_{\mathcal{C}}, =_m)\\
\widetilde{lim}(f) &:\equiv \sum_{c:cn} \prod_{v:cn} \sum_{h:lhomclass_{\mathcal{C}}(\pi_0 v, \pi_0 c)} \lambda h.U[h] \times C[h]
\end{aligned}
$$

where

$$
\begin{aligned}
U[h] &:\equiv \prod (lhomclass_{\mathcal{C}}(\pi_0 v, \pi_0 c), \lambda g.h =_m g)\\
C[h] &:\equiv \prod (ob_{\mathcal{I}}, \lambda i.\pi_1 v(i) =_m (\pi_1 c)(i) \circ h)\\
lim(f) &:\equiv \mathrm{dom}(\widetilde{lim}(f))
\end{aligned}
$$

Note that dom refers to the **EM** axiom in definition 1.0.5(k) not the operation *dom* to extract the domain from a morphism. Of course $lim(f)$ should be called $lim(f, ob_{\mathcal{C}}, \cdots, =_m)$, but this will be omitted when it's clear from context.

The version for colimit is again the same except for the reversal of the unique morphism in U and the equation $(\pi_1 v)(i) =_m h \circ (\pi_1 c)(i)$ in C. □

Example 2.1.32 (Terminal object)**.** Let $\mathcal{C}$ be any category. A *terminal object* in $\mathcal{C}$ is a limit of the empty functor (the functor from the empty category). This means it's any pair $\langle c, p\rangle$ and term h such that c is an object and h returns for any pair $\langle d, q\rangle$ where d is another object the unique morphism $h\langle d, q\rangle : d \overset{m}{\to} c$.

From this description we see that the second component which selects the morphisms of the cone is completely irrelevant since there are no objects to construct morphisms into. In other words if $\langle c, p\rangle$ is a cone, then this is for example also true for the pair $\langle c, 0\rangle$. ◇

Notation 2.1.33. We will write $!_{\mathrm{d}} : d \overset{m}{\to} c$ for the morphism $h\langle d, q\rangle$. ◇

Example 2.1.34 (Product)**.** Let $\mathcal{C}$ be any category. A *product* in $\mathcal{C}$ is a limit for any functor $f = \langle f_o, f_m\rangle$ from the following category into $\mathcal{C}$: Let $\mathbb{Fin}(\mathbb{2})$ be given by the category

$$\begin{aligned}
ob &:\equiv \{0, 1\} \\
mor &:\equiv \{\langle 0, 0, 0\rangle, \langle 1, 1, 1\rangle\} \\
=_o &:\equiv \Delta(ob) \\
=_m &:\equiv \Delta(mor) \\
id(a) &:\equiv \langle a, a, a\rangle \\
f \circ g &:\equiv f
\end{aligned}$$

In other words, for two objects $f_o(0), f_o(1) \mathrel{\dot\in} ob_{\mathcal{C}}$ it is a cone $\langle c, p\rangle$ such that $c \mathrel{\dot\in} ob_{\mathcal{C}}$ and $p(0) : c \overset{m}{\to} f_o(0)$, $p(0) : c \overset{m}{\to} f_o(1)$ and a term h such that the following diagram is satisfied for arbitrary cones $\langle d, q\rangle$ of f:

$$\begin{tikzcd}
 & d \arrow[dl, "q(0)"'] \arrow[d, dotted, "h\langle d,q\rangle"] \arrow[dr, "q(1)"] & \\
f_o(0) & c \arrow[l, "p(0)"] \arrow[r, "p(1)"'] & f_o(1)
\end{tikzcd}$$

◇

Notation 2.1.35. We will write $\langle\!\langle k, g\rangle\!\rangle$ for the morphism $h\langle d, q_{k,g}\rangle$ induced by some $k : d \overset{m}{\to} f_o(0)$ and $g : d \overset{m}{\to} f_o(1)$. Additionally, the projections $p(0)$

and $p(1)$ from the limit will often be written pr_0 and pr_1 or even $pr_0^{x,y}$ and $pr_1^{x,y}$ if $f_o(0) =_o x$ and $f_o(1) =_o y$. Last but not least, we will sometimes overload the syntax for ×. So far we have defined it as a specific name of a class of pairs. If it is understood that we are given objects a, b in an arbitrary (weak) category, we will instead write $a \times b$ for their product. similarly if we have two morphisms $f : a \overset{m}{\rightarrow} c$ and $g : b \overset{m}{\rightarrow} d$ we will write $f \times g$ for the unique morphism $f \times g : a \times b \overset{m}{\rightarrow} c \times d$ which makes the obvious diagram commute. ◇

Example 2.1.36 (Pullback)**.** Let $\mathcal{C}$ be any category. A *pullback* in $\mathcal{C}$ is a limit for any functor $f = \langle f_o, f_m \rangle$ from $\mathcal{D}$ into $\mathcal{C}$ where $\mathcal{D}$ is given by

$$
\begin{aligned}
ob &:\equiv \{0, 1, 2\} \\
mor &:\equiv \{\langle 0,0,0\rangle, \langle 1,1,0\rangle, \langle 2,2,0\rangle, \langle 0,2,0\rangle, \langle 1,2,0\rangle\} \\
=_o &:\equiv \Delta(ob) \\
=_m &:\equiv \Delta(mor) \\
id(a) &:\equiv \langle a, a, 0\rangle \\
f \circ g &:\equiv \begin{cases} f & dom(g) = cod(g) \\ g & otherwise \end{cases}
\end{aligned}
$$

In other words, for three objects $f_o(0), f_o(1), f_o(2) \mathrel{\dot\in} ob_{\mathcal{C}}$ it is a cone $\langle c, p\rangle$ such that $c \mathrel{\dot\in} ob_{\mathcal{C}}$ and $p(0) : c \overset{m}{\rightarrow} f_o(0)$, $p(0) : c \overset{m}{\rightarrow} f_o(1)$ and a term h such that the following diagram is satisfied for arbitrary cones $\langle d, q\rangle$ of f:

$$\begin{tikzcd}
d \arrow[drr, bend left, "q(1)"] \arrow[ddr, bend right, "q(0)"'] \arrow[dr, dotted, "h\langle d,q\rangle"] \arrow[ddrr, dotted, "q(2)"] & & \\
& c \arrow[r, "p(1)"] \arrow[d, "p(0)"'] \arrow[dr, dotted, "p(2)"] & f_o(1) \arrow[d, "f_m(\langle 1,2,0\rangle)"] \\
& f_o(0) \arrow[r, "f_m(\langle 0,2,0\rangle)"'] & f_o(2)
\end{tikzcd}$$

Note that $p(2)$ and $q(2)$ are normally not explicitly given, since they are determined by the other morphisms. ◇

Notation 2.1.37. We will often use the notation $f * g$ for the object-part for the pullback of some morphisms $x \xrightarrow{f} z \xleftarrow{g} y$. Sometimes this will be combined with notation for the projections:

$$x \xleftarrow{f^*(g)} f * g \xrightarrow{g^*(f)} y. \qquad \diamond$$

Example 2.1.38. Let $\mathcal{C}$ be any category. A *coequalizer* is a colimit of a functor from a category of the following form:

$$\begin{aligned}
ob &:\equiv \{0, 1\} \\
mor &:\equiv \{\langle 0,0,0\rangle, \langle 1,1,1\rangle, \langle 0,1,0\rangle, \langle 0,1,1\rangle\} \\
=_o &:\equiv \Delta(ob) \\
=_m &:\equiv \Delta(mor) \\
id(a) &:\equiv \langle a, a, a\rangle \\
f \circ g &:\equiv \begin{cases} f & dom(g) = cod(g) \\ g & otherwise \end{cases}
\end{aligned}$$

So it is given as a pair $\langle c, p\rangle$ and a term h such that it fits the following diagram for all other cocones $\langle a, q\rangle$. Note the reversal of the arrows when compared to limits.

$$\begin{tikzcd}
f_o(0) \arrow[rr, shift left, "f_m(\langle 0,1,0\rangle)"] \arrow[rr, shift right, "f_m(\langle 0,1,1\rangle)"'] \arrow[dr, "p(0)"'] \arrow[ddr, bend right, "q(0)"'] & & f_o(1) \arrow[dl, "p(1)"] \arrow[ddl, bend left, "q(1)"] \\
& c \arrow[d, dotted, "h\langle a,p\rangle"] & \\
& a &
\end{tikzcd}$$

Normally we'll omit the morphisms $p(0)$ and $q(0)$ since they are determined as the compositions. $\diamond$

For the general case of (co)limits of diagrams, see appendix A.2 and in particular example A.2.2. Proofs that the important categories in this thesis have pullbacks are given in proposition 3.1.11 for **EM**, proposition 3.2.7 for **ECB** and theorem 3.3.15 for $\mathcal{C}_{\text{ex}}$ for any $\mathcal{C}$ with finite limits.

2.2. Examples

We have now all of the required ingredients to define categories of objects with additional structure inside another category. Here we present quivers and the category of group-objects in $\mathcal{C}$. Note that quivers as defined here are strongly related to the path categories defined in A.2.1. In general the *category-of-X-objects inside* $\mathcal{C}$ is a way to take structures which normally live inside the category of sets and define them in any sufficiently nice category. This is equivalent to defining categories of structure-preserving functors from some archetypal model of X into $\mathcal{C}$ (see definition 2.1.26). For a complete treatment of functorial semantics see Lawvere's thesis [32].

Example 2.2.1 (The category of quivers in $\mathcal{C}$: $Quiv(\mathcal{C})$)**.** Let $\mathcal{C} :\equiv \langle ob, mor, =_o, =_m, id, \circ\rangle$ be some category. The category of quivers (also known as directed pseudographs) in $\mathcal{C}$ has

- as objects tuples $\langle v, e, s, t\rangle$ such that $v, e \mathrel{\dot\in} ob$ and $s, t : v \overset{m}{\to} e$.

- while the morphisms $g : \langle v, e, s, t\rangle \overset{m}{\to} \langle v', e', s', t'\rangle$ are constructed from pairs $\langle g_0, g_1\rangle$ with $g_0 : v \overset{m}{\to} v'$ and $g_1 : e \overset{m}{\to} e'$ such that the diagrams

$$\begin{tikzcd} v \arrow[r, "g_0"] \arrow[d, "s"] & v' \arrow[d, "s'"'] \\ e \arrow[r, "g_1"] & e' \end{tikzcd} \quad \begin{tikzcd} v \arrow[r, "g_0"] \arrow[d, "t"] & v' \arrow[d, "t'"'] \\ e \arrow[r, "g_1"] & e' \end{tikzcd}$$

  commute. ◇

Example 2.2.2 (The category of group-objects in $\mathcal{C}$: $\mathcal{C}_G$)**.** Let $\mathcal{C} :\equiv \langle ob, mor, =_o, =_m, id, \circ\rangle$ be some category with a terminal object $\mathbb{1} \mathrel{\dot\in} ob$ and all finite products. We define the category $\mathcal{C}_G$ of group-objects and morphisms in $\mathcal{C}$.

Given terms $a, e, \otimes, (\cdot)^{-1}$, there exist elementary formulas for the following commutative diagrams where the only other parameters are taken from $\mathcal{C}$.

$$\begin{tikzcd}
a \times a \times a \arrow[r, "id(a)\times\otimes"] \arrow[d, "\otimes\times id(a)"'] & a \times a \arrow[d, "\otimes"] \\
a \times a \arrow[r, "\otimes"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\langle id(a){,}(\cdot)^{-1}\rangle\circ\Delta(a)"] \arrow[d, "\langle(\cdot)^{-1}{,}id(a)\rangle\circ\Delta(a)"'] \arrow[dr, "e\circ!_a"] & a \times a \arrow[d, "\otimes"] \\
a \times a \arrow[r, "\otimes"'] & a
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \times a \arrow[d, "e\times id(a)"'] \arrow[dr, "\pi_1"] & \\
a \times a \arrow[r, "\otimes"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \times \mathbb{1} \arrow[r, "id(a)\times e"] \arrow[dr, "\pi_0"'] & a \times a \arrow[d, "\otimes"] \\
& a
\end{tikzcd}$$

An object in $\mathcal{C}_G$ is then a tuple $\langle a, e, \otimes, (\cdot)^{-1}\rangle$ such that $a \mathrel{\dot{\in}} ob$, $e : \mathbb{1} \overset{m}{\rightarrow} a$, $\otimes : a \times a \overset{m}{\rightarrow} a$ and $(\cdot)^{-1} : a \overset{m}{\rightarrow} a$, which satisfy the conditions given in the commutative diagrams above.

This is just a reformulation of the usual group-axioms. An (informal! i.e. wrong[3]) translation on elements $a_0, a_1, a_2, \mathbb{1} \mathrel{\dot{\in}} a$ yields:

$$\begin{aligned}
(a_0 \otimes a_1) \otimes a_2 &= a_0 \otimes (a_1 \otimes a_2) \\
a_0^{-1} \otimes a_0 &= \mathbb{1} = a_0 \otimes a_0^{-1} \\
\mathbb{1} \otimes a_0 &= a_0 \\
a_0 \otimes \mathbb{1} &= a_0.
\end{aligned}$$

Morphisms $f : \langle a, e, \otimes, (\cdot)^{-1}\rangle \overset{m}{\rightarrow} \langle b, e', \otimes', (\cdot)^{-1'}\rangle$ determined by $f : a \overset{m}{\rightarrow} b$ in $\mathcal{C}$ such that the following diagrams commute in $\mathcal{C}$.

---

[3] At best this makes sense as equations in the internal logic of $\mathcal{C}$. Otherwise we have to assume that objects are some kind a of classes (for which the element relation makes sense.) and that elements are distinguished by the usual equality. The point of the category of group-objects of $\mathcal{C}$ is exactly that it's not restricted to such assumptions, but instead makes sense for any $\mathcal{C}$ satisfying the requirements!

$$\begin{tikzcd}
a \times a \arrow[r, "f \times f"] \arrow[d, "\otimes"'] & b \times b \arrow[d, "\otimes'"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "(\cdot)^{-1}"'] & b \arrow[d, "(\cdot)^{-1\prime}"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "e'"] \arrow[d, "e"'] & b \\
a \arrow[ur, "f"'] &
\end{tikzcd}$$

Equality on objects is given by equality on the factors

$$\begin{aligned}
&\langle a, e, \otimes, (\cdot)^{-1} \rangle =_o^{\mathcal{C}_{\mathrm{G}}} \langle b, e', \otimes', (\cdot)^{-1\prime} \rangle \\
&\qquad\qquad \leftrightarrow a =_o b \wedge e =_m e' \wedge \otimes =_m \otimes' \wedge (\cdot)^{-1} =_m (\cdot)^{-1\prime}.
\end{aligned}$$

Equality on morphisms is taken from $\mathcal{C}$

$$\begin{aligned}
& f =_m^{\mathcal{C}_{\mathrm{G}}} g \\
& \quad \leftrightarrow \langle \pi_0 dom(f), \pi_0 cod(f), \overline{f} \rangle \\
& \qquad =_m \langle \pi_0 dom(g), \pi_0 cod(g), \overline{g} \rangle.
\end{aligned}$$

The same goes for composition and identity which again just repack domain and codomain and apply the terms from $\mathcal{C}$. ◇

Example 2.2.3 (The category of monoid-objects in $\mathcal{C}$: $\mathcal{C}_M$)**.** Consider the definition of group-objects above. If we forget all mentions of $(\cdot)^{-1}$ (and all diagrams in which it occurs) we get the category $\mathcal{C}_M$ of *monoids* inside $\mathcal{C}$. ◇

Proposition 2.2.4. *Let $\mathcal{C}$ be any category which satisfies the requirements for the construction of $\mathcal{C}_M$ (i.e. existence of binary products and terminal object.) This category still has a terminal object (The terminal object in $\mathcal{C}$) and all finite products (given by the underlying products in $\mathcal{C}$.)*

Then $(\mathcal{C}_M)_M$ the category of monoid-objects in $\mathcal{C}_M$ is the category of commutative monoids of $\mathcal{C}$.

Proof. This is a just the Eckmann-Hilton argument [15]. Because if we have an object $\langle\langle a, e, \otimes\rangle, i, \oplus\rangle$ we get by assumption that $i : \mathbb{1}_{\mathcal{C}_M} \overset{m}{\to} \langle a, e, \otimes\rangle$ and $\oplus : \langle a, e, \otimes\rangle \times \langle a, e, \otimes\rangle \overset{m}{\to} \langle a, e, \otimes\rangle$ both are monoid-morphisms (i.e. morphisms in $\mathcal{C}_M$.) But this is essentially[4] saying that

$$(x \oplus y) \otimes (u \oplus v) = (x \otimes u) \oplus (y \otimes v).$$

From which we can deduce that both multiplications $\otimes$ and $\oplus$ and the unit morphisms e and i are the same. This gets us[5]

$$x \oplus y = (x \otimes e) \oplus (y \otimes e) = (e \otimes x) \oplus (y \otimes e) = (e \oplus y) \otimes (e \oplus x) = y \oplus x.$$

$\square$

Remark 2.2.5. The Eckmann-Hilton argument [15] as used in proposition 2.2.4 also works for group-objects in the category $\mathcal{C}_G$ for some category $\mathcal{C}$ to get category of commutative group-objects in $\mathcal{C}$. ◇

2.3. More Definitions

The *slice category* or *over category* of a fixed category $\mathcal{C}$ with a fixed $Ob_{\mathcal{C}}(x)$ are given by all morphisms of $\mathcal{C}$ with codomain x and morphisms induced by commutative triangles over x. In diagrams:

$$\begin{tikzcd}
a \arrow[d, "u"] \\
x
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a \arrow[rr, "h"] \arrow[dr, "u"] & & b \arrow[dl, "v"] \\
& x &
\end{tikzcd}$$

$u : a \overset{m}{\to} x$, $v : b \overset{m}{\to} x$ and $h : a \overset{m}{\to} b$ are morphisms in $\mathcal{C}$. u and v represent objects in the slice category and the triangle given by $u =_m v \circ h$ represents a morphism.

---

[4] This should again only be read as an idea how to interpret the statement not as an actual equation in explicit mathematics.

[5] See footnote [4].

Definition 2.3.1 (Slice Category)**.** Let $\mathcal{C}$ be a weak category. Let x be an object in $\mathcal{C}$. The slice category $\mathcal{C}/x$ is given by

$$
\begin{aligned}
Ob_{\mathcal{C}/x}(k) &:\equiv Mor_{\mathcal{C}}(k) \wedge cod(k) =_o^{\mathcal{C}} x \\
Mor_{\mathcal{C}/x}(f) &:\equiv \exists k, l, s(f = \langle k, l, s\rangle \\
&\qquad\quad \wedge Mor_{\mathcal{C}}(k) \wedge Mor_{\mathcal{C}}(l) \wedge Mor_{\mathcal{C}}(s) \\
&\qquad\quad \wedge cod(k) =_o^{\mathcal{C}} x \wedge cod(l) =_o^{\mathcal{C}} x \\
&\qquad\quad \wedge dom(s) =_o^{\mathcal{C}} dom(k) \wedge cod(s) =_o^{\mathcal{C}} dom(l) \\
&\qquad\quad \wedge k =_m^{\mathcal{C}} l \circ^{\mathcal{C}} s \\
k =_o^{\mathcal{C}/x} l &:\equiv k =_m^{\mathcal{C}} l \\
f =_m^{\mathcal{C}/x} g &:\equiv dom(f) =_m^{\mathcal{C}} dom(g) \\
&\qquad\quad \wedge cod(f) =_m^{\mathcal{C}} cod(g) \\
&\qquad\quad \wedge \overline{f} =_m^{\mathcal{C}} \overline{g} \\
id_{\mathcal{C}/x}(k) &:\equiv \langle k, k, id_{\mathcal{C}}(dom(k))\rangle \\
k \circ^{\mathcal{C}/x} l &:\equiv \langle dom(l), cod(k), \overline{k} \circ^{\mathcal{C}} \overline{l}\rangle
\end{aligned}
$$

Note that all of the above equations become elementary if we are given a category of the form $\mathcal{C} = \langle ob_{\mathcal{C}}, mor_{\mathcal{C}}, =_o^{\mathcal{C}}, =_m^{\mathcal{C}}, id_{\mathcal{C}}, \circ^{\mathcal{C}}\rangle$. ◇

Definition 2.3.2 (Product Category)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two (weak) categories. The (weak) product category $\mathcal{C} \times \mathcal{D}$ is defined as pairs of objects and morphisms of the form $\langle\langle c_0, d_1\rangle, \langle c_1, d_1\rangle, \langle f, g\rangle\rangle$ where the first and second components are morphisms in $\mathcal{C}$ and $\mathcal{D}$.

$$
\begin{aligned}
Ob(x) &:\equiv \exists a, b(x = \langle a, b\rangle \wedge Ob_{\mathcal{C}}(a) \wedge Ob_{\mathcal{D}}(b)) \\
Mor(f) &:\equiv \exists c_0, c_1, d_0, d_1, f_0, f_1(f = \langle\langle c_0, d_1\rangle, \langle c_1, d_1\rangle, \langle f_0, f_1\rangle\rangle \\
&\qquad \wedge Mor_{\mathcal{C}}(f_{\pi_0}) \wedge Mor_{\mathcal{D}}(f_{\pi_1})) \\
id(x) &:\equiv \langle x, x, \langle \overline{id_{\mathcal{C}}(\pi_0 x)}, \overline{id_{\mathcal{D}}(\pi_1 x)}\rangle\rangle \\
f \circ g &:\equiv \langle dom(g), cod(f), \langle \overline{(f_{\pi_0} \circ^{\mathcal{C}} g_{\pi_0})}, \overline{(f_{\pi_1} \circ^{\mathcal{D}} g_{\pi_1})}\rangle\rangle \\
x =_o y &:\equiv \pi_0 x =_o^{\mathcal{C}} \pi_0 y \wedge \pi_1 x =_o^{\mathcal{D}} \pi_1 y \\
f =_m g &:\equiv f_{\pi_0} =_m^{\mathcal{C}} g_{\pi_0} \wedge f_{\pi_1} =_m^{\mathcal{D}} g_{\pi_1}
\end{aligned}
$$

where

$$f_{\pi_i} :\equiv \langle \pi_i dom(f), \pi_i cod(f), \pi_i(\overline{f})\rangle \qquad \text{for } i = 0, 1. \qquad \diamond$$

Definition 2.3.3 (Finitely complete category)**.** A *finitely complete category* (called cartesian category in the elephant [28] and lex category in other literature) is a category $\mathcal{C}$ which has all finite limits. In other words, given any finite category $\mathcal{I}$ (i.e. a category with finite object and morphism classes) and any functor $f : \mathcal{I} \to \mathcal{C}$, then f has a limit.

A category is finitely complete if and only if it has one of the following properties.

- It has a terminal object, all binary products and all equalizers.
- It as a terminal object and all binary pullbacks. $\diamond$

Definition 2.3.4 (Extensive category [11])**.** A category $\mathcal{C}$ is called *extensive* if it has well-behaved sums in the following sense.

For each pair of objects x and y in $\mathcal{C}$, the canonical functor

$$+ : \mathcal{C}/x \times \mathcal{C}/y \to \mathcal{C}/(x + y)$$

is an equivalence of categories.

That is, there exists a functor q in the other direction and two natural isomorphisms[6] to the identity-functors.

$$(q \circ +) \cong id(\mathcal{C}/x \times \mathcal{C}/y)$$
$$(+ \circ q) \cong id(\mathcal{C}/(x + y)) \qquad \diamond$$

Definition 2.3.5 (Natural numbers object)**.** If we want to do arithmetic in a category $\mathcal{C}$, we require a natural numbers object, which is defined as

---

[6] A natural isomorphism $\eta : f \Rightarrow g$ is a natural transformation where the morphism $\eta(x) : f_o(x) \xrightarrow{m} g_o(x)$ at every object x is iso.

follows:

$$
\begin{aligned}
&(NNO1) \qquad TERMINALOBJECT[\mathcal{C}, \mathbb{1}] \\
&(NNO2) \qquad Ob(n) \wedge zero : \mathbb{1} \overset{m}{\to} n \wedge suc : n \overset{m}{\to} n \\
&(NNO3) \qquad (\forall Ob(a))(\forall g : \mathbb{1} \overset{m}{\to} a)(\forall f : a \overset{m}{\to} a)(\exists! u : n \overset{m}{\to} a) \\
&\qquad\qquad\qquad (g =_m u \circ zero \wedge u \circ suc =_m f \circ u)
\end{aligned}
$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "zero"] \arrow[dr, "g"'] & n \arrow[r, "suc"] \arrow[d, dotted, "u"] & n \arrow[d, dotted, "u"] \\
& n \arrow[r, "f"] & n
\end{tikzcd}$$

◇

Definition 2.3.6 (Projective Object)**.** An object p in a category $\mathcal{C}$ is called *projective,* if it has the left-lifting property with respect to all epimorphisms. That is, for any morphism $f : p \overset{m}{\to} b$ and all epimorphisms $e : q \overset{m}{\to} b$, f factors through e by some morphism $p \overset{m}{\to} q$.

$$\begin{tikzcd}
& q \arrow[d, two heads, "e"] \\
p \arrow[ur, "\exists"] \arrow[r, "f"'] & b
\end{tikzcd}$$

◇

Since the traditional definition of projective objects as given above is not very useful for our needs, we will now weaken it. From now on, every time a *projective object* is mentioned, we will actually mean *regular projective object.*

Definition 2.3.7 (Regular Projective Object)**.** An object is called *regular projective* if it has the left lifting property with respect to all *regular* epimorphisms. (See definition 2.4.1 in the next section for the definition of regular epis.) ◇

2.4. Regular Categories

In chapter 3 we will try to build up to a category which mimics the usual category of sets. Before we do that lets recall some properties many, as Francis Borceux in [8] calls it, “algebraic-like” categories but also the category of sets share. The property of being regular is one half of the definition of Abelian Categories, which is one of the very important classes in algebra for providing a general version of (long) exact sequence in its usual sense and a full and faithful embedding of any small Abelian category into a category Mod_R of modules over some ring R. additivity (which we're not going to define,) being the other part of the definition, is unsuited of our use case but we can still retain quite a bit of the exactness-properties even when it is dropped. For us particularly interesting is the existence of a well-behaved image-factorization of any morphism. In the usual category of sets, any map can be factored into a surjective map followed by an injective one which in that category is equivalent to a factorization into an epi followed by a mono. In general being epi turns out to be a condition which is too weak. For example it is *not* generally true that a morphism which is epic and monic is also an isomorphism which is true for sets. If we however strengthen that requirement we can get something meaningful. As far as the factorization property is concerned, what we are after are *strong epis* but being regular implies being strong. Let's recall the relevant definitions and properties.

Definition 2.4.1 (Regular Epimorphism)**.** A morphism $f : y \overset{m}{\to} x$ in a category $\mathcal{C}$ is called *regular epimorphism* if there exists some object z and a pair of morphisms $p_0, p_1 : z \overset{m}{\to} y$ such that f is the coequalizer of p_0, p_1. ◇

Claim 2.4.2. *Any Regular epimorphism is epi*

Proof. Let $f : a \overset{m}{\to} b$ be the coequalizer of $u, v : d \overset{m}{\to} a$. and let $k, l : b \overset{m}{\to} c$ be morphisms with $k \circ f =_m l \circ f$. Then $k \circ f \circ u =_m l \circ f \circ v$ and so there exists a unique $q : b \overset{m}{\to} c$ with $q \circ f =_m l \circ f =_m k \circ f$. But both k and l satisfy this requirement for q and so $q =_m k =_m l$. ☐

Notation 2.4.3. We will occasionally write $f : x \overset{m}{\twoheadrightarrow} y$ for regular epimorphisms (and *only* for regular epimorphisms) for emphasis. ◇

Definition 2.4.4 (Kernel pair)**.** Let $f : a \overset{m}{\to} b$ be any morphism. If it exists, we call the pair of projections p_0, p_1 from the pullback of f along itself *the kernel pair of f.* ◇

Definition 2.4.5 (Regular category)**.**
A category $\mathcal{C}$ is called *regular* if

- It is finitely complete
- The kernel pair

$$\begin{tikzcd} d \times_c d \arrow[r, "p_1"] \arrow[d, "p_2"] & d \arrow[d, "f"] \\ d \arrow[r, "f"] & c \end{tikzcd}$$

  of any morphism $f : d \overset{m}{\to} c$ has a coequalizer

$$\begin{tikzcd} d \times_c d \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & d \arrow[r] & coeq(p_1, p_2) \end{tikzcd}$$

- the pullback of a regular epimorphism along any morphism is again a regular epimorphism.

Alternatively, $\mathcal{C}$ is called regular, if

- It is finitely complete
- each morphism $f : x \overset{m}{\to} y$ can be factored into $m \circ e$ where $m : im(f) \overset{m}{\rightarrowtail} y$ is the smallest subobject of y through which f factors. So that, given any other factorization $f =_m n \circ d$ with $n : z \overset{m}{\rightarrowtail} y$ then there exists a unique $h : im(g) \overset{m}{\to} z$ such that $m =_m n \circ h$.
- image-factorization is pullback-stable. ◇

These two alternative definitions are equivalent. For a proof of this fact see the appendix A.2. The first one emphasizes the existence of certain quotients (given by the coequalizer) while the second one gives a characterization in terms of the existence of a somewhat well-behaved image for any morphism.

Note that the definition of “images” via the coequalizer of the kernel-pair is normally called the *regular coimage.*[7] For well-behaved categories and a morphism $f : a \overset{m}{\rightarrow} b$ this is just the projection $p : a \overset{m}{\rightarrow} (a/ker(f))$ which is then isomorphic to $im(f)$ via the first isomorphism theorem. The image on the other hand as it is normally defined is the smallest subobject with some property through which the morphism factors. Mostly, this will be the *regular image.* In the category of sets this gives a factorization of any map into a (regular) epimorphism (a surjective map) followed by a (regular) monomorphism (an injective map.) Regular categories have a weaker version of this. Instead of a factorization a regular epi and a regular monomorphism, which would be defined as the equalizer of the pushout of f along itself,[8] it just requires the factorization into a regular epi and *any* monomorphism. Since this provides us with the smallest subobject through which the morphism factors, it's reasonable to call this the *image.*

We have not yet mentioned the stability under pullback. This makes it possible to get an internal logic from this category. Substitution in internal logics of categories is generally interpreted as pullback, so any construction for the interpretation of a formula needs to be stable under pullback to make sure variable substitution is well-defined. For regular categories we get regular logic, which is roughly the $\exists$-$\wedge$-fragment of many-sorted first-order logic. Below we give a sound and complete calculus for this. For the comprehensive presentation of regular logic and a proof of soundness and completeness see [9].

Definition 2.4.6. The language of regular logic is given by induction: Let $\vec{X}$ be a collection of sorts. We write $x : X$ if x is a variable of type X.

1. x is a term if $x : X$.
2. The constant c is a term of type X if $c : X$.
3. If $t_1, \ldots, t_n$ are terms of type $X_1, \ldots, X_n$ and if $f : X_1, \ldots, X_n \rightarrow Y$ is a function symbol, then $f(t_1, \ldots, t_n)$ is a term of type Y.
4. $\top$ is a formula.
5. If $t_1 : X$ and $t_2 : X$ then $t_1 =_X t_2$ is a formula.

---

[7] The regular image of $f^{op} : b \overset{m}{\rightarrow} a$ in the opposite category.

[8] This is the definition of a regular epimorphism if we reverse all involved morphisms, i.e. it's the definition of its dual.

6. If φ and ψ are formulas, then so are $\varphi \wedge \psi$ and $\exists x \varphi$ where x is a variable of some type.
7. If $t_1, \ldots, t_n$ are terms of type $X_1, \ldots, X_n$ and if $R \rightarrowtail X_1, \ldots, X_n$ is a relation symbol, then $R(t_1, \ldots, t_n)$ is a formula. ◇

Definition 2.4.7. Entailment relation $\vdash_F$ between formulas where F is a finite set of typed variables. $\varphi \vdash_F \psi$ is only defined if both the free variables of φ and ψ are contained in the set F.

1. Structural rules
   - 1.1. $p \vdash_F p$
   - 1.2. $\dfrac{p \vdash_F q \qquad q \vdash_F r}{p \vdash_F r}$
   - 1.3. $\dfrac{p \vdash_F q}{p \vdash_{F \cup \{y\}} q}$
   - 1.4. $\dfrac{\varphi(y) \vdash_F \psi(y)}{\varphi(b) \vdash_{F \setminus \{y\}} \psi(b)}$

     where $y : B$ is a variable, b is a term of type B and b is not free in $\varphi(y)$ and $\psi(y)$
2. Logical rules
   - 2.1. $p \vdash_F \top$
   - 2.2. if $r \vdash_F p \wedge q$ then $r \vdash_F p$ and $r \vdash_F q$.

     If both $r \vdash_F p$ and $r \vdash_F q$ then $r \vdash_F p \wedge q$.
   - 2.3. if $\exists y \varphi(y) \vdash_F p$ then $\varphi(y) \vdash_{F \cup \{y\}} p$.

     conversely if $\varphi(y) \vdash_{F \cup \{y\}} p$ then $\exists y \varphi(y) \vdash_F p$.
3. Rules of equality
   - 3.1. $\top \vdash_{\{x\}} x =_X x$
   - 3.2. $x =_X y \vdash_{\{x,y\}} y =_X x$
   - 3.3. $x =_X y \wedge y =_X z \vdash_{\{x,y,z\}} x =_X z$
   - 3.4. $\vec{x} =_{\vec{X}} \vec{y} \vdash_{\{\vec{x},\vec{y}\}} f(\vec{x}) =_{\vec{Z}} f(\vec{y})$

     for each function symbol $f : \vec{X} \overset{m}{\rightarrow} \vec{Z}$ (where $\vec{x} =_{\vec{X}} \vec{y}$ is the conjunction of the equations.)

3.5. $\vec{x} =_{\vec{X}} \vec{y} \wedge R(\vec{x}) \vdash_{\{\vec{x},\vec{y}\}} R(\vec{y})$
for each relation symbol $R \rightarrowtail \vec{X}$ ◇

2.5. Monoidal and Cartesian Closed Categories

A monoidal category is a category with some sort of 'tensor product'. One of the important examples being of course the category of vector spaces with the usual tensor product which allows us to talk about bilinear maps.

Definition 2.5.1. A monoidal category is a category $\mathcal{C}$ equipped with

(a) A functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

(b) some object $Ob_{\mathcal{C}}(1_T)$ called the tensor unit

(c) A natural isomorphism

$$a : ((\cdot) \otimes_o (\cdot)) \otimes_o (\cdot) \overset{\cong}{\Rightarrow} (\cdot) \otimes_o ((\cdot) \otimes_o (\cdot))$$

called the associator

(d) a natural isomorphism

$$\gamma : (1_T \otimes_o (\cdot)) \overset{\cong}{\Rightarrow} (\cdot)$$

called a left unitor

(e) and a natural isomorphism

$$\rho : ((\cdot) \otimes_o 1_T) \overset{\cong}{\Rightarrow} (\cdot)$$

called a right unitor.

Such that the following diagrams commute. Let $Ob_{\mathcal{C}}(x, y, z, w)$.

$$\begin{tikzcd}
(x \otimes 1_T) \otimes y \arrow[rr, "a_{x,1_T,y}"] \arrow[dr, "\rho_x \otimes_m id(y)"'] & & x \otimes (1_T \otimes y) \arrow[dl, "id(x) \otimes_m \gamma_y"] \\
& x \otimes y &
\end{tikzcd}$$

$$\begin{tikzcd}
& (w \otimes x) \otimes (y \otimes z) \arrow[dr, "a_{w,x,y\otimes z}"] & \\
((w \otimes x) \otimes y) \otimes z \arrow[ur, "a_{w\otimes x,y,z}"] \arrow[d, "a_{w,x,y}\otimes_m id(z)"] & & w \otimes (x \otimes (y \otimes z)) \\
(w \otimes (x \otimes y)) \otimes z \arrow[rr, "a_{w,x\otimes y,z}"] & & w \otimes ((x \otimes y) \otimes z) \arrow[u, "id(w)\otimes_m a_{x,y,z}"']
\end{tikzcd}$$

Written as equations, we require

$$\begin{aligned}
(x \otimes 1_T) \otimes y &\cong x \otimes y \cong x \otimes (1_T \otimes y) \\
(w \otimes x) \otimes (y \otimes z) &\cong ((w \otimes x) \otimes y) \otimes z \\
&\cong (w \otimes (x \otimes y)) \otimes z \\
&\cong w \otimes ((x \otimes y) \otimes z) \\
&\cong w \times (x \otimes (y \otimes z))
\end{aligned}$$

Note that, in the case of a, the functors involved are more formally written as

$$\begin{aligned}
((\cdot) \otimes_o (\cdot)) \otimes_o (\cdot) : (\mathcal{C} \times \mathcal{C}) \times \mathcal{C} \to \mathcal{C} \\
\langle \lambda x.\, \otimes_o (\langle \otimes_o(\pi_0 x), \pi_1 x\rangle), \\
\lambda f.\, \otimes_m (\langle \otimes_m(f_{\pi_0}), f_{\pi_1})\rangle\rangle \\
(\cdot) \otimes_o ((\cdot) \otimes_o (\cdot)) : \mathcal{C} \times (\mathcal{C} \times \mathcal{C}) \to \mathcal{C} \\
\langle \lambda x.\, \otimes_o (\langle \pi_0 x, \otimes_o(\pi_1 x)\rangle), \\
\lambda f.\, \otimes_m (\langle f_{\pi_0}, \otimes_m(f_{\pi_1}))\rangle\rangle
\end{aligned}$$

where

$$g_{\pi_i} :\equiv \langle \pi_i dom(g), \pi_i cod(g), \pi_i(\overline{g})\rangle$$

The other functors are built similarly and in particular the codomain of γ and ρ are the identity functors on $\mathcal{C}$. ◇

Definition 2.5.2 (Braided monoidal category)**.** A braided monoidal category is a monoidal category $(\mathcal{C}, \otimes, a, \gamma, \rho)$ with a natural isomorphism

$$b : ((\cdot_0) \otimes_o (\cdot_1)) \overset{\cong}{\Rightarrow} ((\cdot_1) \otimes_o (\cdot_0))$$

called a braiding such that the following two diagrams commute:

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "a_{x,y,z}"] \arrow[d, "b_{x,y}"'] & x \otimes (y \otimes z) \arrow[r, "b_{x,y\otimes z}"] & (y \otimes z) \otimes x \arrow[d, "a_{y,z,x}"] \\
(y \otimes x) \otimes z \arrow[r, "a_{y,x,z}"'] & y \otimes (x \otimes z) \arrow[r, "id(y)\otimes_m b_{x,z}"'] & y \otimes (z \otimes x)
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "a^{-1}_{x,y,z}"] \arrow[d, "id(x)\otimes_m b_{y,z}"'] & (x \otimes y) \otimes z \arrow[r, "b_{x\otimes y,z}"] & z \otimes (x \otimes y) \arrow[d, "a^{-1}_{z,x,y}"] \\
x \otimes (z \otimes y) \arrow[r, "a^{-1}_{x,y,z}"'] & (x \otimes z) \otimes y \arrow[r, "b_{x,z}\otimes_m id(y)"'] & (z \otimes x) \otimes y
\end{tikzcd}$$

◇

Definition 2.5.3 (Symmetric monoidal category)**.** A Symmetric monoidal category is a braided monoidal category $(\mathcal{C}, \otimes, a, \gamma, \rho, b)$ such that the braiding b satisfies

$$b_{y,x} \circ b_{x,y} =_m id(x \otimes y).$$

◇

Definition 2.5.4 (Symmetric closed monoidal category)**.** A symmetric monoidal category $(\mathcal{C}, \otimes, a, \gamma, \rho, b)$ is closed if for all objects $Ob_{\mathcal{C}}(c)$ the functor $(\cdot) \otimes c : \mathcal{C} \to \mathcal{C}$ has a right adjoint $(\cdot)^c : \mathcal{C} \to \mathcal{C}$. ◇

Proposition 2.5.5 (Cartesian monoidal categories)**.** *Any category $\mathcal{C}$ with a terminal object and a specified product for all pairs of objects has the structure of a symmetric monoidal category.*

Here specified product *means that we can construct it in a uniform way, which in particular includes a uniform construction (given by a term) of universal morphisms into all binary products.*

Proof. The terminal object is the tensor unit and the natural isomorphisms can be constructed from the universal morphisms of cartesian products.

$$\begin{aligned}
a(\langle\langle x, y\rangle, z\rangle) &:\equiv \langle\!\langle \pi_0^{x,y} \circ \pi_0^{x\times y,z}, \langle\!\langle \pi_1^{x,y} \circ \pi_0^{x\times y,z}, \pi_1^{x\times y,z} \rangle\!\rangle_{y,z} \rangle\!\rangle_{x\times(y\times x)} \\
\gamma(x) &:\equiv \pi_1^{\mathbb{1},x} \\
\rho(x) &:\equiv \pi_0^{x\times\mathbb{1}} \\
b(\langle x, y\rangle) &:\equiv \langle\!\langle \pi_1^{x,y}, \pi_0^{x,y} \rangle\!\rangle_{y,x}
\end{aligned}$$

The construction of the inverse natural transformations (and of course the verification of this fact) is similar but long and not very enlightening since in the end it is all based on the essential uniqueness of the categorical product. So we only note that for any object x we can show $\mathbb{1} \times x \cong x \cong x \times \mathbb{1}$ directly from the universal property of products. □

Definition 2.5.6 (Cartesian closed category)**.** A Category $\mathcal{C}$ is cartesian closed if it has the structure of a cartesian closed monoidal category. ◇

A more elementary definition which is much simpler to check is the the following:

Definition 2.5.7 (Cartesian closed category [31])**.** Let exp, eps, lam be terms and write

$$x^y :\equiv exp(y, x) \qquad \varepsilon^{x,y} :\equiv eps(x, y) \qquad \Lambda^z_{x,y} :\equiv lam(z, x, y).$$

for their applications to any two (resp. three) arguments. We say x^y is an *exponential object* or just an *exponential*.

We call a category $\mathcal{C}$ with all binary products and a terminal object *cartesian closed*, iff the following axioms are satisfied:

$$(CC1) \qquad Ob(x, y) \to x^y{\downarrow} \wedge Ob(x^y)$$

$$(CC2) \qquad Ob(x, y) \to \varepsilon^{x,y}{\downarrow} \wedge \varepsilon^{x,y} : x^y \times y \overset{m}{\to} x$$

$$(CC3) \qquad (Ob(x, y, z) \wedge h : z \times y \overset{m}{\to} x) \to \Lambda^z_{x,y}(h) : z \overset{m}{\to} x^y$$

$$(CC4) \qquad (h : z \times y \overset{m}{\to} x) \to (\varepsilon^{x,y} \circ \langle\!\langle \Lambda^z_{x,y}(h) \circ pr_0, pr_1 \rangle\!\rangle =_m h)$$

$$(CC5) \qquad (k : z \overset{m}{\to} x^y) \to (\Lambda^z_{x,y}(\varepsilon^{x,y} \circ \langle\!\langle k \circ pr_0, pr_1 \rangle\!\rangle) =_m k)$$

Here

$$\langle\!\langle f, g \rangle\!\rangle : c \overset{m}{\to} x \times y$$

is the unique map into a product and pr_0, pr_1 are the projections. ◇

Definition 2.5.8 (Locally cartesian closed category)**.** A category $\mathcal{C}$ is called locally cartesian closed (LCC) iff for all objects x, the slice category $\mathcal{C}/x$ is cartesian closed.

This is equivalent to requiring a right adjoint to all pullback functors $f^*(\cdot)$ along any $f : y \overset{m}{\to} x$. ◇

Proposition 2.5.9 ([2], proposition 6.6)**.** *The assignment* $b \mapsto b^a$ *is functorial.*

Proof. To show this is a functor we have to define how it acts on morphisms.

$$
\begin{aligned}
&f^a_o(x) :\equiv x^a \\
&f^a_m(g) :\equiv \Lambda^{dom(g)^a}_{a,cod(g)}(g \circ \varepsilon^{dom(g),a})
\end{aligned}
$$

We have to check $f^a_m(g \circ h) =_m f^a_m(g) \circ f^a_m(h)$

$$
\begin{aligned}
f^a_m(g \circ h) &=_m \Lambda^{dom(g\circ h)^a}_{a,cod(g\circ h)}((g \circ h) \circ \varepsilon^{dom(g\circ h),a}) \\
&=_m \Lambda^{dom(g)^a}_{a,cod(h)}((g \circ h) \circ \varepsilon^{dom(g),a})
\end{aligned}
$$

Let $g : y \overset{m}{\to} z$ and $h : x \overset{m}{\to} y$.

$$\begin{tikzcd}
z^a \times a \arrow[r, "\varepsilon^{a,z}"] & z \\
y^a \times a \arrow[u, "f^a_m(g)\times id(a)"] \arrow[r, "\varepsilon^{a,y}"] & y \arrow[u, "g"] \\
x^a \times a \arrow[u, "f^a_m(h)\times id(a)"] \arrow[r, "\varepsilon^{a,x}"] & x \arrow[u, "h"]
\end{tikzcd}$$

The above is a commutative diagram and since it's possible to show that $(u \times id(a)) \circ (v \times id(a)) =_m (u \circ v) \times id(a)$ for any two composable morphisms u and v this means that we have $f^a_m(g \circ h) =_m f^a_m(g) \circ f^a_m(h)$. This follows by uniqueness of transposed morphisms as we have now two different candidates for the transpose of $h \circ g \circ \varepsilon^{a,x}$. One being $f^a_m(g \circ h)$ directly from the definition of the transpose, and $f^a_m(g) \circ f^a_m(h)$ from the above equality.

The fact that identities are preserved can be seen from

$$\begin{tikzcd}
z^a \times a \arrow[r, "\varepsilon^{a,z}"] & z \\
z^a \times a \arrow[u, "f^a_m(id(z)) \times id(a)"] \arrow[r, "\varepsilon^{a,z}"] & z. \arrow[u, "id(z)"']
\end{tikzcd}$$

Since the morphism[9] $id(z^a \times a)$ also fits this square on the left side we get by uniqueness of the transpose that

$$f^a_m(id(z)) =_m id(z^a) =_m id(f^a_o(z)). \qquad \square$$

Sometimes we may not have a cartesian closed category, but we might still construct weak dependent products. With the right additional properties this guarantees the local cartesian closure as described in section 3.3.

Definition 2.5.10 ([46]). For two morphisms $f : x \overset{m}{\to} j$ and $a : j \overset{m}{\to} i$ in a finitely complete category $\mathcal{C}$, a *weak dependent product* of f along a is an object $\xi : z \overset{m}{\to} i$ in $\mathcal{C}/i$ together with a morphism $\varepsilon : a^*\xi \overset{m}{\to} f$ in $\mathcal{C}/j$ such that for any object $m : w \overset{m}{\to} i$ in $\mathcal{C}/i$ together with a morphism $h : a^*m \overset{m}{\to} f$ in $\mathcal{C}/j$ there exists a (not necessarily unique) morphism $g : m \overset{m}{\to} \xi$ in $\mathcal{C}/i$ such that $h =_m \varepsilon \circ a^*g$ in $\mathcal{C}/j$.

$$\begin{tikzcd}
 & a * m \arrow[r] \arrow[ld, bend right, "h"'] \arrow[d, "a^*g"] \arrow[dd, bend right, "a^*m"'] & w \arrow[d, "\exists g"'] \arrow[dd, bend left, "m"] \\
x \arrow[rd, bend right, "f"'] & a * \xi \arrow[l, "\varepsilon"] \arrow[r] \arrow[d] & z \arrow[d, "\xi"'] \\
 & j \arrow[r, "a"'] & i
\end{tikzcd}$$

◇

---

[9] We can get $id(a) \times id(b) =_m id(a \times b)$ from the universal property of products.

3. Towards a Category of Sets

The next few sections will build up to the *Category of Sets*. Clearly it's impossible to capture all aspects of the category of sets as we would define it for example in **ZFC**+(a countable sequence of Grothendieck universes). The most obvious failure in capturing Sets is the lack of power classes. We let

$$\begin{aligned} SPow[a,b] &:\equiv \forall x(x \mathrel{\dot{\in}} b \leftrightarrow x \mathrel{\dot{\subset}} a), \\ WPow[a,b] &:\equiv \forall x(x \mathrel{\dot{\in}} b \rightarrow x \mathrel{\dot{\in}} a) \wedge \forall x(x \mathrel{\dot{\in}} a \rightarrow (\exists y \mathrel{\dot{\in}} b)(x \doteq y)). \end{aligned}$$

The strong and the weak uniform power class axioms are defined as follows:

$$\begin{aligned} &(\text{US-Pow}) \qquad && \forall x(\mathfrak{R}(x) \rightarrow SPow[x, \mathrm{pow}(x)]) \\ &(\text{UW-Pow}) \qquad && \forall x(\mathfrak{R}(x) \rightarrow WPow[x, \mathrm{pow}(x)]). \end{aligned}$$

It's well known that (US-Pow) is inconsistent with **EM** even without the join axiom, while (UW-Pow) is inconsistent once we add join. In fact **EM** with join even proves the negation of the non-uniform versions of these axioms (i.e. shows that they are inconsistent with join.)

3.1. The Category EC

Probably the most natural (weak) category one can define in Explicit Mathematics is the category of classes and operations. In fact this category was one of the initial reasons to build the framework of this thesis. The definition is simple. The weak version uses all names of classes, the regular one is relativized to a given universe in Explicit Mathematics.

Definition 3.1.1 (Weak **EC**)**.**

$$\begin{aligned}
Ob(x) &:\equiv \mathfrak{R}\,(x) \\
Mor(f) &:\equiv \exists x, y, f'(f = \langle x, y, f' \rangle \wedge Ob(x) \wedge Ob(y) \\
&\qquad \wedge (\forall z \mathrel{\dot\in} x)(f'z \mathrel{\dot\in} y)) \\
x =^{\mathbf{EC}}_o y &:\equiv x = y \\
f =^{\mathbf{EC}}_m g &:\equiv dom(f) =^{\mathbf{EC}}_o dom(g) \wedge cod(f) =^{\mathbf{EC}}_o cod(g) \\
&\qquad \wedge (\forall z \mathrel{\dot\in} dom(f))(\overline{f}z = \overline{g}z) \\
f \circ g &:\equiv \langle dom(g), cod(f), \lambda z.\overline{f}(\overline{g}z) \rangle \\
id(x) &:\equiv \langle x, x, \lambda x.x \rangle
\end{aligned}$$

◇

This is the minimal definition to make **EC** work as a weak category. The only point where we have to take some care is composition of morphisms, which has to be well-defined.

Definition 3.1.2 (**EC**)**.** We require the Join axiom. Let u be a universe in the sense of $\mathcal{U}(u)$.

$$\begin{aligned}
ob &:\equiv u \\
mor &:\equiv \sum_{x:u} \sum_{y:u} (x \mathrel{\dot\to} y) \\
x =^{\mathbf{EC}}_o y &:\equiv t_{EQO}(u) \\
f =^{\mathbf{EC}}_m g &:\equiv t_{EQM}(mor, ext(=_o)) \\
f \circ g &:\equiv \langle dom(g), cod(f), \lambda z.\overline{f}(\overline{g}z) \rangle \\
id(x) &:\equiv \langle x, x, \lambda x.x \rangle
\end{aligned}$$

where

$$\begin{aligned}
EQO[u, U] &:\equiv u = \langle \pi_0 u, \pi_1 u \rangle \wedge \pi_0 u \in U \wedge \pi_1 u \in U \\
EQM[u, E] &:\equiv \exists f, g(u = \langle f, g \rangle \wedge \langle f, g, * \rangle \in E) \\
EXT[u, f, g, =_o, X] &:\equiv u = * \wedge dom(f) =_o dom(g) \\
&\qquad \wedge cod(f) =_o cod(g) \\
&\qquad (\forall x \in X)(\overline{f}x = \overline{g}x)
\end{aligned}$$

$$ext(=_o) :\equiv \sum_{f:mor} \sum_{g:mor} t_{EXT}(f, g, =_o, dom(f))$$

It's impossible to define equality on morphisms as equality on the representing terms. The reason for this is the requirement of associativity of the composition. In general it's impossible to prove that for arbitrary $f, g, h \mathrel{\dot{\in}} mor$

$$\overline{((f \circ g) \circ h)} \text{ which is } \lambda w.(\lambda z.\overline{f}(\overline{g}z))(\overline{h}w)$$

is the same as

$$\overline{(f \circ (g \circ h))} \text{ which is } \lambda w.\overline{f}((\lambda z.\overline{g}(\overline{h}z))w). \qquad \diamondsuit$$

The relation $=_m^{\mathbf{EC}}$ is an equivalence and respects morphism composition. This fact is proved in proposition A.5.1.

Proposition 3.1.3 (Verification of Category-properties)**.**

- *$id(x)$ defined as $\langle x, x, \lambda x.x\rangle$ satisfies the identity axioms.*
- *Composition is associative.*

Proof. Let $Ob(x, y, z)$, $f : y \overset{m}{\to} z$, $g : x \overset{m}{\to} y$ and $h : w \overset{m}{\to} x$.

$$\begin{aligned}
id(x) \circ h &=_m \langle w, x, \lambda z.(\lambda x.x)(\overline{h}z)\rangle \\
&=_m \langle w, x, \lambda z.(\overline{h}z)\rangle \\
&=_m \langle w, x, \overline{h}\rangle \\
&=_m h \\
&=_m \langle w, x, \lambda z.(\overline{h})((\lambda w.w)z)\rangle \\
&=_m h \circ id(w).
\end{aligned}$$

For associativity we calculate

$$\begin{aligned}
(f \circ g) \circ h &=_m \langle \mathit{dom}(h), \mathit{cod}(f \circ g), \lambda z.(\overline{(f \circ g)})(\overline{h}z)\rangle \\
&=_m \langle w, z, \lambda z.((\overline{\langle x, z, \lambda z.(\overline{f})(\overline{g}z)\rangle}))(\overline{h}z)\rangle \\
&=_m \langle w, z, \lambda z.(\lambda z.(\overline{f})(\overline{g}z))(\overline{h}z)\rangle \\
&=_m \langle w, z, \lambda z.\overline{f}(\overline{g}(\overline{h}z))\rangle \\
&=_m \langle w, z, \lambda z.\overline{f}((\lambda z.\overline{g}(\overline{h}z))z)\rangle \\
&=_m \langle \mathit{dom}(g \circ h), z, \lambda z.\overline{f}((\overline{\langle w, y, \lambda z.\overline{g}(\overline{h}z)\rangle})z)\rangle \\
&=_m \langle \mathit{dom}(g \circ h), z, \lambda z.\overline{f}(\overline{(g \circ h)}z)\rangle \\
&=_m \langle \mathit{dom}(g \circ h), \mathit{cod}(f), \lambda z.(\overline{f})(\overline{(g \circ h)}z)\rangle \\
&=_m f \circ (g \circ h) \qquad \square
\end{aligned}$$

Remark 3.1.4. The weak category **EC** is proper local without using Join:

- The class of all morphisms between two objects $Ob(x), Ob(y)$ is given by elementary comprehension:

$$\begin{aligned}
A[u, x, y, F] &:\equiv \exists f(u = \langle x, y, f\rangle \wedge f \in F) \\
h(x, y) &:\equiv t_A(x, y, x \dot{\to} y)
\end{aligned}$$

- Restricted to some $h(x, y)$, we can write equality on morphisms as:

$$\begin{aligned}
EXT[u, X, H] &:\equiv \exists f, g(u = \langle f, g\rangle \wedge f, g \in H \\
&\qquad\qquad \wedge (\forall x \in X)(\overline{f}x = \overline{g}x) \\
ext(x, y) &:\equiv t_{EXT}(x, h(x, y)). \qquad \diamond
\end{aligned}$$

Definition 3.1.5 (injectivity)**.** We call a morphism $f : x \overset{m}{\to} y$ *injective* if it satisfies

$$(\forall a, b \,\dot{\in}\, x)((\overline{f}a = \overline{f}b) \to (a = b)) \qquad \diamond$$

Proposition 3.1.6. *Injective operations are monomorphisms*

Proof. Let $g, h : z \overset{m}{\to} x$ be morphisms such that $f \circ g =_m f \circ h$. We have to show $g =_m h$. For $a \dot{\in} z$ we have

$$\overline{(f \circ g)}z = \overline{(f \circ h)}z.$$

So, by definition

$$(\forall a \dot{\in} x)((\overline{f}(\overline{g}z) = \overline{f}(\overline{h}z)).$$

Since f is injective, we get $\overline{g}z = \overline{h}z$ for all $z \dot{\in} a$. This is exactly the definition of $g =_m h$. ☐

Proposition 3.1.7. *Monomorphisms are injective operations*

Proof. Let $f : x \overset{m}{\rightarrowtail} y$ be monic. For any $a \dot{\in} x$ We can define the global element-morphism

$$\{a\} :\equiv \langle \mathbb{1}, x, \lambda x.a \rangle : \mathbb{1} \overset{m}{\to} x.$$

(See definition 1.0.5 for the class $\mathbb{1}$.) Let $a, b \dot{\in} x$ such that $\overline{f}a = \overline{f}b$. By definition of composition we get the following string of equivalences.

$$\overline{(f \circ \{a\})}(*) = \overline{f}(\overline{\{a\}}(*)) = \overline{f}a = \overline{f}b = \overline{f}(\overline{\{b\}}(*)) = \overline{(f \circ \{b\})}(*)$$

But this just means $(f \circ \{a\}) =_m (f \circ \{b\})$. Because f is mono this implies that we have $\{a\} =_m \{b\}$ which can only be true if $a = b$ holds, which is what we wanted to show. ☐

Proposition 3.1.8. *All idempotents in* **EC** *split. That is, for all* $e : x \overset{m}{\to} x$ *with* $e \circ e =_m e$ *there exist* $r : x \overset{m}{\to} y$ *and* $s : y \overset{m}{\to} x$ *with* $s \circ r =_m e$ *and* $r \circ s =_m id(y)$.

Proof. This is just the existence of fixed points as classes.
Let

$$\begin{aligned} FIX[u, e, X] &:\equiv u \in X \wedge \overline{e}u = u \\ fix(e) &:\equiv t_{FIX}(e, dom(e)). \end{aligned}$$

The splitting $(s_e \circ r_e) : dom(e) \overset{m}{\to} fix(e) \overset{m}{\to} cod(e)$ is then given by

$$r_e :\equiv \langle dom(e), fix(e), \overline{e}\rangle$$

and

$$s_e :\equiv \langle fix(e), dom(e), \lambda x.x\rangle.$$

The morphism r_e is well-defined, because for all $a \mathrel{\dot\in} x$ we have $\overline{e}(\overline{e}a) = \overline{e}a$ by assumption and hence $\overline{e}a \mathrel{\dot\in} fix(e)$ holds as required.
Checking the properties, we get for $a \mathrel{\dot\in} x$ that

$$\overline{(s_e \circ r_e)}(a) = \overline{s_e}(\overline{r_e}a) = (\overline{r_e}a) = \overline{e}a$$

and so $(s_e \circ r_e) =_m e$. For $a \mathrel{\dot\in} fix(e)$ we have

$$\overline{(r_e \circ s_e)}(a) = \overline{r_e}(\overline{s_e}a) = \overline{r_e}a = \overline{e}a = a$$

where the last equality is because $a \mathrel{\dot\in} fix(e)$. □

Finite limits in EC and Cartesian Closure

In this section we will be using definition 2.5.7 for cartesian closed categories.

Proposition 3.1.9. *Any one-element class is a terminal object in* **EC**.

Proof. Let $\mathbb{1}$ be the class from definition 1.0.13 with $u \mathrel{\dot\in} \mathbb{1} \leftrightarrow u = *$. For an arbitrary class name z, we have $\langle z, \mathbb{1}, \lambda z.*\rangle : z \overset{m}{\to} \mathbb{1}$. For any other morphism $f : z \overset{m}{\to} \mathbb{1}$ we know that $(\forall x \mathrel{\dot\in} z)(\overline{f}x = *)$, therefore we have the required $f =_m \langle z, \mathbb{1}, \lambda z.*\rangle$. By the same argument, it's clear that any other one-element class[1] is isomorphic to $\mathbb{1}$. □

Proposition 3.1.10. *We can construct arbitrary binary products.*

Proof. Given $Ob(s,t)$, the class named $s \times t$ given in definition 1.0.13 is the required object. Defining for arbitrary $Ob(x,y,z,w)$, $f : z \overset{m}{\to} x$, $g : z \overset{m}{\to} y$,

---

[1] A *one-element class* t here means that we have exactly one explicitly given element in the class as opposed to a description like $(\exists x \mathrel{\dot\in} t)(\forall z \mathrel{\dot\in} t)(z = x)$.

$u : x \overset{m}{\to} z,\ v : y \overset{m}{\to} w$

$$\begin{aligned}
\langle\!\langle f, g\rangle\!\rangle &:\equiv \langle z, x \times y, \lambda z.\langle \overline{f}z, \overline{g}z\rangle\rangle : z \overset{m}{\to} x \times y\\
\mathrm{pr}_i^{x_1,x_2} &:\equiv \langle x_1 \times x_2, x_i, \pi_i\rangle\\
u \times v &:\equiv \langle\!\langle u \circ \mathrm{pr}_0^{x,y}, v \circ \mathrm{pr}_1^{x,y}\rangle\!\rangle
\end{aligned}$$

Note that we have overloaded the $(\cdot \times \cdot)$ notation both for objects and morphisms. We will use this only where the context is clear. To see that the equations hold up to functional extensionality let $w \dot{\in} z$ and $h : z \overset{m}{\to} x \times y$ with $pr_0^{x,y} \circ h =_m f$ and $pr_0^{x,y} \circ h =_m g$.

$$\begin{aligned}
\overline{(pr_0^{x,y} \circ \langle\!\langle f, g\rangle\!\rangle)}w &= \overline{pr_0^{x,y}}(\langle \overline{f}w, \overline{g}w\rangle)\\
&= \pi_0\langle \overline{f}w, \overline{g}w\rangle\\
&= \overline{f}w\\
&= \overline{(pr_0^{x,y} \circ h)}w\\
&= \overline{pr_0^{x,y}}(\overline{h}w).
\end{aligned}$$

Similarly

$$\overline{(pr_1^{x,y} \circ \langle\!\langle f, g\rangle\!\rangle)}w = \overline{pr_1^{x,y}}(\overline{h}w).$$

If we can show that $\langle\!\langle pr_0^{x,y}, pr_1^{x,y}\rangle\!\rangle$ is an isomorphism (and hence monic) we are done.

To this end let $\langle v, w\rangle \dot{\in} x \times y$

$$\begin{aligned}
\overline{\langle\!\langle pr_0^{x,y}, pr_1^{x,y}\rangle\!\rangle}\langle v, w\rangle &= \langle \overline{pr_0^{x,y}}\langle v, w\rangle, \overline{pr_1^{x,y}}\langle v, w\rangle\rangle\\
&= \langle \pi_0\langle v, w\rangle, \pi_1\langle v, w\rangle\rangle\\
&= \langle v, w\rangle
\end{aligned}$$

So we have $id(x \times y) =_m \langle\!\langle pr_0^{x,y}, pr_1^{x,y}\rangle\!\rangle$ which finishes the proof. □

Proposition 3.1.11. **EC** *has pullbacks.*

Proof. We define the following elementary formula

$$\begin{aligned} PB[u, f, g, F, G, Z] :\equiv\ &(\exists a \in F)(\exists b \in G) \\ &(u = \langle a, b\rangle \wedge fa = gb \wedge fa \in Z) \end{aligned}$$

Given $f : x \overset{m}{\to} z$ and $g : y \overset{m}{\to} z$ then the object-part of their pullback is the class[2]

$$\mathit{pullb}(x, z, y, f, g) :\equiv t_{PB}(\overline{f}, \overline{g}, x, y, z)$$

with the obvious projections

$$\begin{aligned} f^* &:\equiv \langle \mathit{pullb}(x, z, y, f, g), x, \pi_0\rangle \\ g^* &:\equiv \langle \mathit{pullb}(x, z, y, f, g), y, \pi_1\rangle \end{aligned}$$

We can give the morphism explicitly:
If e is a class and $u : e \to x$, $v : e \to y$ are morphisms such that $f \circ u =_m g \circ v$, then

$$\langle e, \mathit{pullb}(x, z, y, f, g), \lambda c.\langle \overline{u}c, \overline{v}c\rangle\rangle : e \overset{m}{\to} \mathit{pullb}(x, z, y, f, g).$$

If the pullback happens to be empty, then there are no elements in x and y which get mapped to the same value in z. So there can be no $c \mathrel{\dot{\in}} e$ such that $\overline{f}(\overline{u}c) = \overline{g}(\overline{v}c)$ or rewritten $\overline{(f \circ u)}c = \overline{(g \circ v)}c$. Therefore, using the definition of morphisms in **EC** and the fact that the above is a commuting square, we can deduce that e is also empty. Note however, that we actually don't have to care about this case, since any defined term tracks the morphism from the initial object. Proposition 3.1.17 then tells us that the above term is a representation of the required unique morphism. We still have to show this commutes and is unique. We call the morphism defined above $h : e \to \mathit{pullb}(x, z, y, f, g)$ and let $h' :\equiv \langle e, \mathit{pullb}(x, z, y, f, g), \tilde{h}\rangle : e \to$

---

[2] Note that we will sometimes refer to this object by the name $f * g$.

$pullb(x,z,y,f,g)$ be another such morphism.

$$\begin{aligned} f^* \circ h &= \langle e, x, \lambda c.\overline{(f^*)}(\pi_2\langle e, pullb(x,z,y,f,g), \lambda c.\langle \overline{u}c, \overline{v}c\rangle\rangle c)\rangle \\ &=_m \langle e, x, \lambda c.\overline{(f^*)}((\lambda c.\langle \overline{u}c, \overline{v}c\rangle)c)\rangle \\ &=_m \langle e, x, \lambda c.\overline{(f^*)}\langle \overline{u}c, \overline{v}c\rangle\rangle \\ &=_m \langle e, x, \lambda c.\pi_0\langle \overline{u}c, \overline{v}c\rangle\rangle \\ &=_m \langle e, x, \lambda c.\overline{u}c\rangle \\ &=_m \langle e, x, \overline{u}\rangle =_m u \end{aligned}$$

The case for $g^* \circ h =_m v$ is similar.

To check for uniqueness suppose additionally that the equations

$$f^* \circ h' =_m u \qquad\qquad \text{and} \qquad\qquad g^* \circ h' =_m v$$

hold. Then we have

$$\begin{aligned} h' &=_m \langle e, pullb(x,z,y,f,g), \tilde{h}'\rangle, \\ f^* \circ h' &=_m \langle e, x, \lambda c.\pi_0(\tilde{h}'c)\rangle \text{ and} \\ g^* \circ h' &=_m \langle e, y, \lambda c.\pi_1(\tilde{h}'c)\rangle \end{aligned}$$

From this we get

$$\begin{aligned} &(\forall c \mathrel{\dot{\in}} e)(\exists a \mathrel{\dot{\in}} x)(\exists b \mathrel{\dot{\in}} y)(\tilde{h}c = \langle a, b\rangle) \\ &(\forall c \mathrel{\dot{\in}} e)(\exists a' \mathrel{\dot{\in}} x)(\exists b' \mathrel{\dot{\in}} y)(\tilde{h}'c = \langle a', b'\rangle) \\ &(\forall c \mathrel{\dot{\in}} e)(\pi_0(\tilde{h}'c) = \overline{u}c \wedge \overline{u}c = \pi_1(h'c) \\ &\qquad\qquad \wedge\ \pi_1(\tilde{h}'c) = \overline{v}c \wedge \overline{v}c = \pi_1(h'c)) \end{aligned}$$

and from this it's clear that with the above values

$$\langle a, b\rangle = \langle a', b'\rangle$$

and therefore

$$h =_m h'.$$

□

Conjecture 3.1.12. **EC** apparently does not have arbitrary infinite products. In particular, we conjecture that even when the Join-axiom holds, there exist functors from

$$\bullet \qquad \bullet \qquad \bullet \qquad \bullet \qquad \cdots$$

which more formally are functors from the category $\mathbb{n}$ defined as

$$\mathbb{n} :\equiv \langle nat, t_D, \mathrm{id}(nat), \mathrm{id}(t_D), \lambda n.\langle n, n, *\rangle, \lambda f \lambda g.f\rangle^3$$

$$D[u] :\equiv u = \langle n, n, *\rangle \wedge n \in N$$

which do *not* have a limit. But note that we can *not* prove that

$$\mathbf{EM} \vdash (\forall Ob(x))(x \text{ is not a limit for } Functor[f, \mathbb{n}, \mathbf{EC}]).$$

But on the other hand it's impossible to construct a product in the way we would in set theory.

Given a functor $\langle f_o, f_n\rangle$ from $\mathbb{n}$ into **EC**, in general the only choice for a limiting cone is via the Pi class constructor. But it's enough to treat the case where $f_o(n) = nat$. Then a candidate object for the limiting cone is just $nat \dot{\rightarrow} nat$. Clearly any other cone $\langle c, q\rangle$ has a morphism $h\langle c, q\rangle$ into given by

$$h\langle d, q\rangle :\equiv \langle d, nat \dot{\rightarrow} nat, \lambda d.\lambda n.\overline{(q(n))}d\rangle$$

Since $q(n) : d \overset{m}{\rightarrow} f_o(n)$ holds, we have for any $z \dot{\in} d$ that

$$\lambda n.\overline{(q(n))}z \dot{\in} nat \dot{\rightarrow} nat.$$

However there is no possibility to make this morphism unique. See lemma A.5.2 for a simple counter example. A more tedious calculation shows that in general

$$\mathbf{EM} \nvdash \lambda w.((\lambda z.fz)w) = (\lambda z.fz)$$

since it is false in the term model (see for example [4] VI.6). But for w such

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[3] Here $\mathrm{id}(s)$ is the **EM** class constructor from definition 1.0.5 rather than an identity morphism.

that $(fw)\downarrow$ we have of course

$$(\lambda y.(\lambda z.fz)y)w = (\lambda z.fz)w = fw$$

This means there are an infinite number of non-equal morphisms into $nat \dot\rightarrow nat$ by prepending lambda abstractions and applications: For $z \dot\in d$ we have

$$\begin{aligned}\overline{(\tilde{h}\langle d, q\rangle)}z &:\equiv \overline{\langle d, \prod_{n:N} f_o(n), \lambda d.\lambda n.((\lambda z.\overline{(q(z))}d)n)\rangle}z \\ &= (\lambda d.\lambda n.((\lambda z.\overline{(q(z))}d)n))z \\ &= \lambda n.((\lambda z.\overline{(q(z))}z)n)\end{aligned}$$

and we can not prove that this is equal to

$$\begin{aligned}&\lambda n.(\overline{(q(n))}z) \\ &= \overline{(h\langle d, q\rangle)}z\end{aligned}$$

Since in **EM** it is impossible to form arbitrary indexed classes without the join axiom this means it is unclear how one would construct an infinite product at least in cases where no prejoined class is given. ◇

Corollary 3.1.13. *The class* $\prod_{n:N} \mathbb{1}$ *is not a terminal object.*

Proof. There are two terms $\lambda x.\lambda n.*$ and $\lambda x.\lambda n.(\lambda p.*)n$ which are different in the term model. ☐

Remark 3.1.14. This is not to say that the infinite product of $\mathbb{1}$ does not exist. In fact the product exists and is exactly what it is supposed to be. Namely $\mathbb{1}$ with projections $p(n) :\equiv id(\mathbb{1})$. Clearly $h\langle d, q\rangle :\equiv\, !_d : d \overset{m}{\to} \mathbb{1}$ has the required properties. ◇

Theorem 3.1.15. **EC** *is not cartesian closed*

Proof. Cartesian closure would imply that there is an isomorphism

$$hom_{\mathbf{EC}}(\mathbb{1}, nat^{nat}) \cong hom_{\mathbf{EC}}(\mathbb{1} \times nat, nat) \cong hom_{\mathbf{EC}}(nat, nat)$$

Let

$$s :\equiv \lambda f.\overline{(\Lambda^{\mathbb{1}}_{nat,nat}(\langle \mathbb{1} \times nat, nat, \lambda t.f(\pi_1 t)\rangle))}(*)$$

Then with proposition A.5.3 we have

$$(\forall f, g \in (N \to N))(s(f) = s(g) \leftrightarrow (\forall x \in N)(fx = gx)).$$

But by lemma A.5.4 such a term can not exist. ☐

Theorem 3.1.16. **EC** *has weak dependent products*

Proof. Let $f : x \overset{m}{\to} j$ and $a : j \overset{m}{\to} i$ be two morphisms in **EC**. The weak dependent product (definition 2.5.10) of f along a is given by

$$\begin{aligned} pi_a(f) &:\equiv \sum_{i_0 : i} \prod_{j : a^{-1}\{i\}} f^{-1}\{j\} \\ \Pi_a(f) &:\equiv \langle pi_a(f), i, \pi_0 \rangle \end{aligned}$$

$$\begin{aligned} \varepsilon &: a * \Pi_a(f) \overset{m}{\to} j \\ \varepsilon &:\equiv \langle a * \Pi_a(f), j, \lambda p.(\pi_1(\pi_1 p))(\pi_0 p) \rangle \end{aligned}$$

Note that we write morphisms in the slice category just as morphisms in **EC** to keep the amount of syntax down.

To show ε induces a morphism in $\mathcal{C}/j$ we calculate for $\langle j_0, \langle i_0, q \rangle \rangle \mathrel{\dot\in} a * \Pi_a(f)$:

$$\begin{aligned} \overline{f}(\overline{\varepsilon}\langle j_0, \langle i_0, q \rangle \rangle) &= \overline{f}((\pi_1(\pi_1 \langle j_0, \langle i_0, q \rangle \rangle))(\pi_0 \langle j_0, \langle i_0, q \rangle \rangle)) \\ &= \overline{f}(q(j_0)) \\ &= j_0 \\ &= \pi_0(\langle j_0, \langle i_0, q \rangle \rangle). \end{aligned}$$

The last term is just the projection-morphism to j from the pullback. The third equality holds because

$$\begin{aligned} &\langle j_0, \langle i_0, q \rangle \rangle \mathrel{\dot\in} a * pi_a(f) \\ &\leftrightarrow \overline{a}(j_0) = \overline{(\Pi_a(f))}\langle i_0, q \rangle \wedge j_0 \mathrel{\dot\in} j \wedge \langle i_0, q \rangle \mathrel{\dot\in} pi_a(f) \end{aligned}$$

and hence

$$q \mathrel{\dot\in} \prod_{j : a^{-1}\{i_0\}} f^{-1}\{j\}.$$

But $\overline{a}(j_0) = i_0$ means

$$\begin{aligned} q(j_0) &\mathrel{\dot\in} f^{-1}\{j_0\} \\ &\leftrightarrow \overline{f}(q(j_0)) = j_0 \end{aligned}$$

as required.

In the second part of the proof we will verify the weak universal property. Let $m : w \overset{m}{\to} i$ be an object in $\mathbf{EC}/i$ and and $h : a^*(m) \overset{m}{\to} f$ a morphism in $\mathbf{EC}/j$. We have to construct morphism $g : m \overset{m}{\to} \Pi_a(f)$ in $\mathbf{EC}/i$ and its pullback[4] $a^*g : a^*(m) \overset{m}{\to} a^*(\Pi_a(f))$ in $\mathbf{EC}/j$ such that $h =_m \varepsilon \circ a^*g$ in $\mathbf{EC}/j$. The big picture is best understood from the following diagram:

$$\begin{tikzcd}
 & \langle j_0, w_0 \rangle \arrow[r, mapsto] \arrow[d, mapsto, "a^*g"] \arrow[ld, mapsto, bend right, "h"'] \arrow[d, mapsto, bend right=60, "a^*(m)"'] & w_0 \arrow[d, mapsto, "\exists g"'] \arrow[d, mapsto, bend left=60, "m"] \\
\overline{h}\langle j_0, w_0 \rangle \arrow[rd, mapsto, bend right, "f"'] & \langle j_0, \langle \overline{m}w_0, \lambda j.\overline{h}\langle j, w_0 \rangle \rangle \rangle \arrow[l, mapsto, "\varepsilon"] \arrow[r, mapsto] \arrow[d, mapsto, "a^*(\Pi_a(f))"] & \langle \overline{m}w_0, \lambda j.\overline{h}\langle j, w_0 \rangle \rangle \arrow[d, mapsto, "\Pi_a(f)"'] \\
 & j_0 \arrow[r, mapsto, "a"'] & i
\end{tikzcd}$$

The morphism g is induced by the term $\lambda w.\langle \overline{m}w, \lambda j.\overline{h}\langle j, w \rangle \rangle$. Since the pullback of a is just the second projection this is clearly again a pullback (with an obvious unique morphism into it.)

But we still have to verify this is even well-defined.

(a) We need to show that for any $w_0 \mathrel{\dot\in} w$ the term $\langle \overline{m}w_0, \lambda j.\overline{h}\langle j, w_0 \rangle \rangle$ is an element of $pi_a(f)$. Clearly $\overline{m}w_0 \in i$ so we only have to check that

$$\lambda j.\overline{h}\langle j, w_0 \rangle \in \prod_{j : a^{-1}\{\overline{m}w_0\}} f^{-1}\{j\}.$$

Let $j_0 \mathrel{\dot\in} j$ such that $\overline{a}(j_0) = \overline{m}w_0$. Then we have $\langle j_0, w_0 \rangle \mathrel{\dot\in} a * m$. By assumption this gets us $j_0 = \overline{f}(\overline{h}\langle j_0, w_0 \rangle)$ or taking the inverse $\overline{h}\langle j_0, w_0 \rangle \mathrel{\dot\in} f^{-1}\{j_0\}$. Since we chose $j_0 \mathrel{\dot\in} a^{-1}\{\overline{m}w_0\}$ this shows what we wanted.

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[4] In **EC** along the projection $\Pi_a(f)^*(a)$

(b) We still have to check g induces a morphism in the slice category. $\Pi_a(f)$ is just the first projection and hence we have

$$\overline{m}w = \pi_0(\langle \overline{m}w, \cdots\rangle) = \Pi_a(f)(\langle \overline{m}w, \cdots\rangle)$$

assuming the element is indeed in $pi_a(f)$.

(c) Note that, while we write[5] the pullback of g in **EC** as $a * g$, the chosen pullback is in fact defined as pairs $\langle\langle j_0, \overline{g}w_0\rangle, w_0\rangle$. But this is clearly isomorphic to pairs $\langle j_0, w_0\rangle$ satisfying the required equalities. The morphism a^*g is induced by the term $\lambda r.\langle \pi_0 r, \overline{g}(\pi_1 r)\rangle$, which does reduce in the second component before being applied to g and hence giving an syntactically equal term which we can then project to $\Pi_a(f)$. This is different from just writing out the lambda term with $\pi_1 r$ in place of w_0, which would *not* give the required morphism!

(d) Finally, we have to verify, that the composition $\varepsilon \circ a^*g$ is equal to h in $\mathbf{EC}/j$.

Let $\langle j_0, w_0\rangle \mathrel{\dot\in} a * m$. Then we have

$$\begin{aligned}
&\overline{(\varepsilon \circ a^*g)}\langle j_0, w_0\rangle \\
&= \overline{\varepsilon}((\lambda r.\langle \pi_0 r, \overline{g}(\pi_1 r)\rangle)\langle j_0, w_0\rangle) \\
&= \overline{\varepsilon}\langle \pi_0\langle j_0, w_0\rangle, \overline{g}(\pi_1\langle j_0, w_0\rangle)\rangle \\
&= \overline{\varepsilon}\langle j_0, \overline{g}(w_0)\rangle \\
&= \overline{\varepsilon}\langle j_0, \langle \overline{m}w_0, \lambda j.\overline{h}\langle j, w_0\rangle\rangle\rangle \\
&= (\lambda p.(\pi_1(\pi_1 p))(\pi_0 p))\langle j_0, \langle \overline{m}w_0, \lambda j.\overline{h}\langle j, w_0\rangle\rangle\rangle \\
&= (\lambda j.\overline{h}\langle j, w_0\rangle)(j_0) \\
&= \overline{h}\langle j_0, w_0\rangle
\end{aligned}$$

But this is just $\varepsilon \circ a^*g =_m h$ as morphisms in **EC** which of course means they are also equal as morphisms in the slice category.

With that we are done. In view of theorem 3.1.15 let us note that g is really not unique. Indeed, the term $\lambda w_0.\langle \overline{m}w_0, \lambda j.(\lambda x.x)(\overline{h}\langle j, w_0\rangle)\rangle$, and in fact

---

[5] By abuse of notation, since a not directly involved, but instead its pullback along $\Pi_a(f)$.

any composition of $\lambda j.\overline{h}\langle j, w_0\rangle$ with a finite number of $\lambda x.x$ terms, will give a morphism which is different in the term model. □

Finite Colimits in EC

It is not hard to see that we have no clear way how to define pushouts in a general way. Weaker properties about the existence of quotients will be handled in the following sections. So while there is no hope to show that **EC** is a finitely cocomplete category, we can at least get some results for finite coproducts.

Proposition 3.1.17. *Up to $=_m$ there is exactly one morphism with an empty class as its domain.*

Proof. Let $Ob(\emptyset, z), f, g : \emptyset \to z$ such that $\emptyset$ is a name of empty class for example given by the elementary formula $(1 = 0)$. Clearly we have

$$(\forall u \mathrel{\dot{\in}} \emptyset)(\overline{f}u = \overline{g}u).$$

Construction of any such map concludes the proof:

$$\langle \emptyset, z, (\lambda x.x)\rangle : \emptyset \overset{m}{\to} z.$$ □

Corollary 3.1.18. *Any empty class is an initial object in* **EC**. □

Proposition 3.1.19 (**EC** has binary coproducts)**.**

Proof. Let $Ob(a, b)$ be two classes. We define $a + b$ as

$$\begin{aligned} COP[u, A, B] :\equiv \exists i, x(u = \langle i, x\rangle \wedge N(i) \wedge (i < 2) \\ \wedge (i = 0 \to x \in A) \wedge (i = 1 \to x \in B)) \\ a + b :\equiv t_{COP}(s, t). \end{aligned}$$

The coprojections $inl : a \overset{m}{\to} a + b$, $inr : b \overset{m}{\to} a + b$ are represented by

$$\begin{aligned} inl &:\equiv \langle a, a + b, \lambda x.\langle 0, x\rangle\rangle \\ inr &:\equiv \langle b, a + b, \lambda x.\langle 1, x\rangle\rangle. \end{aligned}$$

Given any cocone

$$\begin{tikzcd}
a \arrow[dr, "f"] & & b \arrow[dl, "g"'] \\
& c &
\end{tikzcd}$$

we need a unique morphism $[\![f,g]\!] : a + b \overset{m}{\to} c$ whose composition with the coprojections commutes with f and g.

$$\overline{[\![f,g]\!]} :\equiv \lambda x. \begin{cases} \overline{f}(\pi_1 x) & \pi_0 x = 0 \\ \overline{g}(\pi_1 x) & \pi_0 x = 1. \end{cases}$$

Let $h : a + b \overset{m}{\to} c$ with $h \circ inl =_m f$ and $h \circ inr =_m g$ and $z \mathrel{\dot{\in}} a + b$. Then we have two cases $z = \langle 0, x\rangle \wedge x \mathrel{\dot{\in}} a$ or $z = \langle 1, y\rangle \wedge y \mathrel{\dot{\in}} b$.

$$\begin{aligned}
\overline{h}\langle 0, x\rangle &= \overline{(h \circ inl)}x \\
&= \overline{f}x \\
&= \overline{([\![f,g]\!] \circ inl)}x \\
&= \overline{[\![f,g]\!]}\langle 0, x\rangle.
\end{aligned}$$

The other case is handled similarly.

Taken together, these show that $(h =_m [\![f,g]\!]) : a + b \overset{m}{\to} c$. □

It should be clear that this construction can be generalized to any finite number. In fact unlike the limit case, we do have a countably infinite coproduct. Furthermore, the coproducts which turn out to exist, actually behave rather well. They are disjoint and stable under pullbacks.

Notation 3.1.20. Given any two coproducts $a + b$ and $x + y$ and any morphisms $f : a \overset{m}{\to} x$ and $g : b \overset{m}{\to} y$, we will call the induced morphism between the coproducts $f \oplus g$.

$$\begin{tikzcd}
a \arrow[r, "inl_{ab}"] \arrow[d, "f"'] & a + b \arrow[d, dotted, "f \oplus g"] & b \arrow[l, "inr_{ab}"'] \arrow[d, "g"] \\
x \arrow[r, "inl_{xy}"'] & x + y & y \arrow[l, "inr_{xy}"]
\end{tikzcd}$$

On chosen coproducts, $f \oplus g$ is represented by the term

$$\lambda z. \begin{cases} \overline{f}(\pi_1 z); & \pi_0 z = 0 \\ \overline{g}(\pi_1 z); & \pi_0 z = 1. \end{cases}$$

◇

Proposition 3.1.21 (Countably infinite coproducts are disjoint.)**.**

Proof. Let $\sum_{k:N} a_k$ be a coproduct, which is at most[6] countably infinite, with coprojections $in_k : a_k \overset{m}{\to} \sum_{k:N} a_k$ induced by

$$\lambda n.\langle a_n, \sum_{l:N} a_l, \lambda x.\langle n, x\rangle\rangle \mathrel{\dot{\in}} \prod_{k:N} hom_{\mathbf{EC}}(a_k, \sum_{l:N} a_l).$$

Consider the pullback

$$\begin{tikzcd}
in_k * in_l \arrow[r, "p_l"] \arrow[d, "p_k"'] & a_l \arrow[d, "in_l"] \\
a_k \arrow[r, "in_k"'] & \sum_{k:N} a_k
\end{tikzcd}$$

By definition we have $z \mathrel{\dot{\in}} (in_k * in_l) \leftrightarrow z = \langle u, v\rangle \wedge \overline{in_k} u = \overline{in_l} v$. But this means that for some $k, l \in N$, $x \mathrel{\dot{\in}} a_k$, and $y \mathrel{\dot{\in}} a_l$ we get

$$k = \pi_0\langle k, x\rangle = \pi_0(\overline{in_k} u) = \pi_0(\overline{in_l} v) = \pi_0\langle l, y\rangle = l.$$

For natural numbers $k \neq l$ this implies $(in_k * in_l) \cong \emptyset$. ☐

Proposition 3.1.22. *It holds that*

(a) For any morphisms $f : a \overset{m}{\to} x$ *and* $g : b \overset{m}{\to} y$ *the diagram*

$$\begin{tikzcd}
a \arrow[r, "inl_{ab}"] \arrow[d, "f"'] & a + b \arrow[d, "f \oplus g"] \\
x \arrow[r, "inl_{xy}"'] & x + y
\end{tikzcd}$$

is a pullback and

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[6] For the finite case, exchange all mentions of the natural numbers N in the proof with the class $fin(j) :\equiv \{u \in N \mid u < j\}$ for some finite $j \in N$.

(b) *for any morphism* $h : z \overset{m}{\to} x{+}y$ *there exist* $h_x : z_x \overset{m}{\to} x$ *and* $h_y : z_y \overset{m}{\to} y$ *such that*

$$\begin{tikzcd}
z_x \arrow[r, "inl_{z_x z_y}"] \arrow[dd, "h_x"'] & z_x + z_y \arrow[d, "t"] & z_y \arrow[l, "inr_{z_x z_y}"'] \arrow[dd, "h_y"] \\
 & z \arrow[d, "h"] & \\
x \arrow[r, "inl_{xy}"'] & x + y & y \arrow[l, "inr_{xy}"]
\end{tikzcd}$$

commutes and $t : z_x + z_y \overset{m}{\to} z$ *is an isomorphism.*

Proof. Both these claims again rely heavily on the specific form of the coproduct and the property of N to have decidable equality. But for any other coproduct, we can compose with the unique isomorphism.

(a) Let $p : z \overset{m}{\to} a + b$ and $q : z \overset{m}{\to} x$ be two morphisms such that $inl_{xy} \circ p =_m (f \oplus g) \circ q$. The required morphism $e : z \overset{m}{\to} a$ is represented by the term $\lambda z.\pi_1 \overline{p}z$. This is not a priori well-defined, as we just assume that g maps only into a. But using the definition of $f \oplus g$, which simply applies f or g depending on the number tag of the given element in the sum,

$$\begin{aligned}
\overline{f \oplus g}(\overline{q}z) &= \overline{((f \oplus g) \circ q)}z \\
&= \overline{(inl_{xy} \circ p)}z \\
&= \overline{inl_{xy}}(\overline{p}z) \\
&= \langle 0, \overline{p}z \rangle
\end{aligned}$$

shows not only that $\overline{q}z$ has been mapped into a but in particular (assuming the chosen coproduct of a and b) that $\pi_1(\overline{q}z) =_m \overline{p}z$.

To check uniqueness we just have to notice that inl_{ab} is injective, hence a monomorphism. Then we get directly that for any $e, e' : z \overset{m}{\to} a$ with $inl_{ab} \circ e =_m p =_m inl_{ab} \circ e'$ we have $e =_m e'$.

(b) For the second statement, construct z_x and z_y as the pullback.

$$\begin{tikzcd}
inl_{xy} * h \arrow[r, "inl_{z_x z_y}"] \arrow[dd, "h^*(inl_{xy})"'] & (inl_{xy} * h) + (inr_{xy} * h) \arrow[d, "t"] & inr_{xy} * h \arrow[l, "inr_{z_x z_y}"'] \arrow[dd, "h^*(inr_{xy})"] \\
 & z \arrow[d, "h"] & \\
x \arrow[r, "inl_{xy}"'] & x + y & y \arrow[l, "inr_{xy}"]
\end{tikzcd}$$

The morphism $t : (inl_{xy} * h) + (inr_{xy} * h) \overset{m}{\to} z$ is induced by the term

$$\lambda w.\pi_1(\pi_1 w).$$

This is because both pullbacks have the form $\langle u, v\rangle$ where $u \mathrel{\dot{\in}} x$ (resp. $u \mathrel{\dot{\in}} y$) and $v \mathrel{\dot{\in}} z$. To show this is an isomorphism, we give a term for the inverse t^{-1} :

$$\lambda w. \begin{cases} \langle 0, \langle \pi_1(\overline{h}w), w\rangle\rangle; & \pi_0(\overline{h}w) = 0 \\ \langle 1, \langle \pi_1(\overline{h}w), w\rangle\rangle; & \pi_0(\overline{h}w) = 1 \end{cases}$$

alternatively, the term

$$\lambda w.\langle \pi_0(\overline{h}w), \langle \pi_1(\overline{h}w), w\rangle\rangle.$$

If we calculate using the second term, we see that

$$\begin{aligned}
\overline{(t \circ t^{-1})}z &= \overline{t}((\lambda w.\langle \pi_0(\overline{h}w), \langle \pi_1(\overline{h}w), w\rangle\rangle)z) \\
&= \overline{t}(\langle \pi_0(\overline{h}z), \langle \pi_1(\overline{h}z), z\rangle\rangle) \\
&= (\lambda w.\pi_1(\pi_1 w))(\langle \pi_0(\overline{h}z), \langle \pi_1(\overline{h}z), z\rangle\rangle) \\
&= \pi_1(\pi_1(\langle \pi_0(\overline{h}z), \langle \pi_1(\overline{h}z), z\rangle\rangle)) \\
&= \pi_1(\langle \pi_1(\overline{h}z), z\rangle) \\
&= z
\end{aligned}$$

and for the left case $\langle 0, \langle x, z\rangle\rangle \mathrel{\dot{\in}} (inl_{xy} * h) + (inr_{xy} * h)$

$$\begin{aligned}
\overline{(t^{-1} \circ t)}(\langle 0, \langle x, z\rangle\rangle) &= \overline{(t^{-1} \circ t)}(\langle 0, \langle x, z\rangle\rangle) \\
&= \overline{t^{-1}}((\lambda w.\pi_1(\pi_1 w))(\langle 0, \langle x, z\rangle\rangle))
\end{aligned}$$

$$\begin{aligned}
&= \overline{t^{-1}}(\pi_1(\pi_1(\langle 0, \langle x, z\rangle\rangle)))) \\
&= \overline{t^{-1}}(\pi_1(\langle x, z\rangle)) = \overline{t^{-1}}(z) \\
&= \langle \pi_0(\overline{h}z), \langle \pi_1(\overline{h}z), z\rangle\rangle = \langle 0, \langle \pi_1(\overline{h}z), z\rangle\rangle \\
&= \langle 0, \langle x, z\rangle\rangle
\end{aligned}$$

where the last two equalities come from the assumption $\overline{inl_{xy}}x = \langle 0, x\rangle = \overline{h}z$. The right case works exactly the same. This shows that the coproduct of the coprojections along any morphism is still a coproduct. ☐

Corollary 3.1.23. *[30, Proposition 1.1] Assuming a category $\mathcal{C}$ with binary sums, Lack and Vitale give the properties (a) and (b) of proposition 3.1.21 as being equivalent*[7] *to $\mathcal{C}$ being extensive in the sense of 2.3.4.* ☐

We have not yet considered unions of subobjects. Having an axiom in **EM** which says that we have binary unions of any class, one might think that this would translate into a statement in category theory. That is, however, not the case. Not in the sense that unions of arbitrary sub*objects* would exist. Instead, we can only show existence of unions for two more restricted cases.

Proposition 3.1.24. *Let $a : dom(a) \overset{m}{\rightarrowtail} x$, and $b : dom(b) \overset{m}{\rightarrowtail} x$ be two subobjects of x.*

(a) The conditions

1. *$dom(a) \mathrel{\dot\subset} x$ and $dom(b) \mathrel{\dot\subset} x$*
2. *$(\forall u \mathrel{\dot\in} a)(\overline{a}u = u)$ and $(\forall v \mathrel{\dot\in} b)(\overline{b}v = v)$*

are sufficient for the existence of $a \cup b \overset{m}{\rightarrowtail} x$.

(b) If a and b are disjoint (i.e. $a * b \cong \emptyset$) then $a \cup b \overset{m}{\rightarrowtail} x$ exists.

Proof. (a) is exactly the case where the **EM** axioms for elementary comprehension state that unions of classes exist: $\langle \mathrm{un}(a, b), x, \lambda x.x\rangle$ is the union-subobject.

(b) is just the case where the disjoint union is already the union. a and b are monic, and the case where we require $\langle 0, a_0\rangle = \langle 1, b_0\rangle$ in $a + b$, because we have $\overline{a}a_0 = \overline{b}b_0$ in x doesn't occur by assumption. ☐

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[7] We have *not* proved that this is still true in our setting.

3.2. The Category ECB

> A set is not an entity which has an ideal existence. A set exists only when it has been defined. To define a set we prescribe, at least implicitly, what we (the constructing intelligence) must do in order to construct an element of the set, and what we must do to show that two elements of the set are equal. (Bishop, [6])

In the above quote of Bishop, it is stated that we “must do [something] to show that two elements of a set are equal.” This leaves open some room for interpretation. If we only have to *show* something, that is we have to prove a proposition, then we end up with a notion of (implicit) Bishop sets as defined in this section. If we, however, assume that we have to *construct* something, this leads to the category $\mathbf{EC}_{\mathrm{ex}}$ described in section 3.3, which we might call explicit Bishop sets since we have to explicitly construct proof objects to show equalities.

As we have seen in section 3.1, the defined category **EC** has problems with products and cartesian closure. To fix this we can augment each class with an equivalence relation. This fixes the problem of infinite products and “function spaces” not being quotiented by functional extensionality.

Maybe even more importantly it gives a well-behaved notion of an *image* of a morphism. This allows for interpretation of *regular logic* as internal logic of the category which is an ∃-∧ fragment of many-sorted first-order logic [9]. In fact **ECB** also has well-behaved binary sums. It is *extensive* in the sense of [11].

Definition 3.2.1. ECB
Weak **ECB** is the category defined in the following way.

$$\begin{aligned}
Ob(x) &:\equiv \mathfrak{R}\,(\|x\|) \wedge \mathfrak{R}\,(x_{\mathrm{E}}) \wedge \mathfrak{EQV}(x)\\
Mor(f) &:\equiv f \mathrel{\dot{\in}} (\|x\| \mathrel{\dot{\to}} \|y\|) \wedge f \mathrel{\dot{\in}} Fun(x_{\mathrm{E}}, y_{\mathrm{E}})\\
x =_o y &:\equiv x = y
\end{aligned}$$

$$\begin{aligned}
f =_m g &:\equiv f, g : x \overset{m}{\to} y \wedge Ext[x, y, f, g]\\
id(x) &:\equiv \langle x, x, \lambda x.x\rangle\\
f \circ g &:\equiv \langle dom(g), cod(f), \lambda z.\overline{f}(\overline{g}z)\rangle
\end{aligned}$$

where we have used the following definitions

$$\begin{aligned}
\|x\| &:\equiv \pi_0 x \\
x_{\mathrm{E}} &:\equiv \pi_1 x \\
Fun[u, R, S] &:\equiv (\forall \langle x, y\rangle \in R)(\langle ux, uy\rangle \in S) \\
Ext[x, y, f, g] &:\equiv (\forall a \mathrel{\dot\in} \|x\|)(fa \mathrel{\dot\in} \|y\| \wedge \langle fa, ga\rangle \mathrel{\dot\in} y_{\mathrm{E}}) \\
rel(x, r) &:\equiv \{0 \mid (\forall q \in R)((q)_0 \in X \wedge (q)_1 \in X)\} \\
refl(x, r) &:\equiv \{0 \mid (\forall a \in X)(\langle a, a\rangle \in R)\} \\
symm(x, r) &:\equiv \{0 \mid (\forall a, b \in X)(\langle a, b\rangle \in R \to \langle b, a\rangle \in R)\} \\
trans(x, r) &:\equiv \left\{\begin{matrix} 0 \mid (\forall a, b, c \in X)(\langle a, b\rangle \in R \wedge \langle b, c\rangle \in R \\ \to \langle a, c\rangle \in R) \end{matrix}\right\} \\
\mathfrak{EQV}(x) &:\equiv 0 \mathrel{\dot\in} rel(\|x\|, r_{\mathrm{E}}) \cap refl(\|x\|, r_{\mathrm{E}}) \\
&\qquad \cap symm(\|x\|, r_{\mathrm{E}}) \cap trans(\|x\|, r_{\mathrm{E}})
\end{aligned}$$

Given $a, b \mathrel{\dot\in} \|x\|$ and $c, d \in XR$ we will from now on write

$$\begin{aligned}
a \sim_x b &:\equiv \langle a, b\rangle \mathrel{\dot\in} x_{\mathrm{E}} \\
c \sim_{XR} d &:\equiv \langle c, d\rangle \in XR \qquad \text{and} \\
Ext[f, g] &:\equiv Ext[dom(f), cod(f), f, g].
\end{aligned}$$

Less formal, we now have objects of pairs $\langle x, r\rangle$ which we will call *implicit Bishop sets*,[8] where x is a class name and r is a name of an equivalence relation in the classical sense (i.e. a class of pairs which is reflexive, symmetric and transitive.)

To define the (non-weak) **ECB**-category we just restrict $Ob(x)$ to a universe $\mathcal{U}(u)$ instead of allowing arbitrary class names. Then all classes defined above will be contained in u. ◇

Proposition 3.2.2. *Composition is compatible with equality.*

Proof. Let $f, f' : y \overset{m}{\to} z$ and $g, g' : x \overset{m}{\to} y$ with $f =_m f'$ and $g =_m g'$.

---

[8] Throughout this section we will mostly omit the *implicit* qualifier and just call them *Bishop Sets.*

By definition we have

$$\begin{aligned} f \circ g &:\equiv \langle x, z, \lambda x.\overline{f}(\overline{g}x) \rangle. \\ Ext[g, g'] &\to (\forall w \mathrel{\dot{\in}} \|x\|)(\overline{g}w \sim_y \overline{g'}w) \end{aligned}$$

Because f and f' are functions we get

$$\begin{aligned} &\overline{f}(\overline{g}w) \sim_z \overline{f}(\overline{g'}w) \wedge \overline{f'}(\overline{g}w) \sim_z \overline{f'}(\overline{g'}w). \\ Ext[f, f'] &\to \overline{f}(\overline{g}w) \sim_z \overline{f'}(\overline{g'}w) \end{aligned}$$

hence

$$f \circ g =_m f' \circ g'.$$

☐

Proposition 3.2.3. **ECB** *has a terminal object.*

Proof. Any one-element Bishop-set is terminal. Set $\mathbb{1} :\equiv \langle \{*\}, \{\langle *, * \rangle\} \rangle$ then $\langle x, \mathbb{1}, \lambda z.* \rangle$ exists for all $Ob(x)$ and is the unique morphism up to functional extensionality.

Note that if we write this as $h(x, p) :\equiv \langle x, \mathbb{1}, \lambda z.* \rangle : x \overset{m}{\to} \mathbb{1}$, we get the unique morphism for any cone over the diagram $\bullet \circlearrowleft id(\bullet)$ in a uniform way which is required by the definition of a limit. Since in Explicit Mathematics it is almost impossible to write down such morphisms in a way which is not uniform, we will generally omit mentioning this part of the proof. ☐

Definition 3.2.4. Injectivity

We call a morphism $f : x \overset{m}{\to} y$ *injective* if the underlying operation is injective up to equivalence.

$$(\forall a, b \mathrel{\dot{\in}} \|x\|)((\overline{f}a \sim_y \overline{f}b) \to (a \sim_x b))$$

◇

Proposition 3.2.5. *Injective functions are monomorphisms.*

Proof. Let $f : y \overset{m}{\to} z$ and $(\forall a, b \mathrel{\dot{\in}} \|y\|)((\overline{f}a \sim_z \overline{f}b) \to (a \sim_y b))$. Given $g, h : x \overset{m}{\to} y$ such that $f \circ g =_m f \circ h$, we have to show $g =_m h$.

$$f \circ g =_m f \circ h \leftrightarrow (\forall a \mathrel{\dot{\in}} \|x\|)(\overline{(f \circ g)}a \sim_z \overline{(f \circ h)}a)$$

The right-hand side of this formula reduces to

$$(\forall a \mathrel{\dot\in} \|x\|)(\overline{f}(\overline{g}a) \sim_z \overline{f}(\overline{h}a)).$$

By injectivity, it follows that $\overline{g}a \sim_y \overline{h}a$ holds for all a in $\|x\|$. This finishes the proof, since through $Ext[\overline{g}, \overline{h}]$ we also have $g =_m h$. □

Proposition 3.2.6. *Monomorphisms are injective.*

Proof. Let $f : x \overset{m}{\rightarrowtail} y$ be monic. For any $a \mathrel{\dot\in} \|x\|$ We can define the global element-morphism

$$\{a\} :\equiv \langle \mathbb{1}, x, \lambda x.a \rangle : \mathbb{1} \overset{m}{\to} x$$

Let $a, b \mathrel{\dot\in} \|x\|$ such that $\overline{f}a \sim_y \overline{f}b$. By definition of composition we get the following string of equivalences.

$$\begin{aligned} &\overline{(f \circ \{a\})}(*) \sim_y \overline{f}(\overline{\{a\}}(*)) \\ &\qquad \sim_y \overline{f}a \sim_y \overline{f}b \sim_y \overline{f}(\overline{\{b\}}(*)) \sim_y \overline{(f \circ \{b\})}(*) \end{aligned}$$

But this just means $(f \circ \{a\}) =_m (f \circ \{b\})$. Because f is mono this implies that we have $\{a\} =_m \{b\}$ which can only be true if $a \sim_x b$ holds, which is what we wanted to show. □

Proposition 3.2.7. **ECB** *is finitely complete.*

Proof. We need to show that we have a terminal object and all binary pullbacks. A terminal exists as shown in prop. 3.2.3. For the pullback of $f : x \overset{m}{\to} a$ and $g : y \overset{m}{\to} a$ we can adapt the usual construction of the category of sets.

$$\begin{aligned}
PB[u,f,g,X,Y,S] &:\equiv \exists x,y(u=\langle x,y\rangle \wedge x\in X \wedge y \in Y\\
&\qquad\qquad \wedge \langle \overline{f}x,\overline{g}y\rangle \in S)\\
\mathit{pullb}(f,g) &:\equiv t_{PB}(f,g,\|\mathit{dom}(f)\|,\|\mathit{dom}(g)\|,\mathit{cod}(f)_{\mathrm{E}})\\
PBEQV[u,f,g,\overrightarrow{X}] &:\equiv (\exists x_1,x_2\in X)(\exists y_1,y_2\in Y)(\\
&\qquad\qquad u=\langle\langle x_1,y_1\rangle,\langle x_2,y_2\rangle\rangle\\
&\qquad\qquad \wedge \langle x_1,x_2\rangle \in R \wedge \langle y_1,y_2\rangle \in Q\\
&\qquad\qquad \wedge \langle \overline{f}x_1,\overline{g}y_1\rangle \in S \wedge \langle \overline{f}x_2,\overline{g}y_2\rangle \in S)
\end{aligned}$$

where $\overrightarrow{X}$ is short for X,Y,R,Q,S

$$\begin{aligned}
\mathit{pullbeqv}(f,g) &:\equiv t_{PBEQV}(f,g,\|\mathit{dom}(f)\|,\|\mathit{dom}(g)\|,\\
&\qquad\qquad \mathit{dom}(f)_{\mathrm{E}},\mathit{dom}(g)_{\mathrm{E}},\mathit{cod}(f)_{\mathrm{E}})\\
f*g &:\equiv \langle \mathit{pullb}(f,g),\mathit{pullbeqv}(f,g)\rangle,
\end{aligned}$$

So far this works, because the equivalence relation is basically the restriction of the relation on products to the carrier.

the actual pullback maps f^*g and g^*f are then just projections.

$$\begin{aligned}
pr_0 &:\equiv \langle f*g,\mathit{dom}(f),\pi_0\rangle\\
pr_1 &:\equiv \langle f*g,\mathit{dom}(g),\pi_1\rangle
\end{aligned}$$

To see that those are functions of Bishop-sets, consider $\langle x_1,y_1\rangle,\langle x_2,y_2\rangle \mathrel{\dot\in} \|f*g\|$ which are related. Then by definition, $\langle x_1,x_2\rangle \mathrel{\dot\in} \mathit{dom}(f)_{\mathrm{E}}$ and $\langle y_1,y_2\rangle \mathrel{\dot\in} \mathit{dom}(g)_{\mathrm{E}}$, but that is exactly what we need for the projections to be functions.

Given morphisms $x \xleftarrow{m} c \xrightarrow{n} y$ with $f\circ m =_m g\circ n$ we can construct

$$\begin{aligned}
&h : c \overset{m}{\rightarrow} f*g\\
&\overline{h} :\equiv \lambda x.\langle \overline{m}x,\overline{n}x\rangle
\end{aligned}$$

Let $u, v \mathrel{\dot{\in}} \|c\|$ and $\langle u, v\rangle \mathrel{\dot{\in}} c_{\mathrm{E}}$. To show h is a well-defined function, we need that $\langle \overline{m}u, \overline{n}u\rangle$ is in $\|f * g\|$ and that the results are related. If we can show that both elements of the pair get mapped to a related result, i.e.

$$\overline{f}(\overline{m}u) \sim_{cod(f)} \overline{g}(\overline{n}u)$$

holds, we are done with the first part. But this can be rewritten as

$$\overline{(f \circ m)}u \sim_a \overline{(g \circ n)}u$$

which is true by assumption.

For the second part, note that both m and n are morphisms so it follows that $\overline{m}u \sim_x \overline{m}v$ and $\overline{n}u \sim_y \overline{n}v$. Those were exactly the missing equations to get

$$\langle \overline{m}u, \overline{n}u\rangle \sim_{f*g} \langle \overline{m}v, \overline{n}v\rangle.$$

Suppose we have another $\tilde{h} : c \xrightarrow{m} f * g$ and $z \mathrel{\dot{\in}} c$.

$$\begin{aligned} pr_0 \circ \tilde{h} =_m m &\leftrightarrow Ext[c, x, \overline{(pr_0 \circ \tilde{h})}, \overline{m}] \\ &\leftrightarrow (\forall z \mathrel{\dot{\in}} c)(\overline{(pr_0 \circ \tilde{h})}z \sim_x \overline{m}z) \end{aligned}$$

Of course it also holds that

$$Ext[c, x, \overline{(pr_0 \circ \tilde{h})}, \overline{\langle c, x, \lambda x.pr_0(\overline{\tilde{h}}x)\rangle}]$$

and so

$$(\forall z \mathrel{\dot{\in}} c)((\lambda x.pr_0(\overline{\tilde{h}}x))z \sim_x \overline{m}z).$$

Since the same can be done for the other projection, this means we have some $a \mathrel{\dot{\in}} x$ and $b \mathrel{\dot{\in}} y$ such that $\overline{\tilde{h}}z \sim_{f*g} \langle a, b\rangle$. But with the first equation it follows, that $\langle a, b\rangle = \langle \overline{m}z, \overline{n}z\rangle$. Hence we are done because with this we obtain $Ext[\tilde{h}, h]$ and so $\tilde{h} =_m h$. □

Cartesian Closure

Proposition 3.2.8. **ECB** *is cartesian closed.*

Proof. We can define the *internal hom* (or *exponential*) as

$$\begin{aligned}
HOM[u, FN, FUN] &:\equiv u \in FN \wedge u \in FUN \\
homCarr(x, y) &:\equiv t_{HOM}(\|x\| \dot{\to} \|y\|, Fun(\|x\|, x_{\mathrm{E}}, \|y\|, y_{\mathrm{E}})) \\
EXTF[u, F, X, R] &:\equiv (\exists f, g \in F) \\
&\qquad\qquad (u = \langle f, g\rangle \wedge (\forall x \in X)(\langle fx, gx\rangle \in R) \\
homeqv(x, y) &:\equiv t_{EXTF}(homCar(x, y), \|x\|, y_{\mathrm{E}}) \\
hombset(x, y) &:\equiv \langle homCarr(x, y), homeqv(x, y)\rangle
\end{aligned}$$

With this we can define

$$\begin{aligned}
x^y &:\equiv hombset(x, y) \\
\varepsilon^{x,y} &:\equiv \langle x^y \times y, x, \lambda p.(pr_0 p)(pr_1 p)\rangle \\
\Lambda^z_{x,y} &:\equiv \lambda h.\langle z, x^y, \lambda z.\lambda y.\overline{h}\langle z, y\rangle\rangle
\end{aligned}$$

Let $\langle f, w\rangle \sim_{x^y \times y} \langle g, z\rangle$. Then we have $Ext[y, x, f, g]$ and $w \sim_y z$. Checking well-definedness of $\varepsilon^{x,y}$ we see that:

$$\begin{aligned}
(\lambda p.(pr_0 p)(pr_1 p))\langle f, w\rangle &= (pr_0\langle f, w\rangle)(pr_1\langle f, w\rangle) \\
&= fw \\
&\sim_x fz \\
&\sim_x gz \\
&= (pr_0\langle g, z\rangle)(pr_1\langle g, z\rangle) \\
&= (\lambda p.(pr_0 p)(pr_1 p))\langle g, z\rangle
\end{aligned}$$

The third line is because $w \sim_y z$ and because f respects equivalences. The fourth line is because f and g are equivalent ($f \sim_{x^y} g$.)

Now we have to check $\Lambda^z_{x,y}(h)$: Let $u \sim_z v$ and $s \sim_y t$:

$$\begin{aligned}
\overline{(\Lambda^z_{x,y}(h))}u &= (\lambda z.\lambda y.\overline{h}\langle z, y\rangle)u \\
&= \lambda y.\overline{h}\langle u, y\rangle \\
&\sim_{x^y} \lambda y.\overline{h}\langle v, y\rangle \\
&\sim_{x^y} \overline{(\Lambda^z_{x,y}(h))}v
\end{aligned}$$

The third equivalence is justified by:

$$\begin{aligned}(\lambda y.\overline{h}\langle u, y\rangle)s &= \overline{h}\langle u, s\rangle \\ &\sim_x \overline{h}\langle v, t\rangle \\ &= (\lambda y.\overline{h}\langle v, y\rangle)t\end{aligned}$$

The second line is because $u \sim_z v$ and $s \sim_y t$ implies $\langle u, s\rangle \sim_{z\times y} \langle v, t\rangle$ and h is a function with domain $z \times y$.

Because this calculation still works if we don't exchange u with v, this shows both that $(\lambda y.\overline{h}\langle u, y\rangle)$ is well-defined as element of x^y and that the partially applied functions are indeed related when we substitute related elements.

Now it remains to check the actual equations. Let $\langle u, w\rangle \mathrel{\dot{\in}} z \times y$

$$(CC4) \qquad (h : z \times y \overset{m}{\to} x) \to (\varepsilon^{x,y} \circ \langle\!\langle \Lambda^z_{x,y}(h) \circ pr_0, pr_1 \rangle\!\rangle =_m h)$$

$$(CC5) \qquad (k : z \overset{m}{\to} x^y) \to (\Lambda^z_{x,y}(\varepsilon^{x,y} \circ \langle\!\langle k \circ pr_0, pr_1 \rangle\!\rangle) =_m k)$$

$$\begin{aligned}&\overline{(\varepsilon^{x,y} \circ \langle\!\langle \Lambda^z_{x,y}(h) \circ pr_0, pr_1 \rangle\!\rangle)}\langle u, w\rangle \\ &\sim_x \overline{(\lambda p.(\overline{(\varepsilon^{x,y})}(\overline{((\langle\!\langle \Lambda^z_{x,y}(h) \circ pr_0, pr_1 \rangle\!\rangle)}p))}\langle u, w\rangle \\ &\sim_x (\overline{(\varepsilon^{x,y})}(\overline{((\langle\!\langle \Lambda^z_{x,y}(h) \circ pr_0, pr_1 \rangle\!\rangle)}\langle u, w\rangle)) \\ &\sim_x \overline{(\varepsilon^{x,y})}\langle \overline{(\Lambda^z_{x,y}(h) \circ pr_0)}\langle u, w\rangle, \overline{pr_1}\langle u, w\rangle\rangle \\ &\sim_x \overline{(\varepsilon^{x,y})}\langle \overline{(\Lambda^z_{x,y}(h))}(\overline{pr_0}\langle u, w\rangle), w\rangle \\ &\sim_x \overline{(\varepsilon^{x,y})}\langle \overline{(\Lambda^z_{x,y}(h))}u, w\rangle \\ &\sim_x \overline{(\varepsilon^{x,y})}\langle \overline{(\lambda y.\overline{h}\langle u, y\rangle)}, w\rangle \\ &\sim_x \overline{(\lambda p.(pr_0 p)(pr_1 p))}\langle \overline{(\lambda y.\overline{h}\langle u, y\rangle)}, w\rangle \\ &\sim_x (pr_0\langle \overline{(\lambda y.\overline{h}\langle u, y\rangle)}, w\rangle)(pr_1\langle \overline{(\lambda y.\overline{h}\langle u, y\rangle)}, w\rangle)) \\ &\sim_x (\overline{(\lambda y.\overline{h}\langle u, y\rangle)}))(w)) \\ &\sim_x \overline{h}\langle u, w\rangle\end{aligned}$$

Because we chose arbitrary elements of $z \times y$, we get by definition of *Ext* that $\varepsilon^{x,y} \circ \langle\!\langle \Lambda^z_{x,y}(h) \circ pr_0, pr_1 \rangle\!\rangle =_m h$.

For the other equation let $u \mathrel{\dot\in} z$.

$$
\begin{aligned}
&\overline{(\Lambda^z_{x,y}(\varepsilon^{x,y} \circ \langle\!\langle k \circ pr_0, pr_1 \rangle\!\rangle)}u \\
&\sim_{x^y} \overline{(\Lambda^z_{x,y}(\varepsilon^{x,y} \circ \langle\!\langle k \circ pr_0, pr_1 \rangle\!\rangle)}u \\
&\sim_{x^y} \overline{(\lambda y.\overline{(\varepsilon^{x,y} \circ \langle\!\langle k \circ pr_0, pr_1 \rangle\!\rangle)}\langle u, y\rangle)}
\end{aligned}
$$

Because we actually want to check equivalence in x we will no supply an additional $w \mathrel{\dot\in} y$.

$$
\begin{aligned}
&\overline{(\lambda y.\overline{(\varepsilon^{x,y} \circ \langle\!\langle k \circ pr_0, pr_1 \rangle\!\rangle)}\langle u, y\rangle)}w \\
&\sim_x \overline{(\varepsilon^{x,y} \circ \langle\!\langle k \circ pr_0, pr_1 \rangle\!\rangle)}\langle u, w\rangle \\
&\sim_x \overline{(\varepsilon^{x,y})}(\overline{(\langle\!\langle k \circ pr_0, pr_1 \rangle\!\rangle)}\langle u, w\rangle) \\
&\sim_x \overline{(\varepsilon^{x,y})}(\langle\overline{(k \circ pr_0)}\langle u, w\rangle, \overline{pr_1}\langle u, w\rangle\rangle) \\
&\sim_x \overline{(\varepsilon^{x,y})}(\langle\bar{k}u, w\rangle) \\
&\sim_x \overline{\lambda p.(pr_0 p)(pr_1 p)}(\langle\bar{k}u, w\rangle) \\
&\sim_x (\bar{k}u)w
\end{aligned}
$$

So, going back to one argument, we get for arbitrary $u \mathrel{\dot\in} z$

$$\overline{(\Lambda^z_{x,y}(\varepsilon^{x,y} \circ \langle\!\langle k \circ pr_0, pr_1 \rangle\!\rangle)}u \sim_{x^y} (\bar{k}u)$$

from this we finally have the required

$$\Lambda^z_{x,y}(\varepsilon^{x,y} \circ \langle\!\langle k \circ pr_0, pr_1 \rangle\!\rangle) =_m k. \qquad \square$$

Proposition 3.2.9. ECB *is locally cartesian closed.*

Proof. We move this proof to the appendix. Definition A.4.1 gives the dependent product, and corollary A.4.4 shows that this gives exponential objects in all slices of **ECB**. □

For completeness' sake we just state the left-adjoint to pullback. This always exists and the verification in Explicit Mathematics is no harder than it is otherwise.

Definition 3.2.10. For any $f : a \xrightarrow{m} b$, the left-adjoint $\Sigma_f \vdash f^*$ on slice-categories is given by composition with f.

$$\begin{aligned} \Sigma_f &: \mathbf{ECB}/a \to \mathbf{ECB}/b \\ \Sigma_f(k) &:\equiv f \circ k \\ \Sigma_{f_m}(\eta) &:\equiv \langle \Sigma_f(dom(\eta)), \Sigma_f(cod(\eta)), \overline{\eta} \rangle \end{aligned} \qquad \diamond$$

Corollary 3.2.11. *The product $f \times g$ in the slices of* **ECB** *can then be written as $\Sigma_f(f^*g) : f * g \xrightarrow{m} cod(f)$ which is just one of the projections from the pullback composed with its morphism.* $\square$

Colimits

Proposition 3.2.12. *Any empty class with an empty equivalence relation is an initial object in* **ECB**.

Proof. The proof of proposition 3.1.17 stating that there is only one morphism with an empty class as its domain, still applies. $\square$

Proposition 3.2.13. **ECB** *has image factorizations.*

Proof. Let $f : x \xrightarrow{m} y$ be any morphism. We now construct
$x \longrightarrow im(f) \rightarrowtail y.$

$$\begin{aligned} IMG[u, g, X] &:\equiv u \in X \\ IMGEQV[u, g, X, S] &:\equiv (\exists a, b \in X)(u = \langle a, b \rangle \wedge \langle \overline{g}a, \overline{g}b \rangle \in S) \\ im(g) &:\equiv \langle t_{IMG}(g, \|dom(g)\|), \\ &\qquad t_{IMGEQV}(g, \|dom(g)\|, cod(g)_{\mathrm{E}}) \rangle \\ \tilde{g} &:\equiv \langle im(g), cod(g), \overline{g} \rangle \\ i_g &:\equiv \langle dom(g), im(g), \lambda x.x \rangle \end{aligned}$$

This is not your usual construction from Sets, but it satisfies the usual properties. Let $a, b \mathrel{\dot{\in}} \|im(g)\|$ with $\overline{g}a \sim_x \overline{g}b$. The relation of $im(g)$ is defined exactly so that $a \sim_{im(g)} b$ holds. This makes $\tilde{g}$ trivially injective and by prop. 3.2.5 also monic. Because g is a morphism, this also makes i_g well-defined, since $im(g)_{\mathrm{E}}$ then has to extend the relation x_{E}. What we still have to show is the following: Given another factorization $h : x \xrightarrow{m} z$, $n : z \overset{m}{\rightarrowtail} y$ there needs to exist a unique morphism $j : im(g) \xrightarrow{m} z$ such that the diagram

$$\begin{tikzcd}
x \arrow[d, "i_g"] \arrow[dr, "h"] & \\
im(g) \arrow[r, dotted, "j"] \arrow[d, tail, "\tilde{g}"] & z \arrow[dl, tail, "n"] \\
y. &
\end{tikzcd}$$

commutes. We can do this, by setting $j :\equiv \langle im(g), z, \overline{h} \rangle$. To check that this is a function, let $a \sim_{im(g)} b$. Then we have $\overline{g}a \sim_y \overline{g}b$. This means we also have $\overline{n}(\overline{h}a) \sim_y \overline{n}(\overline{h}b)$. Prop. 3.2.6 tells us, that n is injective, so it also holds that $(\overline{h}a) \sim_z (\overline{h}b)$. This is exactly what is needed to make h also into a morphism with $im(g)$ as it's domain. Lastly, note that such a morphism is always unique because any other $f : im(g) \xrightarrow{m} z$ with $n \circ f =_m \tilde{g} =_m n \circ j$ is equal to j because n is mono. □

Proposition 3.2.14. *Given any morphism $f : x \xrightarrow{m} y$ and its image factorization $f =_m \tilde{f} \circ i_f$, the morphism i_f will be a regular epi.*

Proof. Let $p_0, p_1 : i_f * i_f \xrightarrow{m} im(f)$ be the projections from the chosen pullback of i_f along itself. We have

$$\langle x_0, x_1 \rangle \mathrel{\dot{\in}} \| i_f * i_f \| \leftrightarrow \overline{f}(\overline{i_f} x_0) \sim_y \overline{f}(\overline{i_f} x_1) \leftrightarrow \overline{f}(x_0) \sim_y \overline{f}(x_1).$$

i_f is then the coequalizer of p_0, p_1. Let $h : x \xrightarrow{m} z$ be another morphism such that $h \circ p_0 =_m h \circ p_1$ holds. This means that for all $\langle x_0, x_1 \rangle \mathrel{\dot{\in}} \| i_f * i_f \|$

$$\begin{aligned}
\overline{h}(x_0) &\sim_z \overline{h}(\overline{p_0}\langle x_0, x_1 \rangle) \\
&\sim_z \overline{h}(\overline{p_1}\langle x_0, x_1 \rangle) \\
&\sim_z \overline{h}(x_1).
\end{aligned}$$

This means h can be reused as $\overline{h} :\equiv \langle im(f), z, \overline{h}\rangle$. Note that $x_0 \sim_{im(f)} x_1 \leftrightarrow \overline{f}x_0 \sim_y \overline{f}x_1 \leftrightarrow \langle x_0, x_1\rangle \dot{\in} \|i_f * i_f\|$. Hence $\overline{h}$ is a function. But clearly $h =_m \overline{h} \circ i_f$ and $\overline{i_f}x_0 \sim_x \overline{i_f}x_1 \leftrightarrow \langle x_0, x_1\rangle \dot{\in} \|i_f * i_f\|$, so point-wise this is really just application of $\overline{h}$ and thus unique.

$$\begin{tikzcd}
i_f * i_f \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & x \arrow[rr, "i_f"] \arrow[dr, "h"'] & & im(f) \arrow[dl, dotted, "\overline{h}"] \\
& & z &
\end{tikzcd}$$

$\square$

Proposition 3.2.15. *Image factorizations are stable under pullback*

Proof. Let $f : x \overset{m}{\to} y$ be a morphism, $\tilde{f} \circ i_f : x \overset{m}{\to} im(f) \overset{m}{\rightarrowtail} y$ its image factorization and $h : z \overset{m}{\to} y$ some other morphism. The pullback of f along h is given by $pr_0 : f * h \overset{m}{\to} z$ where $f * h$ is defined as $pullback(f, h)$.

$$\begin{tikzcd}
& f * h \arrow[r] \arrow[dl, "i_{pr_0}"'] \arrow[d] \arrow[dd, bend right, "pr_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] & x \arrow[d, "i_f"] \\
im(pr_0) \arrow[dr, "\widetilde{pr_0}"'] & \tilde{f} * h \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & im(f) \arrow[d, "\tilde{f}"] \\
& z \arrow[r, "h"] & y
\end{tikzcd}$$

We show that $im(pr_0)$ and $\tilde{f} * h$ are isomorphic and in fact extensionally equal.

- The carriers have the same elements:

  Consider an element $\langle a, w\rangle \dot{\in} \|im(pr_0)\|$. Since by definition of images we have $\|im(pr_0)\| = \|f * h\|$, it also holds that $\langle a, w\rangle \dot{\in} \|f * h\|$. But that means $\overline{f}a \sim_y \overline{h}w$. Because $\overline{f}$ is extensionally equal to $\overline{\tilde{f}}$ this implies $\langle a, w\rangle \dot{\in} \|f * h\|$. The other direction is similar since $\overline{h}w \sim_y \overline{\tilde{f}}a \to \overline{h}w \sim_y \overline{f}a$.

- The equivalence relations are the same:

  Let $\langle a, w\rangle \sim_{im(\widetilde{pr_0})} \langle b, v\rangle$. This is defined as $\overline{pr_0}\langle a, w\rangle \sim_z \overline{pr_0}\langle b, v\rangle$ which reduces to $a \sim_z b$. But $\overline{f}w \sim_y \overline{h}a \sim_y \overline{h}b \sim_y \overline{f}v$ which implies $w \sim_{im(f)} v$. This means these pairs are related in the pullback $\tilde{f} * h$.

The other direction is trivial since $\langle a, w\rangle \sim_{\tilde{f}*h} \langle b, v\rangle$ immediately gives us $a \sim_z b$.

This shows that $im(pr_0)$ and $\tilde{f}*h$ are extensionally equal (so the pullback of an image of some f is the image of the pullback of f.) The unique image-map into other factorizations is then just the one we get from $im(pr_0)$ composed with $\langle \tilde{f}*h, im(pr_0), \lambda x.x\rangle$. ☐

Proposition 3.2.16. *Regular epimorphisms of the form i_h for some morphism h are preserved under pullback up to isomorphism.*

Proof. Let $q : c \overset{m}{\to} im(h)$ be an arbitrary morphism and $q * i_h$ the chosen pullback with g the projection to c, i.e. the pullback of i_h.

$$\begin{tikzcd}
q * i_h \arrow[rr] \arrow[dr, "i_g"] \arrow[dd, "g"] & & a \arrow[dd, "i_h"] \\
& im(g) \arrow[dl, "\tilde{g}"] & \\
c \arrow[rr, "q"] & & im(h)
\end{tikzcd}$$

We have

$$\begin{aligned}
\langle c_0, a_0\rangle \mathrel{\dot{\in}} \|q * i_h\| &\leftrightarrow (\overline{q}c_0 \sim_{im(g)} \overline{i_h}a_0) \\
&\leftrightarrow (\overline{q}c_0 \sim_{im(g)} a_0) \\
&\leftrightarrow (\overline{h}(\overline{q}c_0) \sim_{cod(h)} \overline{h}(a_0)).
\end{aligned}$$

and also

$$\langle\langle c_0, a_0\rangle, \langle c_1, a_1\rangle\rangle \mathrel{\dot{\in}} (im(g))_{\mathrm{E}} \leftrightarrow (c_0 \sim_c c_1).$$

This gives a possibility to construct a section into $im(g)$.

$$\tilde{g}^{-1} :\equiv \langle c, im(g), \lambda c.\langle c, \overline{q}c\rangle\rangle$$

We need to show that this is a function and that

$$\langle c_0, a_0\rangle \sim_{im(g)} \overline{(\tilde{g}^{-1} \circ \tilde{g})}\langle c_0, a_0\rangle.$$

Let $c_0 \sim_c c_1$. Then $\overline{q}c_0 \sim_{im(h)} \overline{q}c_1$. That is equivalent to $\overline{h}(\overline{q}c_0) \sim_{cod(h)} \overline{h}(\overline{q}c_1)$. This implies that

$$\langle c_0, \overline{q}c_0\rangle \sim_{q * i_h} \langle c_1, \overline{q}c_1\rangle.$$

So $\tilde{g}^{-1}$ maps to the correct class and is trivially a function.

$$\begin{aligned}
&\overline{(\tilde{g}^{-1} \circ \tilde{g})}\langle c_0, a_0\rangle \\
&\sim_{im(g)} (\lambda z.((\lambda.c.\langle c, \overline{q}c\rangle)(\lambda w.\pi_0 w)z)\langle c_0, a_0\rangle \\
&\sim_{im(g)} ((\lambda c.\langle c, \overline{q}c\rangle)(\lambda w.\pi_0 w)\langle c_0, a_0\rangle \\
&\sim_{im(g)} ((\lambda c.\langle c, \overline{q}c\rangle)(\pi_0\langle c_0, a_0\rangle) \\
&\sim_{im(g)} (\lambda c.\langle c, \overline{q}c\rangle)c_0 \\
&\sim_{im(g)} \langle c_0, \overline{q}c_0\rangle).
\end{aligned}$$

From the definition of $(im(g))_{\mathrm{E}}$ we then get $\langle c_0, \overline{q}c_0\rangle \sim_{im(g)} \langle c_0, a_0\rangle$. The other direction is trivial as we just add another factor and then project it away again.

Hence the pullback of any i_h is of the form $\tilde{g} \circ i_g$ were $\tilde{g}$ is an isomorphism. It is in particular a regular epimorphism (being the coequalizer of the kernel pair of i_g. (see proposition 3.2.14.) ☐

Theorem 3.2.17. *All regular epimorphisms are up to isomorphism of the form i_h for some $h : x \overset{m}{\to} y$.*

Proof. Let $f : x \overset{m}{\to} z$ be some regular epi. We pull back along $id(z)$ and factor the resulting morphism g. The proof of proposition 3.2.16 tells us that the monomorphism is in fact an iso.

$$\begin{tikzcd}
id(z) * f \arrow[r, "id", "=_m"'] \arrow[d, two heads, "i_g"] & id(z) * f \arrow[r, "\cong"'] \arrow[d, two heads, "g"] & x \arrow[d, two heads, "f"] \\
im(g) \arrow[r, "\cong", "\tilde{g}"'] & z \arrow[r, "=_m", "id"'] & z
\end{tikzcd}$$

Since the pullback on the right is up to isomorphism given by

$$z \xleftarrow{f} x \xrightarrow{id(x)} z,$$

we see that f is indeed up to isomorphism of the form i_g. ☐

Corollary 3.2.18. ECB *is a regular category*

Proof. We can apply proposition A.3.7. We have arbitrary binary pullbacks, as seen in propositions 3.2.13 and 3.2.14 we can factor any morphism into a regular epi followed by a monomorphism and Proposition 3.2.16 and theorem 3.2.17 are used to show that regular epis are stable under pullbacks along arbitrary morphisms. ☐

Proposition 3.2.19. ECB *has finite coproducts*

Proof. We define our chosen coproducts as

$$\begin{aligned}
COP[u, A, B] &:\equiv \exists i, x(u = \langle i, x\rangle \land N(i) \land (i < 2) \\
&\qquad \land (i = 0 \to x \in A) \land (i = 1 \to x \in B)) \\
COPEQV[u, R, S] &:\equiv \exists i, x(u = \langle\langle i, \pi_0 x\rangle, \langle i, \pi_1 x\rangle\rangle \land N(i) \land (i < 2) \\
&\qquad \land (i = 0 \to x \in R) \land (i = 1 \to x \in S) \\
a + b &:\equiv \langle t_{COPR}(\|a\|, \|b\|), t_{COPREQV}(a_{\mathrm{E}}, b_{\mathrm{E}})\rangle
\end{aligned}$$

We need two coprojections $inl : a \overset{m}{\to} a + b$ and $inr : b \overset{m}{\to} a + b$ into the sum:

$$\begin{aligned}
\overline{inl} &:\equiv \lambda x.\langle 0, x\rangle \\
\overline{inr} &:\equiv \lambda x.\langle 1, x\rangle
\end{aligned}$$

Clearly these are functions: if $x \sim_a y$ then by definition $\langle 0, x\rangle \sim_{a+b} \langle 0, y\rangle$ and similarly for related elements in b.

Now we still need the morphism out of the coproduct. Given morphisms $f : a \overset{m}{\to} c$ and $g : b \overset{m}{\to} c$, we define $[\![f, g]\!] : (a + b) \overset{m}{\to} c$ as

$$\overline{[\![f, g]\!]} :\equiv \lambda x. \begin{cases} \overline{f}(\pi_1 x) & \pi_0 x = 0 \\ \overline{g}(\pi_1 x) & \pi_0 x = 1 \end{cases}$$

To check this is a function let $x \sim_{a+b} y$ be two related elements. We have two cases. Either $\pi_0 x = 0 = \pi_0 y$ or $\pi_0 x = 1 = \pi_0 y$. We consider the first case. From this we get a representation $\langle 0, x_0\rangle = x \sim_{a+b} y = \langle 0, y_0\rangle$ with

$x_0 \sim_a y_0$. This yields $\overline{[\![f,g]\!]}\langle 0, x_0\rangle \sim_c \overline{f}x_0 \sim_c \overline{f}y_0 \sim_c \overline{[\![f,g]\!]}\langle 0, y_0\rangle$. The other case, showing point-wise equality to g, is similar. Combined they prove uniqueness. ☐

So far we have tried to do without the use of essential existential quantifiers and disjunctions, since this allows for Feferman's realization interpretation [21]. Also, this made it possible to be agnostic as to whether the system is interpreted intuitionistic or classically. However, for the next definition, we will forgo these restrictions and work in classical logic.

Proposition 3.2.20. ECB *has binary pushouts.*

Proof sketch. Let $f : a \overset{m}{\to} b$ and $g : a \overset{m}{\to} c$ be two functions. We do essentially the usual construction of an equivalence relation generated by some relation E : $E_0 = E \cup \Delta \cup \{\langle v, u\rangle \mid \langle u, v\rangle \in E_0\}$ and then take the transitive closure: $E_n = \{\langle u, w\rangle \mid \langle u, v\rangle, \langle v, w\rangle \in E_{n-1}\}$ with the final relation then given as $R = \bigcup\limits^{\infty} E_n$.

First, we define composition of a relation with itself:

$$C[u, E, R] :\equiv (\exists v, w \in E)(\pi_1 v \sim_R \pi_0 w \wedge u = \langle \pi_0 v, \pi_1 w\rangle)$$
$$comp(r, e) :\equiv t_C(e, r_{\mathrm{E}})$$

As carrier of the pushout we take

$$p(f, g) :\equiv \|cod(f) + cod(g)\|.$$

We define relation on $p(f, g)$ from which we'll generate the transitive closure:

$$\begin{aligned} E_0[u, f, g, A, B, C, R] :\equiv u \in R \vee \\ &(\exists a \in A)(\exists b \in B)(\exists c \in C) \\ &(\pi_0 b = \overline{f}a \wedge \pi_0 c = \overline{g}a \wedge \\ &(u = \langle\langle 0, \pi_1 b\rangle, \langle 1, \pi_1 c\rangle\rangle) \vee \\ &(u = \langle\langle 1, \pi_1 c\rangle, \langle 0, \pi_1 b\rangle\rangle)) \end{aligned}$$

The equivalence relation is then

$$\begin{aligned}
\tilde{e}(h,f,g,a,b,c) &:\equiv \lambda n. \begin{cases} t_{E_0}(f,g,\|a\|,b_{\mathrm{E}},c_{\mathrm{E}},(b+c)_{\mathrm{E}}) & n=0 \\ \mathit{comp}((b+c)_{\mathrm{E}},h(f,g,a,b,c)(n-1)) & n\neq 0 \end{cases}\\
EX[u,E] &:\equiv (\exists n\in N)(\langle n,u\rangle \in E)\\
\mathit{closure}(f,g) &:\equiv \sum_{n:N}(\mathit{fix}(\tilde{e})(f,g,\mathit{dom}(f),\mathit{cod}(f),\mathit{cod}(g),n))\\
e(f,g) &:\equiv t_{EX}(\mathit{closure}(f,g))
\end{aligned}$$

The proof that this is an equivalence relation and that it is minimal can be done the usual way, so we only give the case for symmetry:

$$\begin{aligned}
\mathit{SYMM}[u,C] &:\equiv (u\in N)\wedge \forall p(\langle u,p\rangle \in C \to \langle u,\langle \pi_1 p \pi_0 p\rangle\rangle \in C)\\
\mathit{symm} &:\equiv t_{\mathit{SYMM}}(\mathit{closure}(f,g))
\end{aligned}$$

For $u=0$ we have the diagonal and inverse directly built in. Let $u>0$, $u \mathrel{\dot\in} \mathit{symm}$, and $\langle u+1,p\rangle \in \mathit{closure}(f,g)$. If $\langle u,p\rangle \in \mathit{closure}(f,g)$ then by induction we are done. If $\langle u,p\rangle \notin \mathit{closure}(f,g)$ then there are $k,l<u+1$ $\langle k,v\rangle,\langle l,w\rangle \in \mathit{closure}(f,g)$ such that $\pi_1 v \sim_{(b+c)_{\mathrm{E}}} \pi_0 w$ and $\pi_0 u \sim \pi_0 v \wedge \pi_1 u \sim \pi_1 w$. But by induction, we also get the inverses v^{-1},w^{-1} of v and w in the k-th (resp. the l-th) step, which means their composition is added in step $\mathit{max}(k,l)+1 \leq u+1$. If we now have $\langle\langle i,x\rangle,\langle j,y\rangle\rangle \mathrel{\dot\in} e(f,g)$, i.e. there there exists $k\in N$ such that

$$\langle k,\langle\langle i,x\rangle,\langle j,y\rangle\rangle\rangle \mathrel{\dot\in} \mathit{closure}(f,g).$$

Class induction now implies that we have $k \mathrel{\dot\in} \mathit{symm}$ and hence also $\langle\langle j,y\rangle,\langle i,x\rangle\rangle \mathrel{\dot\in} e(f,g)$.

The pushout in implicit Bishop sets is now given by

$$\begin{aligned}
f\star g &:\equiv \langle p(f,g),e(f,g)\rangle\\
q_0 &:\equiv \langle \mathit{cod}(f),f\star g,\lambda x.\langle 0,x\rangle\rangle\\
q_1 &:\equiv \langle \mathit{cod}(g),f\star g,\lambda x.\langle 1,x\rangle\rangle
\end{aligned}$$

To see these are functions, let $b = cod(f)$, $c = cod(g)$, and $x \sim_b y$. The equivalence relation on $f \star g$ contains $(b+c)_{\mathrm{E}}$. So in particular $\langle 0, x\rangle \sim \langle 0, y\rangle$, which is what we needed to show for q_0. The case for q_1 is the same.

Let $u : b \overset{m}{\to} d$ and $v : c \overset{m}{\to} d$ be two morphisms such that $u \circ f =_m v \circ g$. There is a unique morphism $e : f \star g \overset{m}{\to} d$ making the obvious triangles commute. k is represented by the term

$$\overline{k(u,v)}(x) :\equiv \begin{cases} \overline{u}(\pi_1 x) & \pi_0 x = 0 \\ \overline{v}(\pi_1 x) & \pi_0 x = 1. \end{cases}$$

This is well-defined: Let $\langle i, x\rangle \sim_{f\star g} \langle j, y\rangle$. We have four cases, two of which are symmetric:

(i) $i = 0 = j$ or $i = 1 = j$: Both u and v are functions.

(ii) $i = 0$, $j = 1$: By induction. There is some finite n such that $\langle n, \langle\langle 0, x\rangle, \langle 1, y\rangle\rangle\rangle$. If $n = 0$ then there is some $a_0 \dot{\in} a$ such that $\overline{f}a_0 \sim x \wedge \overline{g}a_0 \sim y$ which yields

$$\overline{u}x \sim \overline{u}(\overline{f}a_0) \sim \overline{v}(\overline{g}a_0) \sim \overline{v}y.$$

Otherwise, there are $j, l < n$ and $w \sim_{b+c} z$ such that $\langle j, \langle\langle 0, x\rangle, w\rangle\rangle$ and $\langle l, \langle z, \langle 1, y\rangle\rangle\rangle$. By induction we have some $a_0, a_1 \dot{\in} a$ with

$$\overline{u}x \sim \overline{u}(\overline{f}a_0) \sim k(u,v)w \sim k(u,v)z \sim \overline{v}(\overline{g}a_1) \sim \overline{v}y.z$$

(iii) $i = 1$, $j = 0$: Symmetric to the above. ☐

Proposition 3.2.21. *Finite coproducts are disjoint*

Proof. Clear: $\langle a, b\rangle \in \|inl * inr\| \leftrightarrow \langle 0, a\rangle \sim_{a+b} \langle 1, b\rangle \leftrightarrow \perp$. ☐

Notation 3.2.22. We need some notation to make the following part more readable. We write the unique morphism between coproducts in the following way:

$$\begin{tikzcd}
a \arrow[r, "inr"] \arrow[d, "f"] & a+b \arrow[d, dotted, "f \oplus g"] & b \arrow[l, "inl"'] \arrow[d, "g"] \\
c \arrow[r, "inr"] & c+d & d \arrow[l, "inl"']
\end{tikzcd}$$

Direct calculation show, that this morphism can be written as

$$f \oplus g :\equiv \left\langle dom(f) + dom(g), cod(f) + cod(g), \lambda z. \begin{cases} \langle 0, \overline{f}(\pi_1 z)\rangle & \pi_0 z = 0 \\ \langle 1, \overline{g}(\pi_1 z)\rangle & \pi_0 z = 1 \end{cases} \right\rangle$$

$\diamond$

Definition 3.2.23. Let $Ob_{\mathbf{ECB}}(x, y)$ and let $\mathbf{ECB}/x \times \mathbf{ECB}/y$ be the product category of the slice categories defined in the obvious way. The canonical functor into the slice of the sum $x + y$ will be called p. In the following we will omit the doubled $\overline{\overline{k}}$ to get the operation of a morphism in a slice and just write $\overline{k}$.

$$\begin{aligned}
p &:\equiv \langle p_o, p_m\rangle : (\mathbf{ECB}/x \times \mathbf{ECB}/y) \to \mathbf{ECB}/(x + y) \\
p_o(\langle f, g\rangle) &:\equiv f \oplus g \\
p_m(\langle k, l\rangle) &:\equiv \langle p_o(\langle dom(k), dom(l)\rangle), p_o(\langle cod(k), cod(l)\rangle), \\
&\qquad \overline{(\overline{k} \oplus \overline{l})}\rangle
\end{aligned}$$

$\diamond$

Proposition 3.2.24. *There exists a functor from* $\mathbf{ECB}/(x + y)$ *to the product* $\mathbf{ECB}/x \times \mathbf{ECB}/y$*.*

Proof. The following classes are essentially given as the pullback of f along *inl* or *inr* or written differently $f^{-1}\{\langle i, -\rangle\}$.

$$\begin{aligned}
C[u,i,f,A] &:\equiv u \in A \wedge \pi_0(\overline{f}u) = i\\
CR[u,i,f,R] &:\equiv u \in R \wedge \pi_0(\overline{f}\pi_0 u) = i\\
\|case(f,i)\| &:\equiv t_C(f, \|dom(f)\|, i)\\
case(f,i)_{\mathrm{E}} &:\equiv t_{CR}(f, dom(f)_{\mathrm{E}}, i)\\
[f|_0 &:\equiv case(f,0)\\
[f|_1 &:\equiv case(f,1)
\end{aligned}$$

The functor is the given by

$$\begin{aligned}
q &:\equiv \langle q_o, q_m\rangle : \mathbf{ECB}/(x+y) \to (\mathbf{ECB}/x \times \mathbf{ECB}/y)\\
q_o(f) &:\equiv \langle\langle [f|_0, x, \lambda z.\pi_1(\overline{f}z)\rangle,\\
&\quad\ \langle [f|_1, y, \lambda z.\pi_1(\overline{f}z)\rangle\rangle\\
q_m(h) &:\equiv \langle q_o(dom(h)), q_o(cod(h)),\\
&\quad\ \langle\langle [dom(h)|_0, [cod(h)|_0, \lambda z.\overline{h}z\rangle,\\
&\quad\ \ \langle [dom(h)|_1, [cod(h)|_1, \lambda z.\overline{h}z\rangle\rangle\rangle
\end{aligned}$$

□

Theorem 3.2.25. *p and q are an equivalence of categories with natural isomorphisms*

$$\begin{aligned}
\eta :\equiv\ &\langle q \circ p, id(\mathbf{ECB}/x \times \mathbf{ECB}/y),\\
&\lambda z.\langle\langle\\
&\langle [p_o(z)|_0, x, \lambda w.\pi_1(\overline{(\pi_0 z \oplus \pi_1 z)}w)\rangle\\
&\langle [p_o(z)|_1, y, \lambda w.\pi_1(\overline{(\pi_0 z \oplus \pi_1 z)}w)\rangle\rangle,\\
&z,\\
&\lambda w.\langle \pi_1(\pi_0 w), \pi_1(\pi_1 w)\rangle\rangle\rangle\\
\xi :\equiv\ &\langle p \circ q, id(\mathbf{ECB}/(x+y)), \lambda h.\langle p_m(q_m(h)), h, \lambda w.\langle \overline{h}w, w\rangle\rangle\rangle
\end{aligned}$$

Proof. Let $\langle f, g\rangle :\equiv \langle\langle a, x, \overline{f}\rangle, \langle b, y, \overline{g}\rangle\rangle$ be an object of the product category

$(\mathbf{ECB}/x \times \mathbf{ECB}/y)$.

$$p_o(\langle f, g\rangle) = \langle a+b, x+y, \lambda z. \begin{cases} \langle 0, \overline{f}(\pi_1 z)\rangle & \pi_0 z = 0 \\ \langle 1, \overline{g}(\pi_1 z)\rangle & \pi_0 z = 1 \end{cases} \rangle$$

In a first step we check when $x \mathrel{\dot{\in}} \|[p_o(\langle f,g\rangle)|_0\|$ holds.

$$\begin{aligned} x \mathrel{\dot{\in}} \|[p_o(\langle f,g\rangle)|_0\| &\leftrightarrow x \mathrel{\dot{\in}} \|a+b\| \wedge \pi_0(\overline{p_o(\langle f,g\rangle)}x) = 0 \\ &\leftrightarrow x \mathrel{\dot{\in}} \|a+b\| \wedge \pi_0 x = 0 \wedge \pi_1 x \mathrel{\dot{\in}} a \\ &\leftrightarrow \pi_0 x = 0 \wedge \pi_1 x \mathrel{\dot{\in}} a. \end{aligned}$$

So there is an isomorphism $[p_o(\langle f,g\rangle)|_0 \cong a$ induced by $\langle 0, x\rangle \mapsto x$ uniform in the argument pair. The other case works similarly, and so the pair of those isomorphisms is an isomorphism in the product category.

$$\begin{aligned} & q_o(p_o\langle f,g\rangle) \\ &= \langle\langle [p_o(\langle f,g\rangle)|_0, x, \lambda z.\pi_1(\overline{p_o(\langle f,g\rangle)}z)\rangle, \\ &\quad \langle [p_o(\langle f,g\rangle)|_1, y, \lambda z.\pi_1(\overline{p_o(\langle f,g\rangle)}z)\rangle\rangle \\ &=_m \langle\langle [p_o(\langle f,g\rangle)|_0, x, \lambda z.\pi_1((\overline{(\overline{k} \oplus \overline{l})})z)\rangle, \\ &\quad \langle [p_o(\langle f,g\rangle)|_1, y, \lambda z.\pi_1((\overline{(\overline{k} \oplus \overline{l})})z)\rangle\rangle \end{aligned}$$

from the above calculation of $[p_o(\langle f,g\rangle)|_i$ we know the form of z and can simplify.

$$\begin{aligned} &=_m \langle\langle [p_o(\langle f,g\rangle)|_0, x, \lambda z.\pi_1(\langle 0, \overline{k}(\pi_1 z)\rangle)\rangle, \\ &\quad \langle [p_o(\langle f,g\rangle)|_1, y, \lambda z.\pi_1(\langle 1, \overline{l}(\pi_1 z)\rangle)\rangle\rangle \\ &=_m \langle\langle [p_o(\langle f,g\rangle)|_0, x, \lambda z.\overline{k}(\pi_1 z)\rangle, \\ &\quad \langle [p_o(\langle f,g\rangle)|_1, y, \lambda z.\overline{l}(\pi_1 z)\rangle\rangle \end{aligned}$$

We have to check this is natural. Let $k : f_0 \overset{m}{\to} f_1$, $l : g_0 \overset{m}{\to} g_1$ be two morphisms in $\mathbf{ECB}/x$ and $\mathbf{ECB}/y$.

$$\begin{aligned}
&q_m(p_m\langle k,l\rangle)\\
&\quad= \langle q_o(dom(p_m\langle k,l\rangle)), q_o(cod(p_m\langle k,l\rangle)),\\
&\qquad \langle\langle [f_0 \oplus g_0|_0, [f_1 \oplus g_1|_0, \lambda z.\overline{(p_m\langle k,l\rangle)}z\rangle,\\
&\qquad \langle [f_0 \oplus g_0|_1, [f_1 \oplus g_1|_1, \lambda z.\overline{(p_m\langle k,l\rangle)}z\rangle\rangle\rangle\\
&\quad=_m \langle q_o(f_0 \oplus g_0), q_o(f_1 \oplus g_1),\\
&\qquad \langle\langle [p_m(\langle dom(k), dom(l)\rangle)|_0, [p_m(\langle cod(k), cod(l)\rangle)|_0,\\
&\qquad \lambda z.\overline{(p_m\langle k,l\rangle)}z\rangle,\\
&\qquad \langle [p_m(\langle dom(k), dom(l)\rangle)|_1, [p_m(\langle cod(k), cod(l)\rangle)|_1,\\
&\qquad \lambda z.\overline{(p_m\langle k,l\rangle)}z\rangle\rangle\rangle\\
&\quad=_m \langle q_o(f_0 \oplus g_0), q_o(f_1 \oplus g_1),\\
&\qquad \langle\langle [p_m(\langle f_0, g_0\rangle)|_0, [p_m(\langle f_1, g_1\rangle)|_0, \lambda z.\overline{(\overline{k} \oplus \overline{l})}z\rangle,\\
&\qquad \langle [p_m(\langle f_0, g_0\rangle)|_1, [p_m(\langle f_1, g_1\rangle)|_1, \lambda z.\overline{(\overline{k} \oplus \overline{l})}z\rangle\rangle\rangle\\
&\quad=_m \langle q_o(f_0 \oplus g_0), q_o(f_1 \oplus g_1),\\
&\qquad \langle\langle [p_m(\langle f_0, g_0\rangle)|_0, [p_m(\langle f_1, g_1\rangle)|_0, \lambda z.\overline{k}z\rangle,\\
&\qquad \langle [p_m(\langle f_0, g_0\rangle)|_1, [p_m(\langle f_1, g_1\rangle)|_1, \lambda z.\overline{l}z\rangle\rangle\rangle
\end{aligned}$$

At this point we should note that $f_0 =_m f_1 \circ \overline{k}$ in particular implies, that in the coproduct $\overline{k}$ does not "switch sides" and so p_m only adds labels to the morphism staying in the correct $[p_m(\langle f_0, g_0\rangle)|_i$. Essentially $(q \circ p)_m$ applied to two morphisms $\langle u, v\rangle$, sends pairs $\langle\langle 0, a\rangle, \langle 1, b\rangle\rangle$ to pairs $\langle\langle 0, \overline{u}a\rangle, \langle 1, \overline{v}b\rangle\rangle$. The identity functor just applies the morphisms point-wise, so $\langle a, b\rangle \mapsto \langle \overline{u}a, \overline{v}b\rangle$. This makes it clear that $\eta : (q \circ p) \Rightarrow id(\mathbf{ECB}/x \times \mathbf{ECB}/y)$ is a natural isomorphism.

For the natural isomorphism we're going sketch the proof. Let $f : a \overset{m}{\to} x + y$ and let $h : f \overset{m}{\to} g$ with $g : b \overset{m}{\to} x + y$. We have

$$\begin{aligned}
p_o(q_o(f)) &=_m p_0(\langle f|_0, f|_1\rangle) =_m f|_0 \oplus f|_1\\
p_m(q_m(h)) &=_m p_m(\langle h|_{0,f,g}, h|_{1,f,g}\rangle) =_m h|_{0,f,g} \oplus h|_{1,f,g}
\end{aligned}$$

where the required morphism-restrictions are constructed as

$$\begin{aligned} f|_0 &:\equiv \langle [f|_0, x, \lambda z.\pi_1(\overline{f}z)\rangle \\ f|_1 &:\equiv \langle [f|_1, y, \lambda z.\pi_1(\overline{f}z)\rangle \\ h|_{0,f,g} &:\equiv \langle [f|_0, [g|_0, \lambda z.\pi_1(\overline{h}z)\rangle \\ h|_{0,f,g} &:\equiv \langle [f|_1, [g|_1, \lambda z.\pi_1(\overline{h}z)\rangle. \end{aligned}$$

Application of a morphism again just gets distributed to the summands and applied there. So given $f : a \overset{m}{\to} x + y$, the natural isomorphism sends $a_0 \dot{\in} a$ to $\langle \pi_0(\overline{f}a_0), a_0\rangle$ □

Corollary 3.2.26. **ECB** *is extensive.*

Proof. By definition this means that canonical functor

$$(\mathbf{ECB}/x \times \mathbf{ECB}/y) \to \mathbf{ECB}/(x + y)$$

is an equivalence of categories. This is theorem 3.2.25. □

Proposition 3.2.27. *Finite coproducts are stable under pullbacks*

Proof. This follows from being extensive.

For a direct proof see proposition 3.1.22 for the same statement in **EC** which can be adapted. □

Additional Properties

Proposition 3.2.28 (**ECB** has a natural numbers object)**.**

Proof. Unsurprisingly, the natural numbers object is the elementary class of natural numbers with the discrete equivalence.

$$\begin{aligned} DISCRETE[u, X] &:\equiv (\exists x \in X)(u = \langle x, x\rangle) \\ \mathbb{n} &:\equiv \langle nat, t_{DISCRETE}(nat)\rangle \\ zero &:\equiv \langle \mathbb{1}, \mathbb{n}, \lambda x.0\rangle \\ suc &:\equiv \langle \mathbb{n}, \mathbb{n}, \lambda x.\mathrm{s}_N x\rangle \end{aligned}$$

To prove that this is actually a NNO we require class induction. We recall the definition:

$$
\begin{aligned}
&(\text{C-I}_\text{N}) \qquad \forall X(0 \in X \wedge (\forall x \in N)(x \in X \to \mathrm{s}_N x \in X) \\
&\qquad\qquad\qquad \to (\forall x \in N)(x \in X))
\end{aligned}
$$

Now suppose we are given two morphisms $z : \mathbb{1} \overset{m}{\to} a$ and $s : a \overset{m}{\to} a$ we will construct a morphism $u : \mathbb{n} \overset{m}{\to} a$ induced by a term $r(z, s)$ which makes the following diagram commute:

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "zero"] \arrow[dr, "z"'] & \mathbb{n} \arrow[r, "suc"] \arrow[d, dotted, "u"] & \mathbb{n} \arrow[d, dotted, "u"] \\
& a \arrow[r, "s"] & a
\end{tikzcd}$$

We can define the required recursive function by

$$
\begin{aligned}
\tilde{r}(z, s) &:\equiv \lambda f.\lambda x. \begin{cases} \overline{z}0 & x = 0 \\ \overline{s}(f(x-1)) & x \neq 0 \end{cases} \\
r(z, s) &:\equiv \operatorname{fix}(\tilde{r}(z, s))
\end{aligned}
$$

We have to show this is always defined and gives the required commuting diagram.

$$
\begin{aligned}
DEF[u, z, s, A] &:\equiv N(u) \wedge (r(z, s)u) \in A \\
COMM[u, z, s] &:\equiv N(u) \wedge \overline{s}(r(z, s)u) = r(z, s)(\overline{suc}u)
\end{aligned}
$$

The fact that $z =_m u \circ zero$ holds is trivial. And so $\overline{zero}(*) = 0 \mathrel{\dot{\in}} t_{DEF}(z, s)$ shows the base case.

Let $N(x) \wedge r(z, s)x \mathrel{\dot{\in}} a$. We can calculate

$$
\begin{aligned}
r(z, s)(x + 1) &\simeq \operatorname{fix}(\tilde{r}(z, s))(x + 1) \\
&\simeq \overline{s}(\operatorname{fix}(\tilde{r}(z, s))x) \\
&\simeq \overline{s}(r(z, s)x)
\end{aligned}
$$

Because we know that $s : a \overset{m}{\to} a$, the assumption implies $r(z, s)(x + 1) \mathrel{\dot{\in}} a$.

And so we have $(\forall x \in N)(x \mathrel{\dot\in} t_{DEF}(z,s) \to (x+1) \mathrel{\dot\in} t_{DEF}(z,s))$ for which, (C-I$_{\mathrm{N}}$) yields $(\forall n \in N)((r(z,s)n) \mathrel{\dot\in} a)$.

For commutativity, consider

$$\begin{aligned}
\overline{s}(r(z,s)(0)) &= \overline{s}(\overline{z}0)\\
&= \overline{s}((\tilde{r}(z,s))(\mathrm{fix}(\tilde{r}(z,s))0))\\
&= \overline{s}(\mathrm{fix}(\tilde{r}(z,s))0)\\
&= \mathrm{fix}(\tilde{r}(z,s))(1)\\
&= r(z,s)(1)\\
&= r(z,s)(\overline{suc}0)
\end{aligned}$$

This is the base case $0 \mathrel{\dot\in} t_{COMM}(z,s)$. Now let

$$N(x) \wedge \overline{s}(r(z,s)x) = r(z,s)(\overline{suc}x).$$

Since we have already shown that $(\forall x \in N)(r(z,s)x \mathrel{\dot\in} a)$ holds, we can calculate

$$\begin{aligned}
\overline{s}(r(z,s)(x+1)) &= \overline{s}(\mathrm{fix}(\tilde{r}(z,s))(x+1))\\
&= (\tilde{r}(z,s)(\mathrm{fix}(\tilde{r}(z,s))))(x+2)\\
&= \mathrm{fix}(\tilde{r}(z,s))(x+2)\\
&= r(z,s)(x+2)\\
&= r(z,s)(\overline{suc}(x+1)).
\end{aligned}$$

Using (C-I$_{\mathrm{N}}$) again, this shows that

$$(\forall n \in N)(\overline{s}(r(z,s)n) = r(z,s)(\overline{suc}(n))).$$

But that is just what we needed to show for $s \circ u = u \circ suc$. Note that the above equations hold because, $n \sim_{\mathbb{n}} m \leftrightarrow n = m$.

With this we have shown that

$$u :\equiv \langle \mathbb{n}, a, r(z,s) \rangle$$

really is a well-defined morphism which commutes for all elements of $\mathbb{n}$. □

Definition 3.2.29 (Sigma Bishop Set)**.**
If we work in $\mathbf{EC} + (J)$ we can define what it means to be a sigma Bishop set.

Intuitively, we want $\sum(a, f, g)$ to be defined in a way such that $\langle u, w\rangle \mathrel{\dot\in} \sum(a, f, g) \leftrightarrow u \mathrel{\dot\in} \|a\| \wedge w \mathrel{\dot\in} \|fx\|$ with equivalence $\langle x, w\rangle \sim_{\sum(a,f,g)} \langle y, v\rangle \leftrightarrow x \sim_a y \wedge (gxy)(w) \sim_{fb} v$ for a transport isomorphism $gxy : f(x) \cong f(y)$.

We use the following formula as a precondition for the construction of our sigma Bishop Set.

$$\begin{aligned} F[a, f, g] :\equiv (\forall x \mathrel{\dot\in} \|a\|)(Ob(fx)) \\ \wedge (\forall x, y \mathrel{\dot\in} \|a\|)(x \sim_a y \\ \to gxy : fx \overset{m}{\to} fy \wedge gxx =_m id(fx) \\ \wedge ISO[gxy, gyx]) \end{aligned}$$

where $ISO[f, g]$ is the elementary statement, that f and g compose to identities up to function extensionality. If we would had already defined what we mean by a universe $u_{\mathbf{ECB}}$ of Bishop sets, we could instead have said that f, g is a functor between a and $u_{\mathbf{ECB}}$ when viewed as categories with the equivalence relations used as morphisms (where $u_{\mathbf{ECB}}$ has isomorphisms of Bishop Sets as equivalence relation). Let $Ob(a)$, f and g be such that $F[a, f, g]$ holds. The sigma Bishop set $\sum_{\mathbf{ECB}}(a, f, g)$ is then given by

$$\begin{aligned} carrier(a, f, g) &:\equiv \sum(\|a\|, \lambda x.\|fx\|) \\ TR[u, g, x, y, w, v, R, S] &:\equiv u = * \wedge x \sim_R y \wedge \overline{(gxy)}w \sim_S v \\ transp(a, f, g, x, y, v, w) &:\equiv t_{TR}(g, x, y, w, v, a_{\mathrm{E}}, (fy)_{\mathrm{E}}) \\ preeqv(a, f, g) &:\equiv \sum(carrier(a, f, g), \\ &\qquad \lambda x.\sum(carrier(a, f, g), \\ &\qquad \lambda y.transp(a, f, g, \pi_0 x, \pi_0 y, \pi_1 x, \pi_1 y))) \\ EQV[u, P] &:\equiv u = \langle\langle x, w\rangle, \langle y, v\rangle\rangle \\ &\qquad \wedge \langle\langle x, w\rangle, \langle y, v\rangle, *\rangle \in P \\ \sum_{\mathbf{ECB}}(a, f, g) &:\equiv \langle carrier(a, f, g), t_{EQV}(preeqv(a, f, g))\rangle \end{aligned}$$

Note that f is not an arbitrary operation, but actually respects a_{E} in the sense that $x \sim_a y \to f(x) \cong f(y)$. So f maps equivalent elements to isomorphic Bishop sets (witnessed by g) and hence it is a morphism into a universe of Bishop Sets with existence of isomorphisms as equivalence relation.
Any universe u in Explicit Mathematics which is closed under join, will contain both $carrier(a, f, g)$ and $t_{EQV}(preeqv(a, f, g))$ if

$$\|a\| \mathrel{\dot\in} u \wedge a_{\mathrm{E}} \mathrel{\dot\in} u \wedge (\forall x \mathrel{\dot\in} \|a\|)(\pi_0(fx) \mathrel{\dot\in} u \wedge \pi_1(fx) \mathrel{\dot\in} u) \qquad \diamondsuit$$

Definition 3.2.30 (Pi Bishop Set)**.**
We define the Pi Bishop set in a similar way.
Let $Ob(a), f$ and g be such that $F[a, f, g]$ holds. $\Pi_{\mathbf{ECB}}(a, f, g)$ is given by

$$\begin{aligned}
c(a, f, g) &:\equiv \prod(\|a\|, \lambda x.\|fx\|) \\
TR[u, g, p, q, R, S] &:\equiv u = * \wedge (\forall x \sim_R y)(\overline{(gxy)}(px) \sim_S (qy)) \\
transp(a, f, g, p, q) &:\equiv t_{TR}(g, p, q, a_{\mathrm{E}}, (fy)_{\mathrm{E}}) \\
preeqv(a, c, f, g) &:\equiv \prod(c(a, f, g), \\
&\qquad \lambda p. \prod(c(a, f, g), \\
&\qquad \lambda q.transp(a, f, g, p, q))) \\
EQV[u, P] &:\equiv u = \langle p, q\rangle \wedge \langle p, q, *\rangle \in P \\
C[u, P] &:\equiv \langle u, u, *\rangle \in P \\
carrier(a, f, g) &:\equiv t_C(preeqv(a, c, f, g)) \\
eqv(a, f, g) &:\equiv t_{EQV}(preeqv(a, carrier(a, f, g), f, g)) \\
\prod_{\mathbf{ECB}}(a, f, g) &:\equiv \langle carrier(a, f, g), eqv(a, f, g))\rangle
\end{aligned}$$

Any universe u in Explicit Mathematics which is closed under join, will contain both $carrier(a, f, g)$ and $t_{EQV}(preeqv(a, f, g))$ if

$$\|a\| \mathrel{\dot\in} u \wedge a_{\mathrm{E}} \mathrel{\dot\in} u \wedge (\forall x \mathrel{\dot\in} \|a\|)(\pi_0(fx) \mathrel{\dot\in} u \wedge \pi_1(fx) \mathrel{\dot\in} u) \qquad \diamondsuit$$

Adding a Choice Principle

For the rest of this section, we're going to add the following version of the axiom of choice.

$$(AC_V) \qquad \forall x \exists y \phi(x, y) \to \exists f \forall x \phi(x, fx)$$

This is consistent, which can be shown by a realizability interpretation; see for example Beeson [4, theorem 4.3].

In particular we can apply this to

$$\forall x \exists y ((\mathfrak{R}(x) \wedge \exists z (z \mathrel{\dot\in} x)) \to y \mathrel{\dot\in} x)$$

then we have some term c_{AC_V} such that for all non-empty classes x we can prove $c_{AC_V}(x) \mathrel{\dot\in} x$.

We are now going to show that in **EM** + (AC_V) internal equivalence relations (see definition 3.3.3) in **ECB** are just equivalence relations in the sense of **ECB**. Having a way to uniformly select an element from any preimage of a monomorphism (which is necessarily unique up to equivalence), lets us do the usual construction in the category of sets to get a quotient. In fact we immediately get lots of properties which are suddenly easy to prove. We can show that all monomorphisms split, which is just a restatement of the above. This immediately implies that any mono $f : a \overset{m}{\to} b$ is regular (as the equalizer of $id(b)$ and $f \circ r$, where $r : b \overset{m}{\to} a$ is the retraction.) And most importantly of course, **EM** + (AC_V) proves that **ECB** is an exact category.

Proposition 3.2.31. *An object in* **ECB** *can be written as an internal equivalence relation.*

Proof. Let $\langle x, r \rangle$ be an object in **ECB** and $\langle \gamma_o, \gamma_m \rangle : \mathbf{EC} \to \mathbf{ECB}$ the embedding which sends classes to discrete Bishop sets Δ_x. If we send x and r to $\gamma_o(x)$ and $\gamma_o(r)$ and $\pi_i : r \overset{m}{\to} x$ to $\gamma_m(\pi_i)$. $\Delta_r \underset{\gamma_m(\pi_1)}{\overset{\gamma_m(\pi_0)}{\rightrightarrows}} \Delta_x$ is then an internal equivalence relation. Intuitively this should be clear, but we can directly verify this:

$$\begin{aligned} refl_{rx} &:\equiv \langle \Delta_x, \Delta_r, \lambda x.\langle x, x\rangle\rangle \\ symm_{rx} &:\equiv \langle \Delta_r, \Delta_r, \lambda p.\langle \pi_1 p, \pi_0 p\rangle\rangle \\ tran_{rx} &:\equiv \langle \gamma_o(\pi_1 * \pi_0), \Delta_r, \lambda z.\langle \pi_0(\pi_0 z), \pi_1(\pi_1 z)\rangle\rangle \end{aligned}$$

The fact that the last one is a function follows directly from

$$\gamma_m(\pi_1) * \gamma_m(\pi_0) \doteq \gamma_o(\pi_1 * \pi_0).$$ □

Proposition 3.2.32. *An internal equivalence relation* $r \rightrightarrows x$ *(in* **ECB***) can be represented by an object* x/r *and a regular epi* $i_{rx} : x \overset{m}{\to} x/r$*.*

Proof. Let $r \overset{p_0}{\underset{p_1}{\rightrightarrows}} x$ with morphisms $refl_{rx}$, $symm_{rx}$, and $tran_{rx}$ be an internal equivalence relation. We define

$$\begin{aligned} \|q\| &:\equiv \|x\| \\ Q[u, X, R] &:\equiv \exists r, s(u = \langle \overline{p_0}r, \overline{p_1}s\rangle \wedge (\overline{p_0}r) \sim_X (\overline{p_1}s) \wedge r \sim_R s) \\ q_{\mathrm{E}} &:\equiv t_Q(x_{\mathrm{E}}, r_{\mathrm{E}}) \end{aligned}$$

To show this is an equivalence relation on $\|x\|$, suppose we have $u, v, w \,\dot{\in}\, \|x\|$: We have $\overline{refl_{rx}}u \sim_r \overline{refl_{rx}}u$. The equation $id(x) =_m p_i \circ refl_{rx}$ then shows reflexivity. Symmetry directly follows from sending the existing witnesses to their images under $symm_{rx}$. Similar for transitivity.

The projection into i_{rx} into x/r is given by

$$i_{rx} :\equiv \langle x, r/x, \lambda x.x\rangle$$

To see that this is well-defined, consider $u \sim_x v$. Then, because $refl_{rx}$ is a function which means it send u and v to equivalent elements, we can reuse the reflexivity-argument:

$$\overline{(p_0 \circ refl_{rx})}u \sim_x u \sim_x v \sim_x \overline{(p_1 \circ refl_{rx})}v$$

Hence, $u \sim_{x/r} v$ holds, and we are done. □

The above construction works already in **EM**. But while it makes sense to call this a representation of the equivalence relation from the outside, it's not clear how to prove such a statement from inside the theory without using some amount choice. If we, however, accept (AC_V) for now; it's easy to prove that p_0, p_1 is the kernel pair of i_{rx} and i_{rx} is its pushout/coequalizer.

Proposition 3.2.33. *For any internal equivalence relation* $r \underset{pr_1}{\overset{pr_0}{\rightrightarrows}} x$, *the function induced by the identity* $i_{rx} : x \overset{m}{\to} x/r$ *has* pr_0, pr_1 *as its kernel pair.*

Proof. Let $u, v : z \overset{m}{\to} x$ be two morphisms such that $i_{rx} \circ u =_m i_{rx} \circ v : z \overset{m}{\to} x/r$. We can construct a function from z to r using (AC_V).

$$\langle\!\langle u, v \rangle\!\rangle :\equiv \langle z, r, \lambda z. c_{AC_V}(\langle\!\langle pr_0, pr_1 \rangle\!\rangle^{-1}\{\langle \overline{u}z, \overline{v}z \rangle\})\rangle$$

Because $\langle\!\langle pr_0, pr_1 \rangle\!\rangle : r \overset{m}{\to} x \times x$ is monic by assumption, we get uniqueness for free and are done if we can show that this is indeed a function.

(a) We have to check, that all preimages really contain an element which we can select.

$$\begin{aligned}
&\overline{(i_{rx} \circ u)}z \sim_{x/r} \overline{(i_{rx} \circ v)}z \\
\leftrightarrow\ &\overline{i_{rx}}(\overline{u}z) \sim_{x/r} \overline{i_{rx}}(\overline{v}z) \\
\leftrightarrow\ &\exists w_0, w_1 (\overline{u}z \sim_x \overline{pr_0}w_0 \wedge \overline{v}z \sim_x \overline{pr_1}w_1 \wedge \\
&\qquad\qquad \overline{pr_0}w_0 \sim_x \overline{pr_1}w_1 \wedge w_0 \sim_r w_1)
\end{aligned}$$

In particular this means we get some $w_0, w_1 \mathrel{\dot{\in}} \langle\!\langle pr_0, pr_1 \rangle\!\rangle^{-1}\{\langle \overline{u}z, \overline{v}z \rangle\}$

(b) Now let $z_0 \sim_z z_1$.

$$\begin{aligned}
\overline{\langle\!\langle u, v \rangle\!\rangle}z_0 &\sim_r (\lambda z. c_{AC_V}(\langle\!\langle pr_0, pr_1 \rangle\!\rangle^{-1}\{\langle \overline{u}z, \overline{v}z \rangle\}))z_0 \\
&\sim_r c_{AC_V}(\langle\!\langle pr_0, pr_1 \rangle\!\rangle^{-1}\{\langle \overline{u}z_0, \overline{v}z_0 \rangle\}) \\
&\sim_r c_{AC_V}(\langle\!\langle pr_0, pr_1 \rangle\!\rangle^{-1}\{\langle \overline{u}z_1, \overline{v}z_1 \rangle\}) \\
&\sim_r \overline{\langle\!\langle u, v \rangle\!\rangle}z_1
\end{aligned}$$

Because u and v are functions, and the equivalence relation on products is defined component-wise we have $\langle \overline{u}z_0, \overline{v}z_0 \rangle \sim_{x \times x} \langle \overline{u}z_1, \overline{v}z_1 \rangle$. But then we can apply lemma 4.1.8 which says that preimages of equivalent values are extensionally equal. Note that $c_{AC_V}(\cdot)$, even when restricted to particular Bishop sets, *doesn't* in general have to be a function.[9] However, because $\langle\!\langle pr_0, pr_1 \rangle\!\rangle$ is monic, it is injective by proposition 3.2.6, and hence the preimage is isomorphic to $\mathbb{1}$. $\square$

Corollary 3.2.34. *x/r is the pushout of p_0, p_1.*

Proof. Let $(f \circ p_0 =_m g \circ p_1) : r \overset{m}{\to} z$. The unique morphism into z is given by

$$h :\equiv \langle x/r, z, \lambda x.\overline{f}x \rangle.$$

for well-definedness we have to check that h is a function and that $g =_m h \circ i_{rx}$. ($f =_m h \circ i_{rx}$ is obvious.)

(a) Let $u \sim_{x/r} v$. There exist $s \sim_m t$ such that $u \sim_x \overline{p_0}s \sim_x \overline{p_1}t \sim_x v$. Therefore, we have $\overline{f}u \sim_z \overline{f \circ p_0}s \sim_z \overline{f \circ p_0}t \sim_z \overline{f}v$. Which is what we had to show.

(b) This makes the diagram commute because

$$\begin{aligned} g &=_m g \circ (p_1 \circ refl_{rx}) \\ &=_m (g \circ p_1) \circ refl_{rx} \\ &=_m (f \circ p_0) \circ refl_{rx} \\ &=_m f \\ &=_m h \circ i_{rx}. \end{aligned}$$

To prove uniqueness, consider some other $k : x/r \overset{m}{\to} z$ with $f =_m k \circ i_{rx} =_m g$. We get for all $u \mathrel{\dot{\in}} \|x/r\|$ that

$$\overline{k}u \sim_z \overline{k}(\overline{i_{rx}}u) \sim_z \overline{f}u \sim_z \overline{h}u. \qquad \square$$

Corollary 3.2.35. *x/r is the coequalizer of p_0, p_1.*

Proof. To construct the extension of some f to the coequalizer, we set $g :\equiv f$ in corollary 3.2.34. $\square$

---

[9] Consider $\mathfrak{2}/(0 \sim 1)$ and the map induced by the identity $p : \mathfrak{2} \overset{m}{\to} \mathfrak{2}/(0 \sim 1)$ if $\lambda x.c_{AC_V}(p^{-1}\{x\})$ is given by the identity it isn't a function.

Proposition 3.2.36. *Assuming classical logic, all monomorphisms are regular.*

Proof. Let $f : a \overset{m}{\rightarrowtail} b$ be a monomorphism and $e_0, e_1 : b \overset{m}{\to} f \star f$ be the coprojections into the pushout. Then f is the equalizer of e_0, e_1. To check the universal property, let $g : w \overset{m}{\to} b$ be an arbitrary morphism with $e_0 \circ g =_m e_1 \circ g$. We can show that g only hits elements in the image of f. Let $z \dot{\in} \|w\|$. The above equation yields $\langle 0, \overline{g}z\rangle \sim_{f\star f} \langle 1, \overline{g}z\rangle$. By construction of $(f \star f)_{\mathrm{E}}$ that means there exists some $x \dot{\in} \|a\|$ with $\overline{f}x \sim_b \overline{g}z$.

The morphism $h : w \overset{m}{\to} a$ is induced by $\lambda w.c_{AC_V}(f^{-1}\{w\})$. This is a unique function because f is monic. In particular this guarantees it doesn't matter what elements c_{AC_V} picks from a preimage. If we have $u \sim_w v$, we will get

$$\overline{h}u \sim_a c_{AC_V}(f^{-1}\{u\}) \sim_a c_{AC_V}(f^{-1}\{v\}) \sim_a \overline{h}v.$$

This shows that $g =_m f \circ h : w \overset{m}{\to} b$ with unique h as required. □

Theorem 3.2.37. *In* **EM** $+ (AC_V)$ *and classical logic, it is provable that* **ECB** *is a finitely cocomplete pretopos with a natural numbers object.*

Proof. A pretopos is by definition an exact and extensive category. Corollary 3.2.35 shows exactness, and corollary 3.2.26 proves extensiveness.
Proposition 3.2.20 which states that (assuming classical logic) arbitrary binary pushouts exist can be combined with the existence of an initial object (proposition 3.2.12), and binary coproducts (proposition 3.1.19) to get any finite colimit. Finally, proposition 3.2.28 provides a natural numbers object. □

Remark 3.2.38. The reason we give Beeson's q-realizability as a reference instead of, for example Feferman's [21], is that Feferman chose a procedure which requires ($\exists$,$\vee$)-free elementary formulas. But while this is fine for almost all other parts of this thesis, the above construction relies in an essential way on the existence quantifier. ◇

While it's not clear how to prove that the above is the only way of showing exactness, it gives a good idea why the construction in the next section really is different. In fact we can easily construct arbitrary equivalence relations, for which it is not clear how to construct a morphism into them to show they are a pullback:

Let $\mathbb{2}$ be the discrete Bishop set with carrier $\{0, 1\}$, and let q be the discrete Bishop set $\{a, b, c, d\}$.

Consider the following situation:

$$\begin{tikzcd}
\mathbb{2} \arrow[rrd, bend left, "i"] \arrow[rd, dotted, "\langle i,1-i\rangle"] \arrow[rdd, bend right, "1-i"'] & & \\
& \mathbb{2} \times \mathbb{2} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & \mathbb{2} \arrow[d, "i"] \\
& \mathbb{2} \arrow[r, "i"'] & \mathbb{2}/(0 \sim 1)
\end{tikzcd}$$

It's perfectly clear how to construct the unique morphism here, but suppose we have some injective morphism $f : q \overset{m}{\rightarrowtail} \mathbb{2} \times \mathbb{2}$. Of course the two different functions of the cone don't matter for the quotient, but they do matter for the map into $\mathbb{2} \times \mathbb{2}$. Most importantly, we have $(\mathit{refl}_{\mathbb{2}\times\mathbb{2}} \circ i) \neq \langle\!\langle i, 1 - i \rangle\!\rangle \neq (\mathit{refl}_{\mathbb{2}\times\mathbb{2}} \circ (1 - i))$. That, however, is the only function we can construct explicitly if we don't know how q looks. There exist 24 different equivalence relations $q_0, q_1 : q \overset{m}{\rightrightarrows} \mathbb{2}$. But even if we know a monomorphism f exists which commutes with the internal equivalence relations, and we are explicitly given refl_q, symm_q, and tran_q we still have no way to construct a morphism into q which makes the diagram commute.

The above is of course not a real counter example. There may be a possibility to make it into one, by a clever selection of the equivalence relation such verification of the universal property of pullbacks would yield some choice principle. Then we would still need a proof that principle gained from this is wrong in some model. For now this is an avenue for future research. So, while we can not prove that **ECB** is not exact in **EM**, we can give a construction of a category which is exact *without* relying on (AC_V) in the next section.

3.3. Exact Completion

As mentioned at the beginning of section 3.2, we can interpret Bishop's quote on sets such that we either show or construct something to describe sets. We have described the first version in the previous section, and we are now turning to explicit Bishop sets. We would like for these to be "the category of sets" in Explicit Mathematics, however it will turn out that there are some difficulties since explicit Bishop sets are, in some sense, *too well* behaved when used with the rest of our framework. For more explanation of this see section 5.2. The category **ECB** while designed to have quotients, turns out to be not enough. The actual construction we're going to use is of course not new at all, in fact is has been well-known for a long time. The following quote is from 1998 and even then it was not new. The main point here is then, that the same construction still goes through in Explicit Mathematics.

> Instead of diving into the diagrammatic definition of what a stable effective quotient is, [. . .] we shall try first to suggest which properties of the standard construction of a set of equivalence classes they single out. Given an equivalence relation $\sim$ on a set S , the quotient S/ $\sim$ is the smallest solution to the problem of identifying elements $s \sim s'$. The property (that one then checks in proving the factorization theorem for set-functions), namely that
>
> $$[x]_\sim = [x']_\sim \Longleftrightarrow x \sim x',$$
>
> can be restated as saying that the kernel equivalence relation induced by the canonical surjection S $\rightarrow$ S/ $\sim$ coincides with the given equivalence relation $\sim$. This makes a quotient of sets effective. Finally, it is a property of the logic that gives stability: any renaming of the equivalence classes $g : X \rightarrow \mathrm{S}/\sim$ is in bijection with the classes for an equivalence relation on $\{(x, s) \mid g(x) = [s]_\sim\}$.
>
> In fact, it is the failure to produce an exact category which suggests where to search for a category-theoretic explanation of the construction of categories of PERs. ([5])

The property described above, namely exactness, was introduced by Barr in [3]. It can be seen as one-half of the requirements to make a category abelian. Taken by itself it is still very useful since it implies that we can write down arbitrary equivalence relations in the internal language of a category and construct their quotient, while in regular categories only equivalence relations generated by a morphism are guaranteed to have quotients.

The following theorem stated in the same paper will serve as a short description of this section:

Theorem 3.3.1 ([5, 12, 10]). *If $\mathcal{C}$ has finite limits, then $\mathcal{C}_{ex}$ is exact and the assignment of the diagonal relation $x \mapsto x \overset{id}{\underset{id}{\rightrightarrows}} x$ to objects of $\mathcal{C}$ defines a full and faithful functor which preserves finite limits and is universal (in the 2-categorical sense) among limit preserving functors from $\mathcal{C}$ into an exact category.*

Note that we will *not* prove this in its entirety, only the part which states that the generated category is exact. In that sense the main result of this section which is proved completely in Explicit Mathematics is theorem 3.3.23.

Before we recall some the notions which we will need to use, let us point out similar work done in a different setting by Emmenegger and Palmgren [17, 16]. Of course properties of exact completions have been studied extensively by several people. See for example [12, 49, 10, 14, 13, 36].

Convention 3.3.2. For the rest of this section we will always assume that any arbitrary categories $\mathcal{C}, \mathcal{D}, \dots$ have all finite limits unless noted otherwise. ◇

Definition 3.3.3. An equivalence relation in the internal sense (a congruence) in a finitely complete category $\mathcal{C}$, is given by a subobject $(\partial_0, \partial_1) : r \overset{m}{\rightarrowtail} x \times x$ equipped with the following morphisms:

- internal reflexivity: $r : x \overset{m}{\to} r$ which is a section both of ∂_0 and of ∂_1. $(\partial_i \circ r =_m id(x))$
- internal symmetry: $s : r \overset{m}{\to} r$ which interchanges ∂_0 and ∂_1, namely $\partial_0 \circ s =_m \partial_1$ and $\partial_1 \circ s =_m \partial_0$;
- internal transitivity: A morphism $t : \partial_1 * \partial_0 \overset{m}{\to} r$ from the pullback of the second projection along the first;

$$\begin{tikzcd}
\partial_1 * \partial_0 \arrow[r, "pr_1"] \arrow[d, "pr_0"] & r \arrow[r, "\partial_1"] \arrow[d, "\partial_0"] & x \\
r \arrow[r, "\partial_1"] \arrow[d, "\partial_0"] & x & \\
x & &
\end{tikzcd}$$

Such that the following equations hold:

$$\partial_1 \circ pr_1 =_m \partial_1 \circ t$$
$$\partial_0 \circ pr_0 =_m \partial_0 \circ t$$

$$\begin{tikzcd}
 & & \partial_1 * \partial_0 \arrow[dl, "pr_1"'] \arrow[dr, "pr_0"] \arrow[dd, "t"] & & \\
 & r \arrow[dl, "\partial_1"'] & & r \arrow[dr, "\partial_0"] & \\
x & & r \arrow[ll, "\partial_1"'] \arrow[rr, "\partial_0"] & & x
\end{tikzcd}$$

◇

In particular, if we have a finitely complete category, every kernel-pair induces a congruence. This explains why regularity can be seen as a weak form of exactness. A regular category has at least those quotients of equivalence relations which were generated by a kernel-pair, while exact categories have all of them.

Proposition 3.3.4. *Let $f : a \xrightarrow{m} b$ any morphism in a finitely complete category and let $p_0, p_1 : f * f \xrightarrow{m} a$ be its kernel-pair. This induces a congruence.*

Proof. We need to show that $\langle\!\langle p_0, p_1 \rangle\!\rangle$ is monic, and that we have morphisms for reflexivity, symmetry and transitivity.

(a) Let $h, h' : c \xrightarrow{m} f * f$ with $\langle\!\langle p_0, p_1 \rangle\!\rangle \circ h =_m \langle\!\langle p_0, p_1 \rangle\!\rangle \circ h'$. We have $h_i, h'_i : c \xrightarrow{m} a$ as compositions of $p_i \circ h$ and $p_i \circ h'$. By assumption these give us two equal commuting diagrams with f. Therefore, the universal property of pullbacks shows that $h =_m h'$.

(b) **Reflexivity:** We have a cone over f :

$$\begin{tikzcd}
 & a \arrow[dl, "id(a)"'] \arrow[dr, "id(a)"] & \\
a & & a
\end{tikzcd}$$

Which gives us a morphism $refl_f$ such that $p_i \circ refl_f =_m id(a)$.

(c) **Symmetry:** We have a cone

$$\begin{tikzcd}
 & f * f \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
a & & a
\end{tikzcd}$$

which gives a morphism $symm_f : f * f \overset{m}{\to} f * f$ with $p_i \circ symm_f =_m p_{1-i}$.

(d) **transitivity:**

Let the following diagram be the pullback of the projections.

$$\begin{tikzcd}
p_1 * p_0 \arrow[r, "p'_1"] \arrow[d, "p'_0"'] & f * f \arrow[d, "p_0"] \\
f * f \arrow[r, "p_1"'] & a
\end{tikzcd}$$

We have a cone

$$\begin{tikzcd}
p_1 * p_0 \arrow[drr, bend left, "p_1 \circ p'_1"] \arrow[ddr, bend right, "p_0 \circ p'_0"'] \arrow[dr, dotted, "trans_f"] & & \\
 & f * f \arrow[r, "p_1"] \arrow[d, "p_0"'] & a \arrow[d, "f"] \\
 & a \arrow[r, "f"'] & b
\end{tikzcd}$$

To see why this commutes, consider the equations which just use the two commuting squares of the pullbacks.

$$\begin{aligned}
f \circ (p_1 \circ p'_1) =_m (f \circ p_1) \circ p'_1 &=_m (f \circ p_0) \circ p'_1 \\
&=_m f \circ (p_1 \circ p'_0) =_m (f \circ p_0) \circ p'_0.
\end{aligned}$$

This gives us a morphism $trans_f : p_1 * p_0 \overset{m}{\to} f * f$ such that

$$p_i \circ p'_i =_m p_i \circ trans_f \qquad \square$$

Definition 3.3.5. A pseudo-equivalence relation is a parallel pair of morphisms $r \underset{r_1}{\overset{r_0}{\rightrightarrows}} x$ which is reflexive, symmetric and transitive, but not necessarily jointly monic. This means, that objects are tuples $\langle r_0, r_1, r, s, t\rangle$ where the morphisms r, s and t do not have to be uniquely determined. $\Diamond$

Definition 3.3.6. The objects of $\mathcal{C}_{\mathrm{ex}}$ are pseudo-equivalence relations in $\mathcal{C}$. The morphisms between $r \underset{r_1}{\overset{r_0}{\rightrightarrows}} x$ and $s \underset{s_1}{\overset{s_0}{\rightrightarrows}} y$ are pairs of morphisms $\langle f, f'\rangle$ in $\mathcal{C}$ such that

$$\begin{tikzcd}
r \arrow[r, "f'"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
x \arrow[r, "f"] & y
\end{tikzcd}$$

commutes component-wise: $f \circ r_i =_m s_i \circ f'$ for $i = 0, 1$. Two parallel morphisms $\langle f, f'\rangle, \langle g, g'\rangle : x \overset{m}{\to} y$ are the same, if we have a morphism $\gamma : x \overset{m}{\to} s$ such that

$$\begin{tikzcd}
r \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
x \arrow[ur, "\gamma"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & y
\end{tikzcd}$$

with $f =_m s_0 \circ \gamma$ and $g =_m s_1 \circ \gamma$.

We will show below that this is an equivalence relation compatible with composition. $\Diamond$

Definition 3.3.7. We formalize these definitions as follows. Let $\mathcal{C}$ be the category $\langle ob_{\mathcal{C}}, mor_{\mathcal{C}}, =^{\mathcal{C}}_o, =^{\mathcal{C}}_m, id_{\mathcal{C}}, \circ^{\mathcal{C}}\rangle$.

$$\begin{aligned}
OB[u, \mathcal{C}] :\equiv u = \langle r_0, r_1, r, s, t\rangle \wedge r_0, r_1, r, s, t \,\dot{\in}\, mor_{\mathcal{C}} \\
\wedge\, dom(r_0) =_o dom(r_1) \wedge cod(r_0) =_o cod(r_1) \\
\wedge\, r : cod(r_0) \overset{m}{\to} dom(r_0) \\
\wedge\, r_0 \circ^{\mathcal{C}} r =^{\mathcal{C}}_m id_{\mathcal{C}}(cod(r_0)) \\
\wedge\, r_1 \circ^{\mathcal{C}} r =^{\mathcal{C}}_m id_{\mathcal{C}}(cod(r_0))
\end{aligned}$$

$$
\begin{aligned}
&\wedge\ s : \mathit{dom}(r_0) \overset{m}{\to} \mathit{dom}(r_0)\\
&\wedge\ r_0 \circ^{\mathcal{C}} s =^{\mathcal{C}}_m r_1 \wedge r_1 \circ^{\mathcal{C}} s =^{\mathcal{C}}_m r_0\\
&\wedge\ \exists pr_0, pr_1(\\
&\quad \mathit{PULLBACK}[\mathit{pullback}(r_1, r_0), r_1, pr_0, pr_1]\\
&\quad \wedge\ t : \mathit{pullback}(r_1, r_0) \overset{m}{\to} \mathit{dom}(r_0)\\
&\quad \wedge\ r_0 \circ^{\mathcal{C}} t =^{\mathcal{C}}_m r_0 \circ pr_0 \wedge r_1 \circ^{\mathcal{C}} t =^{\mathcal{C}}_m r_1 \circ pr_1)\\
\mathit{EQO}[u, ob, \mathcal{C}] :\equiv{}& u = \langle a, b\rangle \wedge a, b \mathrel{\dot\in} ob\\
&\quad \wedge\ \pi_0 a =^{\mathcal{C}}_m \pi_0 b \wedge \pi_1 a =^{\mathcal{C}}_m \pi_1 b\\
&\quad \wedge\ \pi_2 a =^{\mathcal{C}}_m \pi_2 b \wedge \pi_3 a =^{\mathcal{C}}_m \pi_3 b \wedge \pi_4 a =^{\mathcal{C}}_m \pi_4 b\\
\mathit{MOR}[u, ob, \mathcal{C}] :\equiv{}& u = \langle a, b, \langle f, g\rangle\rangle \wedge a, b \mathrel{\dot\in} ob \wedge f, g \mathrel{\dot\in} \mathit{mor}_{\mathcal{C}}\\
&\wedge\ \mathit{dom}(f) =^{\mathcal{C}}_o \mathit{cod}(\pi_0 a) \wedge \mathit{cod}(f) =^{\mathcal{C}}_o \mathit{cod}(\pi_0 b)\\
&\wedge\ \mathit{dom}(g) =^{\mathcal{C}}_o \mathit{dom}(\pi_0 a) \wedge \mathit{cod}(g) =^{\mathcal{C}}_o \mathit{dom}(\pi_0 b)\\
&\wedge\ f \circ^{\mathcal{C}} (\pi_0 a) =^{\mathcal{C}}_m (\pi_0 b) \circ g\\
&\wedge\ f \circ^{\mathcal{C}} (\pi_1 a) =^{\mathcal{C}}_m (\pi_1 b) \circ g\\
\mathit{EQM}[u, mor, \mathcal{C}] :\equiv{}& u = \langle f, g\rangle \wedge f, g \mathrel{\dot\in} mor\\
&\wedge\ \mathit{dom}(f) =_o \mathit{dom}(g) \wedge \mathit{cod}(f) =_m \mathit{cod}(g)\\
&\wedge\ \exists \gamma(\gamma \mathrel{\dot\in} \mathit{mor}_{\mathcal{C}}\\
&\quad \wedge\ \mathit{dom}(\gamma) =^{\mathcal{C}}_o \mathit{cod}(\pi_0(\overline{f}))\\
&\quad \wedge\ \mathit{cod}(\gamma) =^{\mathcal{C}}_o \mathit{dom}(\pi_0(\overline{f}))\\
&\quad (\pi_0 \mathit{cod}(f)) \circ^{\mathcal{C}} \gamma =^{\mathcal{C}}_m \pi_0(\overline{f})\\
&\quad (\pi_1 \mathit{cod}(f)) \circ^{\mathcal{C}} \gamma =^{\mathcal{C}}_m \pi_0(\overline{g}))\\
\mathit{id}(a) :\equiv{}& \langle a, a, \mathit{id}_{\mathcal{C}}(\mathit{cod}(\pi_0 a)), \mathit{id}_{\mathcal{C}}(\mathit{dom}(\pi_0 a))\rangle\\
\circ(f, g) :\equiv{}& \langle \mathit{dom}(g), \mathit{cod}(a), \langle \pi_0 f \circ_{\mathcal{C}} \pi_0 f, \pi_1 f \circ_{\mathcal{C}} \pi_1 f\rangle\rangle
\end{aligned}
$$

Clearly this definition is rather unwieldy, which is why we will mostly use somewhat informal abbreviations:

$$
\begin{aligned}
\langle f, f' \rangle : a \overset{m}{\to} b &:\equiv \langle a, b, \langle f, f' \rangle\rangle : a \overset{m}{\to} b \\
r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x \ \ &:\equiv \langle r_0, r_1, refl_{rx}, symm_{rx}, trans_{rx} \rangle \\
&\quad \text{for } \langle r_0, r_1, refl_{rx}, symm_{rx}, trans_{rx} \rangle \mathrel{\dot{\in}} ob_{\mathcal{C}_{\text{ex}}} \\
&\text{with } r_i =^{\mathcal{C}}_m \langle r, x, \overline{(r_i)} \rangle \mathrel{\dot{\in}} mor_{\mathcal{C}} \qquad \diamond
\end{aligned}
$$

We would like to give a direct construction of a *homset* to use in the exact completion of **EC** which is intended as our category of *Sets*, i.e. the category of explicit Bishop sets. As already meantioned in the introduction of this section, we will see that this is not as widely applicable as we might hope. For example for the category $\mathbf{EC}_{\text{ex}}$ itself (restricted to some universe), we need particular properties which we could not reconstruct directly. Furthermore, section 5.2 gives reasons why the whole approach might not be ideal. Still, it is possible to define such an explicit Bishop set, even if it will turn out to be not as well-behaved as we would wish.

Definition 3.3.8 (The *homset*)**.** Let $\langle ob_{\mathcal{C}}, mor_{\mathcal{C}}, =^{\mathcal{C}}_o, =^{\mathcal{C}}_m, id_{\mathcal{C}}, \circ^{\mathcal{C}} \rangle$ be any category and $a, b \mathrel{\dot{\in}} ob_{\mathcal{C}}$.

$$
\begin{aligned}
H[u, a, b, O, M, =^{\mathcal{C}}_o] &:\equiv u \in M \wedge dom(u) =^{\mathcal{C}}_o a \wedge cod(u) =^{\mathcal{C}}_o b \\
R[u, a, b, O, M, =^{\mathcal{C}}_o, =^{\mathcal{C}}_m] &:\equiv \exists f, g(\langle f, g \rangle = u \wedge f, g \in M \\
&\qquad\qquad \wedge dom(f) =^{\mathcal{C}}_o a \wedge cod(f) =^{\mathcal{C}}_o b \wedge f =^{\mathcal{C}}_m g \\
r(a, b) &:\equiv t_R(a, b, ob_{\mathcal{C}}, mor_{\mathcal{C}}, =^{\mathcal{C}}_o, =^{\mathcal{C}}_m) \\
h(a, b) &:\equiv t_H(a, b, ob_{\mathcal{C}}, mor_{\mathcal{C}}, =^{\mathcal{C}}_o) \\
h_{ab0} &:\equiv \langle r(a, b), h(a, b), \pi_0 \rangle \\
h_{ab1} &:\equiv \langle r(a, b), h(a, b), \pi_1 \rangle \\
refl_{abrh} &:\equiv \langle h(a, b), r(a, b), \lambda f.\langle f, f \rangle\rangle \\
symm_{abrh} &:\equiv \langle r(a, b), r(a, b), \lambda p.\langle \pi_1 p, \pi_0 p \rangle\rangle \\
trans_{abrh} &:\equiv \langle pullback(h_{ab1}, h_{ab0}), r(a, b), \\
&\qquad \lambda p.\langle \pi_0(\pi_0 p), \pi_{(}\pi_1 p)\rangle\rangle \\
hom_{\mathcal{C}}(a, b) &:\equiv \langle h_{ab0}, h_{ab1}, refl_{abrh}, symm_{abrh}, trans_{abrh} \rangle
\end{aligned}
$$

Since we already have equivalence relations as part of the definition of a category, we can just reuse them. So for any map $f : h(a,b) \overset{m}{\to} h(u,v)$ between the classes used as homsets, we just have to show that it respects the equivalence relations. Then we get a lifting to the equivalence relations $f \times f : r(a,b) \overset{m}{\to} r(u,v)$ and hence a morphism $\langle f, f \times f \rangle : hom_{\mathcal{C}}(a,b) \overset{m}{\to} hom_{\mathcal{C}}(u,v)$ in $\mathbf{EC}_{\text{ex}}$. ◇

Example 3.3.9. Let $f : a \overset{m}{\to} b$ be a morphism in some category $\mathcal{C}$.

$$y_{\mathbf{EC}}(c)(f) :\equiv \langle h(b,c), h(a,c), \lambda h.h \circ f \rangle$$

then we can extend this to

$$\begin{aligned} y_m(c)(f) :\equiv \langle \overline{y_{\mathbf{EC}}(f)}, \overline{y_{\mathbf{EC}}(f)} \times \overline{y_{\mathbf{EC}}(f)} \rangle \\ : hom_{\mathcal{C}}(b,c) \overset{m}{\to} hom_{\mathcal{C}}(a,c) \end{aligned}$$

We can do this, because $g =^{\mathcal{C}}_m h$ implies $g \circ f =^{\mathcal{C}}_m h \circ f$. ◇

We will now get back to establishing properties which hold in $\mathcal{C}_{\text{ex}}$ for arbitrary finitely complete categories $\mathcal{C}$:

Proposition 3.3.10. *Any two parallel morphisms $\langle f, u \rangle, \langle f, w \rangle$ between two objects $r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x$ and $s \overset{s_0}{\underset{s_1}{\rightrightarrows}} y$ in $\mathcal{C}_{ex}$ are equal.*

Proof. By definition the equation

$$s_i \circ refl_{sy} =_m id(y)$$

holds, so we can use the witness $\gamma :\equiv (refl_{sy} \circ f)$ to get $s_i \circ \gamma =_m f$. □

Proposition 3.3.11. *Equality of morphisms in the exact completion is symmetric.*

Proof. Let

$$\langle f, f' \rangle, \langle g, g' \rangle : \left(r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x \right) \overset{m}{\to} \left(s \overset{r_0}{\underset{r_1}{\rightrightarrows}} y \right)$$

be two morphisms with some $\gamma : x \overset{m}{\to} s$ witnessing $\langle f, f' \rangle =_m \langle g, g' \rangle$. So

$$
\begin{aligned}
&s_0 \circ \gamma =_m f \\
&s_1 \circ \gamma =_m g.
\end{aligned}
$$

Then

$$
\begin{aligned}
&s_0 \circ sym_s \circ \gamma =_m s_1 \circ \gamma =_m g \\
&s_1 \circ sym_s \circ \gamma =_m s_0 \circ \gamma =_m f
\end{aligned}
$$

and $(sym_s \circ \gamma)$ is a witness of $\langle g, g' \rangle =_m \langle f, f' \rangle$ ☐

Proposition 3.3.12. *Equality of morphisms in the exact completion is transitive.*

Proof. Let $\langle f, f' \rangle, \langle g, g' \rangle, \langle h, h' \rangle : (\ r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x\ ) \overset{m}{\to} (\ s \overset{r_0}{\underset{r_1}{\rightrightarrows}} y\ )$ be three morphisms with some $\gamma, \delta : x \overset{m}{\to} s$ witnessing $\langle f, f' \rangle =_m \langle g, g' \rangle =_m, \langle h, h' \rangle$. So

$$
\begin{aligned}
&s_0 \circ \gamma =_m f && s_1 \circ \gamma =_m g \\
&s_0 \circ \delta =_m g && s_1 \circ \delta =_m h.
\end{aligned}
$$

Consider the pullback

$$\begin{tikzcd}
s * s \arrow[r, "\overline{s}_1"] \arrow[d, "\overline{s}_0"'] & s \arrow[d, "s_0"] \\
s \arrow[r, "s_1"'] & y
\end{tikzcd}$$

We get a cone $$\begin{tikzcd}
& x \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\
s & & s
\end{tikzcd}$$ and hence a morphism $h : x \overset{m}{\to} s * s$ with $\overline{s}_0 \circ h =_m \gamma$ and $\overline{s}_1 \circ h =_m \delta$. But then we can apply transitivity of s to get.

$$
\begin{aligned}
&s_0 \circ trans_s \circ h =_m s_0 \circ \overline{s}_0 \circ h =_m s_0 \circ \gamma =_m f \\
&s_1 \circ trans_s \circ h =_m s_1 \circ \overline{s}_1 \circ h =_m s_1 \circ \delta =_m h.
\end{aligned}
$$

This makes $(trans_s \circ h) : x \overset{m}{\to} s$ into a witness for $\langle f, f' \rangle =_m \langle h, h' \rangle$. ☐

Proposition 3.3.13. *Equality of morphisms in the exact completion is compatible with composition.*

Proof. Let $\langle f, f' \rangle, \langle g, g' \rangle : (\; r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x \;) \overset{m}{\to} (\; s \overset{r_0}{\underset{r_1}{\rightrightarrows}} y \;)$, $\langle k, k' \rangle, \langle l, l' \rangle : (\; s \overset{s_0}{\underset{s_1}{\rightrightarrows}} y \;) \overset{m}{\to} (\; t \overset{t_0}{\underset{t_1}{\rightrightarrows}} z \;)$ be four morphisms with some $\gamma : x \overset{m}{\to} s$, $\delta : y \overset{m}{\to} t$ witnessing $\langle f, f' \rangle =_m \langle g, g' \rangle$ and $\langle k, k' \rangle =_m \langle l, l' \rangle$.

$$\begin{tikzcd}
r \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & t \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
x \arrow[ur, "\gamma"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & y \arrow[ur, "\delta"] \arrow[r, shift left, "k"] \arrow[r, shift right, "l"'] & z
\end{tikzcd}$$

So we have

$$s_0 \circ \gamma =_m f \qquad s_1 \circ \gamma =_m g$$
$$t_0 \circ \delta =_m k \qquad t_1 \circ \delta =_m l.$$

and we want to show $\langle k, k' \rangle \circ \langle f, f' \rangle =_m \langle l, l' \rangle \circ \langle g, g' \rangle$. From compositions of both sides we calculate

$$t_0 \circ \delta \circ g =_m k \circ g =_m k \circ s_1 \circ \gamma =_m t_1 \circ k' \circ \gamma$$

Consider the pullback

$$\begin{tikzcd}
t * t \arrow[r, "\bar{t}_1"] \arrow[d, "\bar{t}_0"'] & s \arrow[d, "t_0"] \\
s \arrow[r, "t_1"'] & y.
\end{tikzcd}$$

With the above equation we get a cone
$$\begin{tikzcd}
& x \arrow[dl, "k' \circ \gamma"'] \arrow[dr, "\delta \circ g"] & \\
s & & s
\end{tikzcd}$$
which provides us with a morphism $h : x \overset{m}{\to} t * t$ such that $\bar{t}_0 \circ h =_m k' \circ \gamma$ and $\bar{t}_1 \circ h =_m \delta \circ g$.

But that means

$$\begin{aligned}
&t_0 \circ trans_t \circ h =_m t_0 \circ \bar{t}_0 \circ h =_m t_0 \circ k' \circ \gamma =_m k \circ s_0 \circ \gamma =_m k \circ f \\
&t_1 \circ trans_t \circ h =_m t_1 \circ \bar{t}_1 \circ h =_m t_1 \circ \delta \circ g =_m l \circ g
\end{aligned}$$

and so $trans_t \circ h : x \overset{m}{\rightarrow} t$ is the witness we were looking for. If we set $\tilde{\delta} :\equiv \delta \circ sym_t$, the proof for $\langle l, l' \rangle \circ \langle f, f' \rangle =_m \langle k, k' \rangle \circ \langle g, g' \rangle$ is up to relabeling exactly the same. ☐

Finite Limits in the Exact Completion

Proposition 3.3.14. *$\mathcal{C}_{ex}$ has a terminal object.*

Proof. Let $\mathbb{1}$ be the terminal in $\mathcal{C}$ and $!_x : x \overset{m}{\rightarrow} \mathbb{1}$ the unique morphism. Then $\mathbb{1} \underset{id(\mathbb{1})}{\overset{id(\mathbb{1})}{\rightrightarrows}} \mathbb{1}$ is terminal in $\mathcal{C}_{\mathrm{ex}}$. The reflexivity, symmetry, and transitivity maps are all trivial and given any object $r \underset{r_1}{\overset{r_0}{\rightrightarrows}} x$ we have the morphism

$$\langle !_r, !_x \rangle : (\; r \underset{r_1}{\overset{r_0}{\rightrightarrows}} x \;) \overset{m}{\rightarrow} (\; \mathbb{1} \underset{id(\mathbb{1})}{\overset{id(\mathbb{1})}{\rightrightarrows}} \mathbb{1} \;).$$

Uniqueness is clear as there can be only one morphism to $\mathbb{1}$ in $\mathcal{C}$ and so $!_x$ is a homotopy between $\langle !_r, !_x \rangle$ and any other morphism between these objects. ☐

Theorem 3.3.15 ([49])**.** *$\mathcal{C}_{ex}$ has all binary pullbacks.*

Proof. Let

$$\begin{tikzcd}
r \arrow[r, "f'"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & t \arrow[l, "g'"'] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
x \arrow[r, "f"] & y & z \arrow[l, "g"']
\end{tikzcd}$$

be two morphisms.

$\mathcal{C}$ has all finite limits so we can consider limits for the following index categories.

$$\begin{tikzcd}
\bullet \arrow[d] \arrow[dr] & \bullet \arrow[dl] \arrow[dr] & \bullet \arrow[dl] \arrow[dr] & \bullet \arrow[dl] \arrow[d] & & \bullet \arrow[d] & & \bullet \arrow[d] \\
\bullet & \bullet & \bullet & \bullet & & \bullet & \bullet \arrow[l] \arrow[r] & \bullet
\end{tikzcd}$$

Let p be given by the following limit:

$$\begin{tikzcd}
x \arrow[d, "f"'] & p \arrow[l, "p_0"'] \arrow[r, "p_1"] \arrow[d, "\varphi"] & z \arrow[d, "g"] \\
y & s \arrow[l, "s_0"] \arrow[r, "s_1"'] & y
\end{tikzcd}$$

Let e be the limit of the lower part of the following diagram. (As usual we omit the operations for the cone and into e.)

$$\begin{tikzcd}
 & & & e \arrow[dlll, "\delta"] \arrow[dl, "e_0"] \arrow[dr, "e_1"] \arrow[drrr, "\tau"] & & & \\
r \arrow[d, "r_0"] \arrow[drr, "r_1"] & & p \arrow[dll, "p_0"] \arrow[drr, "p_1"] & & p \arrow[dll, "p_0"] \arrow[drr, "p_1"] & & t \arrow[dll, "t_0"] \arrow[d, "t_1"] \\
x & & x & & z & & z
\end{tikzcd}$$

From this we can construct the required pseudo-equivalence relation

$$\begin{tikzcd}
e \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] & p
\end{tikzcd} .$$

Reflexivity is given by the morphism $r : p \overset{m}{\to} e$ induced by the cone

$$\begin{tikzcd}
 & & & p \arrow[dlll, "refl_{rx} \circ p_0"'] \arrow[dl, "id(p)"] \arrow[dr, "id(p)"'] \arrow[drrr, "refl_{tz} \circ p_1"] & & & \\
r & & p & & p & & t
\end{tikzcd}$$

over the diagram defining e. This works, because

$$r_i \circ refl_{rx} \circ p_0 =_m id(x) \circ p_0 =_m p_0 \circ id(p)$$
$$t_i \circ refl_{tz} \circ p_1 =_m id(x) \circ p_1 =_m p_1 \circ id(p)$$

and it is reflexivity, since universality implies in particular $e_i \circ r =_m id(p)$.

Symmetry is defined in a similar way by the morphism $s : e \overset{m}{\to} e$ induced by the cone

$$\begin{tikzcd}
 & & & e \arrow[dlll, "symm_{rx}\circ\delta"'] \arrow[dl, "e_1"] \arrow[dr, "e_0"'] \arrow[drrr, "symm_{tz}\circ\tau"] & & & \\
r & & p & & p & & t
\end{tikzcd}$$

over the diagram defining e. This works, because

$$r_0 \circ symm_{rx} \circ \delta =_m r_1 \circ \delta =_m e_1 \circ p_0$$
$$r_1 \circ symm_{rx} \circ \delta =_m r_0 \circ \delta =_m e_0 \circ p_0$$

and similarly for t_i. This means that $e_i \circ s =_m e_{1-i}$ as required.

Transitivity requires some more verification. Consider the pullbacks:

$$\begin{tikzcd}
r * r \arrow[r, "\overline{r}_1"] \arrow[d, "\overline{r}_0"] \arrow[dr, phantom, "\lrcorner", very near start] & r \arrow[d, "r_0"] & e * e \arrow[r, "\overline{e}_1"] \arrow[d, "\overline{e}_0"] \arrow[dr, phantom, "\lrcorner", very near start] & e \arrow[d, "e_0"] & t * t \arrow[r, "\overline{t}_1"] \arrow[d, "\overline{t}_0"] \arrow[dr, phantom, "\lrcorner", very near start] & t \arrow[d, "t_0"] \\
r \arrow[r, "r_1"] & x & e \arrow[r, "e_1"] & p & t \arrow[r, "t_1"] & z
\end{tikzcd}$$

We get that $r \xleftarrow{\delta} e \xleftarrow{\overline{e}_0} e * e \xrightarrow{\overline{e}_1} e \xrightarrow{\delta} r$ is a cone over $r * r$ and hence a morphism $\overline{\delta} : e * e \overset{m}{\to} r * r$ with $\overline{r}_i \circ \overline{\delta} =_m \delta \circ \overline{e}_i$ since

$$r_1 \circ \delta \circ \overline{e}_0 =_m p_0 \circ e_1 \circ \overline{e}_0 =_m p_0 \circ e_0 \circ \overline{e}_1 =_m r_0 \circ \delta \circ \overline{e}_1.$$

Analogously we have $\overline{\tau} : e * e \overset{m}{\to} t * t$ with $\overline{t}_i \circ \overline{\tau} =_m \tau \circ \overline{e}_i$. From this we can construct a cone over the diagram defining e :

$$\begin{tikzcd}
 & & e * e \arrow[lld, "trans_{rx}\circ\overline{\delta}"'] \arrow[ld, "e_0\circ\overline{e}_0"] \arrow[rd, "e_1\circ\overline{e}_1"'] \arrow[rrd, "trans_{tz}\circ\overline{\tau}"] & & \\
r & p & & p & t.
\end{tikzcd}$$

Verification of one case:

$$r_0 \circ trans_{rx} \circ \overline{\delta} =_m r_0 \circ \overline{r}_0 \circ \overline{\delta} =_m r_0 \circ \delta \circ \overline{e}_0 =_m p_0 \circ e_0 \circ \overline{e}_0$$

$$r_1 \circ trans_{rx} \circ \overline{\delta} =_m r_1 \circ \overline{r}_1 \circ \overline{\delta} =_m r_1 \circ \delta \circ \overline{e}_1 =_m p_0 \circ e_1 \circ \overline{e}_1.$$

Now we have a universal morphism $t : e * e \overset{m}{\to} e$ with $e_i \circ t =_m e_i \circ \overline{e}_i$. This shows that $e \overset{e_0}{\underset{e_1}{\rightrightarrows}} p$ is a pseudo-equivalence.

Going back to the main part of the proof and putting everything together, we have

$$\begin{tikzcd}
s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & t \arrow[l, "g'"'] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] & e \arrow[l, "\tau"'] \arrow[r, "\delta"] \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] & r \arrow[r, "f'"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
y & z \arrow[l, "g"] & p \arrow[l, "p_1"] \arrow[r, "p_0"'] & x \arrow[r, "f"'] & y
\end{tikzcd}$$

To show that this commutes, we need a homotopy $\gamma : p \overset{m}{\to} s$. But the canonical morphism φ is exactly such a witness that

$$\langle g, g' \rangle \circ \langle p_1, \tau \rangle =_m \langle f, f' \rangle \circ \langle p_0, \delta \rangle.$$

To verify the universal property, suppose we have some

$$\begin{tikzcd}
r \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & h \arrow[l, "l'"] \arrow[r, "k'"] \arrow[d, shift right, "h_0"'] \arrow[d, shift left, "h_1"] & t \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
x & c \arrow[l, "l"] \arrow[r, "k"] & z
\end{tikzcd}$$

with $\langle f, f' \rangle \circ \langle l, l' \rangle =_m \langle g, g' \rangle \circ \langle k, k' \rangle$. That means there is some γ :

$c \overset{m}{\to} s$ with $s_0 \circ \gamma =_m f \circ l$ and $s_1 \circ \gamma =_m g \circ k$. That gives us a cone

$$\begin{tikzcd}
 & c \arrow[dl, "l"'] \arrow[d, "\gamma"] \arrow[dr, "k"] & \\
x & s & z
\end{tikzcd}$$

over p and hence a morphism $u : c \overset{m}{\to} p$ such that $p_0 \circ u =_m l$ and $p_1 \circ u =_m k$. To construct the second part, consider

$$\begin{tikzcd}
 & & h \arrow[dll, "l'"'] \arrow[dl, "u\circ h_0"] \arrow[dr, "u\circ h_1"'] \arrow[drr, "k'"] & & \\
r & p & & p & t
\end{tikzcd}$$

This is a cone over e since we have

$$\begin{aligned}
r_0 \circ l' &=_m l \circ h_0 =_m p_0 \circ u \circ h_0 \\
r_1 \circ l' &=_m l \circ h_1 =_m p_0 \circ u \circ h_1 \\
t_0 \circ k' &=_m k \circ h_0 =_m p_1 \circ u \circ h_0 \\
t_1 \circ k' &=_m k \circ h_1 =_m p_1 \circ u \circ h_1.
\end{aligned}$$

This gives a morphism $q : h \overset{m}{\to} e$ in $\mathcal{C}$ with $u \circ h_i =_m e_i \circ q$ .and so a morphism $\langle u, q \rangle$ in $\mathcal{C}_{\text{ex}}$.

$$\begin{tikzcd}
h \arrow[r, "q"] \arrow[d, shift right, "h_0"'] \arrow[d, shift left, "h_1"] & e \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] \\
c \arrow[r, "u"] & p
\end{tikzcd}$$

We need morphisms $c \overset{m}{\to} r$ and $c \overset{m}{\to} t$ showing equalities $\langle p_0, \delta \rangle \circ \langle u, q \rangle =_m \langle l, l' \rangle$ and $\langle p_1, \tau \rangle \circ \langle u, q \rangle =_m \langle k, k' \rangle$. But as show above, any two morphisms with equal first components are homotopic. But we have

$$p_0 \circ u =_m l$$
$$p_1 \circ u =_m k$$

And so $refl_E \circ p_i \circ u$ provides the required homotopies.

Finally, we have to show that $\langle p_0, \delta\rangle$ and $\langle p_1, \tau\rangle$ are a monomorphic pair. This implies the uniqueness of $\langle u, q\rangle$.

Let $\langle a, a'\rangle, \langle b, b'\rangle : (\ h \rightrightarrows c\ ) \xrightarrow{m} (\ e \rightrightarrows p\ )$ be two morphisms with $\langle p_0, \delta\rangle \circ \langle a, a'\rangle =_m \langle p_0, \delta\rangle \circ \langle b, b'\rangle$ and $\langle p_1, \tau\rangle \circ \langle a, a'\rangle =_m \langle p_1, \tau\rangle \circ \langle b, b'\rangle$. There exist homotopies $\alpha : c \xrightarrow{m} r$ and $\beta : c \xrightarrow{m} t$ with

$$r_0 \circ \alpha =_m p_0 \circ a \qquad r_1 \circ \alpha =_m p_0 \circ b$$
$$t_0 \circ \beta =_m p_1 \circ a \qquad t_1 \circ \beta =_m p_1 \circ b$$

This is a cone over the diagram of e

$$\begin{tikzcd}
& & h \arrow[lld, "\alpha"'] \arrow[ld, "a"] \arrow[rd, "b"'] \arrow[rrd, "\beta"] & & \\
r & p & & p & t
\end{tikzcd}$$

and so we get $\epsilon : h \xrightarrow{m} e$ with $e_0 \circ \epsilon =_m a$ and $e_1 \circ \epsilon =_m b$ and so the constructed morphism is unique. This concludes the proof of existence of pullbacks. □

Theorem 3.3.16. *[49] $\mathcal{C}_{ex}$ has equalizers.*

Proof. Consider two morphisms $\langle f, f'\rangle, \langle g, g'\rangle : (\ r \underset{r_1}{\overset{r_0}{\rightrightarrows}} x\ ) \xrightarrow{m} (\ s \underset{s_1}{\overset{s_0}{\rightrightarrows}} y\ )$. We construct the limit over the diagram

$$\begin{tikzcd}
x \arrow[d, "f"'] \arrow[rd, "g" near start] & s \arrow[d, "s_1"] \arrow[ld, "s_0"' near start] \\
y & y
\end{tikzcd}$$

as

$$\begin{tikzcd}
 & q \arrow[dl, "e"'] \arrow[dr, "\varphi"] & \\
x \arrow[d, "f"'] \arrow[ddrr, phantom] \arrow[drr, "g"] & & s \arrow[d, "s_1"] \arrow[dll, "s_0"'] \\
y & & y
\end{tikzcd}$$

and the limit over $\begin{tikzcd} q \arrow[d, "e"'] & & q \arrow[d, "e"] \\ x & r \arrow[l, "r_0"] \arrow[r, "r_1"'] & x \end{tikzcd}$ as

$$\begin{tikzcd}
q \arrow[d, "e"'] & h \arrow[l, "h_0"'] \arrow[r, "h_1"] \arrow[d, "e'"] & q \arrow[d, "e"] \\
x & r \arrow[l, "r_0"] \arrow[r, "r_1"'] & x
\end{tikzcd}$$

The equalizer is then given by

$$\begin{tikzcd}
h \arrow[r, "e'"] \arrow[d, shift right, "h_0"'] \arrow[d, shift left, "h_1"] & r \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] \\
q \arrow[r, "e"] & x.
\end{tikzcd}$$

The rest of the proof is similar to the case for pullbacks.

Reflexivity is given by the morphism $r_{hq} : q \overset{m}{\to} h$ induced by the cone

$$\begin{tikzcd}
 & q \arrow[dl, "id(q)"'] \arrow[d, "refl_{rx} \circ e"'] \arrow[dr, "id(q)"] & \\
q & r & q
\end{tikzcd}$$

which has the property $h_i \circ r_{hq} =_m id(q)$.

Symmetry is given by the morphism $s_{hq} : h \overset{m}{\to} h$ induced by the cone

$$\begin{tikzcd}
 & h \arrow[dl, bend right, "h_1"'] \arrow[d, "symm_{rx} \circ e'"'] \arrow[dr, bend left, "h_0"] & \\
q & r & q
\end{tikzcd}$$

which has the property $h_i \circ s_{hq} =_m h_{1-i}$.

Transitivity requires again a bit more work.

Let

$$\begin{tikzcd}
h * h \arrow[r, "\overline{h}_1"] \arrow[d, "\overline{h}_0"] \arrow[dr, phantom, "\lrcorner", very near start] & h \arrow[d, "h_0"] \\
h \arrow[r, "h_1"] & q
\end{tikzcd}
\qquad
\begin{tikzcd}
r * r \arrow[r, "\overline{r}_1"] \arrow[d, "\overline{r}_0"] \arrow[dr, phantom, "\lrcorner", very near start] & r \arrow[d, "r_0"] \\
r \arrow[r, "r_1"] & x
\end{tikzcd}$$

be the pullbacks of h_0 along h_1 and r_0 along r_1. We have

$$(r_0 \circ e') \circ \overline{h}_1 =_m (e \circ h_0) \circ \overline{h}_1 =_m e \circ (h_0 \circ \overline{h}_1) =_m e \circ (h_1 \circ \overline{h}_0) =_m (r_1 \circ e') \circ \overline{h}_0$$

and so get a morphism $\overline{e}' : h * h \overset{m}{\to} r * r$ with $\overline{r}_i \circ \overline{e}' =_m e' \circ \overline{h}_i$.

We can use this to construct a cone over the diagram defining h

$$\begin{tikzcd}
 & h * h \arrow[dl, bend right, "h_0 \circ \overline{h}_0"'] \arrow[d, "trans_{rx} \circ \overline{e}'"] \arrow[dr, bend left, "h_1 \circ \overline{h}_1"] & \\
q & r & q
\end{tikzcd}$$

This commutes, because we have

$$e \circ h_i \circ \overline{h}_i =_m r_i \circ e' \circ \overline{h}_i =_m r_i \circ \overline{r}_i \circ \overline{e}' =_m r_i \circ trans_{rx} \circ \overline{e}'.$$

Since we get from this some $t_{hq} : h * h \overset{m}{\to} h$ with $h_i \circ t_{hq} =_m h_i \circ \overline{h}_i$, we have now shown that $h \underset{h_1}{\overset{h_0}{\rightrightarrows}} q$ is a pseudo-equivalence relation.

Returning to the main part of the proof, $\varphi : q \overset{m}{\to} s$ provides the witness, that $\langle f, f' \rangle \circ \langle e, e' \rangle =_m \langle g, g' \rangle \circ \langle e, e' \rangle$.

Suppose there is any other $\langle p, p' \rangle : (\ t \underset{t_1}{\overset{t_0}{\rightrightarrows}} z\ ) \overset{m}{\to} (\ r \underset{r_1}{\overset{r_0}{\rightrightarrows}} x\ )$ with $\gamma : z \overset{m}{\to} s$ witnessing $\langle f, f' \rangle \circ \langle p, p' \rangle =_m \langle g, g' \rangle \circ \langle p, p' \rangle$.

This directly gives a cone over the diagram defining q:

$$\begin{tikzcd}
 & z \arrow[dl, "p"'] \arrow[dr, "\gamma"] & \\
x & & s
\end{tikzcd}$$

from which we get a morphism $k : z \overset{m}{\to} q$ with $e \circ k =_m p$ and $\varphi \circ k =_m \gamma$.

For the second part we have

$$\begin{tikzcd}
 & t \arrow[dl, "k\circ t_0"'] \arrow[d, "p'"] \arrow[dr, "k\circ t_1"] & \\
q & r & q
\end{tikzcd}$$

which is a cone over the diagram defining h. This means we have $k' : t \overset{m}{\to} h$ such that $h_i \circ k' =_m k \circ t_i$ which is exactly what we need to get a morphism in $\mathcal{C}_{\text{ex}}$

$$\begin{tikzcd}
t \arrow[r, "k'"] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] & h \arrow[d, shift right, "h_0"'] \arrow[d, shift left, "h_1"] \\
z \arrow[r, "k"] & q.
\end{tikzcd}$$

Finally have to verify that $\langle e, e' \rangle$ is mono. Suppose we have $\langle p, p' \rangle, \langle s, s' \rangle$: $\left(t \underset{t_1}{\overset{t_0}{\rightrightarrows}} z \right) \overset{m}{\to} \left(h \underset{h_1}{\overset{h_0}{\rightrightarrows}} q \right)$ with $\delta : z \overset{m}{\to} r$ witnessing

$$\langle e, e' \rangle \circ \langle p, p' \rangle =_m \langle e, e' \rangle \circ \langle s, s' \rangle.$$

As such we have $e \circ p =_m r_0 \circ \delta$ and $e \circ s =_m r_1 \circ \delta$.
Hence

$$\begin{tikzcd}
 & z \arrow[dl, "p"'] \arrow[d, "\delta"] \arrow[dr, "s"] & \\
q & r & q
\end{tikzcd}$$

is a cone over the diagram defining h and we get a morphism $\tau : z \overset{m}{\to} h$ with $h_0 \circ \tau =_m p$ and $h_1 \circ \tau =_m s$ which means exactly $\langle p, p' \rangle =_m \langle q, q' \rangle$. ☐

Regularity and Exactness

Lemma 3.3.17. *Reminder: If a morphism is the coequalizer of some kernel-pair, then it's the coequalizer of its kernel-pair if it exists. In other words, if a category has binary pullbacks, then regular epimorphisms are the coequalizers of their kernel-pairs.*

Proof. Let $p_0, p_1 : x * x \overset{m}{\to} x$ be the kernel-pair of $g : x \overset{m}{\to} z$ and $f : x \overset{m}{\to} y$ be the coequalizer of p_0, p_1.

Since $g \circ p_0 =_m g \circ p_1$ the universal property of coequalizers provides us with some $h : y \overset{m}{\to} z$ such that $h \circ f =_m g$. This means the following augmented diagram of the kernel-pair of g commutes.

$$\begin{tikzcd}
x * x \arrow[rr, "p_1"] \arrow[dd, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & & x \arrow[dl, "f"] \arrow[dd, "g"] \\
& y \arrow[dr, dotted, "h"] & \\
x \arrow[ur, "f"'] \arrow[rr, "g"'] & & z
\end{tikzcd}$$

Hence, if we are given some morphisms $e_0, e_1 : e \overset{m}{\to} x$ such that $f \circ e_0 =_m f \circ e_1$ we also have $h \circ f \circ e_0 =_m h \circ f \circ e_1$ and so there is a unique morphism $q : e \overset{m}{\to} x * x$ such that $p_i \circ q =_m e_i$. But that is what is required for $x * x$ to be a pullback of f along itself. That means $\langle p_0, p_1\rangle$ is also kernel-pair of f. Here we need the existence of pullbacks to be able to write

$$\begin{tikzcd}
\widetilde{x * x} \arrow[r, "h_1"] \arrow[d, "h_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "f"] \\
x \arrow[r, "f"'] & y
\end{tikzcd}$$

for the kernel-pair of f which is then of course isomorphic to $x * x$. ☐

Lemma 3.3.18 ([49]). *Every morphism in $\mathcal{C}_{ex}$ can be factored into a regular epimorphism and monomorphism.*

Proof. Let $\langle f, f'\rangle : \left(\, r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x \, \right) \overset{m}{\to} \left(\, s \overset{s_0}{\underset{s_1}{\rightrightarrows}} y \, \right)$ be some morphism in $\mathcal{C}_{\mathrm{ex}}$. We construct the limit i in $\mathcal{C}$:

$$\begin{tikzcd}
x \arrow[d, "f"'] & i \arrow[l, "i_0"'] \arrow[r, "i_1"] \arrow[d, "\varphi"'] & x \arrow[d, "f"] \\
y & s \arrow[l, "s_0"] \arrow[r, "s_1"'] & y
\end{tikzcd}$$

Since $\langle f, f' \rangle$ is a morphism in $\mathcal{C}_{\mathrm{ex}}$,

$$\begin{tikzcd}
 & r \arrow[dl, "r_0"'] \arrow[d, "f'"] \arrow[dr, "r_1"] & \\
x & s & x
\end{tikzcd}$$

is a cone on this diagram. This means we have some $t : r \overset{m}{\to} i$ with $i_0 \circ t =_m r_0$ and $i_1 \circ t =_m r_1$. The required factorization of $\langle f, f' \rangle$ is then given by

$$\begin{tikzcd}
r \arrow[r, "t"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & i \arrow[r, "\varphi"] \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] & s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
x \arrow[r, "id(x)"'] & x \arrow[r, "f"'] & y
\end{tikzcd}$$

We can show **Reflexivity** of $i \overset{i_0}{\underset{i_1}{\rightrightarrows}} x$ as in previous proofs, by observing that

$$\begin{tikzcd}
 & x \arrow[dl, bend right, "id(x)"'] \arrow[d, "refl_{sy} \circ f"] \arrow[dr, bend left, "id(x)"] & \\
x & s & x
\end{tikzcd}$$

is a cone which provides us with an arrow $r : x \overset{m}{\to} i$ such that $i_0 \circ r =_m id(x) =_m i_1 \circ r$.

Symmetry works the same way:

$$\begin{tikzcd}
 & i \arrow[dl, bend right, "i_1"'] \arrow[d, "symm_{sy} \circ \varphi"] \arrow[dr, bend left, "i_0"] & \\
x & s & x
\end{tikzcd}$$

Transitivity: Let

$$\begin{tikzcd}
i * i \arrow[r, "\bar{i}_1"] \arrow[d, "\bar{i}_0"] \arrow[dr, phantom, "\lrcorner", very near start] & i \arrow[d, "i_0"] \\
i \arrow[r, "i_1"] & x
\end{tikzcd}
\qquad
\begin{tikzcd}
s * s \arrow[r, "\bar{s}_1"] \arrow[d, "\bar{s}_0"] \arrow[dr, phantom, "\lrcorner", very near start] & s \arrow[d, "s_0"] \\
s \arrow[r, "s_1"] & y
\end{tikzcd}$$

As in the case of equalizers,

$$\begin{tikzcd}
& i * i \arrow[ld, bend right, "\varphi \circ \overline{i}_0"'] \arrow[rd, bend left, "\varphi \circ \overline{i}_1"] & \\
s & & s
\end{tikzcd}$$

is a cone over the diagram defining $s * s$:

$$s_1 \circ \varphi \circ \overline{i}_0 =_m f \circ i_1 \circ \overline{i}_0 =_m f \circ i_0 \circ \overline{i}_1 =_m s_0 \circ \varphi \circ \overline{i}_1.$$

From this we have a morphism $\overline{\varphi} : i * i \xrightarrow{m} s * s$ with $\overline{s}_j \circ \overline{\varphi} =_m \varphi \circ \overline{i}_j$ and so we can construct a cone

$$\begin{tikzcd}
& i * i \arrow[ld, bend right, "i_0 \circ \overline{i}_0"'] \arrow[d, "trans_{sy} \circ \overline{\varphi}"] \arrow[rd, bend left, "i_1 \circ \overline{i}_1"] & \\
x & s & x
\end{tikzcd}$$

$$f \circ i_j \circ \overline{i}_j =_m s_j \circ \varphi \circ \overline{i}_j =_m s_j \circ \overline{s}_j \circ \overline{\varphi} =_m s_j \circ trans_{sy} \circ \overline{\varphi}.$$

This finishes the verification of transitivity, since we get some $t : i * i \xrightarrow{m} i$ with $i_j \circ t =_m i_j \circ \overline{i}_j$.

To see that $\langle f, \varphi \rangle$ is mono, suppose we have two morphisms

$$\langle u, u' \rangle, \langle v, v' \rangle : (\ t \rightrightarrows z\ ) \xrightarrow{m} (\ i \rightrightarrows x\ )$$

with $\langle f, \varphi \rangle \circ \langle u, u' \rangle =_m \langle f, \varphi \rangle \circ \langle v, v' \rangle$ witnessed by $\gamma : z \xrightarrow{m} s$. We get again a cone over the diagram defining i. (The property of being a homotopy: $f \circ u =_m s_0 \circ \gamma$ and $f \circ v =_m s_1 \circ \gamma$.)

$$\begin{tikzcd}
& z \arrow[ld, "u"'] \arrow[d, "\gamma"] \arrow[rd, "v"] & \\
x & s & x
\end{tikzcd}$$

This provides us with yet another morphism $\delta : z \xrightarrow{m} i$ with $i_0 \circ \delta =_m u$ and $i_1 \circ \delta =_m v$. But this just means $\langle u, u' \rangle =_m \langle v, v' \rangle$.

It remains to show that the morphism on the left-hand side is regular epic which we prove in Lemma 3.3.20. □

Corollary 3.3.19. *The usual universal property of images is satisfied.*

Proof. Let $\langle f, f \rangle$ be any morphism and $\langle m, m' \rangle \circ \langle e, e' \rangle =_m \langle f, f' \rangle$ some other epi-mono factorization. Additionally, let $\langle i_0, \delta \rangle, \langle i_1, \tau \rangle$ be the kernel-pair of $\langle id(x), t \rangle$. Clearly we have

$$\langle m, m' \rangle \circ \langle e, e' \rangle \circ \langle i_0, \delta \rangle =_m \langle m, m' \rangle \circ \langle e, e' \rangle \circ \langle i_1, \tau \rangle.$$

$\langle m, m' \rangle$ being mono implies that $\langle e, e' \rangle \circ \langle i_0, \delta \rangle =_m \langle e, e' \rangle \circ \langle i_1, \tau \rangle$. Then, by the universal property of coequalizers, there is a morphism $\langle g, g' \rangle$ such that $\langle e, e' \rangle =_m \langle g, g' \rangle \circ \langle id(x), t \rangle$. ☐

The following two Lemmas 3.3.20 and3.3.21 say that morphisms of the form $\langle id(x), g \rangle$ are regular epi and stable under pullback, and Theorem 3.3.22 tells us, that in fact all regular epimorphisms are of this form up to isomorphism.

Lemma 3.3.20. *Morphisms in $\mathcal{C}_{ex}$ of the form $\langle id(x), q \rangle$ for some object $x \mathrel{\dot\in} \mathcal{C}$ are regular epi.*

Proof. Using the construction in the proof of Theorem 3.3.15, the kernel pair of

$$\begin{tikzcd}
r \arrow[r, "q"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & i \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] \\
x \arrow[r, "id(x)"'] & x
\end{tikzcd}$$

is given by

$$\begin{tikzcd}
e \arrow[r, shift left, "\delta"] \arrow[r, shift right, "\tau"'] \arrow[d, shift right] \arrow[d, shift left] & r \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] \\
i \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & x.
\end{tikzcd}$$

To see this, remember that the pullback of a morphism $k : a \overset{m}{\to} b$ along

$id(b) : b \overset{m}{\to} b$ remains unchanged. In particular the object-part of the pullback is a. Here, we calculate a more general limit given by $(p; p_0, \varphi, p_1)$, but as it turns out, $(p; p_0, \varphi, p_1)$ is still just $(i; i_0, id(i), i_1)$ because we essentially pull back twice along $id(x)$ and then once more combining the projections.

To verify the universal property of coequalizers, Consider some morphism

$$\begin{tikzcd}
r \arrow[r, "w'"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & t \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
x \arrow[r, "w"'] & z
\end{tikzcd}$$

with $\langle w, w' \rangle \circ \langle i_0, \delta \rangle =_m \langle w, w' \rangle \circ \langle i_1, \tau \rangle$ witnessed by $\gamma : i \overset{m}{\to} t$, we immediately get a morphism in $\mathcal{C}_{\text{ex}}$

$$\begin{tikzcd}
i \arrow[r, "\gamma"] \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] & t \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
x \arrow[r, "w"'] & z
\end{tikzcd}$$

which commutes because γ is required to satisfy $t_j \circ \gamma =_m w \circ i_j$ for $j = 0, 1$. Clearly, we have $\langle w, \gamma \rangle \circ \langle id(x), q \rangle =_m \langle w, w' \rangle$ since the lower part of the morphisms are equal in $\mathcal{C}$ and Proposition 3.3.10 tells us that the composition of w with $refl_{tz}$ provides a homotopy.

Furthermore, $\langle w, \gamma \rangle$ is unique with this property. This follows since $\langle id(x), q \rangle$ is epi (the homotopy of the composition works as is) and hence any two morphisms which both compose to $\langle w, w' \rangle$ are equal. $\square$

Lemma 3.3.21. *Regular epimorphisms of the form $\langle id(x), f' \rangle$ in $\mathcal{C}_{ex}$ are pullback-stable.*

Proof. Let $\langle g, g' \rangle : \left(t \overset{t_0}{\underset{t_1}{\rightrightarrows}} z \right) \overset{m}{\to} \left(r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x \right)$ be any morphism in $\mathcal{C}_{\text{ex}}$ with the codomain $\left(r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x \right) =_o cod(\langle id(x), f' \rangle)$. Using the description of pullbacks from Theorem 3.3.15, the resulting morphism of pulling $\langle id(x), f' \rangle$ back along $\langle g, g' \rangle$ looks like

$$\begin{tikzcd}
e \arrow[r, "\tau"] \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] & t \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
p \arrow[r, "p_1"'] & z
\end{tikzcd}$$

where

$$\begin{tikzcd}
p \arrow[r, "p_1"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\lrcorner", very near start] & z \arrow[d, "g"] \\
r \arrow[r, "r_1"'] & x
\end{tikzcd}$$

is a pullback in $\mathcal{C}$.

We now construct the regular epi-mono factorization of $\langle p_1, \tau \rangle$.

$$\begin{tikzcd}
e \arrow[r, "q"] \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] & i \arrow[r, "\delta"] \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] & t \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] \\
p \arrow[r, "id(p)"'] & p \arrow[r, "p_1"'] & z
\end{tikzcd}$$

where i in

$$\begin{tikzcd}
p \arrow[d, "p_1"'] & i \arrow[l, "i_0"'] \arrow[r, "i_1"] \arrow[d, "\delta"'] & p \arrow[d, "p_1"] \\
z & t \arrow[l, "t_0"] \arrow[r, "t_1"'] & z
\end{tikzcd}$$

is a limit.

If we can show that $\langle p_1, \delta \rangle$ is an isomorphism, this implies that up to iso the pullback of a morphism of the form $\langle id(x), f' \rangle$ remains in that form which means it is still a regular epimorphism.

We construct a left-inverse to $\langle p_1, \delta\rangle$. We can use $refl_{tz} : z \overset{m}{\to} t$ to construct a cone over the diagram defining p. The resulting universal arrow l satisfies $\varphi \circ l =_m g' \circ refl_{tz}$ and $p_1 \circ l =_m id(z)$. (indeed $r_1 \circ g' \circ refl_{tz} =_m g \circ t_1 \circ refl_{tz} =_m g \circ id(z)$.)

$$\begin{tikzcd}
z \arrow[d, "refl_{tz}"'] \arrow[dr, dotted, "l"] \arrow[drr, bend left, "id(z)"] & & \\
t \arrow[dr, bend right, "g'"'] & p \arrow[r, "p_1"'] \arrow[d, "\varphi"] \arrow[dr, phantom, "\lrcorner", very near start] & z \arrow[d, "g"] \\
 & r \arrow[r, "r_1"] & x
\end{tikzcd}$$

From this we get cone over the diagram defining i.

$$\begin{tikzcd}
 & t \arrow[dl, bend right, "l \circ t_0"'] \arrow[d, "id(t)"] \arrow[dr, bend left, "l \circ t_1"] & \\
p & t & p
\end{tikzcd}$$

This works because we have

$$\begin{aligned}
p_1 \circ l \circ t_0 &=_m id(z) \circ t_0 =_m t_0 \circ id(t) \\
p_1 \circ l \circ t_1 &=_m id(z) \circ t_1 =_m t_1 \circ id(t)
\end{aligned}$$

So there exists a universal morphism $l' : t \overset{m}{\to} i$ such that $i_0 \circ l' =_m l \circ t_0$, $i_1 \circ l' =_m l \circ t_1$ and $\delta \circ l' =_m id(t)$. In other words, we have a morphism

$$\begin{tikzcd}
t \arrow[r, "l'"] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] & i \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] \\
z \arrow[r, "l"'] & p
\end{tikzcd}$$

Since we have

$$p_1 \circ l =_m id(z),$$

we get with Prop. 3.3.10 that $\langle p_1, \delta'\rangle \circ \langle l, l'\rangle =_m id(\ t \rightrightarrows z\ )$. In the other direction we have that

$$\begin{aligned}\langle p_1, \delta\rangle \circ id(\ i \rightrightarrows p\ ) &=_m \langle p_1, \delta\rangle \\ &=_m id(\ t \rightrightarrows z\ ) \circ \langle p_1, \delta\rangle \\ &=_m (\langle p_1, \delta'\rangle \circ \langle l, l'\rangle) \circ \langle p_1, \delta\rangle \\ &=_m \langle p_1, \delta\rangle \circ (\langle l, l'\rangle \circ \langle p_1, \delta\rangle)\end{aligned}$$

And since $\langle p_1, \delta\rangle$ is mono, this implies $id(\ i \rightrightarrows p\ ) =_m \langle l, l'\rangle \circ \langle p_1, \delta\rangle$ which shows that $\langle p_1, \delta\rangle$ and $\langle l, l'\rangle$ are inverses and iso. But with this we are done, since we have shown, that the pullback $\langle p_1, \tau\rangle$ is up to isomorphism the same as the regular epi $\langle id(p), q\rangle$ obtained from its regular epi-mono factorization. ☐

Theorem 3.3.22. *All regular epimorphisms are up to isomorphism of the form $\langle id(x), f'\rangle$ for some object x in $\mathcal{C}_{ex}$.*

Proof. This follows directly from the proof of Lemma 3.3.21. Given a regular epimorphism $\langle f, f'\rangle : (\ t \overset{t_0}{\underset{t_1}{\rightrightarrows}} z\ ) \overset{m}{\twoheadrightarrow} (\ r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x\ )$, we pull back along $id(\ r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x\ )$ and factor the resulting morphism $\langle g, g'\rangle$. As seen in that proof, we have an isomorphism between $cod(\langle g, g'\rangle)$ and the image of $\langle g, g'\rangle$ which we can write as some $dom(\langle id(p), h'\rangle)$. But because we pulled back along the identity-morphism, we get by composition that $\langle id(p), h'\rangle$ is isomorphic to $\langle f, f'\rangle$.

$$\begin{tikzcd}
(e \overset{e_0}{\underset{e_1}{\rightrightarrows}} p) \arrow[r, "=_m"] \arrow[d, two heads, "\langle id(p){,}h'\rangle"'] & (e \overset{e_0}{\underset{e_1}{\rightrightarrows}} p) \arrow[r] \arrow[d, two heads, "\langle g{,}g'\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & (t \overset{t_0}{\underset{t_1}{\rightrightarrows}} z) \arrow[d, two heads, "\langle f{,}f'\rangle"] \\
(i \overset{i_0}{\underset{i_1}{\rightrightarrows}} p) \arrow[r, "\cong"] & (r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x) \arrow[r, "=_m"] & (r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x)
\end{tikzcd}$$

☐

Theorem 3.3.23. *[49] $\mathcal{C}_{ex}$ is exact. That is, $\mathcal{C}_{ex}$ is regular and every equivalence relation in $\mathcal{C}_{ex}$ is the kernel-pair of some morphism. (An equivalence relation which has a kernel-pair is also called* effective*.)*

Proof. Given lemma 3.3.18 we still have to verify the last step, which is to show that an equivalence relation in $\mathcal{C}_{\text{ex}}$ (not it $\mathcal{C}$!) has a coequalizer and it is the kernel pair of this coequalizer.

Let

$$\begin{tikzcd}
r \arrow[r, shift left, "h_0'"] \arrow[r, shift right, "h_0'"'] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
x \arrow[r, shift left, "h_0"] \arrow[r, shift right, "h_1"'] & y.
\end{tikzcd}$$

be an equivalence relation. To construct its coequalizer consider the following limit v in $\mathcal{C}$.

$$\begin{tikzcd}
s \arrow[d, "s_1"'] & v \arrow[l, "p_0"'] \arrow[r, "p_1"] \arrow[d, "\nu"] & s \arrow[d, "s_1"] \\
y & x \arrow[l, "h_0"] \arrow[r, "h_1"'] & y
\end{tikzcd}$$

We call $v_i :\equiv s_0 \circ p_i$ but postpone the proof that $v \overset{v_0}{\underset{v_1}{\rightrightarrows}} y$ is indeed a pseudo-equivalence relation. For now we need to show that there is indeed a morphism $q : s \xrightarrow{m} v$ such that

$$\begin{tikzcd}
s \arrow[r, "q"] \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & v \arrow[d, shift right, "v_0"'] \arrow[d, shift left, "v_1"] \\
y \arrow[r, "id(y)"'] & y
\end{tikzcd}$$

is an arrow in $\mathcal{C}_{\text{ex}}$. Reflexivity of $\langle\langle h_0, h_0'\rangle, \langle h_1, h_1'\rangle\rangle$ means, there is a morphism $\langle r, r'\rangle$ such that there exist $\gamma_0, \gamma_1 : y \xrightarrow{m} s$ with

$$\begin{array}{ll}
s_0 \circ \gamma_0 =_m h_0 \circ r & \qquad s_1 \circ \gamma_0 =_m id(y) \\
s_0 \circ \gamma_1 =_m h_1 \circ r & \qquad s_1 \circ \gamma_1 =_m id(y)
\end{array}$$

witnessing $\langle h_0, h_0' \rangle \circ \langle r, r' \rangle =_m \langle id(s), id(y) \rangle =_m \langle h_1, h_1' \rangle \circ \langle r, r' \rangle$.

Consider the pullback

$$\begin{tikzcd}
s * s \arrow[r, "\overline{s}_1"] \arrow[d, "\overline{s}_0"'] & s \arrow[d, "s_0"] \\
s \arrow[r, "s_1"'] & y
\end{tikzcd}$$

Because we have

$$s_1 \circ \gamma_1 \circ s_0 =_m id(s) \circ s_0$$

we get a morphism $c : s \overset{m}{\to} s * s$ with $\overline{s}_0 \circ c =_m \gamma_1 \circ s_0$ and $\overline{s}_1 \circ c =_m id(s)$. From that we can construct a cone over the diagram defining v.

$$\begin{tikzcd}
 & s \arrow[dl, "symm_{sy} \circ \gamma_0 \circ s_0"'] \arrow[d, "r \circ s_0"] \arrow[dr, "symm_{sy} \circ trans_{sy} \circ c"] & \\
s & x & s
\end{tikzcd}$$

$$\begin{aligned}
s_1 \circ symm_{sy} \circ \gamma_0 \circ s_0 &=_m s_0 \circ \gamma_0 \circ s_0 =_m h_0 \circ r \circ s_0 \\
h_1 \circ r \circ s_0 &=_m s_0 \circ \gamma_1 \circ s_0 =_m s_0 \circ \overline{s}_0 \circ c \\
&=_m s_0 \circ trans_{ys} \circ c \\
&=_m s_1 \circ symm_{sy} \circ trans_{sy} \circ c
\end{aligned}$$

The universal morphism $q : s \overset{m}{\to} v$ is what we were looking for since

$$\begin{aligned}
v_0 \circ q &=_m s_0 \circ p_0 \circ q \\
&=_m s_0 \circ symm_{ys} \circ \gamma_0 \circ s_0 \\
&=_m s_1 \circ \gamma_0 \circ s_0 \\
&=_m id(y) \circ s_0 \\
v_1 \circ q &=_m s_0 \circ p_1 \circ q \\
&=_m s_0 \circ symm_{sy} \circ trans_{sy} \circ c \\
&=_m s_1 \circ trans_{sy} \circ c \\
&=_m s_1 \circ \overline{s}_1 \circ c \\
&=_m s_1 \circ id(s) \\
&=_m id(y) \circ s_1
\end{aligned}$$

Coming back to the verification that $v \underset{v_1}{\overset{v_0}{\rightrightarrows}} y$ is a pseudo-equivalence relation, we check **Reflexivity**:

$$v_i \circ q \circ refl_{ys} =_m id(y) \circ s_i \circ refl_{ys} =_m id(y) \circ id(y) =_m id(y)$$

So $r : y \overset{m}{\to} v$ can just be defined as $q \circ refl_{ys}$.

Symmetry is harder to check. Symmetry of $\langle\langle h_0, h'_0\rangle, \langle h_1, h'_1\rangle\rangle$ means we have a (non-commuting) diagram

$$\begin{tikzcd}
r \arrow[r, "r'_h"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & r \arrow[r, shift left, "h'_0"] \arrow[r, shift right, "h'_1"'] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
x \arrow[r, "r_h"'] \arrow[urr, bend right, "\varphi_0{,}\varphi_1"'] & x \arrow[r, shift left, "h_0"] \arrow[r, shift right, "h_1"'] & y
\end{tikzcd}$$

with $\varphi_i : x \overset{m}{\to} s$ the homotopies witnessing

$\langle h_i, h'_i\rangle \circ \langle r_h, \overline{r}_h\rangle =_m \langle h_{1-i}, h'_{1-i}\rangle$, such that

$$\begin{aligned} s_0 \circ \varphi_0 &=_m h_0 \circ r_h & \qquad s_1 \circ \varphi_0 &=_m h_1 \\ s_0 \circ \varphi_1 &=_m h_1 \circ r_h & \qquad s_1 \circ \varphi_1 &=_m h_0. \end{aligned}$$

We have

$$s_0 \circ symm_{sy} \circ \varphi_1 \circ \nu =_m s_1 \circ \varphi_1 \circ \nu =_m h_0 \circ \nu =_m s_1 \circ p_0$$

hence there is some $\alpha_0 : v \overset{m}{\to} s * s$ such that $\overline{s}_0 \circ \alpha_0 =_m p_0$ and $\overline{s}_1 \circ \alpha_0 =_m symm_{sy} \circ \varphi_1 \circ \nu$. Similarly

$$s_0 \circ symm_{sy} \circ \varphi_0 \circ \nu =_m s_1 \circ \varphi_0 \circ \nu =_m h_1 \circ \nu =_m s_1 \circ p_1$$

hence there is some $\alpha_1 : v \overset{m}{\to} s * s$ such that $\overline{s}_0 \circ \alpha_1 =_m p_1$ and $\overline{s}_1 \circ \alpha_1 =_m symm_{sy} \circ \varphi_0 \circ \nu$. Which gives a cone over the diagram defining v in the following way:

$$\begin{tikzcd}
 & v \arrow[ld, bend right, "trans_{sy}\circ\alpha_1"'] \arrow[d, "r_h\circ\nu"] \arrow[rd, bend left, "trans_{sy}\circ\alpha_0"] & \\
s & x & s
\end{tikzcd}$$

$$\begin{aligned} s_1 \circ trans_{sy} \circ \alpha_1 &=_m s_1 \circ \overline{s}_1 \circ \alpha_1 =_m s_1 \circ symm_{sy} \circ \varphi_0 \circ \nu \\ &=_m s_0 \circ \varphi_0 \circ \nu =_m h_0 \circ r_h \circ \nu \\ s_1 \circ trans_{sy} \circ \alpha_0 &=_m s_1 \circ \overline{s}_1 \circ \alpha_0 =_m s_1 \circ symm_{sy} \circ \varphi_1 \circ \nu \\ &=_m s_0 \circ \varphi_1 \circ \nu =_m h_1 \circ r_h \circ \nu \end{aligned}$$

with

$$s_0 \circ trans_{sy} \circ \alpha_i =_m s_0 \circ \overline{s}_0 \circ \alpha_i =_m s_0 \circ p_i.$$

So $s_{vy} : v \overset{m}{\to} v$ satisfies $p_0 \circ s_{vy} =_m trans_{sy} \circ \alpha_1$ and $p_1 \circ s_{vy} =_m trans_{sy} \circ \alpha_0$

which means that

$$\begin{aligned} v_i \circ s_{vy} &=_m s_0 \circ p_i \circ s_{vy} =_m s_0 \circ trans_{sy} \circ \alpha_{1-i} \\ &=_m s_0 \circ \overline{s}_0 \circ \alpha_{1-i} =_m s_0 \circ p_{1-i} =_m v_{1-i}. \end{aligned}$$

And hence $s_{vy} : v \overset{m}{\to} v$ is the required morphism showing Symmetry.

For **Transitivity** of $v \rightrightarrows y$ we again have to look at transitivity of $\langle\langle h_0, h'_0\rangle, \langle h_1, h'_1\rangle\rangle$.

Suppose $e \overset{e_0}{\underset{e_1}{\rightrightarrows}} p$ is the pullback of $\langle h_0, h'_0\rangle$ along $\langle h_1, h'_1\rangle$, then there is a morphism $\langle t_h, t'_h\rangle : (\ e \overset{e_0}{\underset{e_1}{\rightrightarrows}} p\ ) \overset{m}{\to} (\ r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x\ )$ in $\mathcal{C}_{\mathrm{ex}}$ with $\psi_0, \psi_1 : p \overset{m}{\to} s$ witnessing the equalities

$$\begin{aligned} &\langle h_0, h'_0\rangle \circ \langle t_h, t'_h\rangle =_m \langle h_0, h'_0\rangle \circ \langle \overline{h}_1, \overline{h'}_1\rangle \\ &\langle h_1, h'_1\rangle \circ \langle t_h, t'_h\rangle =_m \langle h_1, h'_1\rangle \circ \langle \overline{h}_0, \overline{h'}_0\rangle. \end{aligned}$$

that is

$$\begin{aligned} s_0 \circ \psi_0 &=_m h_0 \circ t_h & \qquad s_1 \circ \psi_0 &=_m h_0 \circ \overline{h}_1 \\ s_0 \circ \psi_1 &=_m h_1 \circ t_h & \qquad s_1 \circ \psi_1 &=_m h_1 \circ \overline{h}_0. \end{aligned}$$

Where, by construction of pullbacks,

$$\begin{tikzcd} x \arrow[d, "h_0"] & p \arrow[l, "\overline{h}_0"'] \arrow[r, "\overline{h}_1"] \arrow[d, "\zeta"] & x \arrow[d, "h_1"] \\ y & s \arrow[l, "s_0"] \arrow[r, "s_1"'] & y \end{tikzcd}$$

is a limit in $\mathcal{C}$.

Consider the pullback in $\mathcal{C}$

$$\begin{tikzcd}
v * v \arrow[r, "u_0"] \arrow[d, "u_1"'] & v \arrow[d, "v_1"] \\
v \arrow[r, "v_0"'] & s
\end{tikzcd}$$

We're looking for a morphism $t_{vs} : v * v \overset{m}{\to} v$ in $\mathcal{C}$ such that $v_i \circ t_{vs} =_m v_i \circ u_i$.

We have the chain of equalities

$$s_1 \circ symm_{ys} \circ p_0 \circ u_1 =_m s_0 \circ p_0 \circ u_1 =_m v_0 \circ u_1 =_m v_1 \circ u_0 =_m s_0 \circ p_1 \circ u_0$$

and hence get a morphism $a : v * v \overset{m}{\to} s * s$ such that

$$\overline{s}_0 \circ a =_m symm_{ys} \circ p_0 \circ u_1 \qquad \overline{s}_1 \circ a =_m p_1 \circ u_0.$$

We can construct a cone over the diagram defining p

$$\begin{tikzcd}
 & v * v \arrow[dl, bend right, "\nu \circ u_1"'] \arrow[d, "trans_{ys} \circ a"] \arrow[dr, bend left, "\nu \circ u_0"] & \\
s & x & s
\end{tikzcd}$$

because we have equations

$$\begin{aligned}
h_0 \circ \nu \circ u_1 &=_m s_1 \circ p_0 \circ u_1 =_m s_0 \circ symm_{sy} \circ p_0 \circ u_1 \\
&=_m s_0 \circ \overline{s}_0 \circ a =_m s_0 \circ trans_{ys} \circ a \\
h_1 \circ \nu \circ u_0 &=_m s_1 \circ p_1 \circ u_0 =_m s_1 \circ \overline{s}_1 \circ a \\
&=_m s_1 \circ trans_{ys} \circ a.
\end{aligned}$$

That means, we have a morphism $\beta : v * v \overset{m}{\to} p$ such that

$$\overline{h}_0 \circ \beta =_m \nu \circ u_1 \qquad \overline{h}_1 \circ \beta =_m \nu \circ u_0.$$

Continuing our calculation, we get

$$\begin{aligned} s_0 \circ symm_{ys} \circ \psi_0 \circ \beta &=_m s_1 \circ \psi_0 \circ \beta \\ &=_m h_0 \circ \overline{h}_1 \circ \beta \\ &=_m h_0 \circ \nu \circ u_0 \\ &=_m s_1 \circ p_0 \circ u_0 \end{aligned}$$

which implies that there is a morphism $b : v * v \overset{m}{\to} s * s$ such that

$$\overline{s}_0 \circ b =_m p_0 \circ u_0 \qquad\qquad \overline{s}_1 \circ b =_m symm_{sy} \circ \psi_0 \circ \beta.$$

This represents the left-hand side of a cone over v. For the right-hand side we get

$$\begin{aligned} s_0 \circ symm_{ys} \circ \psi_1 \circ \beta &=_m s_1 \circ \psi_1 \circ \beta \\ &=_m h_1 \circ \overline{h}_0 \circ \beta \\ &=_m h_1 \circ \nu \circ u_1 \\ &=_m s_1 \circ p_1 \circ u_1 \end{aligned}$$

which implies that there is a morphism $d : v * v \overset{m}{\to} s * s$ such that

$$\overline{s}_0 \circ d =_m p_1 \circ u_1 \qquad\qquad \overline{s}_1 \circ d =_m symm_{sy} \circ \psi_1 \circ \beta.$$

Putting this together there exists an arrow $t_{vs} : v * v \overset{m}{\to} v$

$$\begin{tikzcd} & v * v \arrow[dl, bend right, "trans_{sy} \circ b"'] \arrow[d, "t_h \circ \beta"] \arrow[dr, bend left, "trans_{sy} \circ d"] & \\ s & x & s \end{tikzcd}$$

Which can be seen by calculating

$$\begin{aligned} s_1 \circ trans_{ys} \circ b =_m s_1 \circ \overline{s}_1 \circ b &=_m s_1 \circ symm_{ys} \circ \psi_0 \circ \beta \\ =_m s_0 \circ \psi_0 \circ \beta &=_m h_0 \circ t_h \circ \beta \end{aligned}$$

$$\begin{aligned} s_1 \circ trans_{ys} \circ d =_m s_1 \circ \overline{s}_1 \circ d =_m s_1 \circ symm_{ys} \circ \psi_1 \circ \beta \\ =_m s_0 \circ \psi_1 \circ \beta =_m h_1 \circ t_h \circ \beta. \end{aligned}$$

This means, we have equalities

$$p_0 \circ t_{vs} =_m trans_{ys} \circ b \qquad\qquad p_1 \circ t_{vs} =_m trans_{ys} \circ d.$$

From this it's not immediately clear, that we indeed have a morphism for transitivity, so we're going to check this

$$\begin{aligned} v_0 \circ t_{vs} &=_m s_0 \circ p_0 \circ t_{vs} =_m s_0 \circ trans_{ys} \circ b \\ &=_m s_0 \circ \overline{s}_0 \circ b \\ &=_m s_0 \circ p_0 \circ u_0 =_m v_0 \circ u_0 \\ v_1 \circ t_{vs} &=_m s_0 \circ p_1 \circ t_{vs} =_m s_0 \circ trans_{ys} \circ d \\ &=_m s_0 \circ \overline{s}_0 \circ d \\ &=_m s_0 \circ p_1 \circ u_1 =_m v_1 \circ u_1. \end{aligned}$$

This finishes the proof that $v \overset{v_0}{\underset{v_1}{\rightrightarrows}} y$ is a pseudo-equivalence relation.

If we can show that $\langle h_0, h'_0 \rangle, \langle h_1, h'_1 \rangle$ is the kernel-pair of $\langle id(y), q \rangle$ we are done. This follows from Lemma 3.3.20 which says that $\langle id(y), q \rangle$ is regular epi and Lemma 3.3.17 which then implies that it is the coequalizer of $\langle h_0, h'_0 \rangle, \langle h_1, h'_1 \rangle$.

To that end, we need a witness $\epsilon : x \overset{m}{\to} v$ which verifies, that

$$\langle id(y), q \rangle \circ \langle h_0, h'_0 \rangle =_m \langle id(y), q \rangle \circ \langle h_1, h'_1 \rangle.$$

Because we have

$$\begin{aligned} s_1 \circ refl_{sy} \circ h_0 =_m h_0 \\ s_1 \circ refl_{sy} \circ h_1 =_m h_1 \end{aligned}$$

we get a cone

$$\begin{tikzcd} & x \arrow[ld, bend right, "refl_{ys} \circ h_0"'] \arrow[d, "id(x)"] \arrow[rd, bend left, "refl_{ys} \circ h_1"] & \\ s & x & s \end{tikzcd}$$

which gives as a result $\epsilon : x \overset{m}{\to} v$ with $v_i \circ \epsilon =_m s_0 \circ p_i \circ \epsilon =_m s_0 \circ refl_{ys} \circ h_i =_m h_i$ as required.

To show this is indeed a pullback we have to check the universal property. Given any two morphisms in $\mathcal{C}_{\text{ex}}$

$$\begin{tikzcd}
s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & t \arrow[l, "z_0'"'] \arrow[r, "z_1'"] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] & r \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
y & z \arrow[l, "z_0"] \arrow[r, "z_1"'] & y
\end{tikzcd}$$

such that $\chi : z \overset{m}{\to} v$ ($v_0 \circ \chi =_m z_0$ and $v_1 \circ \chi =_m z_1$) is a homotopy for the equality $\langle id(y), q \rangle \circ \langle z_0, z_0' \rangle =_m \langle id(y), q \rangle \circ \langle z_1, z_1' \rangle$, we have to provide a unique morphism into $r \rightrightarrows x$.

Because we assumed $\langle \langle h_0, h_0' \rangle, \langle h_1, h_1' \rangle \rangle$ to be an equivalence relation (not just a pseudo-equivalence relation) it suffices to find any

$$\begin{tikzcd}
t \arrow[r, "z'"] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] & r \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] \\
z \arrow[r, "z"'] & x
\end{tikzcd}$$

which satisfies $\langle h_i, h_i' \rangle \circ \langle z, z' \rangle =_m \langle z_i, z_i' \rangle$.
This suffices since $\langle \langle h_0, h_0' \rangle, \langle h_1, h_1' \rangle \rangle$ is jointly monic thus making a morphism with these properties unique.

For now we just assume there will be such a z' and set

$$z :\equiv \nu \circ \chi.$$

To show that $\langle h_0, h_0' \rangle \circ \langle z, z' \rangle =_m \langle z_0, z_0' \rangle$ we use $p_0 \circ \chi : z \overset{m}{\to} s$ and calculate

$$s_0 \circ p_0 \circ \chi =_m v_0 \circ \chi =_m z_0$$

$$s_1 \circ p_0 \circ \chi =_m h_0 \circ \nu \circ \chi =_m h_0 \circ z$$

To show that $\langle h_1, h'_1\rangle \circ \langle z, z'\rangle =_m \langle z_1, z'_1\rangle$ we use $p_1 \circ \chi : z \overset{m}{\to} s$ and calculate

$$s_0 \circ p_1 \circ \chi =_m v_1 \circ \chi =_m z_1$$

$$s_1 \circ p_1 \circ \chi =_m h_1 \circ \nu \circ \chi =_m h_1 \circ z.$$

Now we actually have to show existence of such a morphism $\langle z, z'\rangle$ (resp. of a morphism z' satisfying the required equations.) We still need some intermediate morphisms before we can define z'. For that observe that

$$s_1 \circ symm_{ys} \circ p_0 \circ \chi \circ t_0 =_m s_0 \circ p_0 \circ \chi \circ t_0 =_m v_0 \circ \chi \circ t_0 =_m z_0 \circ t_0 =_m s_0 \circ z'_0$$

This provides us with some $a : t \overset{m}{\to} s * s$ such that $\overline{s}_1 \circ a =_m z'_0$ and $\overline{s}_0 \circ a =_m symm_{ys} \circ p_0 \circ \chi \circ t_0$. That means

$$s_1 \circ trans_{sy} \circ a =_m s_1 \circ \overline{s}_1 \circ a =_m s_1 \circ z'_0 =_m z_0 \circ t_1 =_m v_0 \circ \chi \circ t_1 =_m s_0 \circ p_0 \circ \chi \circ t_1$$

which gives another morphism $b : t \overset{m}{\to} s * s$ with

$$\overline{s}_0 \circ b =_m trans_{sy} \circ a \qquad\qquad \overline{s}_1 \circ b =_m p_0 \circ \chi \circ t_1.$$

But now we can construct two morphisms in $\mathcal{C}_{\mathrm{ex}}$ with homotopy $trans_{sy} \circ b$:

$$\begin{tikzcd}
t \arrow[r, shift left, "refl_{rx}\circ\nu\circ\chi\circ t_0"] \arrow[r, shift right, "refl_{rx}\circ\nu\circ\chi\circ t_1"'] \arrow[d, shift right, "id(t)"'] \arrow[d, shift left, "id(t)"] & r \arrow[r, "h'_0"] \arrow[d, shift right, "r_0"'] \arrow[d, shift left, "r_1"] & s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
t \arrow[r, shift left, "\nu\circ\chi\circ t_0"] \arrow[r, shift right, "\nu\circ\chi\circ t_1"'] & x \arrow[r, "h_0"'] & y
\end{tikzcd}$$

$$\begin{aligned}
s_0 \circ trans_{sy} \circ b &=_m s_0 \circ \overline{s}_0 \circ b \\
&=_m s_0 \circ trans_{sy} \circ a \\
&=_m s_0 \circ \overline{s}_0 \circ a \\
&=_m s_0 \circ symm_{ys} \circ p_0 \circ \chi \circ t_0 \\
&=_m s_1 \circ p_0 \circ \chi \circ t_0 \\
&=_m h_0 \circ \nu \circ \chi \circ t_0
\end{aligned}$$

$$\begin{aligned} s_1 \circ trans_{sy} \circ b &=_m s_1 \circ \overline{s}_1 \circ b \\ &=_m s_1 \circ p_0 \circ \chi \circ t_1 \\ &=_m h_0 \circ \nu \circ \chi \circ t_1. \end{aligned}$$

The same way we get morphisms composed with $\langle h_1, h'_1 \rangle$.

But $\langle \langle h_0, h'_0 \rangle, \langle h_1, h'_1 \rangle \rangle$ is jointly monic which implies that

$$\begin{aligned} &\langle \nu \circ \chi \circ t_0, refl_{ys} \circ \nu \circ \chi \circ t_0 \rangle \quad \text{and} \\ &\langle \nu \circ \chi \circ t_1, refl_{ys} \circ \nu \circ \chi \circ t_1 \rangle \end{aligned}$$

are the same morphism in $\mathcal{C}_{\text{ex}}$ and so there exists a homotopy $z' : t \overset{m}{\to} r$ such that $r_0 \circ z' =_m \nu \circ \chi \circ t_0 =_m z \circ t_0$ and $r_1 \circ z' =_m \nu \circ \chi \circ t_1 =_m z \circ t_1$. This concludes the proof. ☐

Proposition 3.3.24. *The functor $\langle \gamma_o, \gamma_m \rangle : \mathcal{C} \to \mathcal{C}_{ex}$ which assigns the diagonal relation is full and faithful.*

Proof.

$$\begin{aligned} \gamma_o &:\equiv \lambda a. \langle id(a), id(a), id(a), id(a), pr_0 \rangle \\ \gamma_m &:\equiv \lambda f. \langle \gamma_o(dom(f)), \gamma_o(cod(f), \lambda x. \overline{f} x, \lambda p. \overline{f} p \rangle \end{aligned}$$

where pr_0 is one of the projections from the pullback of $id(a)$ along itself in $\mathcal{C}$. Let $f, g : a \overset{m}{\to} b$ be two morphisms in $\mathcal{C}$ with $\gamma_m(f) =_m \gamma_m(g)$ and $\delta : a \overset{m}{\to} b$ the homotopy with $id(b) \circ \delta =_m f$ and $id(b) \circ \delta =_m g$. But clearly this means that we have in $\mathcal{C}$ it holds that $f =_m \delta =_m g$. which means $\langle \gamma_o, \gamma_m \rangle$ is faithful.

To check that $\langle \gamma_o, \gamma_m \rangle$ is full consider an arbitrary $\langle f, f' \rangle : \gamma_o(a) \overset{m}{\to} \gamma_o(b)$. By definition we have that

$$f \circ id(a) =_m id(b) \circ f'$$

which reduces to $f =_m f'$. But this is the image of $\gamma_m(f)$. ☐

Proposition 3.3.25. *Every object of $\mathcal{C}_{ex}$ is a colimit of objects in the image of $\langle \gamma_o, \gamma_m \rangle$ in a uniform way.*

Proof. Consider an arbitrary object $r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x$. Mapping the objects and morphisms appearing in this with $\langle \gamma_o, \gamma_m \rangle$, we get

$$\begin{tikzcd}
r \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] \arrow[d, shift right, "id(r)"'] \arrow[d, shift left, "id(r)"] & x \arrow[d, shift right, "id(x)"'] \arrow[d, shift left, "id(x)"] \\
r \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & x
\end{tikzcd}$$

We can deduce, that $\langle id(x), refl_{rx} \rangle : \gamma_o(x) \overset{m}{\rightarrow} (r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x)$ is the coequalizer of this pair of morphisms.

If $\langle g, g' \rangle : \gamma_o(x) \overset{m}{\rightarrow} (t \overset{t_0}{\underset{t_1}{\rightrightarrows}} z)$ is another morphism, such that $\langle g, g' \rangle \circ \gamma_m(r_0) =_m \langle g, g' \rangle \circ \gamma_m(r_1)$, then there exists a homotopy $\delta : r \overset{m}{\rightarrow} t$ with

$$t_0 \circ \delta =_m g \circ r_0 \qquad\qquad t_1 \circ \delta =_m g \circ r_1 .$$

But then we immediately get a commuting diagram

$$\begin{tikzcd}
\gamma_o(r) \arrow[r, shift left, "\gamma_m(r_0)"] \arrow[r, shift right, "\gamma_m(r_1)"'] & \gamma_o(x) \arrow[rr, "\langle id(x), refl_{rx} \rangle"] \arrow[dr, "\langle g, g' \rangle"] & & (r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x) \arrow[dl, "\langle g, \delta \rangle"] \\
& & (t \overset{t_0}{\underset{t_1}{\rightrightarrows}} z) &
\end{tikzcd}$$

Since we already know that morphisms of the form $\langle id(x), q \rangle$ are regular-epic (Thm. 3.3.22) and hence in particular epic, we get uniqueness of $\langle g, \delta \rangle$. This is exactly a coequalizer diagram taken from the image of $\langle \gamma_o, \gamma_m \rangle$ with colimit $r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x$.

In fact, given any morphism $\langle f, f' \rangle : (r \overset{r_0}{\underset{r_1}{\rightrightarrows}} x) \overset{m}{\rightarrow} (s \overset{s_0}{\underset{s_1}{\rightrightarrows}} y)$ the following is a commuting diagram with $\langle f, f' \rangle$ being the unique morphism from the coequalizer.

$$\begin{tikzcd}
\gamma_o(r) \arrow[r, shift left, "\gamma_m(r_0)"] \arrow[r, shift right, "\gamma_m(r_1)"'] \arrow[d, "\gamma_m(f')"] & \gamma_o(x) \arrow[r, "\langle id(x), refl_{rx}\rangle"] \arrow[d, "\gamma_m(f)"] & (r \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & x) \arrow[d, "\langle f, f'\rangle"] \\
\gamma_o(s) \arrow[r, shift left, "\gamma_m(s_0)"] \arrow[r, shift right, "\gamma_m(s_1)"'] & \gamma_o(y) \arrow[r, "\langle id(y), refl_{sy}\rangle"] & (s \arrow[r, shift left, "s_0"] \arrow[r, shift right, "s_1"'] & y)
\end{tikzcd}$$

This is the case because $f' : r \xrightarrow{m} s$ is a homotopy between $\langle id(y), refl_{ys}\rangle \circ \gamma_m(f) \circ \gamma_m(r_0)$ and $\langle id(y), refl_{ys}\rangle \circ \gamma_m(f) \circ \gamma_m(r_1)$ and since $\langle f, f'\rangle$ fits the coequalizer diagram, we are done.

$$\begin{tikzcd}
 & & & s \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] \\
r \arrow[rrru, bend left, "f'"] \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & x \arrow[r, "f"] & y \arrow[r, "id(y)"] & y
\end{tikzcd}$$

□

In fact, Carboni and Vitale [14] gave a characterization of categories with all finite weak limits as projectives of their exact completion.

Theorem 3.3.26 ([14], Theorem 16)**.** *Exact categories with enough projectives are the exact completions of the weakly lex categories of their projectives (more generally, of any of their projective covers) and, conversely, each weakly lex category in which idempotents split appears as the full subcategory of the projectives of an exact category with enough projectives, names of its exact completions.*

This theorem, which we shall not prove inside explicit math, means we don't have to worry about whether **EC** (or more precisely its image under $\langle \gamma_o, \gamma_m\rangle$) is the full subcategory of projectives or not. This follows from the facts that **EC** has all finite limits (propositions 3.1.9, 3.1.10 and 3.1.11) and that all idempotents split (proposition 3.1.8.)

Carboni also stated the following result about coproducts:

Proposition 3.3.27 ([10]). *If $\mathcal{C}$ is a finitely complete category and has disjoint, pullback stable coproducts, then $\mathcal{C}_{ex}$ also has finite coproducts and $\langle \gamma_o, \gamma_m \rangle$ preserves them.*

Proof. We will only give a sketch. To show that $\langle \gamma_o, \gamma_m \rangle$ preserves binary sums, only the verification of the universal property needs some work, the rest is trivial. For the first part, suppose $\mathcal{C}_{\text{ex}}$ has binary coproducts and $r \underset{r_1}{\overset{r_0}{\rightrightarrows}} x$, $s \underset{s_1}{\overset{s_0}{\rightrightarrows}} y$ are two objects. Let $l_r : r_l \overset{m}{\to} r+s$, $l_s : r_s \overset{m}{\to} r+s$, and $l_x : r_x \overset{m}{\to} x+y$, $l_y : r_y \overset{m}{\to} x+y$ be the coproducts in $\mathcal{C}$. Then, if $\langle \gamma_o, \gamma_m \rangle$ preserves coproduct, we get by an interchange argument, that the coproduct has to be isomorphic to $r+s \underset{r_1 \oplus s_1}{\overset{r_0 \oplus s_0}{\rightrightarrows}} x+y$.

$$\begin{tikzcd}
\gamma_o(r) \arrow[r, "\gamma_m(l_r)"] \arrow[d, shift right, "\gamma_m(r_0)"'] \arrow[d, shift left, "\gamma_m(r_1)"] & \gamma_o(r+s) \arrow[d, shift right, "\gamma_m(r_0 \oplus s_0)"'] \arrow[d, shift left, "\gamma_m(r_1 \oplus s_1)"] & \gamma_o(s) \arrow[l, "\gamma_m(r_s)"'] \arrow[d, shift right, "\gamma_m(s_0)"'] \arrow[d, shift left, "\gamma_m(s_1)"] \\
\gamma_o(x) \arrow[r, "\gamma_m(l_x)"] \arrow[d, shift right, "\gamma_m(x_0)"'] \arrow[d, shift left, "\gamma_m(x_1)"] & \gamma_o(x+y) \arrow[d, "e"] & \gamma_o(y) \arrow[l, "\gamma_m(r_y)"'] \arrow[d, shift right, "\gamma_m(y_0)"'] \arrow[d, shift left, "\gamma_m(y_1)"] \\
r \Rightarrow x \arrow[r, "l"'] & (r \Rightarrow x) + (s \Rightarrow y) & s \Rightarrow y \arrow[l, "r"]
\end{tikzcd}$$

The required reflexivity, symmetry, and transitivity morphisms can be constructed from the summands (e.g. $refl :\equiv refl_{rx} \oplus refl_{sy}$.) ☐

Lemma 3.3.28 ([12]). *Let $\mathcal{C}$ be a finitely complete category and $\mathcal{D}$ an exact category. Let $\langle g_o, g_m \rangle : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite limits. Then we can extend $\langle g_o, g_m \rangle$ to a functor $\langle \hat{g}_o, \hat{g}_m \rangle : \mathcal{C}_{ex} \to \mathcal{D}$ which is exact.*

Proof Idea. For our purposes we don't care too much whether $\mathcal{C}_{\text{ex}}$ is in fact a completion. Since the main point of this lemma is to establish that fact, we won't actually prove this completely (but the conjecture is that the proof of Carboni and Magno, and their observation that the described extension preserves monomorphisms, goes through in Explicit Mathematics.)

We write $coeq_{\mathcal{D}}(f, g)$ for the coequalizing morphism of f and g in $\mathcal{D}$ and $coeq_{\mathcal{D}}(f, g, h)$ for the unique morphism which extends h to the coequalizer.

That is

$$dom(coeq_{\mathcal{D}}(f,g,h)) =_o^{\mathcal{D}} cod(coeq_{\mathcal{D}}(f,g))) \text{ and}$$
$$coeq_{\mathcal{D}}(f,g,h) \circ coeq_{\mathcal{D}}(f,g) =_m h.$$

$$\begin{aligned}
\hat{g}_o(\langle r_0, r_1, r, s, t\rangle) &:\equiv coeq_{\mathcal{D}}(g_m(r_0), g_m(r_1)) \\
\hat{g}_m(\langle f, f'\rangle) &:\equiv coeq_{\mathcal{D}}(g_m(r_0) \\
&\qquad , g_m(r_1) \\
&\qquad , coeq_{\mathcal{D}}(g_m(s_0), g_m(s_1)) \circ g_m(f))
\end{aligned}$$

where $\langle f, f'\rangle$ is an abbreviation for the four-tuple

$$\langle f, f'\rangle : (\ r \underset{r_1}{\overset{r_0}{\rightrightarrows}} x\ ) \overset{m}{\to} (\ s \underset{s_1}{\overset{s_0}{\rightrightarrows}} y\ ).$$

What we get of objects, is a diagram like the one in the proof of prop. 3.3.25 when mapped to $\mathcal{D}$. If

$$\begin{tikzcd}
r \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] \arrow[d, shift right, "id(r)"'] \arrow[d, shift left, "id(r)"] & x \arrow[d, shift right, "id(x)"'] \arrow[d, shift left, "id(x)"] \\
r \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & x
\end{tikzcd}$$

is $\gamma_o(r) \underset{\gamma_m(r_1)}{\overset{\gamma_m(r_0)}{\rightrightarrows}} \gamma_o(x)$ then it has a coequalizer in $\mathcal{C}_{\text{ex}}$ (proposition 3.3.25), since this is a commuting diagram, we can map it with $\langle g_o, g_m\rangle$ to another commuting diagram. And since $\mathcal{D}$ is exact we can consider the image of $\langle\!\langle g_m(r_0), g_m(r_1)\rangle\!\rangle$, i.e. the monic part of its factorization, which is an equivalence relation in $\mathcal{D}$. But then it does have a coequalizer for this relation which is the also the coequalizer of $g_m(r_0)$ and $g_m(r_1)$. Hence $\langle \hat{g}_o, \hat{g}_m\rangle$ is a well defined. $\square$

Exponentials in $\mathrm{EC}_{\mathsf{ex}}$

From now on we will be working in $\mathbf{EC}_{\mathrm{ex}}$. As described in theorem 3.1.16 we know how to construct weak dependent products. We're now going show for the case of cartesian closure, how they transfer into an exponential in $\mathbf{EC}_{\mathrm{ex}}$. The weak exponential is of course constructed as usual by what is normally written as $\Pi_A(A^*B)$ for B^A. That is, $\Pi_{!_a}(!_a^*\pi_b)$ where $!_a : a \overset{m}{\to} \mathbb{1}$ and $\pi_b : a \times b \overset{m}{\to} a$. Remember that the reason this is only a weak exponential, is that we have many operations for one morphism. As in **ECB** we can fix this by giving an equivalence relation and taking the quotient.

$$\begin{aligned} R[u, i, \varepsilon, J, W] &:\equiv u = \langle\langle i, q_0\rangle, \langle i, q_1\rangle\rangle \wedge \langle i, q_0\rangle \in W \wedge \langle i, q_1\rangle \in W \\ &\qquad \wedge (\forall j \in J)(\overline{\varepsilon}\langle j, \langle i, q_0\rangle\rangle = \overline{\varepsilon}\langle j, \langle i, q_1\rangle\rangle) \\ r(a, f) &:\equiv \sum_{i:cod(a)} t_R(i, \varepsilon_{a,f}, dom(a), \Pi_a(f)) \end{aligned}$$

Clearly this is an equivalence relation $r(a, f) \rightarrowtail \Pi_a(f) \times \Pi_a(f)$ in **EC**. Transferring this to $\mathbf{EC}_{\mathrm{ex}}$ we take the coequalizer

$$\begin{tikzcd} \gamma_o(r(!_a, !_a^*\pi_b))) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \gamma_o(\Pi_{!_a}(!_a^*\pi_b)) \arrow[r, "coeq"] & a^b \end{tikzcd}$$

This gets rid of the multiple operations per morphism and shows that

Proposition 3.3.29.
All projective objects in $\mathbf{EC}_{ex}$ have an exponential. □

For the general case of exponentials of arbitrary objects in $\mathbf{EC}_{\mathrm{ex}}$ we follow [16] (see also [13])

Proposition 3.3.30. *Any two objects a, b in $\mathbf{EC}_{ex}$ have an exponential a^b.*

Proof sketch. Let $\gamma_o(a_r) \rightrightarrows \gamma_o(a_c) \twoheadrightarrow a$ and $\gamma_o(b_r) \rightrightarrows \gamma_o(b_c) \twoheadrightarrow b$ be exact. We first construct an exponential of b by a projective object. For any $x \dot{\in} ob_{\mathbf{EC}}$ let e be the coequalizer

$$b_r^x \rightrightarrows b_c^x \twoheadrightarrow e.$$

Projections are just the given ones of b precomposed with the projection from b_r^x. The evaluation map is given by the universal property of coequalizers. More specifically, it is the map $\varepsilon_{(x,b)} :\equiv \langle \varepsilon_{(x,b_c)}), \varepsilon_{(x,b_r)})\rangle$:

$$\begin{tikzcd}
\gamma_o(b_r^x) \times \gamma_o(x) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\gamma_m(\varepsilon_{(x,b_r)})"] & \gamma_o(b_c^x) \times \gamma_o(x) \arrow[r, two heads] \arrow[d, "\gamma_m(\varepsilon_{(x,b_c)})"] & e \times \gamma_o(x) \arrow[d, dotted, "\varepsilon_{(x,b)}"] \\
b_r \arrow[r, shift left] \arrow[r, shift right] & b_c \arrow[r, two heads] & b
\end{tikzcd}$$

We write $b^x :\equiv e$ for the exponential of b by any projective object x. To get b^a for an arbitrary object a in $\mathbf{EC}_{\mathrm{ex}}$, we define the equalizer

$$\begin{tikzcd}
b^a \arrow[r, tail, "e"] & b^{a_c} \arrow[r, shift left] \arrow[r, shift right] & b^{a_r}
\end{tikzcd}$$

the parallel maps of which are induced by the morphisms in **EC** given as

$$b^{a_i} :\equiv \Lambda(\varepsilon_{(a_c,b_c)} \circ (id(b_c^{a_c}) \times a_i)) : b_c^{a_c} \overset{m}{\to} b_c^{a_r}$$

which are the transpose of $b_c^{a_c} \times a_r \xrightarrow{id(b_c^{a_c}) \times a_i} b_c^{a_c} \times a_c \xrightarrow{\varepsilon_{(a_c,b_c)}} b_c$ (resp. $b_r^{a_c} \times a_r \xrightarrow{id(b_r^{a_c}) \times a_i} b_r^{a_c} \times a_c \xrightarrow{\varepsilon_{(a_c,b_r)}} b_r$). Note that Λ as an operation as defined in **ECB** (proposition 3.2.8) and interpreted as morphism in **EC** doesn't give a unique morphism, but extended to the coequalizer in $\mathbf{EC}_{\mathrm{ex}}$ this works out again since the above exponentials are essentially exponentials of the discrete implicit Bishop sets embedded into $\mathbf{EC}_{\mathrm{ex}}$.

$$\begin{tikzcd}
& b^a \times \gamma_o(a_r) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail, "e \times id(\gamma_o(a_r))"] \arrow[dl, bend right, "(b^{a_i} \circ e) \times id(\gamma_o(a_r))"'] & b^a \times \gamma_o(a_c) \arrow[r, two heads] \arrow[d, tail] & b^a \times a \arrow[d, dotted, "\varepsilon_{(a,b)}"] \\
b^{a_r} \times \gamma_o(a_r) \arrow[rrr, bend right, "(*)", "\varepsilon_{(a_r,b)}"'] & b^{a_c} \times \gamma_o(a_r) \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] & b^{a_c} \times \gamma_o(a_c) \arrow[r, "\varepsilon_{(a_c,b)}"] & b
\end{tikzcd}$$

The above construction works, because the projectives in $\mathbf{EC}_{\mathrm{ex}}$ (that is **EC**) are closed under products.

Given another object d and some $f : d \times a \overset{m}{\to} b$ we need a morphism $g : d \overset{m}{\to} b^a$ such that $\varepsilon_{(a,b)} \circ (g \times id(a)) =_m f$.

Because b^{a_c} is an exponential we have a unique $\Lambda(f) : d \overset{m}{\to} b^{a_c}$ constructed by applying Λ to the component-morphisms $\langle f_0, f_1 \rangle$ of f between projective objects in $\mathbf{EC}_{\mathrm{ex}}$. (given by the following diagram and a similar one for

$\gamma_o(b_r^{a_c}))$, and then taking the coequalizer extension to $d \overset{m}{\to} b^{a_c}$.

$$\begin{tikzcd}
\gamma_o(b_r^{a_c}) \times \gamma_o(a_c) \arrow[r, "\varepsilon_{(a_c,b_r)}"] & \gamma_o(b_r) \\
\gamma_o(d_r) \times \gamma_o(a_c) \arrow[u, "\Lambda(f_1)\times id(a_c)"] \arrow[ur, "f_1"] &
\end{tikzcd}$$

We have the following commutative diagram where $\overset{\twoheadrightarrow}{u}$: $u_c \overset{m}{\to} u$ represents the morphism into the object u as a coequalizer of projective objects.

$$\begin{tikzcd}
d \times \gamma_o(a_r) \arrow[r, "\Lambda(f)\times id(\gamma_o(a_r))"] \arrow[d, shift right, "id(d)\times\gamma_m(a_0)"'] \arrow[d, shift left, "id(d)\times\gamma_m(a_1)"] & b^{a_c} \times \gamma_o(a_r) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right, "id(d^{a_c})\times\gamma_m(a_0)"'] \arrow[d, shift left, "id(d^{a_c})\times\gamma_m(a_1)"] & b^{a_r} \times \gamma_o(a_r) \arrow[dd, "\varepsilon_{(a_r,b)}"] \\
d \times \gamma_o(a_c) \arrow[r, "\Lambda(f)\times id(\gamma_o(a_c))"'] \arrow[d, two heads, "id(d)\times\overset{\twoheadrightarrow}{a}"'] & b^{a_c} \times \gamma_o(a_c) \arrow[dr, "\varepsilon_{(a_c,b)}"] & \\
d \times a \arrow[rr, "f"] & & b
\end{tikzcd}$$

The lower left trapezoid commutes by definition of $\Lambda(f)$ and exponentials. The trapezoid on the right is just a reshaped version of $(*)$ in the diagram defining $\varepsilon_{(a,b)}$.

Because b^a is an equalizer and because the above commute, we get a unique morphism $g : d \overset{m}{\to} b^a$.

$$\begin{tikzcd}
b^a \times \gamma_o(a_r) \arrow[r, tail, "e"] & b^{a_r} \times \gamma_o(a_r) \arrow[r, shift left] \arrow[r, shift right] & b^{a_r} \times \gamma_o(a_r) \\
d \times \gamma_o(a_r). \arrow[u, dotted, "g\times\gamma_m(id(a_r))"] \arrow[ur, "\Lambda(f)\times\gamma_m(id(a_r))"'] & &
\end{tikzcd}$$

This works because b^{a_r} is an exponential and $f \circ (id(d) \times (\overset{\twoheadrightarrow}{a} \circ \gamma_m(a_i)))$ is the (unique) transpose of both morphisms in the first row of the big square above. Finally, this means the diagram below commutes.

$$\begin{tikzcd}
d\times\gamma_o(a_c) \arrow[rrr, two heads, "id(d)\times\overset{\twoheadrightarrow}{a}"] \arrow[d, two heads, "id(d)\times\overset{\twoheadrightarrow}{a}"'] \arrow[dr, "\Lambda(f)\times id(\gamma_o(a_c))"] \arrow[drr, dotted, "g\times id(\gamma_o(a_c))"] & & & d\times a \arrow[dd, "g\times id(a)"'] \\
d\times a \arrow[d, "f"'] & b^{a_c}\times\gamma_o(a_c) \arrow[dl, "\varepsilon_{(a_c,b)}"] & b^{a}\times\gamma_o(a_c) \arrow[l, tail, "e\times id(\gamma_o(a_c))"'] \arrow[dr, two heads, "id(b^a)\times\overset{\twoheadrightarrow}{a}"'] & \\
b & & & b^a\times a \arrow[lll, "\varepsilon_{(a,b)}"]
\end{tikzcd}$$

$id(d)\times\overset{\twoheadrightarrow}{a}$ is an epimorphism. Hence the required equation for exponentials holds:

$$\varepsilon_{(a,b)}\circ(g\times id(a)) =_m f.$$

g is unique because $\Lambda(f)$ is unique and because e is monic. □

We have seen three different candidate-categories for sets, for which we were able to prove progressively more properties of the usual category of sets. Others have constructed similar categories, in particular [17]. Here we have some notable differences. Emmenegger and Palmgren work in MLTT[10] and note the following:

> this construction has been extensively studied and has a robust theory [...], at least when $\mathbb{C}$ has finite limits, whereas its behaviour is less understood when $\mathbb{C}$ is only assumed to have weak finite limits. The relevance of the latter case comes from the fact that setoids in Martin-Löf type theory arise as the exact completion of the category of closed types, which does have finite products but only weak equalizers (what we will call a quasi-cartesian category), meaning that a universal arrow exists but not necessarily uniquely.

Our construction of Bishop sets is similar, but because of how the system is set up, we get honest equalizers, essentially by the usual set construction $\{x \mathrel{\dot\in} dom(f) | fx = gx\}$. Emmenegger and Palmgren on the other hand would be able to prove Uniqueness of Identity Proofs (UIP) from equalizers.

---

[10] Martin-Löf Type Theory

For convenience we replicate their argument: The (weak) equalizer of some $f, g : X \to Y$ in MLTT is given by the type $\sum\limits_{a:X}(fa = ga)$ together with the first projection. Since equalizers are preserved by the embedding into Bishop sets, if it was an honest equalizer it would imply that, setting X to $\mathbb{1}$, that there would exist a *mere equivalence relation*[11] $(\cdot \sim_Y \cdot) : Y \to Y \to Type$ on Y. Putting this together with a map in $\prod_{a,b:\mathbb{1}}(fa \sim_Y fb)$ (stating that $f \sim f$,) would then yield a term in $\prod_{a,b:\mathbb{1}}(fa \sim_Y fb) \to fa =_Y fb$. Emmenegger and Palmgren then use theorem 7.2.2 of the HoTT book [43] to prove (UIP) for the type X.

This construction actually yields some insight into why **ECB** and $\mathbf{EC}_{\text{ex}}$ really should be different categories. If we restrict equivalence relations to subsets of mere equivalence relations i.e. a class of pairs $r \dot{\subsetneq} x \times x$ as opposed to some arbitrary equivalence relation $r \rightrightarrows x$, we really seem to lose some essential information.

A hint that this is the case is given in subsection 3.2. To show that every internal equivalence relation $pr_0, pr_1 : \langle r, p\rangle \to \langle s, x\rangle$ has a quotient, i.e. its projections are also projections of the pullback of some $f : \langle s, x\rangle \overset{m}{\rightrightarrows} \langle t, q\rangle$, we either need to somehow construct a reason why this would be the case, or we need a way to have something select such a reason for us. In the case of implicit Bishop sets that amounts to a way to uniformly select elements from preimages. This is exactly the term c_{AC_V} we added through an additional axiom there.

Having shown that explicit Bishop sets are indeed exact (theorem 3.3.23) using a proof which does not need choice or classical logic At this point we would like to start proving the usual theorems about the category of sets. However, as described in section 5.2, just using $\mathbf{EC}_{\text{ex}}$ is not enough to get a practical replacement for the category of sets. It turns out that our definition of a category itself is not sufficiently well-behaved for usage with explicit Bishop sets. Since morphism-equality is given as a mere proposition we often seem to lack the ability construct new morphisms from old ones as soon as we start working in $\mathbf{EC}_{\text{ex}}$. This shows up in particular for morphisms in small versions of $\mathbf{EC}_{\text{ex}}$ in itself. We require witnesses to conclude that two given morphisms are the same, but in general we have no way to construct them.

---

[11] This is HoTT-speak for equivalence relations R on X such that for all $x, y : X$ we have that $R(x, y)$ is an element of the type of propositions *Prop*.

This suggests that the correct framework would be to force categories to have such witnesses by construction. The way to do this is to work in categories enriched in $\mathbf{EC}_{\mathrm{ex}}$. The additional structure should provide the necessary ingredients to prove the usual results like the Yoneda Lemma for $\mathbf{EC}_{\mathrm{ex}}-$enriched Categories where $\mathbf{EC}_{\mathrm{ex}}$ is viewed as enriched over itself.

For this thesis however, we have instead opted to prove the Yoneda Lemma for implicit Bishop sets (section 4.1). This shows that Explicit Mathematics can in principle work even for such "set-theoretical" results. Although the formulation in this setting is not as nice, or rather not as "powerful",[12] as it would be when done in enriched categories. One interpretation of the Yoneda Lemma, namely that certain collections of natural transformations are part of a universe, would however still not hold if we used enriched categories. As long as our universes come from Explicit Mathematics (definition 1.0.8) we would still have to problem that these are not closed under isomorphisms. For this reason we now describe a construction of a universe for implicit Bishop sets which satisfies this and other category theoretically desirable requirements. The main downside of this is, that the element-of relation with respect to the universe is not the usual $\dot{\in}$. This can't be avoided, since it is inconsistent with closure under isomorphisms.

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[12] In a non-technical sense

4. Applications of ECB and $\mathrm{EC}_{\mathbf{ex}}$

We will now describe some constructions and applications of **ECB** and $\mathbf{EC}_{\mathrm{ex}}$. The first one is a universe in a category as described in [38]. Unlike universes constructed the usual way in Explicit Mathematics, this construction is closed under isomorphisms, which makes it much better behaved when working with categories. The other construction is an explicit Bishop set of Cardinal numbers (i.e. an object in $\mathbf{EC}_{\mathrm{ex}}$) with elements which allow (some[1]) cardinal arithmetic.

4.1. Universes

Definition 4.1.1 (Predicative Universe in a category [38]).
Let $\mathcal{C}$ be a locally cartesian closed category, *el* be some morphism in $\mathcal{C}$ and $\mathcal{S}[x]$ be a formula. We call $\mathcal{S}$ a universe in $\mathcal{C}$ if the following axioms hold.

---

[1] Only those cardinal numbers can be shown to exist which are constructible within the setting of elementary comprehension and the join axiom.

$(U1)$ $Mor(a) \wedge Mor(f) \wedge \mathcal{S}[a] \rightarrow (PB[a, f, pr_0, pr_1] \rightarrow \mathcal{S}[pr_0])$

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "h"] \\
\bullet \arrow[r] & \bullet
\end{tikzcd} \qquad \mathcal{S}[h] \Rightarrow \mathcal{S}[g]$$

$(U2)$ $Mor(a) \wedge MONO[a] \rightarrow \mathcal{S}[a]$

$(U3)$ $f : b \overset{m}{\rightarrow} c \wedge g : a \overset{m}{\rightarrow} b \wedge \mathcal{S}[f] \wedge \mathcal{S}[g] \rightarrow \mathcal{S}[\Sigma_f g]$

$(U4)$ $f : a \overset{m}{\rightarrow} i \wedge g : b \overset{m}{\rightarrow} a \wedge \mathcal{S}[f] \wedge \mathcal{S}[g] \rightarrow \mathcal{S}[\Pi_f g]$

$(U5)$ $Mor(a) \wedge \mathcal{S}[a]$

$\rightarrow \exists f, pr_1(f : cod(a) \overset{m}{\rightarrow} cod(el) \wedge PB[f, el, a, pr_1])$

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner", very near start] & e \arrow[d, "el"] \\
\bullet \arrow[r, "\exists f"] & u
\end{tikzcd}$$

Where $PB[f, g, pr_0, pr_1]$ means that pr_0, pr_1 build a limiting cone over the diagram built from f and g. Σ_f, and Π_f are the left-, and right-adjoints of the pullback functor $f^* : \mathcal{C}/c \rightarrow \mathcal{C}/b$ of the morphism $f : b \overset{m}{\rightarrow} c$ when presented as morphisms in $\mathcal{C}$ i.e. $\Sigma_f(g) =_m f \circ g : dom(g) \overset{m}{\rightarrow} c$. ◇

Theorem 4.1.2 (Universes in **EC** are not classes)**.**
EC *is the category of names (allowing the* (J) *axiom) and operations up to functional extensionality. If we add a universe* $\mathcal{S}$ *to* **EC** *we can show, that* $\mathcal{S}$ *is not a class (i.e. it is not an elementary formula).*

Proof. Suppose $\mathcal{S}$ is a class and there is some morphism $\mathcal{S}[x \overset{m}{\rightarrow} \mathbb{1}]$. By $(U2)$ we also have $\mathcal{S}[y \overset{m}{\rightarrowtail} x]$ for any monic arrow. By $(U3)$ it follows that $\mathcal{S}\left[\Sigma_{x \overset{m}{\rightarrow} \mathbb{1}} (y \overset{m}{\rightarrowtail} x)\right]$ also holds. From the universal property of terminal objects we know that $\Sigma_{x \overset{m}{\to} \mathbb{1}} (y \overset{m}{\rightarrowtail} x) =_m y \overset{m}{\to} \mathbb{1}$. Suppose $\mathfrak{R}(x, X)$. This means, that for all z such that $\mathfrak{R}(z, X)$ we also have this z in the universe. (The identity operation $\lambda x.x$ is a morphism between x and z.) But then we can build

$$u = \{dom(x) \mid x \in \mathcal{S}\}$$

Clearly U only contains names so we can mimic the proof of Lemma 6 in [27] to construct the class of all names of x in the following way:

$$\begin{aligned} v &:\equiv \sum (u, \lambda x.x) \\ R[u, y, X, V] &:\equiv \forall z(\langle y, z\rangle \in V \leftrightarrow z \in X) \end{aligned}$$

Then, because of $\mathcal{S}[y \overset{m}{\to} \mathbb{1}]$, we have

$$\begin{aligned} \mathfrak{R}(y, X) &\leftrightarrow 0 \mathbin{\dot\in} t_R(y, x, v) \\ N[u, X, V] &:\equiv R[u, u, X, V] \end{aligned}$$

and so for $\mathfrak{R}(x, X)$

$$u \mathbin{\dot\in} t_N(x, v) \leftrightarrow \mathfrak{R}(u, X).$$

But this is a contradiction and so the class $\mathcal{S}$ can not exist. □

We will instead do something different, which is more in the spirit of explicit mathematics but still represents a similar idea to the above universe. We construct a collection of morphisms with small preimages.

We make the following definitions

Definition 4.1.3 (Explicit Mathematics Universe of Bishop sets)**.** Given some universe $\hat{u}$ with $\mathcal{U}(\hat{u})$ in explicit mathematics, we construct a universe of implicit Bishop sets.

$$\begin{aligned} UNIV[u, U] &:\equiv \exists a, v(u = \langle a, v\rangle \wedge a \in U \wedge v \in U \\ &\qquad \wedge \mathfrak{ECB}(\langle a, v\rangle)) \\ u &:\equiv t_{UNIV}(\hat{u}). \end{aligned}$$

◇

Definition 4.1.4 (Preimages)**.**
We explicitly construct a Bishop set for Preimages.

$$\begin{aligned} INV[u, f, z, A, R] &:\equiv u \in A \wedge \overline{f}u \sim_R z \\ inv(f, z) &:\equiv t_{INV}(f, z, \|dom(f)\|, cod(f)_{\mathrm{E}}) \\ INVEQV[u, I, R] &:\equiv u \in R \wedge \pi_0 u \in I \wedge \pi_1 u \in I \end{aligned}$$

Given a morphism f and $z \dot{\in} \|cod(f)\|$ this lets us define

$$f^{-1}\{z\} :\equiv \langle inv(f, z), t_{INVEQV}(inv(f, z), dom(f)_{\mathrm{E}})\rangle.$$

A symmetrical definition of iso for two morphisms is given by

$$\begin{aligned} EXT[f, g, X, R] &:\equiv (\forall x \in X)(\overline{f}x \sim_{YR} \overline{g}x) \\ ISO[u, f, g, a, b, \overrightarrow{A}] &:\equiv EXT[f \circ g, id(b), B, BR] \\ &\quad \wedge EXT[g \circ f, id(a), A, AR] \\ iso(f, g) &:\equiv * \dot{\in} t_{ISO}(f, g, dom(f), dom(g), \\ &\qquad \|dom(f)\|, \|dom(g)\|, cod(f)_{\mathrm{E}}, cod(g)_{\mathrm{E}}) \\ \overrightarrow{A} &:\equiv A, B, RA, RB \end{aligned}$$

◇

Definition 4.1.5 (Categorical Universe of Bishop sets)**.**
Now we say a morphism is part of the categorical universe ($\mathfrak{CU}$) relative to u if the following formula is true.

$$\begin{aligned} \mathfrak{CU}[f, u] :\equiv \exists h, h^{-1}\exists g(\forall x \dot{\in} cod(f))(g[x] \dot{\in} u \\ \wedge (\forall y \dot{\in} cod(f))(x \sim_{cod(f)} y \rightarrow \forall z(z \dot{\in} g[x] \leftrightarrow z \dot{\in} g[y])) \\ (h[x] : f^{-1}\{x\} \overset{m}{\to} g[x] \wedge (\forall y \dot{\in} cod(f)) \\ (x \sim_{cod(f)} y \rightarrow (\forall z \dot{\in} f^{-1}\{x\})(\overline{(h[x])}z \sim_{g[x]} \overline{(h[y])}z))) \\ \wedge h^{-1}[x] : g[x] \overset{m}{\to} f^{-1}\{x\} \wedge (\forall y \dot{\in} cod(f)) \\ (x \sim_{cod(f)} y \rightarrow \\ (\forall z \dot{\in} g[x])(\overline{(h^{-1}[x])}z \sim_{(f^{-1}\{x\})} \overline{(h^{-1}[y])}z))) \\ \wedge iso(h[x], h^{-1}[x])) \end{aligned}$$

Here we have written the application gx, hx and $h^{-1}x$ as $g[x], h[x]$ and $h^{-1}[x]$ to make the following proofs more readable. $\Diamond$

We recall some properties about preimages which we will need in to prove the closure theorem.

Proposition 4.1.6 (Preimages are pullbacks)**.**

Let $f : a \overset{m}{\to} b$ *and* $z \mathrel{\dot\in} \|b\|$*.* $\left(\mathbb{1} \xleftarrow{pr_0} f^{-1}\{z\} \xrightarrow{pr_1} a \right)$ *is a limiting cone over the diagram* $\left(\mathbb{1} \xrightarrow{z} b \xleftarrow{f} a \right)$*. Meaning* $f^{-1}\{z\}$ *is the pullback along the constant function which picks out* $z \mathrel{\dot\in} b$*.*

Proof. This can easily be seen by constructing the isomorphism which sends x to the pair of x with the unique point (up to equivalence relation) in $\mathbb{1}$.

$$x \mapsto \langle x, * \rangle : f^{-1}\{z\} \overset{m}{\to} z * f$$

for the element-morphism from $\mathbb{1}$ which picks out z from b. $\square$

Proposition 4.1.7. *Isomorphisms over a common base are fiberwise iso. That is, given a commutative diagram*

$$\begin{tikzcd}
a \arrow[r, leftrightarrow, "f", "\cong"'] \arrow[d, "p"] & b \arrow[d, "q"] \\
x \arrow[r, "="] & x
\end{tikzcd}$$

there are induced isomorphisms $m_z : q^{-1}\{z\} \overset{m}{\to} p^{-1}\{z\}$*.*

Proof. For arbitrary $z \mathrel{\dot\in} \|x\|$ we get an isomorphism $p^{-1}\{z\} \cong q^{-1}\{z\}$ by universality. Let j_p and j_q be the projections out of the preimage.

$$\begin{tikzcd}
q^{-1}\{z\} \arrow[r, dotted, "m"] \arrow[rrr, bend left, "j_q"] \arrow[dr, bend right, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & p^{-1}\{z\} \arrow[r, "j_p"] \arrow[d, "!"] \arrow[dr, phantom, "\lrcorner", very near start] & a \arrow[r, leftrightarrow, "f", "\cong"'] \arrow[d, "p"] & b \arrow[d, "q"] \\
 & \mathbb{1} \arrow[r, "z"] & x \arrow[r, "="] & x
\end{tikzcd}$$

Consider $\mathbb{1} \xleftarrow[!]{} q^{-1}\{z\} \xrightarrow{f \circ j_q} a$. This gets us a commuting square with $\mathbb{1} \xrightarrow{z} b \xleftarrow[p]{} a$ because $q =_m id(x) \circ p \circ f$. Hence there exists by universality a unique morphism

$$m : q^{-1}\{z\} \overset{m}{\to} p^{-1}\{z\}.$$

Since everything is symmetric, this works also the other way around with some n. Composing them, we have some map $m \circ n : p^{-1}\{z\} \overset{m}{\to} p^{-1}\{z\}$. Again by universality, this has to be equal to $id(p^{-1}\{z\})$. □

Of course the above proposition is just a reformulation of the fact, that all limiting cones over some diagram are isomorphic.

Lemma 4.1.8 (Preimages are invariant under substitution)**.**
Let $f : a \overset{m}{\to} b$ *and* $z, w \mathrel{\dot\in} \|b\|$ *with* $z \sim_b w$*.*
Then $f^{-1}\{z\}$ *and* $f^{-1}\{w\}$ *are extensionally equal.*

Proof. By expanding definitions:

$$\begin{aligned} x \mathrel{\dot\in} \|f^{-1}\{z\}\| &\leftrightarrow x \mathrel{\dot\in} \|a\| \wedge \overline{f}x \sim_b z \\ &\leftrightarrow x \mathrel{\dot\in} \|a\| \wedge \overline{f}x \sim_b w \\ x \mathrel{\dot\in} \|f^{-1}\{w\}\| & \end{aligned}$$

The second line is because $\sim_b$ is an equivalence relation and the assumption $z \sim_b w$.

Using this for the equivalence relation $\sim_{f^{-1}\{z\}}$, we can show

$$\begin{aligned} x \sim_{f^{-1}\{z\}} y &\leftrightarrow x \mathrel{\dot\in} \|f^{-1}\{z\}\| \wedge y \mathrel{\dot\in} \|f^{-1}\{z\}\| \wedge x \sim_a y \\ &\leftrightarrow x \mathrel{\dot\in} \|f^{-1}\{w\}\| \wedge y \mathrel{\dot\in} \|f^{-1}\{w\}\| \wedge x \sim_a y \\ x \sim_{f^{-1}\{w\}} y. & \end{aligned}$$

This is of course symmetric in z and w. □

Lemma 4.1.9 (Closure under chosen pullback)**.**
If we have that u is a universe of **ECB**, *f, g morphisms in* **ECB** *with common codomain and $\mathfrak{CU}[f, u]$. Then the projection pr_0 from the chosen pullback $g * f$ belongs to $\mathfrak{CU}[\cdot, u]$:*

$$pr_0 : g * f \overset{m}{\to} dom(g) \wedge \mathfrak{CU}[pr_0, u]$$

Proof. For the first part suppose we have $pr_0 : g * f \overset{m}{\to} dom(g)$ the projection from the chosen pullback. We need to show that there are h, h^{-1} and s such that for any $y \dot\in \|dom(g)\|$ we have $h[y] : pr_0^{-1}\{y\} \overset{m}{\to} s[y], h[y] : sy \overset{m}{\to} pr_0^{-1}\{y\}$ and $iso(h[y], h^{-1}[y])$ such that g is maps equivalent values to extensionally equal classes and h, h^{-1} don't shuffle these classes between equivalent values. From the assumption $\mathfrak{CU}[f, u]$ we get terms h_f, h_f^{-1}, and s_f such that $(\forall y \in \|cod(f)\|)(h_f[y] : f^{-1}\{y\} \overset{m}{\to} s_f[y] \wedge h_f^{-1}[y] : s_f[y] \overset{m}{\to} f^{-1}\{y\} \wedge iso(h_f[y], h_f^{-1}[y])$. Furthermore for $x \sim_{cod(f)} y$ we have $z \dot\in s_f[x] \leftrightarrow z \dot\in s_f[y]$ and for $z \dot\in f^{-1}\{x\}$ it holds that $\overline{h_f[x]}z \sim_{(s_f[x])} \overline{h_f[y]}z$. So if we can construct an isomorphism

$$i[y] : pr_0^{-1}\{y\} \overset{m}{\to} f^{-1}\{\overline{g}y\} : i^{-1}[y]$$

in a uniform way (such that $(z \dot\in pr_0^{-1}\{y\}) \to (\overline{i[x]}z \sim_{(f^{-1}\{\overline{g}y\})} \overline{i[y]}z)$ for any other $x \dot\in \|dom(g)\|$ with $x \sim_{dom(g)} y$) then we are done because we can then just compose with h_f.

$$\begin{aligned} s[y] &:\equiv s_f[\overline{g}y] \\ h[y] &:\equiv h_f[\overline{g}y] \circ i[y] : pr_0^{-1}\{y\} \overset{m}{\to} s_f[\overline{g}y] \\ h^-1[y] &:\equiv i^{-1}[y] \circ h_f^{-1}[\overline{g}y]. \end{aligned}$$

To see this, suppose that $x \sim_{dom(g)} y$ and $z \dot\in pr_0^{-1}\{x\}$:

$$\begin{aligned} \overline{(h[x])}z &= \overline{(h_f[\overline{g}x] \circ i[x])}z \\ &\sim_{s[x]} h_f[\overline{g}x](\overline{(i[x])}z) \end{aligned}$$

$$\begin{aligned}
&\sim_{s[x]} h_f[\overline{g}y](\overline{(i[x])}z) && \text{by assumption about } h_f[\cdot]\\
&\sim_{s[x]} h_f[\overline{g}y](\overline{(i[y])}z) && \text{by assumption about } i[\cdot]\\
&\sim_{s[x]} \overline{(h[y])}z.
\end{aligned}$$

The fact that g maps equivalent values to extensionally equal classes follows because this is true for s_f and because g is a function.

(i) Note, that

$$\begin{aligned}
u \mathrel{\dot\in} \|pr_0^{-1}\{y\}\| &\leftrightarrow u \mathrel{\dot\in} \|g * f\| \wedge \overline{(pr_0)}u \sim_{dom(g)} y\\
&\leftrightarrow u = \langle y_0, z_0\rangle \wedge y_0 \mathrel{\dot\in} dom(g) \wedge z_0 \mathrel{\dot\in} dom(f)\\
&\qquad \wedge \overline{g}y_0 \sim_{cod(g)} \overline{f}z_0 \wedge \overline{(pr_0)}u \sim_{dom(g)} y\\
&\leftrightarrow u = \langle y_0, z_0\rangle \wedge y_0 \mathrel{\dot\in} dom(g) \wedge z_0 \mathrel{\dot\in} dom(f)\\
&\qquad \wedge \overline{g}y_0 \sim_{cod(g)} \overline{f}z_0 \wedge y_0 \sim_{dom(g)} y.
\end{aligned}$$

We define the following morphisms:

$$\begin{aligned}
i[y] &:\equiv \langle pr_0^{-1}\{y\}, f^{-1}\{\overline{g}y\}, \lambda x.\pi_1 x\rangle\\
i^{-1}[y] &:\equiv \langle f^{-1}\{\overline{g}y\}, pr_0^{-1}\{y\}, \lambda x.\langle y, x\rangle\rangle
\end{aligned}$$

To see that these are well-defined, just expand $f^{-1}\{\overline{g}y\}$ in a similar way as above.

Because we have $\langle y_0, z_0\rangle \mathrel{\dot\in} \|pr_0^{-1}\{y\}\|$ and so $\overline{g}y \sim_{cod(g)} \overline{g}y_0 \sim_{cod(g)} \overline{f}z_0$ we also get $z_0 \mathrel{\dot\in} \|f^{-1}\{\overline{g}y\}\|$. Because both pullback and inverse image classes inherit the equivalence relation from $dom(f)$ we are done for $i[y]$.

For the other direction, we only need that $\langle y, z_0\rangle \mathrel{\dot\in} \|pr_0^{-1}\{y\}\|$ for any $z_0 \mathrel{\dot\in} \|f^{-1}\{\overline{g}y\}\|$. So for $x_0 \sim_{f^{-1}\{\overline{g}y\}} x_1$

$$\langle y, x_0\rangle \sim_{g*f} \langle y, x_1\rangle$$

holds.

(ii) We now show that

$$i[y] \circ i^{-1}[y] : f^{-1}\{\overline{g}y\} \overset{m}{\to} f^{-1}\{\overline{g}y\} =_m id(f^{-1}\{\overline{g}y\})$$

$$i^{-1}[y] \circ i[y] : pr_0^{-1}\{y\} \overset{m}{\to} pr_0^{-1}\{y\} =_m id(pr_0^{-1}\{y\}).$$

Let $\langle y_0, w_0\rangle \sim_{pr_0^{-1}\{y\}} \langle y_1, w_1\rangle \mathrel{\dot\in} pr_0^{-1}\{y\}$. We calculate

$$\overline{(i^{-1}[y] \circ i[y])}\langle y_0, w_0\rangle = \langle y, \pi_1\langle y_0, w_0\rangle\rangle = \langle y, w_0\rangle,$$

but because y_0 is in the preimage of y, we have $y_0 \sim_{dom(g)} y$ by definition. As elements of the pullback this means $\langle y_0, w_0\rangle \sim_{pr_0^{-1}\{y\}} \langle y, w_0\rangle$.

For $w_0 \sim_{f^{-1}\{\overline{g}y\}} w_1$ we get

$$\overline{(i[y] \circ i^{-1}[y])}w_0 = \pi_1\langle y, w_0\rangle = w_0$$

and so we have $\langle f^{-1}\{\overline{g}y\}, i[y], i^{-1}[y], 0\rangle \mathrel{\dot\in} iso(pr_0, y, u)$. □

Theorem 4.1.10 (Closure under arbitrary Pullbacks).
Let u be a universe of **ECB** *and f a morphism with $\mathfrak{CU}[f, u]$. For arbitrary other morphisms g between implicit Bishop sets with $cod(f) =_o cod(g)$ the pullback of f along g belongs to the universe, meaning that $\mathfrak{CU}[g^*f, u]$ holds.*

Proof. We have to show, that for any projection from a limit g^*f, and any $z \mathrel{\dot\in} \|dom(g)\|$ we have an isomorphic copy of $(g^*f)^{(-1)}\{z\}$ in u.

This can be shown by applying proposition 4.1.7 (to the isomorphism of the two pullbacks) and Lemma 4.1.9 to get uniform isomorphisms$^{(*)}$ with

$$(\forall z \mathrel{\dot\in} dom(g))(c[z] : (g^*f)^{(-1)}\{z\} \cong pr_0^{(-1)}\{z\}) \wedge \mathfrak{CU}[pr_0, u]$$

where $pr_0 : g * f \overset{m}{\to} dom(g)$ is the projection from the chosen pullback.

Note that we have to show something for the above $(*)$. A priori it's not clear that we can construct such an iso *uniformly in z* and not just uniform in the cones which we can always do. We know from the proof of prop. 4.1.7 that the required isomorphism $(g^*f)^{(-1)}\{z\} \cong pr_0^{(-1)}\{z\}$ for any $z \mathrel{\dot\in} \|dom(g)\|$ is just given by the mapping from arbitrary cones into the preimages. But these we know how to construct (it's just application of the respective projection.) Hence we can actually write down a term which constructs these morphisms uniformly in z.

To check the requirement for $h_{(g^*f)}$ which is given as $h_{(g^*f)}[y](z) :\equiv h_{pr_0}[y] \circ c[y]$, suppose we have $x \sim_{dom(g)} y$ and $z \mathrel{\dot\in} (g^*f)^{(-1)}\{y\}$

$$\begin{aligned}
\overline{(h_{(g^*f)}[y])}z &\sim_{g_{pr_0}[y]} \overline{(h_{pr_0}[y] \circ c[y])}z \\
&\sim_{g_{pr_0}[y]} \overline{(h_{pr_0}[y])}(\overline{(c[y])}z) \\
&\sim_{g_{pr_0}[y]} \overline{(h_{pr_0}[x])}(\overline{(c[y])}z) \qquad \text{assumption about } h_{pr_0}[\cdot] \\
&\sim_{g_{pr_0}[y]} \overline{(h_{pr_0}[x])}(\overline{(c[x])}z) \qquad \text{see below} \\
&\sim_{g_{pr_0}[y]} \overline{(h_{(g^*f)}[x])}z.
\end{aligned}$$

The proof of lemma 4.1.7 shows, that $c[y]$ is just the restriction of the isomorphism it is applied to and since we know that

$$\langle g^*f, g * f, \lambda x.\langle\overline{(g^*f)}x, \overline{(f^*g)}x\rangle\rangle$$

is the isomorphism into the chosen pullback, we are done since

$$\begin{aligned}
\overline{(c[x])} &\equiv \overline{(\langle g^*f^{-1}\{x\}, pr_0^{(-1)}\{x\}, \lambda z.\langle\overline{(g^*f)}z, \overline{(f^*g)}z\rangle\rangle)} \\
&= \lambda z.\langle\overline{(g^*f)}z, \overline{(f^*g)}z\rangle \\
&= \overline{(\langle g^*f^{-1}\{y\}, pr_0^{(-1)}\{y\}, \lambda z.\langle\overline{(g^*f)}z, \overline{(f^*g)}z\rangle\rangle)} \\
&\equiv \overline{(c[y])}
\end{aligned}$$

□

Proposition 4.1.11 ($\mathfrak{CU}$ contains all identity morphisms)**.**

Proof. Let a be an object. $id(a)$ is given explicitly as $\langle a, a, \lambda x.x\rangle$. For all $z \mathrel{\dot\in} \|a\|$ we have all z_0 with $z_0 \sim_a z$ in the preimage. So $(id(a))^{(-1)}\{z\} \cong \mathbb{1}$ by the morphisms $x \mapsto *$ and $* \mapsto z$. But a class isomorphic to $\mathbb{1}$ is always in u (just take any class c in the (Explicit Mathematics) universe and $i \mathrel{\dot\in} c$ then $\langle t_A(i,c), t_A(i,c) \times t_A(i,c)\rangle$ for $A[u,i,C] = (u \in C \wedge u = i)$ is an element in u. □

Proposition 4.1.12 ($\mathfrak{CU}$ contains all isomorphisms)**.** *Let* $f : a \overset{m}{\to} b$ *be a morphism and* $g : b \overset{m}{\to} a$ *its inverse such that* $ISO[f,g]$ *holds. Then* $\mathfrak{CU}[f,u]$ *and* $\mathfrak{CU}[g,u]$*.*

Proof. Let $g_{\mathbb{1}} \mathrel{\dot\in} u$ with some $w \mathrel{\dot\in} \|g_{\mathbb{1}}\|$. We define

$$\begin{aligned}
g_f[x] &:\equiv g_{\mathbb{1}} \\
h_f[x] &:\equiv \langle f^{-1}\{x\}, g_{\mathbb{1}}, \lambda z.w\rangle \\
h_f^{-1}[x] &:\equiv \langle g_{\mathbb{1}}, f^{-1}\{x\}, \lambda z.\overline{g}z\rangle \\
g_g[y] &:\equiv g_{\mathbb{1}} \\
h_g[y] &:\equiv \langle g^{-1}\{y\}, g_{\mathbb{1}}, \lambda z.w\rangle \\
h_g^{-1}[y] &:\equiv \langle g_{\mathbb{1}}, g^{-1}\{y\}, \lambda z.\overline{f}z\rangle
\end{aligned}$$

Clearly all preimages of an iso are up to iso just the terminal Bishop set:

Given $x \mathrel{\dot\in} a \wedge \overline{f}x \sim_b b_0$ we have $\overline{f}x \sim_b \overline{f}(\overline{g}b_0)$ and so by composition $\overline{g}(\overline{f}x) \sim_a \overline{g}(\overline{f}(\overline{g}b_0))$ which by definition of isomorphisms means $x \sim_a \overline{g}b_0$. Since all Bishop sets $g_f[x]$ and $g_g[y]$ are the same (w.r.t to $=$) we are done. ☐

Proposition 4.1.13 ($\mathfrak{CU}$ is closed under composition)**.**

Let $v : a \xrightarrow{m} b$ and $u : b \xrightarrow{m} c$ with $\mathfrak{CU}[v, t]$ and $\mathfrak{CU}[u, t]$ Then the composition $(u \circ v)$ is also in the universe.

Proof. By assumption there exist h_v, h_v^{-1}, g_v and h_u, h_u^{-1}, g_u. We have to construct their composed versions. Since composition is basically taking the family of preimages of v indexed by the preimages of u. Because we do have a pre-joined set of all $g_v[x]$'s we need the join axiom and more specifically a sigma Bishop set (definition 3.2.29) to construct this:

The proof has five steps.

(a) Verifying that $g_{u\circ v}[x]$ satisfies the properties of a sigma Bishop set.

$$\begin{aligned}
F[a, f, g] :\equiv\ & (\forall x \mathrel{\dot\in} \|a\|)(Ob(fx)) \\
& \wedge (\forall x, y \mathrel{\dot\in} \|a\|)(x \sim_a y \\
& \quad \to gxy : fx \xrightarrow{m} fy \wedge ISO[gxy, gyx])
\end{aligned}$$

(b) Showing that $\forall z(z \mathrel{\dot\in} g_{u\circ v}[x] \leftrightarrow z \mathrel{\dot\in} g_{u\circ v}[y])$ for $z \mathrel{\dot\in} cod(u)$.

(c) Constructing functions $h_{u\circ v}$ and $h_{u\circ v}^{-1}$.

(d) Showing that $\overline{h_{u\circ v}[x]}z \sim_{g_{u\circ v}[x]} \overline{h^{-1}_{u\circ v}[y]}z$ for $x \sim_{cod(u)} y$.

(e) Checking that they are iso.

$$\begin{aligned}
g_{u\circ v}[x] &:\equiv \sum_{ECB} (g_u[x], \lambda y.g_v[\overline{(h_u^{-1}x)y}], q(x)) \\
\text{where } q(x) &:\equiv \lambda yz.h_v[\overline{(h_u^{-1}[x])z}] \\
&\quad \circ\, preimg(\overline{(h_u^{-1}[x])z}, \overline{(h_u^{-1}[x])y}) \\
&\quad \circ\, h_v^{-1}[\overline{(h_u^{-1}[x])y}] \\
preimg(x,y) &:\equiv \langle v^{-1}\{x\}, v^{-1}\{y\}, \lambda x.x\rangle
\end{aligned}$$

$preimg(x,y)$ is a valid isomorphism if $x \sim_{cod(v)} y$ holds (Lemma 4.1.8).

(a) We have to verify this is well-defined. $q(x)$ is constructed by composition of morphisms, so we don't have to verify it is a function. The fact that domain and codomain match the required objects can be directly read off the supplied arguments to h_v, h_v^{-1}.

We do, however, have to check given $m \sim_{g_u[x]} n$, we have an isomorphism $q(x)mn : g_v[\overline{(h_u^{-1}x)m}] \overset{m}{\to} g_v[\overline{(h_u^{-1}x)n}]$. Indeed, the morphism for $q(x)$ falls out of the proof that

$$g_v[\overline{(h_u^{-1}[x])m}] \cong g_v[\overline{(h_u^{-1}[x])n}].$$

$$\begin{aligned}
g_v[\overline{(h_u^{-1}[x])m}] &\cong_{(1)} v^{-1}\{\overline{(h_u^{-1}[x])m}\} \\
&\cong_{(2)} v^{-1}\{\overline{(h_u^{-1}[x])n}\} \\
&\cong_{(3)} g_v[\overline{(h_u^{-1}[x])n}]
\end{aligned}$$

For (1) and (3) we have $h_v^{-1}[\overline{(h_u^{-1}[x])m}]$ and $h_v[\overline{(h_u^{-1}[x])n}]$ as the required isomorphisms. The iso for (2) is because $h_u^{-1}[x]$ is a morphism and hence

$$(m \sim_{g_u[x]} n) \to (\overline{(h_u^{-1}[x])}m \sim_{u^{-1}\{x\}} \overline{(h_u^{-1}[x])}n).$$

But elements of $u^{-1}\{x\}$ are also elements of b. This means that Lemma 4.1.8 applies to v and the above mapped elements with $preimg(\cdot,\cdot)$ being the representation of this lemma. It remains to check that $q(x)zz$ is the identity for any $z \mathrel{\dot{\in}} \|g_u[x]\|$.

$$\begin{aligned}
q(x)zz =_m\ & h_v[\overline{(h_u^{-1}[x])}z] \\
&\quad \circ preimg(\overline{(h_u^{-1}[x])}z, \overline{(h_u^{-1}[x])}z) \\
&\quad \circ h_v^{-1}[\overline{(h_u^{-1}[x])}z] \\
=_m\ & \langle g_v[\overline{(h_u^{-1}[x])}z], \\
&\quad g_v[\overline{(h_u^{-1}[x])}z], \\
&\quad \lambda k.(\overline{h_v^{-1}[\overline{(h_u^{-1}[x])}z]})(\overline{(h_v[\overline{(h_u^{-1}[x])}z])}k)\rangle \\
=_m\ & \langle g_v[\overline{(h_u^{-1}[x])}z], g_v[\overline{(h_u^{-1}[x])}z], \lambda k.k\rangle \\
=_m\ & id(g_v[\overline{(h_u^{-1}[x])}z])
\end{aligned}$$

The second to last step follows from $h_v[e] \circ h_v^{-1}[e] =_m id(g_v[e])$ for any $e \mathrel{\dot{\in}} \|cod(v)\|$. Note that we have in fact shown a much stronger property. Since h_v respects equivalence and $h_u^{-1}[x]$ is a function, any $q(x)zw$ with $z \sim_{g_u[x]} w$ is in fact the identity and we could have defined $q(x)zw :\equiv id(g_v[\overline{(h_u^{-1}[x])}z])$.

This proves that $q(x)$ has the required properties of

$$F[g_u[x], \lambda y.g_v[\overline{(h_u^{-1}[x])}y], q(x)].$$

(b) Let $x \sim_{cod(u)} y$. By assumption we have $\forall z(z \mathrel{\dot{\in}} g_u[x] \leftrightarrow z \mathrel{\dot{\in}} g_u[y])$ and, because $h_u^{-1}[x]$ is a function, also for $w \mathrel{\dot{\in}} g_u[x]$ that $\forall z(z \mathrel{\dot{\in}} g_v[\overline{(h_u^{-1}[x])}w] \leftrightarrow z \mathrel{\dot{\in}} g_v[\overline{(h_u^{-1}[y])}w])$. Putting this together, we have $\langle w, z\rangle \mathrel{\dot{\in}} g_{u\circ v}[x] \leftrightarrow \langle z, w\rangle \mathrel{\dot{\in}} g_{u\circ v}[y]$ as required.

(c) For h and h^{-1} we can then set

$$\begin{aligned} h_{u\circ v}[x] : (u\circ v)^{-1} &\overset{m}{\to} \sum_{ECB} (g_u[x], \lambda y.g_v[\overline{(h_u^{-1}[x])}y], q(x)) \\ \overline{(h_{u\circ v}[x])}(z) &:\equiv \langle w, \overline{(h_v[\overline{(h_u^{-1}[x])}(w)])}z\rangle \\ \text{where } w &:\equiv \overline{(h_u[x])}(\overline{v}z) \\ \overline{(h_{u\circ v}^{-1}[x])}(\langle y, z\rangle) &:\equiv \overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}y])}z. \end{aligned}$$

To check $h_{u\circ v}[x]$, we first calculate the class of the first component w :

$$\begin{aligned} &z \mathrel{\dot{\in}} \|(u\circ v)^{-1}\{x\}\| \\ &\Rightarrow \overline{v}z \mathrel{\dot{\in}} \|u^{-1}\{x\}\| \\ &\Rightarrow \overline{(h_u[x])}(\overline{v}z) \mathrel{\dot{\in}} \|g_u[x]\|. \end{aligned}$$

For the second component of $h_{u\circ v}(x)$ we have

$$\begin{aligned} \overline{v}z &\sim_b \overline{(h_u^{-1}[x])}(\overline{(h_u[x])}(\overline{v}z)) \\ &= \overline{(h_u^{-1}[x])}(w) \end{aligned}$$

and so with Lemma 4.1.8

$$z \mathrel{\dot{\in}} \|v^{-1}\{\overline{v}z\}\| \leftrightarrow z \mathrel{\dot{\in}} \|v^{-1}\{\overline{(h_u^{-1}[x])}(w)\}\|$$

which is exactly what is required to apply

$$h_v[\overline{(h_u^{-1}[x])}(w)] : v^{-1}\{\overline{(h_u^{-1}[x])}(w)\} \overset{m}{\to} g_v(\overline{(h_u^{-1}[x])}(w)).$$

This is a function since given $z \sim_{(u\circ v)^{-1}\{x\}} q$ implies that $w(z) \sim_{g_u[x]} w(q)$ and similarly for the second component which is again built up from morphisms and the above three equations provide the argument if we use that $\overline{v}z \sim_b \overline{v}q$.

To check $h^{-1}_{u\circ v}$, suppose we are given $\langle y, z\rangle \mathrel{\dot\in} g_{u\circ v}[x]$:

$$\begin{aligned} &z \mathrel{\dot\in} \|g_v[\overline{(h_u^{-1}[x])}y]\| \\ &\quad\Rightarrow \overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}y])}z \mathrel{\dot\in} v^{-1}\{\overline{(h_u^{-1}[x])}y\}. \end{aligned}$$

Because we know that $\overline{(h_u^{-1}[x])}y \mathrel{\dot\in} \|u^{-1}\{x\}\|$ we get

$$\forall a \mathrel{\dot\in} \|v^{-1}\{\overline{(h_u^{-1}[x])}y\}\| \Rightarrow a \mathrel{\dot\in} \|(u\circ v)^{-1}\{x\}\|$$

so $h^{-1}_{u\circ v}[x]$ maps elements to the correct class.

(d) Let $x \sim_{cod(u)} y$ and $k \mathrel{\dot\in} u^{-1}\{x\}$. We have $\overline{(h_u[x])}k \sim_{g_u[x]} \overline{(h_u[y])}k$. In particular this means that for $z \mathrel{\dot\in} (u\circ v)^{-1}\{x\}$ we get

$$\overline{(h_u[x])}(\overline{v}z) \sim_{g_u[x]} \overline{(h_u[y])}(\overline{v}z)$$

which means the first factor $w :\equiv \overline{(h_u[x])}(\overline{v}z)$ behaves as required. But then we also have

$$\overline{(h_u^{-1}[x])}(w[x]) \sim_{u^{-1}\{x\}} \overline{(h_u^{-1}[y])}(w[x]) \sim_{u^{-1}\{x\}} \overline{(h_u^{-1}[y])}(w[y]).$$

That gets us

$$\begin{aligned} &\overline{(h_v[\overline{(h_u^{-1}[x])}(w[x])])}z \\ \sim_{g_v[\overline{(h_u^{-1}[x])}(w[x])]}\ &\overline{(h_v[\overline{(h_u^{-1}[y])}(w[y])])}z. \end{aligned}$$

We're not quite done, since we need for $z \mathrel{\dot\in} (u\circ v)^{-1}\{x\}$ that

$$\overline{(h_{u\circ v}[x])}z \sim_{g_{u\circ v}[x]} \overline{(h_{u\circ v}[y])}z$$

which is

$$\begin{aligned} &w[x] \sim_{g_u[x]} w[y] \\ &\quad\wedge q(x)(w[x])(w[y])(\overline{(h_v[\overline{(h_u^{-1}[x])}(w[x])])}z) \\ &\quad\sim_{g_v[\overline{(h_u^{-1}[y])}(w[y])]} \overline{(h_v[\overline{(h_u^{-1}[y])}(w[y])])}z. \end{aligned}$$

But if we calculate this it turns out that this reduces to the equivalence we have just shown above:

Let $p :\equiv g_v[\overline{(h_u^{-1}[y])}(w[y])]$ and $r :\equiv \overline{(h_v[\overline{(h_u^{-1}[x])}(w[x])])}z$

$$\begin{aligned}
&\overline{(q(x)(w[x])(w[y]))}r\\
&\sim_p \overline{(h_v[\overline{(h_u^{-1}[x])}(w[y])])}(\overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}(w[x])])}(r))\\
&\sim_p \overline{(h_v[\overline{(h_u^{-1}[x])}(w[y])])}(\overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}(w[y])])}(r))\\
&\sim_p r)
\end{aligned}$$

and as we have seen above, r is equivalent to $\overline{(h_v[\overline{(h_u^{-1}[y])}(w[y])])}z$.

(e) Lastly, we have to check, that the compositions are equal to the identity on both sides. For this let

$$p :\equiv \overline{(h_u^{-1}[x])}(\overline{(h_u[x])}(\overline{v}z)).$$

Direct calculation shows then

$$\begin{aligned}
&\overline{(h_{u\circ v}^{-1}[x]\circ h_{u\circ v}[x])}(z)\\
&\sim_{g_{u\circ v}[x]} \overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}(\pi_0(\overline{(h_{u\circ v}[x])}(z)))])}\\
&\qquad\qquad (\pi_1(\overline{(h_{u\circ v}[x])}(z)))\\
&\sim_{g_{u\circ v}[x]} \overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}(\overline{(h_u[x])}(\overline{v}z))])}(\pi_1(\overline{(h_{u\circ v}[x])}(z)))\\
&\sim_{g_{u\circ v}[x]} \overline{(h_v^{-1}[p])}(\pi_1(\overline{(h_{u\circ v}[x])}(z)))\\
&\sim_{g_{u\circ v}[x]} \overline{(h_v^{-1}[p])}(\overline{(h_v[\overline{(h_u^{-1}[x])}(\overline{(h_u[x])}(\overline{v}z))])}z)\\
&\sim_{g_{u\circ v}[x]} \overline{(h_v^{-1}[p])}(\overline{(h_v[p])}z)\\
&\sim_{g_{u\circ v}[x]} z.
\end{aligned}$$

The other direction needs some more annotations. Let

$$\begin{aligned} \langle y, z\rangle \mathrel{\dot\in} \sum_{ECB} (g_u[x], \lambda y.g_v[\overline{(h_u^{-1}[x])y}], q(x)) \text{ and define}\\ t :\equiv \overline{(h_u^{-1}[x])y}\\ s :\equiv \overline{(h_u[x])}(\overline{v}(\overline{(h_v^{-1}[t])z})). \end{aligned}$$

Then we have $\overline{(h_{u\circ v}^{-1}[x])}(\langle y, z\rangle)$ in $(u \circ v)^{-1}\{x\}$
and hence also $\overline{v}(\overline{(h_v^{-1}[t])}(z)) = \overline{v}(\overline{(h_{u\circ v}^{-1}[x])}(\langle y, z\rangle))$ in $u^{-1}\{x\}$.
Let $x \mathrel{\dot\in} cod(u)$ and $z \mathrel{\dot\in} g_{u\circ v}[x]$.

$$\begin{aligned}
&\overline{h_{u\circ v}[x] \circ h_{u\circ v}^{-1}[x]}(z)\\
&\sim_{g_{u\circ v}[x]} \overline{(h_{u\circ v}[x])}(\overline{(h_{u\circ v}^{-1}[x])}(z))\\
&\sim_{g_{u\circ v}[x]} \langle\overline{(h_u[x])}(\overline{v}(\overline{(h_{u\circ v}[x])}z)),\\
&\qquad \overline{(h_v[\overline{(h_u^{-1}[x])}(\overline{(h_u[x])}(\overline{v}(\overline{(h_{u\circ v}^{-1})}\langle y, z\rangle))))])}\\
&\qquad \overline{(h_{u\circ v}^{-1}[x])}\langle y, z\rangle\rangle\\
&\qquad \overline{(h_{u\circ v}(x) \circ h_{u\circ v}^{-1}(x))}(\langle y, z\rangle)\\
&\sim_{g_{u\circ v}[x]} \langle\overline{(h_u[x])}(\overline{v}(\overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}y])z})),\\
&\qquad \overline{(h_v[\overline{(h_u^{-1}[x])}(\overline{(h_u[x])}(\overline{v}(\overline{h_v^{-1}[\overline{(h_u^{-1}[x])y}])z})))])}\\
&\qquad \overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}y])}z\rangle\\
&\sim_{g_{u\circ v}[x]} \langle s, \overline{(h_v[\overline{(h_u^{-1}[x])}(s)])}(\overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}y])}z)\rangle\\
&\sim_{g_{u\circ v}[x]} \langle y, \overline{(h_v[\overline{(h_u^{-1}[x])}y])}(\overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}y])}z)\rangle\\
&\sim_{g_{u\circ v}[x]} \langle y, \overline{(h_v[t])}(\overline{(h_v^{-1}[t])}z)\rangle\\
&\sim_{g_{u\circ v}[x]} \langle y, z\rangle
\end{aligned}$$

For the third equation from below, note that

$$\overline{(h_v^{-1}[t])}z = \overline{h_v^{-1}[\overline{(h_u^{-1}[x])}y]}z \mathrel{\dot\in} v^{-1}\{\overline{(h_u^{-1}[x])y}\}$$

and hence we know what value this takes when it's mapped by v :

$$\overline{v}(\overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}y])}z) \sim_b \overline{(h_u^{-1}[x])}y.$$

So, using that $y \mathrel{\dot{\in}} g_u[x]$ implies $\overline{(h_u^{-1})}y \mathrel{\dot{\in}} u^{-1}\{x\} \mathrel{\dot{\subset}} b$ we get the needed

$$\begin{aligned}
&\overline{(h_u[x])}(\overline{v}(\overline{(h_v^{-1}[\overline{(h_u^{-1}[x])}y])}z))\\
&\sim_{u^{-1}\{x\}} \overline{(h_u[x])}(\overline{(h_u^{-1}[x])}y)\\
&\sim_{u^{-1}\{x\}} y
\end{aligned}$$

which was all we had to show. □

Proposition 4.1.14 ($\mathfrak{CU}$ is closed under Π_f)**.** *Let* $\mathfrak{CU}[f : y \overset{m}{\to} x]$ *and* $\mathfrak{CU}[p : dom(p) \overset{m}{\to} y]$, *then we also have* $\mathfrak{CU}[pr : \Pi_f(p) \overset{m}{\to} x]$.

Proof. The explicit construction of Π_f without the join-axiom is given in definition A.4.1. Just like in the case for the left-adjoint to the pullback functor (proposition 4.1.13) the construction of an element isomorphic to $(\Pi_f(p))^{-1}\{y\}$ in the universe can't be replicated as we don't get a pre-joined class of all $g_p[y]$ which could be used to replace $dom(p)$. We will instead use the Pi Bishop set given in definition 3.2.30 which is contained in the universe if the parameters are.

Since the proof of proposition 4.1.13 already showed that the definition of $q(x)$ via the isomorphisms reduces to identity, we will now use this directly.

$$\begin{aligned}
g_{\Pi_f(p)}[y] &:\equiv \prod_{\mathbf{ECB}} (g_f[y], \lambda x.g_p[\overline{(h_f^{-1}[y])}x], q(y))\\
q(y) &:\equiv \lambda xz.\langle g_p[\overline{(h_f^{-1}[y])}x], g_p[\overline{(h_f^{-1}[y])}z], \lambda x.x\rangle\\
h_{\Pi_f(p)}[y] &: pr^{-1}\{y\} \overset{m}{\to} \prod_{\mathbf{ECB}} (g_f[y], \lambda x.g_p[\overline{(h_f^{-1}[y])}x], q(y))\\
h_{\Pi_f(p)}[y](k) &:\equiv \lambda z.\overline{(h_p[\overline{(h_f^{-1}[y])}z])}((\pi_1 k)(\overline{(h_f^{-1}[y])}(z)))\\
h_{\Pi_f(p)}^{-1}[y](k) &:\equiv \langle y, \lambda z.\overline{(h_p^{-1}[\overline{(h_f^{-1}[y])}(\overline{(h_f[y])}z)])}(k(\overline{(h_f[y])}(z)))\rangle
\end{aligned}$$

We divide the proof into the the following steps:

(a) $g_{\Pi_f(p)}$ respects equivalences.

(b) $h_{\Pi_f(p)}$ is well-defined.

(c) $h^{-1}_{\Pi_f(p)}$ is well-defined.

(d) $h_{\Pi_f(p)}$ respects equivalences.

(e) $h^{-1}_{\Pi_f(p)}$ respects equivalences.

(f) $h_{\Pi_f(p)^{-1}}[y] \circ h_{\Pi_f(p)}[y] =_m id(\Pi_f(p))$

(g) $h_{\Pi_f(p)}[y] \circ h^{-1}_{\Pi_f(p)}[y] =_m id(pr^{-1}\{y\})$

(a) Let $x \sim_{cod(f)} y$. As in the proof of proposition 4.1.13 we have $\forall z(z \dot{\in} \|g_f[x]\| \leftrightarrow z \dot{\in} \|g_f[y]\|)$ and for any $z \dot{\in} \|g_f[x]\|$ it holds that

$$\forall w(w \dot{\in} \|g_p[\overline{(h_f^{-1}[x])z}]\| \leftrightarrow w \dot{\in} \|g_p[\overline{(h_f^{-1}[y])z}]\|).$$

Now let $s \dot{\in} \|g_{\Pi_f(p)}\|$. By definition we get for all $z \dot{\in} \|g_f[y]\|$ that $sz \dot{\in} \|g_p[\overline{(h_f^{-1}[y])z}]\|$ but then s is also an element of $\|g_{\Pi_f(p)}[y]\|$.

(b) Reminder: With abbreviation $\overline{X}$ for X, XR, Y, YR, A, AR the morphism pr is given as the first projection from the class defined by

$$\begin{aligned} PI[u, f, p, \overline{X}] :\equiv \exists y_0, q_0(u = \langle y_0, q_0\rangle \wedge y_0 \in Y \wedge \\ &(\forall x \in X)(\overline{f}x \sim_{YR} y_0 \\ &\qquad \to (q_0x\downarrow \wedge q_0x \in A \wedge \overline{p}(q_0x) \sim_{XR} x)) \\ &(\forall x_0, x_1 \in X)((\overline{f}x \sim_{YR} y_0 \wedge x_0 \sim_{XR} x_1) \\ &\qquad \to (qx_0 \sim_{AR} qx_1))) \\ pi(f,p) :\equiv t_{PI}(f, p, \|dom(f)\|, dom(f)_{\mathrm{E}}, \\ &\qquad \|cod(f)\|, cod(f)_{\mathrm{E}}, \|dom(p)\|, dom(p)_{\mathrm{E}}). \end{aligned}$$

As a diagram $h_{\Pi_f(p)}$ sends an element $\langle y_0, q_0\rangle$ of $pr^{-1}\{y\}$ (where $y_0 \sim_{cod(f)} y$ holds) to the following composition

$$\begin{tikzcd}
z \arrow[dd, mapsto] & g_f[y] \arrow[d, "h_f^{-1}[y]"'] & g_p[\overline{(h_f^{-1}[y])}z] & h_p[\overline{(h_f^{-1}[y])}z](\ldots) \\
 & f^{-1}\{y\} \arrow[r, "q_0"] & p^{-1}\{\overline{(h_f^{-1}[y])}z\} \arrow[u, "h_p{[\overline{(h_f^{-1}[y])}z]}"'] & \\
\overline{(h_f^{-1}[y])}z \arrow[rrr, mapsto] & & & q_0(\overline{(h_f^{-1}[y])}z) \arrow[uu, mapsto]
\end{tikzcd}$$

Note that, although we have written $q_0 : f^{-1}\{y\} \overset{m}{\to} p^{-1}\{\overline{(h_f^{-1}[y])}z\}$, this is of course a dependent function and $p^{-1}\{\cdot\}$ varies with the element q_0 is applied to. Let $\langle y_1, q_1\rangle \sim_{pr^{-1}\{y\}} \langle y_0, q_0\rangle$ be an equivalent pair. This implies in particular that $z \dot{\in} f^{-1}\{y_1\}$ implies $q_0 z \sim_{dom(p)} q_1 z$ and hence $q_0(\overline{(h_f^{-1}[y])}z) \sim_{dom(p)} q_1(\overline{(h_f^{-1}[y])}z)$. Because the other two transformations involved are morphisms by assumption, we know that for all $w \dot{\in} g_f[y_0]$, we have

$$\overline{h_{\Pi_f(p)}[y]}\langle y_0, q_0\rangle w \sim_{g_p[\overline{(h^{-1}[y])}w]} \overline{h_{\Pi_f(p)}[y]}\langle y_1, q_1\rangle w.$$

But this is the required equation to show that the terms given from two different elements of the dependent product are mapped to equivalent elements of $g_{\Pi_f(p)}[y]$, assuming the transport morphism q is just the identity, which is the case here.

(c) For the other direction, suppose we have $k \dot{\in} \overline{g_{\Pi_f(p)}[y]}$. We have to check, that this is mapped to a pair $\langle y_0, q_0\rangle \dot{\in} \|pr^{-1}\{y\}\| \dot{\subset} \|\Pi_f(p)\|$. In particular, that means $(\forall x \dot{\in} f^{-1}\{y_0\})(q_0 x \dot{\in} dom(p) \wedge \overline{p}(q_0 x) \sim_{f^{-1}\{y_0\}} x)$. For any $z \dot{\in} \|f^{-1}\{y\}\|$ we have to show that we get a partial section of p above y. The following starts from the definition of k and applies the appropriate h_p^{-1}. Then we use the fact that h_f is an isomorphism and preimages are extensionally equal for equivalent elements by proposition 4.1.8.

$$\begin{aligned}
k(\overline{h_f[y]}z) &\dot{\in} g_p[\overline{h_f^{-1}[y]}(\overline{h_f[y]}z)] \\
\overline{h_p^{-1}[\overline{h_f^{-1}[y]}(\overline{h_f[y]}z)]}(k(\overline{h_f[y]}z)) &\dot{\in} p^{-1}\{\overline{h_f^{-1}[y]}(\overline{h_f[y]}z)\} \\
\overline{h_p^{-1}[\overline{h_f^{-1}[y]}(\overline{h_f[y]}z)]}(k(\overline{h_f[y]}z)) &\dot{\in} p^{-1}\{z\}
\end{aligned}$$

But this shows that the term on the left as an operation of z, composes with p to the identity. It remains to make sure that for $w \sim_{f^{-1}\{y\}} z$ this gets mapped to an equivalent value, but by definition of Pi Bishop sets we have for

$$\overline{h_f[y]}z \sim_{g_f[y]} \overline{h_f[y]}w$$

that k respects this equivalence:

$$k(\overline{h_f[y]}z) \sim_{g_p[\overline{h_f^{-1}[y]}(\overline{h_f[y]}w)]} k(\overline{h_f[y]}w).$$

For the first component of $h_{\Pi_f(p)}^{-1}$, we directly get the correct term y supplied as an argument.

(d) Let $x \sim_{cod(f)} y$. It can be already be seen from the diagram above that these maps respect identity. $h_{\Pi_f(p)}$ is made up of compositions from the argument and h_f^{-1} and h_p which respect equivalence by assumption. We have to check the following: (We write h for $h_{\Pi_f(p)}$.) For $\langle y_0, k_0\rangle :\equiv k \mathrel{\dot{\in}} pr^{-1}\{x\}$ (so $y_0 \sim_{cod(f)} x$.) and any $z \sim_{f^{-1}\{x\}} w$ it holds that $\overline{h[x]}(k)(z)$ and $\overline{h[y]}(k)(w)$ are equivalent.

$$\begin{aligned}
&\overline{h[x]}(k)(z)\\
&= (\lambda z.\overline{(h_p[\overline{(h_f^{-1}[x])}z])}((\pi_1 k)(\overline{(h_f^{-1}[x])}(z))))z\\
&\sim_{g_p[\overline{(h_f^{-1}[x])}z]} \overline{(h_p[\overline{(h_f^{-1}[x])}z])}((\pi_1 k)(\overline{(h_f^{-1}[x])}(z)))\\
&\sim_{g_p[\overline{(h_f^{-1}[x])}z]} \overline{(h_p[\overline{(h_f^{-1}[y])}z])}((\pi_1 k)(\overline{(h_f^{-1}[x])}(z)))\\
&\sim_{g_p[\overline{(h_f^{-1}[x])}z]} \overline{(h_p[\overline{(h_f^{-1}[y])}w])}((\pi_1 k)(\overline{(h_f^{-1}[x])}(z)))\\
&\sim_{g_p[\overline{(h_f^{-1}[x])}z]} \overline{(h_p[\overline{(h_f^{-1}[y])}w])}((\pi_1 k)(\overline{(h_f^{-1}[y])}(z)))\\
&\sim_{g_p[\overline{(h_f^{-1}[x])}z]} \overline{(h_p[\overline{(h_f^{-1}[y])}w])}((\pi_1 k)(\overline{(h_f^{-1}[y])}(w)))\\
&\sim_{g_p[\overline{(h_f^{-1}[x])}z]} \overline{h[y]}(k)(w)
\end{aligned}$$

Remember that we actually have to check equivalence in the Pi Bishop set which means in this case

$$\overline{(q[y](z)(w))}(\overline{h[x]}(k)(z)) \sim_{g_p[\overline{(h_f^{-1}[y])}w])} \overline{h[y]}(k)(w)$$

but q is the identity, so we can ignore that part.

(e) For h^{-1} we only consider the second component, and just mention that it works for the same reason as for the other direction. Let $x \sim_{cod(f)} y$ and $k \dot{\in} g_{\Pi_f(p)}[x]$ and $z \sim_{g_f[y]} w$.

$$\begin{aligned}
&(\pi_0 \overline{h^{-1}[x]}(k))(z) \\
&\sim_{g_p[\overline{(h_f^{-1}[x])}z} \overline{(h_p^{-1}[\overline{(h_f^{-1}[x])}(\overline{(h_f[x])}z)])}(k(\overline{(h_f[x])}(z))) \\
&\sim_{g_p[\overline{(h_f^{-1}[y])}z} \overline{(h_p^{-1}[\overline{(h_f^{-1}[y])}(\overline{(h_f[y])}z)])}(k(\overline{(h_f[y])}(z))) \\
&\sim_{g_p[\overline{(h_f^{-1}[y])}w} \overline{(h_p^{-1}[\overline{(h_f^{-1}[y])}(\overline{(h_f[y])}w)])}(k(\overline{(h_f[y])}(w))) \\
&\sim_{g_p[\overline{(h_f^{-1}[y])}w} (\pi_0 \overline{h^{-1}[y]}(k))(w)
\end{aligned}$$

The second equivalence is because h_f and the family of classes g_p respect equivalences. The third one holds because both $h_f[x]$ and k are assumed to be functions.

(f) Finally, we have to check that h and h^{-1} really are inverses. Let $\tilde{p} :\equiv \langle y_0, p\rangle \dot{\in} pr^{-1}\{x\}$:

$$\begin{aligned}
&\overline{(h^{-1}[x] \circ h[x])}(\tilde{p}) \\
&\sim_{pr^{-1}\{x\}} \langle x, \lambda z.\overline{(h_p^{-1}[\overline{(h_f^{-1}[y])}(\overline{(h_f[y])}z)])}(\\
&\qquad \overline{(\lambda z.(h_p[\overline{(h_f^{-1}[x])}z])}((\pi_1\langle y_0, p\rangle)(\overline{(h_f^{-1}[x])}(z)))) \\
&\qquad (\overline{(h_f[y])}(z)))\rangle \\
&\sim_{pr^{-1}\{x\}} \langle x, \lambda z.\overline{(h_p^{-1}[\overline{(h_f^{-1}[y])}(\overline{(h_f[y])}z)])}(\\
&\qquad (\overline{(h_p[\overline{(h_f^{-1}[x])}(\overline{(h_f[y])}(z))])}(\\
&\qquad p(\overline{(h_f^{-1}[x])}((\overline{(h_f[y])}(z)))))))\rangle
\end{aligned}$$

$$
\begin{aligned}
&\sim_{pr^{-1}\{x\}} \langle x, \lambda z.\overline{(h_p^{-1}[z])}(\\
&\qquad (\overline{(h_p[z])}(p(\overline{(h_f^{-1}[x])}(((\overline{(h_f[y])}(z)))))))\rangle \\
&\sim_{pr^{-1}\{x\}} \langle x, \lambda z.\overline{(h_p^{-1}[z])}(\\
&\qquad (\overline{(h_p[z])}(p(\overline{(h_f^{-1}[x])}(\overline{(h_f[x])}(z))))))\rangle \\
&\sim_{pr^{-1}\{x\}} \langle x, \lambda z.\overline{(h_p^{-1}[z])}(\overline{(h_p[z])}(p(z)))\rangle \\
&\sim_{pr^{-1}\{x\}} \langle x, \lambda z.p(z)\rangle
\end{aligned}
$$

(g) For the other direction suppose we are given $k \mathrel{\dot{\in}} g_{\Pi_f(p)}[y]$.

$$
\begin{aligned}
&\overline{(h[y] \circ h^{-1}[y])}k \\
&\sim_{g_{\Pi_f(p)}[y]} \lambda z.\overline{h_p[\overline{(h_f^{-1}[y])}z]}((\pi_1 \\
&\qquad\qquad (\langle y, \lambda z.\overline{(h_p^{-1}[\overline{(h_f^{-1}[y])}(\overline{(h_f[y])}z)])}(\\
&\qquad\qquad k(\overline{(h_f[y])}(z)))\rangle))(\overline{(h_f^{-1}[y])}z)) \\
&\sim_{g_{\Pi_f(p)}[y]} \lambda z.\overline{h_p[\overline{(h_f^{-1}[y])}z]}((\lambda z.\overline{(h_p^{-1}[\overline{(h_f^{-1}[y])}(\overline{(h_f[y])}z)])}(\\
&\qquad\qquad k(\overline{(h_f[y])}(z))))(\overline{(h_f^{-1}[y])}z)) \\
&\sim_{g_{\Pi_f(p)}[y]} \lambda z.\overline{h_p[\overline{(h_f^{-1}[y])}z]} \\
&\qquad\qquad (\overline{(h_p^{-1}[\overline{(h_f^{-1}[y])}(\overline{(h_f[y])}(\overline{(h_f^{-1}[y])}z))])}(\\
&\qquad\qquad k(\overline{(h_f[y])}\overline{(h_f^{-1}[y])}z))) \\
&\sim_{g_{\Pi_f(p)}[y]} \lambda z.\overline{h_p[\overline{(h_f^{-1}[y])}z]}(\overline{(h_p^{-1}[\overline{(h_f^{-1}[y])}z])}(kz)) \\
&\sim_{g_{\Pi_f(p)}[y]} \lambda z.(kz) \\
&\sim_{g_{\Pi_f(p)}[y]} k
\end{aligned}
$$

Basically all of the above boils down to reducing compositions of isomorphisms to identity enabled the fact that $h_f[\cdot]$ and $h_p[\cdot]$ respect equivalent elements. □

Proposition 4.1.15. *Given two universes* $u \mathrel{\dot\in} u_1$ *in the sense of* $\mathcal{U}(u) \wedge \mathcal{U}(u_1)$ *there are Bishop sets and a morphism in the following way:*

- $\mathfrak{u} :\equiv \langle u_{\mathbf{ECB}}, ext(u_{\mathbf{ECB}})\rangle$ *is a Bishop set of Bishop sets up to extensional equality w.r.t. class membership in the category* $\mathbf{ECB}(u_1)$ *where*

$$\begin{aligned} BEXT[u, X, R, Y, S] &:\equiv u = 0 \wedge \forall z(z \in X \leftrightarrow z \in Y \\ &\qquad \wedge z \mathrel{\dot\in} R \leftrightarrow Z \mathrel{\dot\in} S) \\ \widetilde{ext}(u_{\mathbf{ECB}}) &:\equiv \sum_{x:u_{\mathbf{ECB}}} \sum_{y:u_{\mathbf{ECB}}} t_{BEXT}(\|x\|, x_E, \|y\|, y_E) \\ EXT[u, E] &:\equiv \langle \pi_0 u, \pi_1 u, 0\rangle \in E \\ ext(u_{\mathbf{ECB}}) &:\equiv t_{EXT}(\widetilde{ext}(u_{\mathbf{ECB}})). \end{aligned}$$

- $\sum_{\mathbf{ECB}}(\mathfrak{u}, \lambda x.x, \lambda x.\lambda y.\langle x, y, \lambda z.z\rangle)$ *is a Sigma Bishop set.*
- *The first projection from tuples* π_0 *induces a morphism*

$$el : \sum_{\mathbf{ECB}} (\mathfrak{u}, \lambda x.x, \lambda x.\lambda y.\langle x, y, \lambda z.z\rangle) \overset{m}{\to} \mathfrak{u}.$$

We will write $x \overset{E}{=} y$ *for* $x \sim_{\mathfrak{u}} y$ *and* Σ *for the domain of el.*

Proof. For the first part just note that $x \overset{E}{=} y \leftrightarrow \forall z(z \mathrel{\dot\in} \|x\| \leftrightarrow z \mathrel{\dot\in} \|y\|) \wedge \forall u, v(\langle u, v\rangle \mathrel{\dot\in} x_{\mathrm{E}} \leftrightarrow \langle u, v\rangle \mathrel{\dot\in} y_{\mathrm{E}})$ is clearly an equivalence relation.

For the second part we have to check the requirements for Sigma Bishop sets. We need $F[\mathfrak{u}, \lambda x.x, \lambda x.\lambda y.\langle x, y, \lambda z.z\rangle]$. All elements of a universe are Bishop sets which are just mapped by identity. So we only have to check the second operation. $\langle x, y, \lambda z.z\rangle$ is always a morphism by extensionality: For $x \overset{E}{=} y$ we have the same equivalence relation in the domain and codomain, so $\lambda z.z$ is a function and isomorphic. In fact if $x = y$ holds then it is exactly $id(x)$.

The last part is similarly easy: $\langle x, w\rangle \sim_{\Sigma} \langle y, v\rangle$ implies in particular $x \overset{E}{=} y$ so *el* is a function. □

Proposition 4.1.16. *Let* $\mathfrak{u} :\equiv \langle u, ext(u)\rangle$ *and let* $f : a \overset{m}{\rightarrow} b$ *be any morphism such that* $\mathfrak{CU}[f, u]$*. Then* f *is a pullback of the morphism* $el : \Sigma \overset{m}{\rightarrow} \mathfrak{u}$ *given in proposition 4.1.15 along some* $g : b \overset{m}{\rightarrow} \mathfrak{u}$*.*

Proof. Let $h_f[x] : f^{-1}\{x\} \overset{m}{\rightarrow} g_f[x]$ be the isomorphism which exists because $\mathfrak{CU}[f, u]$.

We will show that that following diagram describes a pullback.

$$\begin{tikzcd}
a \arrow[r, "q"] \arrow[d, "f"'] & \Sigma \arrow[d, "el"] \\
b \arrow[r, "g"'] & \mathfrak{u}
\end{tikzcd}$$

where

$$q :\equiv \langle a, \Sigma, \lambda x.\langle g_f[\overline{f}x], h_f[\overline{f}x](x)\rangle\rangle$$
$$g :\equiv \langle b, \mathfrak{u}, \lambda x.g_f[x]\rangle.$$

We have to show the following

(a) g is a function,

(b) q is a function,

(c) For any $k : c \overset{m}{\rightarrow} a$, $l : c \overset{m}{\rightarrow} \Sigma$ with $g \circ k =_m el \circ l$ there exists $r : c \overset{m}{\rightarrow} a$ with $f \circ r =_m k$ and $q \circ r =_m l$.

(d) This r is unique.

(a) Let $x \sim_b y$. $\mathfrak{CU}[f, u]$ requires that $z \mathrel{\dot\in} g_f[x] \leftrightarrow z \mathrel{\dot\in} g_f[y]$. In fact $g_f[x] \overset{E}{=} g_f[y]$ has to hold since we require that

$$(\forall z \mathrel{\dot\in} g[x])(\overline{(h_f[x])}z \sim_{g_f[x]} \overline{(h_f[y])}z)$$

is "well-typed" and true. Because we chose $ext(u)$ as equivalence for $\mathfrak{u}$ this is a function.

(b) Let $x \sim_a y$ we have to show $\overline{q}x \sim_\Sigma \overline{q}y$. The equivalence relation in Σ amounts to the conjunction of component-wise equivalences:

$$\langle m, w\rangle \sim_\Sigma \langle n, v\rangle \leftrightarrow m \overset{E}{=} n \wedge w \sim_n v.$$

But the argument from the previous claim still applies. $g_f[\cdot]$ and $h_f[\cdot]$ respect equivalence and hence

$$\overline{q}x = \langle g_f[\overline{f}x], \overline{(h_f[\overline{f}x])}(x)\rangle \sim_\Sigma \langle g_f[\overline{f}y], h_f[\overline{f}y](y)\rangle = \overline{q}y.$$

(c) Let $k : c \overset{m}{\rightarrow} a$ and $l : c \overset{m}{\rightarrow} \Sigma$ be morphisms such that $g \circ k =_m el \circ l$.

We define $r : c \overset{m}{\rightarrow} a$ as

$$r :\equiv \langle c, a, \lambda c.\overline{(h_f^{-1}[\overline{k}c])}(\pi_1(\bar{l}c))\rangle.$$

We have to check this is well-defined. Let us remember the following signature

$$h_f^{-1}[\overline{k}c] : g_f[\overline{k}c] \overset{m}{\rightarrow} f^{-1}\{\overline{k}c\}.$$

By assumption we have $g \circ k =_m el \circ l$ and so in particular $g[\overline{k}c] \overset{E}{=} \pi_0(\bar{l}c)$. This implies that $\pi_1(\bar{l}c) \dot{\in} \|g[\overline{k}c]\|$. Since $\|f^{-1}\{\overline{k}c\}\| \dot{\subset} a$ this is an operation which is defined on the whole domain.

Now let $c_0 \sim_c c_1$ be two equivalent elements. As in the previous claims we have $g_f[\overline{k}c_0] \overset{E}{=} g_f[\overline{k}c_1]$ and $\overline{(h_f^{-1}[\overline{k}c_0])}(\pi_1(\bar{l}c_0)) \sim_a \overline{(h_f^{-1}[\overline{k}c_1])}(\pi_1(\bar{l}c_0))$. Since $h^{-1}[\overline{k}c_0]$ and l are functions, and since the equivalence of Σ as stated above is define basically component-wise we also get

$$\overline{(h_f^{-1}[\overline{k}c_0])}(\pi_1(\bar{l}c_0)) \sim_a \overline{(h_f^{-1}[\overline{k}c_1])}(\pi_1(\bar{l}c_1)).$$

Note that the a in the equivalence holds because $(f^{-1}\{b_0\})_{\mathrm{E}} \dot{\subset} a$ for all $b_0 \dot{\in} \|b\|$.

With this we have shown that r is a function. It remains to show uniqueness.

(d) Let $r_0, r_1 : c \overset{m}{\to} a$ be morphisms with $f \circ r_i =_m k$ and $q \circ r_i =_m l$ for $i = 0, 1$.

For arbitrary $x \dot{\in} c$ we have to show $\overline{r_0}x \sim_a \overline{r_1}x$.

Consider $\overline{q}(\overline{r_i}x)$.

$$\overline{q}(\overline{r_i}x) = \langle g_f[\overline{f}(\overline{r_i}x)], \overline{(h_f[\overline{f}(\overline{r_i}x)])}(\overline{r_i}x)\rangle$$

We have $\overline{f}(\overline{r_0}x) \sim_b \overline{k}x \sim_b \overline{f}(\overline{r_0}c_1)$ from $f \circ r_i =_m k$. The other equation yields $\overline{q}(\overline{r_0}x) \sim_\Sigma \overline{l}x \sim_\Sigma \overline{q}(\overline{r_1}x)$. From this we can look at $\overline{q}(\overline{r_i}x)$ component-wise:

$$g_f[\overline{f}(\overline{r_0}x)] \overset{E}{=} g_f[\overline{k}x] \overset{E}{=} g_f[\overline{f}(\overline{r_1}x)]$$

and

$$\begin{aligned}
\overline{(h_f[\overline{k}x])}(\overline{r_0}x) &\sim_{g_f[\overline{k}x]} \overline{(h_f[\overline{f}(\overline{r_0}x)])}(\overline{r_0}x) \\
&\sim_{g_f[\overline{k}x]} \pi_1(\overline{q}(\overline{r_0}x)) \\
&\sim_{g_f[\overline{k}x]} \pi_1(\overline{l}x) \\
&\sim_{g_f[\overline{k}x]} \pi_1(\overline{q}(\overline{r_1}x)) \\
&\sim_{g_f[\overline{k}x]} \overline{(h_f[\overline{f}(\overline{r_1}x)])}(\overline{r_1}x) \\
&\sim_{g_f[\overline{k}x]} \overline{(h_f[\overline{k}x])}(\overline{r_1}x)
\end{aligned}$$

This works of again only because the equivalence relation of Σ is defined component-wise. So $q \circ r_i =_m l$ tells us in particular that, applied to any elements of the domain, the second components of the result in Σ are equivalent. But now we are done. $h_f[\overline{k}x]$ is an isomorphism, hence we can apply $h_f^{-1}[\overline{k}x]$ which yields

$$\overline{r_0}x \sim_a \overline{r_1}x$$

which just means that $r_0 =_m r_1$. □

Definition 4.1.17 (Weak predicative universe in a category)**.**
Let $\mathcal{C}$ be a locally cartesian closed category, el be some morphism in $\mathcal{C}$ and $\mathcal{S}[x]$ be a formula. We call $\mathcal{S}$ a weak universe in $\mathcal{C}$ if the following axioms hold.

- $(U1), (U3), (U4), (U5)$ from definition 4.1.1.
- $(U2W)$ $\quad \forall f, g(Mor(f,g) \wedge dom(f) =_o cod(g)$
  $\wedge\, cod(f) =_o dom(g) \wedge ISO[f,g] \rightarrow \mathcal{S}[f] \wedge \mathcal{S}[g]$

  The closure under identity morphisms and isomorphisms. $\diamond$

Theorem 4.1.18. $\mathfrak{CU}$ *satisfies the closure conditions of the weak universe defined in definition 4.1.17.*

Proof. This is just a direct consequence of the propositions proved above.

- $(U1)$ Closure under pullbacks is theorem 4.1.10.
- $(U2W)$ Closure under isomorphisms is proposition 4.1.12.
- $(U3)$ Closure under Sigma is proposition 4.1.13.
- $(U4)$ Closure under Pi is proposition 4.1.14.
- $(U5)$ Existence of a universal morphism is proposition 4.1.16. $\square$

Theorem 4.1.19. *All Bishop sets* $b \mathrel{\dot\in} u$ *are contained in* $\mathfrak{CU}[\cdot, u]$ *via the morphism* $!_b : b \overset{m}{\rightarrow} \mathbb{1}$.

Proof. We define

$$\begin{aligned} g[*] &:\equiv b \\ h[*] &:\equiv \langle (!_b)^{-1}\{*\}, g[*], \lambda x.x \rangle \\ h^{-1}[*] &:\equiv \langle g[*], (!_b)^{-1}\{*\}, \lambda x.x \rangle \end{aligned}$$

h, h^{-1} are well-defined, because $(!_b)^{-1}\{*\}$ is always extensionally equal to b. Furthermore h is clearly an isomorphism, and g maps all elements of $\mathbb{1}$ into u. But that is exactly what we have to show for $\mathfrak{CU}[!_b, u]$. $\square$

4.2. Cardinal Numbers

Notation 4.2.1. We will now adopt similar notation as for **ECB**. If $x :\equiv\ a \overset{a_0}{\underset{a_1}{\rightrightarrows}} b$ is an object in $\mathcal{C}_{\text{ex}}$, we will write $\|x\|$ for b and x_{E} for a.

For $\langle w, refl_x \circ w\rangle, \langle z, refl_x \circ z\rangle : \mathbb{1} \overset{m}{\to} x$ and some $\gamma : \mathbb{1}_{\mathcal{C}} \overset{m}{\to} x_{\mathrm{E}}$ which witnesses $\langle w, refl_x \circ w\rangle =^{\mathcal{C}_{\text{ex}}}_{m} \langle z, refl_x \circ z\rangle$ we write

$$w \sim_x z$$

or

$$\gamma : w \sim_x z$$

if we need to remember the proof object.

Specialized to $\mathbf{EC}_{\text{ex}}$ this just means that γ selects some $r \mathrel{\dot\in} x_{\mathrm{E}}$ such that $\overline{a_0}(r) = \overline{w}(*)$ and $\overline{a_1}(r) = \overline{z}(*)$. ◇

Example 4.2.2 (Cardinal numbers)**.** Let $u_0 \mathrel{\dot\in} u_1$ be two universes and $\mathbf{EC}(u_0)$ the category **EC** restricted to $Ob(x) \leftrightarrow x \mathrel{\dot\in} u_0$.

$$\begin{aligned}
I[u, \mathcal{C}] :\equiv u = \langle x, f, g, y\rangle \wedge x, y \in Ob \\
\wedge f : x \overset{m}{\to} y \wedge g : y \overset{m}{\to} x \wedge ISO[f, g] \\
i_0 :\equiv \langle i(\mathbf{EC}(u_0)), u_0, \pi_0\rangle \\
i_1 :\equiv \langle i(\mathbf{EC}(u_0)), u_0, \pi_3\rangle \\
r_\kappa :\equiv \langle u_0, i(\mathbf{EC}(u_0)), \lambda x.\langle x, id(x), id(x), x\rangle\rangle
\end{aligned}$$

Symmetry s_κ and transitivity t_κ are tracked by terms which perform the following shuffling.

$$\begin{aligned}
&\langle x, f, g, y\rangle \mapsto \langle y, g, f, x\rangle \\
&\langle\langle x, f, g, y\rangle, \langle y, h, k, z\rangle\rangle \mapsto \langle x, h \circ f, g \circ k, z\rangle
\end{aligned}$$

We call the object $\kappa :\equiv\ i(\mathbf{EC}(u_0)) \overset{i_0}{\underset{i_1}{\rightrightarrows}} u_0$ in $\mathbf{EC}(u_1)_{\text{ex}}$ the cardinals.

Cardinal addition is given by the coproduct:

$$\begin{aligned}
\tilde{c}(\langle x, y\rangle) &:\equiv x + y \\
c &:\equiv \langle u_0 \times u_0, u_0, \tilde{c}\rangle \\
\tilde{r}(\langle\langle a, f, g, b\rangle, \langle c, h, k, d\rangle\rangle) &:\equiv \langle a + c, f \oplus h, g \oplus k, b + d\rangle \\
r &:\equiv \langle i(\mathbf{EC}(u_0)) \times i(\mathbf{EC}(u_0)), i(\mathbf{EC}(u_0)), \tilde{r}\rangle \\
+_\kappa &:\equiv \langle \kappa \times \kappa, \kappa, \langle c, r\rangle\rangle
\end{aligned}$$

Cardinal multiplication is constructed by the product.

$$\begin{aligned}
\tilde{p}(\langle x, y\rangle) &:\equiv x \times y \\
p &:\equiv \langle u_0 \times u_0, u_0, \tilde{p}\rangle \\
\tilde{r}(\langle\langle a, f, g, b\rangle, \langle c, h, k, d\rangle\rangle) &:\equiv \langle a + c, \langle\!\langle f, h\rangle\!\rangle, \langle\!\langle g, k\rangle\!\rangle, b + d\rangle \\
r &:\equiv \langle i(\mathbf{EC}(u_0)) \times i(\mathbf{EC}(u_0)), i(\mathbf{EC}(u_0)), \tilde{r}\rangle \\
\cdot_\kappa &:\equiv \langle \kappa \times \kappa, \kappa, \langle p, r\rangle\rangle
\end{aligned}$$

Finite cardinals are given by finite classes, represented by

$$[k] :\equiv \{n \mid n \in N \wedge n < k\}. \qquad \diamond$$

Proposition 4.2.3 (Properties of cardinal addition)**.** *Let* $\xi, \eta, \zeta \mathrel{\dot{\in}} \|\kappa\|$.

(a) $\xi +_\kappa [0] \sim_\kappa \xi$

(b) $([1] +_\kappa [1]) \sim_\kappa [2]$

(c) $\xi +_\kappa \eta \sim_\kappa \eta +_\kappa \xi$

(d) $(\xi +_\kappa \eta) +_\kappa \zeta \sim_\kappa \xi +_\kappa (\eta +_\kappa \zeta)$

Proof. Zero is the additive neutral element. $\langle \xi, [0]\rangle$ is mapped to $\xi + [0]$. But $[0]$ has no elements, so we can proof that

$$\begin{aligned}
h &:\equiv \langle \xi, \xi + [0], \lambda x.\langle 0, x\rangle\rangle \\
h^{-1} &:\equiv \langle \xi + [0], \xi, \lambda x.\pi_1 x\rangle
\end{aligned}$$

are inverses since there are no elements $\langle 1, z\rangle \mathrel{\dot{\in}} \xi + [0]$ as this would imply $z \mathrel{\dot{\in}} \emptyset$. And so $\langle \xi, h, h^{-1}, \xi + [0]\rangle$ is a witness for $\xi \sim_\kappa (\xi +_\kappa [0])$.

For $([1] +_\kappa [1]) \sim_\kappa [2]$ we can write down the isomorphism

$$\begin{aligned} f &:\equiv \langle [1] + [1], [2], \lambda x.\pi_0 x\rangle \\ f^{-1} &:\equiv \langle [2], [1] + [1], \lambda x.\langle x, 0\rangle\rangle \end{aligned}$$

which shows $([1] +_\kappa [1]) \sim_\kappa [2]$.

Commutativity is easy to see since $\xi +_\kappa \eta$ is mapped to $\xi + \eta$ which is of course isomorphic to $\eta + \xi$ through the morphism $inr \oplus inl$.

For associativity we note just note that we can use definition by numerical cases to map

$$\begin{pmatrix} \langle 0, \langle 0, x\rangle\rangle \\ \langle 0, \langle 1, y\rangle\rangle \\ \langle 1, z\rangle \end{pmatrix} \leftrightarrow \begin{pmatrix} \langle 0, x\rangle \\ \langle 1, \langle 0, y\rangle\rangle \\ \langle 1, \langle 1, z\rangle\rangle . \end{pmatrix}$$

□

Proposition 4.2.4 (Properties of cardinal multiplication)**.** *Let* $\xi, \eta, \zeta \mathrel{\dot\in} \|\kappa\|$.

(a) $\xi \cdot_\kappa [0] \sim_\kappa [0]$

(b) $(\xi \cdot_\kappa [1]) \sim_\kappa \xi$

(c) $\xi \cdot_\kappa \eta \sim_\kappa \eta \cdot_\kappa \xi$

(d) $(\xi \cdot_\kappa \eta) \cdot_\kappa \zeta \sim_\kappa \xi \cdot_\kappa (\eta \cdot_\kappa \zeta)$

Proof. $\xi \times [0]$ contains pairs $\langle x, y\rangle$ with $x \mathrel{\dot\in} \xi \wedge y \mathrel{\dot\in} [0]$. Since $y \mathrel{\dot\in} [0]$ implies $y \in N \wedge y < 0$ no such pairs can exist.

$\xi \times [1]$ is build from pairs $\langle x, 0\rangle$ which we can clearly map to x and back.

The two other claims follow from the existence of the obvious universal maps into products which provide isomorphisms. □

Remark 4.2.5. The construction above, while being a partition of the universe, doesn't behave exactly how we would expect from a definition of cardinals in a strong enough, classical system.

We can for example for any two element class $\{a, b\}$ construct a surjection into it from $[2]$, however constructing an inverse may or may not be possible. If we consider the Church Booleans $\{\mathtt{tt}, \mathtt{ff}\}$ where $\mathtt{tt} :\equiv \lambda xy.x$ and $\mathtt{ff} :\equiv \lambda xy.y$ we can construct an inverse with $\lambda x.x01$ which sends $\mathtt{tt}$ to 0 and $\mathtt{ff}$ to 1. In general it's however not possible to give such a map even when we know there exist exactly two elements in a class. The main reason for this is that the operation for definition by cases is restricted to natural number arguments. This would again be different, if we were to allow (AC_V) as in subsection 3.2. ◇

4.3. Combinatory Logic

As a second example we will give an explicit Bishop set of terms modulo variable renaming of combinatory logic. For this we're going to simplify our notation of operations to allow operations which are essentially specified like Haskell-terms; i.e. several lines per operation with pattern-matching on all terms. We will also not give full details, since such notation is valid as long as we restrict ourselves to natural numbers, which is what we're going to do by using A.1.1 and a coding for terms.

Definition 4.3.1. We call the elements of the class below *pre-terms* of our combinatory logic.

$$\begin{aligned}
A_0[u] &:\equiv u = \langle|1,0|\rangle \vee u = \langle|2,0|\rangle \vee (u = \langle|3,l|\rangle \wedge l \in N) \\
A_n[u, A_{n-1}] &:\equiv u = \langle|4,a,b|\rangle \wedge a,b \in A_{n-1} \\
g(f,x) &:\equiv \begin{cases} t_{A_0} & x = 0 \\ t_{A_n}(f(x-1)) & x \neq 0 \end{cases} \\
a &:\equiv (fixg)
\end{aligned}$$

We have terms s, k, variables x_k and applications $(a\ b)$. We provide a translation $(\cdot)^*$ into pre-terms:

$$\begin{aligned}
k^* &:\equiv \langle|1,0|\rangle \\
s^* &:\equiv \langle|2,0|\rangle \\
x_l^* &:\equiv \langle|3,l|\rangle \\
(ab)^* &:\equiv \langle|4,a^*,b^*|\rangle
\end{aligned}$$

◇

Definition 4.3.2. Substitutions are given by

$$\begin{aligned} S_0[u] &:\equiv u = \langle|1,0|\rangle \\ S_n[u, S_{n-1}] &:\equiv u = \langle|2, \langle|k,l|\rangle, s|\rangle \wedge k,l \in N \wedge s \in S_{n-1} \\ \mathit{subst} &:\equiv \text{defined similar to the class } a \text{ for pre-terms} \end{aligned}$$

We give again a translation:

$$\begin{aligned} \varepsilon^* &:\equiv \langle|0,1|\rangle \\ [k \mapsto l]s^* &:\equiv \langle|2, \langle|k,l|\rangle, s|\rangle \end{aligned}$$

◇

We have a concatenation $+\!\!+$ operation defined on Substitutions:

$$\begin{aligned} \varepsilon^* +\!\!+ \varepsilon^* &:\equiv \varepsilon^* \\ [u \mapsto v]w^* +\!\!+ \varepsilon^* &:\equiv [u \mapsto v]w^* \\ \varepsilon^* +\!\!+ [u \mapsto v]w^* &:\equiv [u \mapsto v]w^* \\ [u \mapsto v]w^* +\!\!+ [l \mapsto m]n^* &:\equiv [u \mapsto v](w^* +\!\!+ [l \mapsto m]n^*), \end{aligned}$$

the obvious substitution-reversal $(\cdot)^{-1}$ and there are operations to apply a substitution to a pre-term and to check of this substitution is valid for a given term.

$$\begin{aligned} \theta(x_l^*, \varepsilon^*) &:\equiv x_l^* \\ \theta(x_l^*, [l \mapsto m]p^*) &:\equiv \theta(x_m^*, p) \\ \theta(x_k^*, [l \mapsto m]p^*) &:\equiv \theta(x_k^*, p) && k \neq l \\ \theta(k^*, p) &:\equiv k^* \\ \theta(s^*, p) &:\equiv s^* \\ \theta((ab)^*, [l \mapsto m]p^*) &:\equiv (\theta(a^*, [l \mapsto m]p^*)\theta(b^*, [l \mapsto m]p^*))^* \end{aligned}$$

By validity, we mean

$$\mathit{valid}(t^*, s) :\equiv \begin{cases} 1 & \mathrm{lh}(fv(t)) = \mathrm{lh}(fv(\theta(t,s))) \\ 0 & \text{otherwise} \end{cases}$$

where

$$fv(t^*) :\equiv \text{list of all unique variables in } t.$$

Clearly, such a term exists. (See the `Data.List` module [24] of the Haskell programming language standard library for explicit function-definitions for manipulating lists.) Note that this is just a restriction to stop identifications like

$$(xy)^* \sim (zz)^*.$$

In other words we make sure substitutions are always injective, since otherwise symmetry would fail as the example clearly shows.

Example 4.3.3. Terms are given by the explicit Bishop Set defined as follows.

$$\begin{aligned}
R[u, P, S] &:\equiv u = \langle| x, s, y |\rangle \wedge x, y \in P \wedge s \in S \\
&\qquad\qquad \wedge\ valid(x, s) = 1 \wedge \theta(x, s) = y \\
r &:\equiv t_R(a, s) \\
r_0 &:\equiv \langle r, a, \pi_0^{\mathrm{N}} \rangle \\
r_1 &:\equiv \langle r, a, \pi_2^{\mathrm{N}} \rangle \\
refl_{ra} &:\equiv \langle a, r, \lambda x. \langle| x, \varepsilon^*, x |\rangle \rangle \\
symm_{ra} &:\equiv \langle r, r, \lambda \langle| x, s, y |\rangle . \langle| y, (s)^{-1}, x |\rangle \rangle \\
tran_{ra} &:\equiv \langle r_1 * r_0, r, \lambda \langle| \langle| x, s, y |\rangle, \langle| y, t, z |\rangle |\rangle . \langle| x, s \mathbin{+\!\!+} t, z |\rangle \rangle
\end{aligned}$$

We call $\tau :\equiv\ r \underset{r_1}{\overset{r_0}{\rightrightarrows}} a$ the explicit Bishop set of terms. Note that $tran_{ra}$ is only well-defined, because concatenation of valid substitutions is always valid for a given term. ◇

We have a small-step evaluation morphism (small-step operational semantics, see [37].)

$$\begin{aligned}
e(((ka)b)^*) &:\equiv a^* \\
e((((sa)b)c)^*) &:\equiv ((ac)(bc))^* \\
e((ab)^*) &:\equiv \begin{cases} \langle|4, a^*, e(b^*)|\rangle & e(a^*) = a^* \\ \langle|4, e(a^*), b^*|\rangle & \text{otherwise} \end{cases} \\
e(x_l^*) &:\equiv x_l^* \\
e(k^*) &:\equiv k^* \\
e(s^*) &:\equiv s^*
\end{aligned}$$

The full morphism is then

$$\begin{aligned}
ev'_\tau &:\equiv \langle r, r, \lambda p.\langle|e(\pi_0^{\mathrm{N}} p), \pi_1^{\mathrm{N}} p, e(\pi_2^{\mathrm{N}} p)|\rangle\rangle \\
ev_\tau &:\equiv \langle a, a, e\rangle
\end{aligned}$$

$$\langle|ev_\tau, ev'_\tau|\rangle : \tau \overset{m}{\rightarrow} \tau$$

Well-definedness can be proved by case-analysis: Only if two pre-terms in a contain an application of s or k of the correct form does something change, but both of those evaluation-steps only ever delete variables from a term (in case of k) or copy variables which already existed in case of s. But since no new variables are ever created by evaluation, any substitution remains valid. This only works because we don't require a substitution to be minimal in any sense. This shows in particular, that τ is a pseudo-equivalence relation but not an equivalence relation.

Definition 4.3.4 (Quotient up to finite evaluation steps)**.** We define an n-step evaluation:

$$\begin{aligned}
en(0) &:\equiv \lambda x.x \\
en(l) &:\equiv \lambda t.en(l-1)(e(t))
\end{aligned}$$

Let

$$\begin{aligned}
Q[u,A] &:\equiv u = \langle|t,l,v|\rangle \wedge t,v \in A \wedge l \in N \wedge (en(l)t = en(l)v)\\
S[u,Q,R] &:\equiv u = \langle|\langle|t_0,l,v_0|\rangle, s_0, s_1, \langle|t_1,m,v_1|\rangle|\rangle\\
&\qquad \wedge \langle|t_0,s_0,t_1|\rangle, \langle|v_0,s_1,v_1|\rangle \in R \wedge (\pi_0^{\mathrm{N}}u), (\pi_3^{\mathrm{N}}u) \in Q\\
q &:\equiv t_Q(a)\\
s &:\equiv t_S(q,r)\\
s_0 &:\equiv \langle s, q, \pi_0^{\mathrm{N}}\rangle\\
s_1 &:\equiv \langle s, q, \pi_3^{\mathrm{N}}\rangle\\
refl_{sq} &:\equiv \langle q, s, \lambda x.\langle|x, \varepsilon^*, \varepsilon^*, x|\rangle\rangle\\
symm_{sq} &:\equiv \langle s, s, \lambda\langle|\langle|t_0,l,v_0|\rangle, s_0, s_1, \langle|t_1,m,v_1|\rangle|\rangle.\\
&\qquad \langle|\langle|t_1,m,v_1|\rangle, (s_0)^{-1}, (s_1)^{-1}, \langle|t_0,l,v_0|\rangle|\rangle\rangle\\
tran_{sq} &:\equiv \langle s * s, s,\\
&\qquad \lambda\langle\langle|\langle|t_0,l,v_0|\rangle, s_0, s_1, \langle|t_1,m,v_1|\rangle|\rangle,\\
&\qquad \langle|\langle|t_1,m,v_1|\rangle, s_2, s_3, \langle|t_2,n,v_2|\rangle|\rangle\rangle.\\
&\qquad \langle|\langle|t_0,l,v_0|\rangle, s_0 \mathbin{+\!\!+} s_2, s_1 \mathbin{+\!\!+} s_3, \langle|t_2,n,v_2|\rangle|\rangle\rangle
\end{aligned}$$

Finally, we get $\langle\!\langle q_0, q_1\rangle\!\rangle : im(\ s \overset{s_0}{\underset{s_i}{\rightrightarrows}} q\ ) \overset{m}{\rightarrowtail} \tau \times \tau$ with induced equivalence relation morphisms; i.e. the eval-relation on q. ◇

Since all equivalence relations in $\mathbf{EC}_{\mathrm{ex}}$ have a coequalizer, we can get the explicit Bishop set $im(\ s \overset{s_0}{\underset{s_i}{\rightrightarrows}} q\ ) \overset{m}{\rightarrowtail} \tau \times \tau \overset{m}{\twoheadrightarrow} e$ of renaming-invariant terms modulo finite applications of s and k terms.

5. Standard Theorems and Future Directions

5.1. Yoneda Embedding in ECB

We now set out to prove the Yoneda Lemma with **ECB** as a stand-in for the Category of Sets.

We're going rename some things which were already defined to make this section more readable and define some new things.

Let $u_0 \dot{\in} u_1$ be universes in the sense of $\mathcal{U}(u_0)$,$\mathcal{U}(u_1)$. $u_{\mathbf{ECB}}$ a universe of implicit Bishop Sets constructed from u_0 and let $\mathcal{C}$ bet an arbitrary u_0-proper local category.

Definition 5.1.1 (Locally small category)**.** If a category $\mathcal{C}$ is u_0-proper local (That is, it has for any two objects c, d classes $lhomclass(c, d)$ containing all morphisms from c to d and $ecm(c, d)$ of the restricted equivalence relation on those morphisms in u_0, then the hom-implicit Bishop set is given by the following two classes:

$$\begin{aligned} \|hom_{\mathcal{C}}(c, d)\| &:\equiv lhomclass(c, d) \\ hom_{\mathcal{C}}(c, d)_{\mathrm{E}} &:\equiv ecm(c, d) \end{aligned}$$

We call a (non-weak!) u_0-proper local category from now on *locally small.* ◇

Notation 5.1.2. We're going to use the following notation during this section. $\mathbf{ECB}_u$ for the non-weak version of **ECB** restricted to $u_{\mathbf{ECB}}$ and

$$\begin{aligned} [\mathcal{C}^{op}, \mathbf{ECB}_u] &:\equiv \mathbf{ECB}_u^{\mathcal{C}^{op}}, \\ [\mathcal{C}^{op}, \mathbf{ECB}_u](a, b) &:\equiv hom_{\mathbf{ECB}_u^{\mathcal{C}^{op}}}(a, b). \end{aligned}$$ ◇

Definition 5.1.3 (hom-functor)**.** Let $\mathcal{C}$ be a locally small category and $c \mathrel{\dot\in} ob_{\mathcal{C}}$.

$$\begin{aligned}\mathcal{C}(\cdot, c)_o &:\equiv \lambda d.\mathcal{C}(d, c)\\ \mathcal{C}(\cdot, c)_m &:\equiv \lambda f.\langle \mathcal{C}(dom(f), c), \mathcal{C}(cod(f), c), \lambda g.g \circ f^{op}\rangle\end{aligned}$$

where

$$f^{op} :\equiv \langle cod(f), dom(f), \overline{f}\rangle.$$

We're going to write $\mathcal{C}(\cdot, c)$ for the functor

$$\langle \mathcal{C}(\cdot, c)_o, \mathcal{C}(\cdot, c)_m\rangle : \mathcal{C}^{op} \to \mathbf{ECB}_u. \qquad \diamond$$

Definition 5.1.4 (Yoneda embedding)**.** Let $\mathcal{C}$ be locally small. Using the contravariant hom-functor ($\mathcal{C}(\cdot, c) \mathrel{\dot\in} [\mathcal{C}^{op}, \mathbf{ECB}_u]$) for any $c \mathrel{\dot\in} ob_{\mathcal{C}}$, we can construct a functor $y : \mathcal{C} \to [\mathcal{C}^{op}, \mathbf{ECB}_u]$.

$$\begin{aligned}y(c) &:\equiv \mathcal{C}(\cdot, c)\\ y_m(f) &:\equiv \langle y(dom(f)), y(cod(f)),\\ &\qquad \lambda x.\langle \mathcal{C}(\cdot, dom(f))_o(x), \mathcal{C}(\cdot, cod(f))_o(x), \lambda g.f \circ g\rangle\rangle.\end{aligned}$$

The fact that this is an *embedding* is the content of the Yoneda Lemma below. $\diamond$

Lemma 5.1.5. *Let $\mathcal{C}$ be a locally small category, $c \mathrel{\dot\in} \mathcal{C}$ and $\langle f_o, f_m\rangle :\equiv f : \mathcal{C}^{op} \to \mathbf{ECB}_u$ a functor. Then there is a natural isomorphism*

$$[\mathcal{C}^{op}, \mathbf{ECB}_u](y(c), f) \cong f_o(c).$$

More formally, Let the following define two functors z and w :

$$\begin{aligned}\langle z_o, z_m\rangle &: \mathcal{C}^{op} \times [\mathcal{C}^{op}, \mathbf{ECB}_u] \to \mathbf{ECB}\\ z_o(c, f) &:\equiv [\mathcal{C}^{op}, \mathbf{ECB}_u](y(c), f)\\ z_m(g, \vartheta) &:\equiv \langle z_o(dom(g), dom(\vartheta)), z_o(cod(g), cod(\vartheta)),\\ &\qquad \lambda\eta.\vartheta \circ \eta \circ y(g^{op})\rangle\end{aligned}$$

$$\begin{aligned}
\langle w_o, w_m\rangle &: \mathcal{C}^{op} \times [\mathcal{C}^{op}, \mathbf{ECB}_u] \to \mathbf{ECB} \\
w_o(c,f) &:\equiv f_o(c) \\
w_m(g,\vartheta) &:\equiv \langle dom(\vartheta)_o(dom(g)), cod(\vartheta)_o(cod(g)), \\
&\qquad \lambda w.\overline{\vartheta}(cod(g))(\overline{dom(\vartheta)_m(g)}w)\rangle.
\end{aligned}$$

There exists a natural isomorphism $\varphi : z \cong w : \psi$.

Proof.
Note that z_m and w_m could be more readably written as

$$z_m(k : c \overset{m}{\to} d, \vartheta : f \Rightarrow g) :\equiv \lambda\eta.\vartheta \circ \eta \circ y(k^{op}) : z_o(c,f) \overset{m}{\to} z_o(d,g)$$

$$w_m(k : c \overset{m}{\to} d, \vartheta : f \Rightarrow g) :\equiv \lambda w.\overline{\vartheta}(d)(\overline{f_m(k)}w) : f_o(c) \overset{m}{\to} g_o(d).$$

and that $\langle z_o, z_m\rangle$ is just the composition of the functors

$$\begin{tikzcd}
\mathcal{C}^{op} \times [\mathcal{C}^{op}, \mathbf{ECB}_u] \arrow[d, "y^{op} \times id"'] \arrow[dr, "z"] & \\
{[\mathcal{C}^{op}, \mathbf{ECB}_u]^{op} \times [\mathcal{C}^{op}, \mathbf{ECB}_u]} \arrow[r, "{[\mathcal{C}^{op}, \mathbf{ECB}_u](\cdot,\cdot)}"'] & \mathbf{ECB}.
\end{tikzcd}$$

The required natural transformations φ and ψ are then given by

$$\begin{aligned}
\varphi(c,f) &:\equiv \langle z_o(c,f), w_o(c,f), \lambda\eta.\overline{\eta(c)}(id(c))\rangle \\
\psi(c,f) &:\equiv \langle w_o(c,f), z_o(c,f), \lambda a.\langle y(c), f, \\
&\qquad \lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{f_m(h^{op})}a\rangle\rangle\rangle
\end{aligned}$$

To see that these are inverses let $c \mathrel{\dot\in} \mathcal{C}$, $f : \mathcal{C}^{op} \to \mathbf{ECB}_u$, and $\eta : y(c) \Rightarrow f$.
We need to check the following two equations:

$$\begin{aligned}
&\overline{(\varphi(c,f) \circ \psi(c,f))}a \sim_{f_o(c)} a \\
&\overline{(\psi(c,f) \circ \varphi(c,f))}\eta \sim_{z_o(c,f)} \eta
\end{aligned}$$

$$
\begin{aligned}
&\overline{(\varphi(c,f) \circ \psi(c,f))}a \\
&\sim_{f_o(c)} \overline{(\langle z_o(c,f), w_o(c,f), \lambda\eta.\overline{\overline{\eta}(c)}(id(c))\rangle)} \\
&\qquad ((\lambda a.\langle y(c), f, \lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{f_m(h^{op})}a\rangle\rangle)a) \\
&\sim_{f_o(c)} (\lambda\eta.\overline{\overline{\eta}(c)}(id(c)))(\langle y(c), f, \lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{f_m(h^{op})}a\rangle\rangle) \\
&\sim_{f_o(c)} \overline{\overline{(\langle y(c), f, \lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{f_m(h^{op})}a\rangle\rangle)}(c)}(id(c)) \\
&\sim_{f_o(c)} \overline{\lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{f_m(h^{op})}a\rangle(c)}(id(c)) \\
&\sim_{f_o(c)} \overline{\langle \mathcal{C}(c,c), f_o(c), \lambda h.\overline{f_m(h^{op})}a\rangle}(id(c)) \\
&\sim_{f_o(c)} (\lambda h.\overline{f_m(h^{op})}a)(id(c)) \\
&\sim_{f_o(c)} \overline{f_m((id(c))^{op})}a \\
&\sim_{f_o(c)} \overline{id(f_o(c))}a \\
&\sim_{f_o(c)} a
\end{aligned}
$$

$$
\begin{aligned}
&\overline{(\psi(c,f) \circ \varphi(c,f))}\eta \\
&\sim_{z_o(c,f)} (\lambda a.\langle y(c), f, \lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{f_m(h^{op})}a\rangle\rangle) \\
&\qquad (\overline{\langle z_o(c,f), w_o(c,f), \lambda\eta.\overline{\overline{\eta}(c)}(id(c))\rangle}\eta) \\
&\sim_{z_o(c,f)} (\lambda a.\langle y(c), f, \lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{f_m(h^{op})}a\rangle\rangle) \\
&\qquad ((\lambda\eta.\overline{\overline{\eta}(c)}(id(c)))\eta) \\
&\sim_{z_o(c,f)} (\lambda a.\langle y(c), f, \lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{f_m(h^{op})}a\rangle\rangle)(\overline{\overline{\eta}(c)}(id(c))) \\
&\sim_{z_o(c,f)} \langle y(c), f, \lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{f_m(h^{op})}(\overline{\overline{\eta}(c)}(id(c)))\rangle\rangle \\
&\sim_{z_o(c,f)} \langle y(c), f, \lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{\overline{\eta}(d)}(\overline{\mathcal{C}(\cdot,c)_m(h^{op})}(id(c)))\rangle\rangle \\
&\sim_{z_o(c,f)} \langle y(c), f, \lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{\overline{\eta}(d)}(id(c) \circ ((h^{op})^{op}))\rangle\rangle \\
&\sim_{z_o(c,f)} \langle y(c), f, \lambda d.\langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{\overline{\eta}(d)}(h)\rangle\rangle \\
&\sim_{z_o(c,f)} \eta
\end{aligned}
$$

The fourth equivalence from below in direction $(\psi(c,f)\circ\varphi(c,f))$ is an application of the naturality of $\eta : y(c) \Rightarrow f$:

$$\begin{tikzcd}
\mathcal{C}(c,c) \arrow[r, "\eta(c)"] \arrow[d, "\mathcal{C}(\cdot,c)_m(h)"'] & f_o(c) \arrow[d, "f_m(h)"] \\
\mathcal{C}(d,c) \arrow[r, "\eta(d)"'] & f_o(d)
\end{tikzcd}$$

We still need to show that φ and ψ are actually functional and natural.

Let $a \sim_{f_o(c)} b$ and $d \mathrel{\dot\in} ob_{\mathcal{C}}$.

$$\overline{(\overline{\psi(c,f)}a)}d =_m^{\mathbf{ECB}} \langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{f_m(h^{op})}a\rangle$$

But $f_m(h^{op})$ is a function for any $h \mathrel{\dot\in} hom_{\mathcal{C}}$, and so $\overline{f_m(h^{op})}(a) \sim_{f_o(d)} \overline{f_m(h^{op})}(b)$ holds, which lifts to

$$\begin{aligned}
&=_m^{\mathbf{ECB}} \langle \mathcal{C}(d,c), f_o(d), \lambda h.\overline{f_m(h^{op})}b\rangle \\
&=_m^{\mathbf{ECB}} \overline{(\overline{\psi(c,f)}b)}d.
\end{aligned}$$

Since this is again just pointwise equivalence as a natural transformation, this proves the claim $\overline{(\overline{\psi(c,f)}a)} \sim_{z_o(c,f)} \overline{(\overline{\psi(c,f)}b)}$.

For the reverse, let $\zeta \sim_{[\mathcal{C}^{op},\mathbf{ECB}_u](y(c),f)} \xi$.

$$\begin{aligned}
\overline{\varphi(c,f)}\zeta &\sim_{f_o(c)} (\lambda\eta.\overline{\overline{\eta}(c)}(id(c)))\zeta \\
&\sim_{f_o(c)} \overline{\overline{\zeta}(c)}(id(c)) \\
&\sim_{f_o(c)} \overline{\overline{\xi}(c)}(id(c)) \\
&\sim_{f_o(c)} \overline{\varphi(c,f)}\xi
\end{aligned}$$

The third equivalence follows from the assumption that equivalent natural transformations are pointwise equal: $\overline{\zeta}(c) =_m \overline{\xi}(c)$ which yields two pointwise equivalent functions.

Finally, we have to check naturality. for $h : c \overset{m}{\to} d$ in $\mathcal{C}^{op}$, $\vartheta : f \Rightarrow g$ in $[\mathcal{C}^{op}, \mathbf{ECB}_u](f,g)$ and $a, c, d \mathrel{\dot\in} ob_{\mathcal{C}^{op}}$

$$\begin{tikzcd}
z_o(c,f) = [\mathcal{C}^{op}, \mathbf{ECB}_u](y(c), f) \arrow[r, shift left, "\varphi(c,f)"] \arrow[d, "z_m(h,\vartheta)"'] & w_o(c,f) = f_o(c) \arrow[l, shift left, "\psi(c,f)"] \arrow[d, "w_m(h,\vartheta)"] \\
z_o(d,g) = [\mathcal{C}^{op}, \mathbf{ECB}_u](y(d), g) \arrow[r, leftrightarrow, "\varphi(d,g)", "\psi(d,g)"'] & = g_o(d)
\end{tikzcd}$$

$$\begin{tikzcd}
\eta : y(c) \Rightarrow f \arrow[r, mapsto, "\eta\mapsto\overline{\overline{\eta}(c)}(id(c))"] \arrow[dd, mapsto, "\eta\mapsto\vartheta\circ\eta\circ y_m(h^{op})"'] & \overline{\overline{\eta}(c)}(id(c)) : f_o(c) \arrow[d, mapsto, "w\mapsto\overline{\overline{\vartheta}(d)}(\overline{f_m(h)}(w))"] \\
 & \overline{\overline{\vartheta}(d)}(\overline{f_m(h)}(\overline{\overline{\eta}(c)}(id(c)))) \\
\vartheta\circ\eta\circ y_m(h^{op}) : y(d) \Rightarrow g \arrow[r, mapsto] & \sim_{g_o(d)} \\
 & \overline{\overline{(\vartheta\circ\eta\circ y(c)_m(h^{op}))}(d)}(id(d))
\end{tikzcd}$$

The equivalence on the lower right stems from the following derivation

$$\begin{aligned}
&\overline{\overline{(\vartheta\circ\eta\circ y_m(h^{op}))}(d)}(id(d)) \\
&\sim_{g_o(d)} (\overline{\overline{\vartheta}(d)} \circ \overline{\overline{(\eta\circ y_m(h^{op}))}(d)})(id(d)) \\
&\sim_{g_o(d)} (\overline{\overline{\vartheta\circ\eta}(d)} \circ \overline{\overline{y_m(h^{op})}(d)})(id(d)) \\
&\sim_{g_o(d)} (\overline{\overline{\vartheta\circ\eta}(d)}\circ \\
&\qquad \overline{\overline{(\lambda x.\langle\mathcal{C}(\cdot, dom(h^{op}))_o(x), \mathcal{C}(\cdot, cod(h^{op}))_o(x), \lambda g.h^{op}\circ g\rangle)}(d)}) \\
&\qquad (id(d)) \\
&\sim_{g_o(d)} (\overline{\overline{\vartheta\circ\eta}(d)}\circ \\
&\qquad \overline{\langle\mathcal{C}(\cdot, dom(h^{op}))_o(d), \mathcal{C}(\cdot, cod(h^{op}))_o(d), \lambda g.h^{op}\circ g\rangle})(id(d)) \\
&\sim_{g_o(d)} (\overline{\overline{\vartheta\circ\eta}(d)}\circ \\
&\qquad \overline{\langle\mathcal{C}(d, dom(h^{op})), \mathcal{C}(d, cod(h^{op})), \lambda g.h^{op}\circ g\rangle})(id(d)) \\
&\sim_{g_o(d)} (\overline{\overline{\vartheta\circ\eta}(d)}\circ \\
&\qquad \overline{\langle\mathcal{C}(d,d), \mathcal{C}(d,c), \lambda g.h^{op}\circ g\rangle})(id(d)) \\
&\sim_{g_o(d)} \overline{\overline{\vartheta\circ\eta}(d)}(h^{op}) \\
&\sim_{g_o(d)} \overline{\overline{\vartheta}(d)}(\overline{\overline{\eta}(d)}(h^{op})) \\
&\sim_{g_o(d)} \overline{\overline{\vartheta}(d)}(\overline{\overline{\eta}(d)}(id(c)\circ h^{op}))
\end{aligned}$$

$$
\begin{aligned}
&\sim_{g_o(d)} \overline{\overline{\vartheta}(d)}(\overline{\overline{\eta}(d) \circ \mathcal{C}(\cdot,c)_o(h)})(id(c))\\
&\sim_{g_o(d)} \overline{\overline{\vartheta}(d)}(\overline{f_m(h) \circ \overline{\eta}(c)})(id(c))\\
&\sim_{g_o(d)} \overline{\overline{\vartheta}(d)}(\overline{f_m(h)}(\overline{\overline{\eta}(c)}(id(c))))
\end{aligned}
$$

Th sixth equivalence holds because we have $h^{op} : d \overset{m}{\rightarrow} c$ in $\mathcal{C}$, and the third equivalence from below follows from $\eta : y(c) \Rightarrow f$, that is, from $\eta : \mathcal{C}(\cdot, c) \Rightarrow f$. □

Remark 5.1.6. The Yoneda Lemma as stated above is in some sense strictly weaker than the usual one. In fact, normally we would use it to prove that a set of natural transformations is small, i.e. part of the universe. Here, this not the case! $[\mathcal{C}^{op}, \mathbf{ECB}_u](y(c), f)$ can't possibly be contained in u, since that name was used to define the class. So here we have an important difference about universes in Explicit Mathematics compared to theories like ZFC. An isomorphism to an element of a universe does *not* imply inclusion in said universe. This is one of the reasons to consider different notions of universes like the one given in section 4.1 which is closed under isomorphisms (see proposition 4.1.12.) ◇

Corollary 5.1.7 (Yoneda Embedding)**.** *For any locally small categories $\mathcal{C}$, the functor*

$$y : \mathcal{C} \rightarrow [\mathcal{C}^{op}, \mathbf{ECB}_u]$$

is an embedding. More formally, for any two objects c, d in $\mathcal{C}$ we have an isomorphism $\mathcal{C}(c,d) \overset{m}{\rightarrow} [\mathcal{C}^{op}, \mathbf{ECB}_u](y(c), y(d))$ of implicit Bishop Sets.

Proof. Let $c, d \mathrel{\dot\in} ob_{\mathcal{C}}$ and $h \mathrel{\dot\in} \mathcal{C}(c,d)$. We can calculate

$$h =_m h \circ id(c) =_m \overline{\overline{y_m(h)}(c)}(id(c)) =_m \overline{\varphi(c, \mathcal{C}(\cdot, d))}(y_m(h)).$$

Hence, by the Yoneda Lemma we get an isomorphism

$$
\begin{aligned}
&i(c,d) : \mathcal{C}(c,d) \overset{m}{\rightarrow} [\mathcal{C}^{op}, \mathbf{ECB}_u](y(c), y(d))\\
&i(c,d) :\equiv \overline{\psi(c, y(d))}(c) = \overline{\psi(c, \mathcal{C}(\cdot, d))}(c).
\end{aligned}
$$

□

Corollary 5.1.8 (Covariant Yoneda Embedding)**.** *Similarly we can show that for any locally small categories $\mathcal{C}$, the functor*

$$k : \mathcal{C} \to [\mathcal{C}, \mathbf{ECB}_u]$$

which uses the functor $\mathcal{C}(c, \cdot)$ and its induced natural transformation, is an embedding.

Corollary 5.1.9. *Let $\mathcal{C}$ be a locally small category. $\mathcal{C}(\cdot, x) \cong \mathcal{C}(\cdot, y)$ implies $x \cong y$.*

Proof. Reminder: Full and faithful functors reflect isomorphisms: Suppose

$$ff_{a,b}(\langle f_o, f_m \rangle) : \mathcal{C}(a, b) \overset{\cong}{\to} \mathcal{D}(f_o(a), f_o(b))$$

is the isomorphism with inverse $ff^{-1}_{a,b}(\langle f_o, f_m \rangle)$ such that

$$ff^{-1}_{a,b}(\langle f_o, f_m \rangle)(f_m(h)) =_m h.$$

Then for $h : f_o(a) \cong f_o(b) : h^{-1}$ we get

$$\begin{aligned}
id(a) &=_m ff^{-1}_{a,b}(\langle f_o, f_m \rangle)(id_{f_o(a)}) \\
&=_m ff^{-1}_{a,b}(\langle f_o, f_m \rangle)(h^{-1} \circ h) \\
&=_m ff^{-1}_{a,b}(\langle f_o, f_m \rangle) \\
&\qquad (f_m(ff^{-1}_{a,b}(\langle f_o, f_m \rangle)(h)) \circ f_m(ff^{-1}_{a,b}(\langle f_o, f_m \rangle)(h^{-1}))) \\
&=_m ff^{-1}_{a,b}(\langle f_o, f_m \rangle) \\
&\qquad quad(f_m(ff^{-1}_{a,b}(\langle f_o, f_m \rangle)(h) \circ ff^{-1}_{a,b}(\langle f_o, f_m \rangle)(h^{-1}))) \\
&=_m ff^{-1}_{a,b}(\langle f_o, f_m \rangle)(h) \circ ff^{-1}_{a,b}(\langle f_o, f_m \rangle)(h^{-1})
\end{aligned}$$

For the other direction the proof is similar. This shows that

$$ff^{-1}_{a,b}(\langle f_o, f_m \rangle)(h) : a \cong b.$$

Applying this to the Yoneda Embedding y, we get from every

$$\eta(a) : \mathcal{C}(a, x) \cong \mathcal{C}(a, y)$$

natural in a, an isomorphism $x \cong y$ in $\mathcal{C}$. ☐

Corollary 5.1.10. *Let $\mathcal{C}$ be a locally small category. $\mathcal{C}(x, \cdot) \cong \mathcal{C}(y, \cdot)$ implies $x \cong y$.* ☐

Proposition 5.1.11. *Let $\mathcal{I}$ and $\mathcal{C}$ be proper local categories and $q : \mathcal{I} \to \mathcal{C}$ a functor such that the limit $\varprojlim_i q$ exists in $\mathcal{C}$. This implies that we have an isomorphism $\mathcal{C}(c, \varprojlim_i q) \cong \varprojlim_i \mathcal{C}(c, q_o(i))$ natural in c.*

Proof. Let the limit be given as $\langle h, \langle \varprojlim_i q, \eta \rangle \rangle$. A cone is essentially just a natural transformation from the constant functor $\Delta(x)_o(y) :\equiv x$ to q; $\eta : \Delta(x) \Rightarrow q$. A limit then just comes with an additional operation that sends any other natural transformation to the unique, commuting morphism from x into the limit.

$$\begin{tikzcd}
\varprojlim_i q \arrow[r, "\eta(i)"] \arrow[d, equal] & q_o(i) \arrow[d, "q_m(f)"] \\
\varprojlim_i q \arrow[r, "\eta(j)"] & q_o(j)
\end{tikzcd}$$

But then we get a new natural transformation $\tilde{\eta} : \mathcal{C}(c, \varprojlim_i q) \Rightarrow \mathcal{C}(\cdot, \varprojlim_i q)$ given by $\tilde{\eta}(i) :\equiv \lambda h.\eta(i) \circ h$. For any other cone $\langle x, s \rangle$ we have

$$\begin{tikzcd}
\mathcal{C}(c, \varprojlim_i q) \arrow[r, "\tilde{\eta}(i)"] & \mathcal{C}(c, q_o(i)) \arrow[d, equal] \\
x \arrow[u, dotted, "\tilde{h}"] \arrow[r, "s(i)"] & \mathcal{C}(c, q_o(i))
\end{tikzcd}$$

The morphism $\tilde{h}$ is induced by the term $\lambda x_0.\overline{h\langle(\lambda j.\overline{s(j)}x_0), x\rangle}$. This is well-defined, because for any $x_0 \sim_x x_1 \mathrel{\dot{\in}} x$ we have that $\overline{s(i)}x_0 =_m \overline{s(i)}x_1 : c \overset{m}{\to} q_o(i)$ is a function of **ECB** and h then selects the unique morphism making the pointwise diagram commute, hence $\tilde{h}$ will be unique. But this shows that $\mathcal{C}(c, \varprojlim_i q)$ is the limit of $q \circ \mathcal{C}(c, \cdot)$, and so we are done. ☐

Corollary 5.1.12. *This finally allows us to conveniently prove some standard results we have omitted so far. Let $\mathcal{C}$, $\mathcal{D}$ be two locally small categories and $f : \mathcal{C} \to \mathcal{D} \dashv g : \mathcal{D} \to \mathcal{C}$ a pair of adjoint functors. If $\mathcal{I}$ is* *a finite category, and* $q : \mathcal{I} \to \mathcal{D}$ *is a functor with* $\varprojlim_i q$ *its limit, then* $g_o(\varprojlim_i q) \cong \varprojlim_i (g \circ q)$. *Short: RAPL (Right Adjoints Preserve Limits)*

Proof. Proposition 5.1.11 states, that we have a natural isomorphism

$$\mathcal{D}(c, \varprojlim_i q) \cong \varprojlim_i \mathcal{D}(c, q_o(i)).$$

Using the definition of adjoint functors, we then get

$$\begin{aligned}\mathcal{C}(a, g_o(\varprojlim_i q)) &\cong \mathcal{D}(f_o(a), \varprojlim_i q)\\ &\cong \varprojlim_i \mathcal{D}(f_o(a), q_o(i))\\ &\cong \varprojlim_i \mathcal{C}(a, g_o(q_o(i)))\\ &\cong \mathcal{C}(a, \varprojlim_i g_o(q_o(i)))\end{aligned}$$

But these are all natural isomorphisms, and so corollary 5.1.10 yields

$$g_o(\varprojlim_i q) \cong \varprojlim_i (g \circ q). \qquad \square$$

5.2. Enriched Categories

As it turns out, there is a mismatch between the definitions of categories and functors, which essentially live in **ECB**, and our candidate for the category sets $\mathbf{EC}_{\mathrm{ex}}$, for which we would like to prove the standard theorems like the Yoneda Lemma. Since $\mathbf{EC}_{\mathrm{ex}}$ is so much better behaved than the other categories, it actually turns out that things can't be proved, because we don't have enough structure in our other definitions.

As an example, consider the case of the proof of the Yoneda Lemma (see lemma 5.1.5 for the version in **ECB**.) Given a functor $f : \mathcal{C} \to \mathbf{EC}_{\mathrm{ex}}$, It seems impossible to directly write down the natural transformation $\psi : f_o(c) \Rightarrow hom_{(\mathbf{EC}_{\mathrm{ex}})^{\mathcal{C}^{op}}}(y(c), f)$ in $\mathbf{EC}_{\mathrm{ex}}$: That is, we need to construct a natural transformation $hom_{\mathcal{C}}(\cdot, c) \Rightarrow f$. For a fixed element a in $f_o(c)$ This is normally just defined as $h \mapsto f_m(h)(a)$.

Because of how $=^{\mathcal{C}}_m$ is defined, $hom_{\mathcal{C}}(d, c)$ has just an equivalence relation in the sense of pairs even in the exact completion. But that would mean that for arbitrary morphisms $h =_m{}^{\mathcal{C}} h'$ we have to explicitly compute a proof-object $r(h, h')$ in the relation of $f_o(d)$ from essentially nothing. In particular if $\mathcal{C}$ was $\mathbf{EC}_{\mathrm{ex}}$ then $f_m(h) =_m f_m(h')$ means there *exists* some $\gamma : \|(f_o(c))\| \xrightarrow{m} (f_o(d))_{\mathrm{E}}$. A map from the carrier to the pseudo-equivalence relation to show that $f_m(h)$ and $f_m(h')$ are the same. This is the map which would (point-wise) provide the second component of a morphism $hom_{\mathcal{C}}(d, c) \xrightarrow{m} f_o(d)$ in $\mathbf{EC}_{\mathrm{ex}}$. In other words, we would need a global term ψ' such that $\psi'(h, h') =_m^{\mathbf{EC}} \gamma_{h,h'}$ which we could then compose with the constant map for a.

We could add an ad-hoc definition to fix this:

Definition 5.2.1. For a locally small category $\mathcal{C}$, a functor $f : \mathcal{C}^{op} \to \mathbf{EC}_{\mathrm{ex}}$ is called an *effective functor* if $f = \langle\langle f_o, f_m\rangle, \gamma\rangle$ such that the following three conditions hold.

$$(EF1) \qquad \mathrm{Functor}_{\mathcal{C}^{op}, \mathbf{EC}_{\mathrm{ex}}}[\langle f_o, f_m\rangle]$$

$$(EF2) \qquad \gamma \,\dot{\in} \prod_{(d,c:ob_{\mathcal{C}})} \prod_{(\langle h,h'\rangle : hom_{\mathcal{C}}(d,c)_{\mathrm{E}})} hom_{\mathcal{C}}(\|f_o(c)\|, (f_o(d))_{\mathrm{E}})$$

$$\begin{aligned} (EF3) \qquad & (\forall c, d \,\dot{\in}\, ob_{\mathcal{C}})(\forall h =_m^{\mathcal{C}} h' : d \xrightarrow{m} c)(\\ & \qquad (r_0^{f_o(d)} \circ \gamma(d,c)(\langle h, h'\rangle) =_m^{\mathcal{C}} \pi_0(f_m(h^{op})) \\ & \qquad \wedge\, r_1^{f_o(d)} \circ \gamma(d,c)(\langle h, h'\rangle) =_m^{\mathcal{C}} \pi_0(f_m(h'^{op})))) \qquad \diamond \end{aligned}$$

But this would only address half of the problem. In fact, it's not even clear how to prove that $hom_{\mathcal{C}}(\cdot, c)$ satisfies this. The above described problem is of course still there, because if we want to prove the Yoneda Lemma we need to show at the very least that the hom-functor is effective. But that was essentially the problem we had at the beginning.

Fortunately, there exists already a big, and very robust theory on how to handle this problem. We can introduce categories, functors and natural transformations enriched in $\mathbf{EC}_{\mathrm{ex}}$; i.e. categories which already start out with hom-objects in the enriching category instead of (in our case) plain classes/formulas of $\mathbf{EM}$. So the main difference of some $\mathbf{EC}_{\mathrm{ex}}$-enriched category $\mathcal{C}$ to the categories as defined before, is that $hom_{\mathcal{C}}(c, d)$ is should now be an explicit Bishop set and composition is now required to be a morphism in $\mathbf{EC}_{\mathrm{ex}}$ instead of just an operation. Since $\mathbf{EC}_{\mathrm{ex}}$ can be viewed as being enriched over itself, this should yield a proof of the Yoneda Lemma as described, for example, by Kelly [29]. In the rest of this section, we give only the required definitions for enriched categories, functors and natural transformations. This shows, that explicit mathematics is, in principle, perfectly capable to reason about enriched categories. Investigation about the practicality and the limits of this framework in **EM** is left to future research.

Definition 5.2.2. Let $\langle \mathcal{V}, \otimes, a, \gamma, \rho, b \rangle$ be a symmetric monoidal category. We call $\mathcal{A}$ a $\mathcal{V}$-category or a *category enriched in* $\mathcal{V}$ if the following is given.

(a) $\mathcal{A} = \langle ob_{\mathcal{A}}, hom_{\mathcal{A}}, \circ, id \rangle$

(b) $ob_{\mathcal{A}}$ a class of objects.

(c) For all $a, b \mathrel{\dot\in} ob_{\mathcal{A}}$ an object $hom_{\mathcal{A}}(a,b) \mathrel{\dot\in} ob_{\mathcal{V}}$.

(d) For all $a, b, c \mathrel{\dot\in} ob_{\mathcal{A}}$ a morphism in $\mathcal{V}$

$$\circ_{a,b,c} : hom_{\mathcal{A}}(b,c) \otimes hom_{\mathcal{A}}(a,b) \overset{m}{\to} hom_{\mathcal{A}}(a,c)$$

(e) For all $a \mathrel{\dot\in} ob_{\mathcal{A}}$ a morphism in $\mathcal{V}$

$$id_{\mathcal{A}}(a) : 1_{\mathcal{V}} \overset{m}{\to} hom_{\mathcal{A}}(a,a)$$

(f) The following commuting diagrams unitality and associativity for all $a, b, c, d \mathrel{\dot\in} ob_{\mathcal{A}}$ where $\mathcal{A}(a,b) :\equiv hom_{\mathcal{A}}(a,b)$:

$$\begin{tikzcd}
\mathcal{A}(c,d) \otimes (\mathcal{A}(b,c) \otimes \mathcal{A}(a,b)) \arrow[r, "id_{\mathcal{V}} \otimes_m \circ"] \arrow[dd, "a"'] & \mathcal{A}(c,d) \otimes \mathcal{A}(a,c) \arrow[d, "\circ"] \\
 & \mathcal{A}(a,d) \\
(\mathcal{A}(c,d) \otimes \mathcal{A}(b,c)) \otimes \mathcal{A}(a,b) \arrow[r, "\circ \otimes_m id_{\mathcal{V}}"'] & \mathcal{A}(b,d) \otimes \mathcal{A}(a,b) \arrow[u, "\circ"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A}(a,b) \otimes \mathcal{A}(b,b) \arrow[r, "\circ"] & \mathcal{A}(a,b) & \mathcal{A}(a,a) \otimes \mathcal{A}(a,b) \arrow[l, "\circ"'] \\
\mathcal{A}(a,b) \otimes 1_{\mathcal{V}} \arrow[u, "id_{\mathcal{V}} \otimes_m id_{\mathcal{A}}(b)"] \arrow[ur, bend right, "\rho"'] & & 1_{\mathcal{V}} \otimes \mathcal{A}(a,b) \arrow[u, "id_{\mathcal{A}}(a) \otimes_m id_{\mathcal{V}}"'] \arrow[ul, bend left, "\gamma"]
\end{tikzcd}$$

◇

Remark 5.2.3. We don't need to give morphism equality directly. Since every morphism $f : a \overset{m}{\to} b$ in $\mathcal{A}$ is represented by some $\tilde{f} : 1_{\mathcal{V}} \overset{m}{\to} \mathcal{A}_{\mathcal{A}}(a, b)$, we get $f =^{\mathcal{A}}_{m} g$ as $\tilde{f} =^{\mathcal{V}}_{m} \tilde{g}$. ◇

Definition 5.2.4. Let $\mathcal{A}, \mathcal{B}$ be two $\mathcal{V}$-categories. We call $f = \langle f_o, f_m \rangle$ a $\mathcal{V}$-functor from $\mathcal{A}$ to $\mathcal{B}$ if it satisfies the following properties.

$$(EF1) \quad (\forall a \mathrel{\dot\in} ob_{\mathcal{A}})(f_o(a) \mathrel{\dot\in} ob_{\mathcal{B}})$$

$$\begin{aligned}
&(EF2) \quad (\forall a, b \mathrel{\dot\in} ob_{\mathcal{A}})(f_m(a, b) : \mathcal{A}(a, b) \overset{m}{\to} \mathcal{B}(f_o(a), f_o(b)))\\
&(EF3) \quad (\forall a \mathrel{\dot\in} ob_{\mathcal{A}})(f_m(a, a) \circ^{\mathcal{V}} id_{\mathcal{A}}(a) =^{\mathcal{V}}_{m} id_{\mathcal{B}}(f_o(a)))\\
&(EF4) \quad (\forall a, b, c \mathrel{\dot\in} ob_{\mathcal{A}})(f_m(a, c) \circ^{\mathcal{V}} \left(\circ^{\mathcal{A}}_{a,b,c}\right)\\
&\qquad\qquad =^{\mathcal{V}}_{m} \left(\circ^{\mathcal{B}}_{f_o(a), f_o(b), f_o(c)}\right) \circ^{\mathcal{V}} f_m(b, c) \otimes_m f_m(a, b))
\end{aligned}$$

The last axiom $(EF4)$ is the equivalent of the usual composition axiom $f_m(h \circ g) =_m f_m(h) \circ f_m(g)$ which is better written as a diagram:

$$\begin{tikzcd}
\mathcal{A}(b, c) \otimes \mathcal{A}(a, b) \arrow[rr, "\circ^{\mathcal{A}}"] \arrow[d, "f_m(b,c) \otimes_m f_m(a,b)"'] & & \mathcal{A}(a, c) \arrow[d, "f_m(a,c)"] \\
\mathcal{B}(f_o(b), f_o(c)) \otimes \mathcal{B}(f_o(a), f_o(b)) \arrow[rr, "\circ^{\mathcal{B}}"] & & \mathcal{B}(f_o(a), f_o(c))
\end{tikzcd}$$ ◇

Definition 5.2.5 (Enriched natural transformation)**.** For $\mathcal{V}$-functors $t, s : \mathcal{A} \to \mathcal{B}$, a $\mathcal{V}$-natural transformation $\alpha : t \Rightarrow s$ is an $ob_{\mathcal{A}}$ indexed family $\alpha(a) : 1_{\mathcal{V}} \overset{m}{\to} \mathcal{B}(t_o(a), s_o(a))$ satisfying the $\mathcal{V}$-naturality condition

$$\begin{tikzcd}
\mathcal{A}(a, b) \arrow[d, "\gamma^{-1}"'] \arrow[dr, "\rho^{-1}"] & \\
1_{\mathcal{V}} \otimes \mathcal{A}(a, b) \arrow[d, "\alpha(b) \otimes_m t(a,b)"'] & \mathcal{A}(a, b) \otimes 1_{\mathcal{V}} \arrow[d, "s(a,b) \otimes_m \alpha(a)"] \\
\mathcal{B}(t_o(b), s_o(b)) \otimes \mathcal{B}(t_o(a), t_o(b)) \arrow[dr, "\circ^{\mathcal{B}}_{t_o(a), t_o(b), s_o(b)}"'] & \mathcal{B}(s_o(a), s_o(b)) \otimes \mathcal{B}(t_o(a), s_o(a)) \arrow[d, "\circ^{\mathcal{B}}_{t_o(a), s_o(a), s_o(b)}"] \\
& \mathcal{B}(t_o(a), s_o(b))
\end{tikzcd}$$

Vertical composition[1] $\beta \cdot \alpha : t \Rightarrow r$ of $\alpha : t \Rightarrow s$ and $\beta : s \Rightarrow r$ is defined as

$$(\beta \cdot \alpha)(a) :\equiv \circ^{\mathcal{B}}_{t_o(a), s_o(a), r_o(a)} \circ^{\mathcal{V}} \beta(a) \otimes_m \alpha(a) \circ^{\mathcal{V}} \rho^{-1}_{1_{\mathcal{V}}}$$

where $\rho^{-1}_{1_{\mathcal{V}}} : 1_{\mathcal{V}} \cong 1_{\mathcal{V}} \otimes 1_{\mathcal{V}}$. In diagram-form:

$$\begin{tikzcd}
1_{\mathcal{V}} \arrow[d, "\rho^{-1}_{1_{\mathcal{V}}}"] \\
1_{\mathcal{V}} \otimes 1_{\mathcal{V}} \arrow[d, "\beta(a) \otimes_m \alpha(a)"] \\
\mathcal{B}(s_o(a), r_o(a)) \otimes \mathcal{B}(t_o(a), s_o(a)) \arrow[d, "\circ^{\mathcal{B}}"] \\
\mathcal{B}(t_o(a), r_o(a))
\end{tikzcd}$$

◇

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[1] Vertical with respect to the usual naturality square

A. Appendix

A.1. Lists of Natural Numbers

Definition A.1.1. Let $(n_0, n_1) \mapsto \langle| n_0, n_1 |\rangle : \mathbb{N}^2 \to \mathbb{N}$ denote any of the usual primitive recursive injective pairing functions on the natural numbers. This can be represented by a term $t_{\langle||\rangle}$ in **BON** and we will write $\langle| n_0, n_1 |\rangle$ for the term resulting from the application $t_{\langle||\rangle} n_0 n_1$.

To encode finite lists of natural numbers by terms $\langle| n_0, \langle| n_1, \dots, n_k |\rangle |\rangle$ which we will abbreviate as $\langle| n_0, n_1, \dots, n_k |\rangle$. We add the usual primitive recursive functions for projection $(m, k) \mapsto \pi^{\mathrm{N}}_k m$, length of a list $(m) \mapsto \mathrm{lh}(m)$ and concatenation $(m, n) \mapsto m * n$.

We demand the following standard properties from our lists:

- $\langle| n_1, \cdots, n_k |\rangle = 0$ if and only if $k = 0$
- If $n \neq 0$ holds, then there is exactly one $k \neq 0$ and natural numbers $n_0, \dots, n_k$ such that $n = \langle| n_0, \dots, n_k |\rangle$ holds,
- $(\pi^{\mathrm{N}}_i n) < n$ for all $i < \mathrm{lh}(n)$,
- $\mathrm{lh}(\langle| n_1, \cdots, n_k |\rangle) = k$,
- $\pi^{\mathrm{N}}_i \langle| n_1, \dots, n_k |\rangle = n_i$ for $0 \leq i \leq k$,
- $\langle| n_1, \dots, n_k |\rangle * \langle| m_0, \dots, m_l |\rangle = \langle| n_1, \dots, n_k, m_1, \dots, m_l |\rangle$.

Furthermore, we will add the abbreviation $\mathrm{last}(m)$ to access the last element of a list given by $(m) \mapsto \pi^{\mathrm{N}}_{\mathrm{lh}(m)}$. ◇

A.2. Path Categories

Categories which are freely generated from a finite diagram may seem entirely trivial and in fact their construction is standard. But note that a finite diagram does *not* imply that the free category generated from it is also finite. Consider the diagram with only one object and one non-identity arrow $\begin{tikzcd}\bullet \arrow[loop right, "f"]\end{tikzcd}$ In the category freely generated from these data, it should not hold that $f \circ f \circ \cdots \circ f = f$ is true for any number of compositions. This implies an infinite number of morphisms even for this simple case. If fact, the only time this is not required is if the diagram is a DAG; a directed acyclic graph. The free category generated from a finite graph (also called path category[1]) is defined in the following manner.

Definition A.2.1. Let $v, e \mathrel{\dot{\subset}} nat$ be finite subsets of the natural numbers describing vertices and edges and $s, t : e \mathrel{\dot{\to}} v$ two operations which give source and target of any edge. The free category on $g = \langle v, e, s, t\rangle$ is given by v as objects and paths in g as morphisms.

$$\begin{aligned} ob &:\equiv v \\ mor &:\equiv \{\langle \pi_0^{\mathrm{N}} p, \mathrm{last}(p), p\rangle \mid p \mathrel{\dot{\in}} \widehat{mor}\} \end{aligned}$$

where

$$\begin{aligned} mor_0 &:\equiv \{\langle | m, m | \rangle \mid m \mathrel{\dot{\in}} v\} \\ M2[u, s, t, V, P] &:\equiv \exists p (p \in P \wedge u = \langle | tf, f, sf | \rangle \wedge f \in V \wedge \mathrm{lh}(p) = 2) \\ M3[u, s, t, V, P] &:\equiv \exists p (p \in P \wedge u = \langle | tf, f, x | \rangle \\ &\qquad \wedge f \in V \wedge p = \langle | sf, x | \rangle \wedge \mathrm{lh}(p) > 2) \\ mor_i(f, n) &:\equiv t_{M2}(s, t, v, f(n-1)) \cup t_{M3}(s, t, v, f(n-1)) \\ m(f, n) &:\equiv \begin{cases} mor_0 & n = 0 \\ mor_i(f, n) & n \neq 0 \end{cases} \\ \widehat{mor} &:\equiv \left\{ p \mid \exists n \in N \wedge \langle n, p\rangle \mathrel{\dot{\in}} \sum (nat, \lambda n.(fix(m))n) \right\} \end{aligned}$$

---

[1] The name *Path Category* is also used by van den Berg and Moerdijk [47, 48]. That notion, short for *category with path objects*, which is a strengthening of a *category of fibrant objects* from homotopy theory is unrelated to the free category generated from a graph.

Composition is essentially given by concatenation of paths and identity $id(k) :\equiv \langle k, k, \langle | k, k | \rangle \rangle$. This may seem a bit redundant, but the additional tupling is just an artifact from the fixed form of morphisms in our categories which is not a very good match for the usual presentation of path categories. ◇

Example A.2.2. Most of the important shapes of diagrams used are diagrams of free categories. This includes in particular the diagrams for

(i) initial/terminal objects

The empty Diagram,

(ii) products/coproducts

$$\begin{tikzcd} \bullet & \bullet \end{tikzcd},$$

(iii) equalizers/coequalizers

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd},$$

(iv) pullbacks/pushouts

$$\begin{tikzcd} \bullet \arrow[r] & \bullet & \bullet \arrow[l] \end{tikzcd} \text{ and }$$

$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd},$$

(v) sequential limits/sequential colimits.

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \cdots \bullet \arrow[r] & \bullet \end{tikzcd}$$

◇

Definition A.2.3 (Diagrams in a weak category $\mathcal{C}$)**.** Let $\mathcal{J}$ be a category with finite object and morphism classes. A *diagram in* $\mathcal{C}$ is just a functor $f : \mathcal{J} \to \mathcal{C}$. Given an actual diagram in the sense of a graphic consisting of points and arrows, we take it as a graph and construct the free category (see definition A.2.1.) This is then taken as the domain of a functor into $\mathcal{C}$. ◇

Remark A.2.4. Diagrams in the sense of definition A.2.3 are not commuting. For *commuting* diagrams defined in this style, see definition A.2.5. ◇

Definition A.2.5 (Commutative Diagram in $\mathcal{C}$)**.**
Given an index category $\mathcal{I}$, A possible definition of a commutative diagram is a functor $\mathcal{I} \Rightarrow \mathcal{C}$ which factors through a *thin* category (that is a category, for which for all objects a and b it holds that $(f : a \overset{m}{\to} b \wedge g : a \overset{m}{\to} b) \to f =_m g$.)

Here we will define a commutative diagram in the following sense. Given any directed graph $g :\equiv \langle v, e, s, t \rangle$ the commutative diagram specified by g is a functor from $\mathcal{CD}(\mathrm{g})$ to $\mathcal{C}$ where $\mathcal{CD}(\mathrm{g})$ is the free category on g with a new equality on morphisms which collapses all parallel morphisms which are longer than one arrow. The reason for the special case is that there is basically never a case where we want to draw a diagram containing a loop of one edge, which should collapse to identity. This means it's the quotient category given by the equivalence relation

$$\begin{aligned} f =_m^{\mathcal{CD}(\mathrm{g})} g :\equiv (\mathrm{lh}(\overline{f}) \leq 2 \to f =_m g) \wedge& \\ (\mathrm{lh}(\overline{f}) > 2 \to \pi_0^{\mathrm{N}} \overline{f} = \pi_0^{\mathrm{N}} \overline{g}& \\ \wedge \, \mathrm{last}(\overline{f}) = \mathrm{last}(\overline{g}))& \qquad \diamond \end{aligned}$$

A.3. Equivalence of Definitions of a Regular Category

Definition A.3.1 (Strong Epimorphism)**.** An epimorphism $f : a \overset{m}{\to} b$ is called a *strong epimorphism* when, for every commutative square $z \circ u =_m v \circ f$

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "u"'] & b \arrow[d, "v"] \arrow[dl, dotted, "w"] \\ x \arrow[r, "z"'] & y \end{tikzcd}$$

with $z : x \overset{m}{\to} y$ a monomorphism, there exists a (unique) morphism $w : b \overset{m}{\to} x$ such that $w \circ f =_m u$, $z \circ w =_m v$. Formally

$$\begin{aligned}
(STRONG) \quad & EPI[f] \wedge \\
& (\forall Ob(x,y))(\forall u : dom(f) \overset{m}{\to} x)(\forall v : cod(f) \overset{m}{\to} y) \\
& \quad (\forall z : x \overset{m}{\to} y) \\
& \qquad (MONO[z] \to (\exists w : cod(f) \overset{m}{\to} x)(\\
& \qquad w \circ f =_m u \wedge z \circ w =_m v \\
& \qquad \wedge (\forall q : cod(f) \overset{m}{\to} x))(\\
& \qquad q \circ f =_m u \wedge z \circ q =_m v \to q =_m w))
\end{aligned}$$

The uniqueness condition is redundant since by assumption, w is an epimorphism and z is a monomorphism. ◇

Proposition A.3.2. *Any morphism $f : a \overset{m}{\to} b$ which is a strong epi and monic, is an isomorphism.*

Proof. This follows directly from the following commutative diagram.

$$\begin{tikzcd}
a \arrow[r, tail, "f"] \arrow[d, equal, "id(a)"'] & b \arrow[d, equal, "id(b)"] \arrow[dl, dotted, "\exists g"'] \\
a \arrow[r, tail, "f"'] & b
\end{tikzcd}$$

□

Proposition A.3.3 ([7])**.** *If $f \circ g$ is a strong epi, then f is a strong epi.*

Proof. Let i and j be any two morphism and u a mono such that $u \circ i =_m j \circ f$ and consider

$$\begin{tikzcd}
a \arrow[r, "g"] \arrow[dr, "i \circ g"'] & b \arrow[r, "f"] \arrow[d, "i"] & c \arrow[d, "j"] \arrow[dl, dotted, "h"] \\
& d \arrow[r, tail, "u"'] & e
\end{tikzcd}$$

Because $f \circ g$ is strong there is a unique $h : c \overset{m}{\to} d$ such that $u \circ h =_m j$ and $h \circ (f \circ g) =_m i \circ g$. Clearly this is also a factorization for f and unique because u is a mono. ☐

Proposition A.3.4 (Regular epis are strong)**.**

Proof. Let f be the coequalizer of p, q and $z : x \overset{m}{\to} y$ a mono with $v \circ f =_m z \circ u$.

$$\begin{tikzcd}
t \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & y \arrow[r, two heads, "f"] \arrow[d, "u"'] & x \arrow[d, "v"] \arrow[dl, dotted, "w"] \\
 & b \arrow[r, "z"'] & c
\end{tikzcd}$$

We have $f \circ p =_m f \circ q$ and hence

$$z \circ u \circ p =_m v \circ f \circ p =_m v \circ f \circ q =_m z \circ u \circ q.$$

Because z is mono, we get $u \circ p =_m u \circ q$ and so, by the universal property of coequalizers, there exists some unique $w : x \overset{m}{\to} b$ with $w \circ f =_m u$. From $z \circ w \circ f =_m z \circ u =_m v \circ f$ we get $z \circ w =_m v$ because f is an epi. ☐

Proposition A.3.5. *Let $f : a \overset{m}{\to} b$ be a morphism in a category where the pullback $f * f$ exists. The kernel-pair of f is given by identical projections u, u for some $u : f * f \overset{m}{\to} a$ iff f is mono.*

Proof. Let $h, l : k \overset{m}{\to} a$ such that

$$f \circ h =_m f \circ l$$

then we have the following diagram:

$$\begin{tikzcd}
k \arrow[dr, dotted, "q"] \arrow[drr, bend left, "l"] \arrow[ddr, bend right, "h"'] & & \\
 & f * f \arrow[r, "u"] \arrow[d, "u"'] & a \arrow[d, "f"] \\
 & a \arrow[r, "f"'] & b
\end{tikzcd}$$

Hence we get

$$h =_m u \circ q =_m l.$$

The other direction is even more trivial: Let f be monic. For the kernel-pair u, v it holds that $f \circ u =_m f \circ v$ which implies $u =_m v$. □

Proposition A.3.6 ([7] (2.5.7))**.** *If a coequalizer has a kernel pair, it is the coequalizer of this kernel pair.*

Proof. Let f be a regular epimorphism coequalizing p, q which has a kernel pair u, v and let g be a morphism such that $g \circ u =_m g \circ v$. p, q are by assumption a cone over the kernel pair so there is a unique $h : a \overset{m}{\to} z$ with $u \circ h =_m p$ and $v \circ h =_m q$. But then

$$g \circ u =_m g \circ v \to g \circ p =_m g \circ u \circ h =_m g \circ v \circ h =_m g \circ q.$$

$$\begin{tikzcd}
 & a \arrow[dl, dotted, "h"'] \arrow[d, shift right, "q"'] \arrow[d, shift left, "p"] & \\
z \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & y \arrow[r, two heads, "f"] \arrow[d, "g"] & x \arrow[dl, dotted, "t"] \\
 & s &
\end{tikzcd}$$

The universal property of coequalizers yields then a unique $t : x \overset{m}{\to} s$. □

Proposition A.3.7 ([8] (2.2.1))**.** *If a category $\mathcal{C}$ satisfies the following conditions, it is regular (In the sense of the first definition.)*

(a) It is finitely complete

(b) every morphism f can be factored as $f =_m m \circ e$ with m a monomorphism and e a regular epimorphism;

(c) the pullback of a regular epimorphism along any morphism is a regular epimorphism.

Proof. Consider a morphism f with a kernel pair u, v and a factorization $f =_m m \circ e$. m is mono so u, v is also the kernel pair of e ($m \circ e \circ u =_m m \circ e \circ v \to e \circ u =_m e \circ v$.) e is a regular epimorphism so using proposition A.3.6 it is the coequalizer of u and v. ☐

Proposition A.3.8. *If $f : a \overset{m}{\to} b$ is a morphism in a regular category (in the sense of the first definition), $u, v : f * f \overset{m}{\to} a$ is the kernel-pair of f, then we get a factorization of f into a regular epi followed by a monomorphism.*

Proof. Let $coeq(u, v) : a \overset{m}{\to} i$ be the coequalizer of u and v. The universal property of coequalizers provides a unique $m : i \overset{m}{\to} b$ such that $f =_m m \circ coeq(u, v)$. We have to show that m is monic. Let $s, t : m * m \overset{m}{\to} i$ be the kernel-pair of m. We have the following coequalizer diagram:

$$\begin{tikzcd}
f * f \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[ddr, dotted, "h"'] & a \arrow[rr, "f"] \arrow[dr, "coeq(u,v)"'] & & b \\
& & i \arrow[ur, dotted, "m"'] & \\
& m * m \arrow[ur, shift left, "s"] \arrow[ur, shift right, "t"'] & &
\end{tikzcd}$$

Here $h : f * f \overset{m}{\to} m * m$ is the morphism provided by

$$\begin{tikzcd}
f * f \arrow[drr, bend left, "co(u,v)\circ u"] \arrow[dr, dotted, "h"] \arrow[ddr, bend right, "co(u,v)\circ v"'] & & \\
& m * m \arrow[r, "s"] \arrow[d, "t"] & i \arrow[d, "m"] \\
& i \arrow[r, "m"] & m
\end{tikzcd}$$

But because this is the unique morphism with this property, we know that proposition A.3.11 applies if we use $f * f$ as $(m \circ coeq(u, v)) * (m \circ coeq(u, v))$. This shows that h is epi.

From the second diagram we get

$$t \circ h =_m co(u, v) \circ v \qquad s \circ h =_m co(u, v) \circ u$$

and by definition of coequalizers we have

$$co(u, v) \circ v =_m co(u, v) \circ u$$

hence, we get

$$t \circ h =_m s \circ h$$

which implies that the kernel-pair s, t is equal and with proposition A.3.5 this finishes the proof. □

Proposition A.3.9. *If a morphism $f : a \overset{m}{\to} b$ is factored into $m \circ e$ where m is the smallest subobject of b through which f factors and if $m' \circ e'$ is a factorization of e with m' mono, then m' is an isomorphism and so e factors only trivially.*

Proof. Consider the diagram

$$\begin{tikzcd}
a \arrow[rd, "e"] \arrow[rrd, bend left, "e'"] \arrow[dd, "f"'] & & \\
& im(f) \arrow[ld, tail, "m"] \arrow[r, dotted, bend left, "q"] & c \arrow[l, tail, bend left, "m'"] \arrow[lld, tail, bend left, "m\circ m'"] \\
b & &
\end{tikzcd}$$

Because the factorization $e =_m m' \circ e'$ can also be seen as a factorization of f by composing with m, we get a unique morphism $q : im(f) \overset{m}{\to} c$. But now we can calculate

$$m \circ id(im(f)) =_m m =_m m \circ m' \circ q$$

Because m is mono, we get

$$id(im(f)) =_m m' \circ q.$$

Using this equation, we have

$$m' \circ id(c) =_m m' =_m id(im(f)) \circ m' =_m (m' \circ q) \circ m'$$

Because m' is also mono, we get

$$id(c) =_m q \circ m'.$$

Hence, m' is an isomorphism and so e factors only trivially. □

Proposition A.3.10. *If the morphism $f : a \overset{m}{\to} b$ in some finitely complete category $\mathcal{C}$ only factors trivially ($f =_m e \circ m$ with m mono implies that m is iso) then it is epi.*

Proof. Let $u, v : b \overset{m}{\to} c$ be any morphisms such that $u \circ f =_m v \circ f$. Since equalizers exist we can consider

$$\begin{tikzcd}
a \arrow[rr, "f"] \arrow[dr, dotted] & & b \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & c \\
& eq(u,v) \arrow[ur, tail, "e"] & &
\end{tikzcd}$$

All equalizers are mono, so this is a factorization of f and hence by proposition A.3.9 e is iso (and in particular epi.) But then we get from $u \circ e =_m v \circ e$ that $u =_m v$ as required. □

Proposition A.3.11 ([8])**.** *Let $f : a \overset{m}{\to} b$ be a regular epi in a regular category using the first definition and let $g : b \overset{m}{\to} c$ be an arbitrary morphism. Then a morphism $h : (g \circ f) * (g \circ f) \overset{m}{\to} g * g$ exists and is epi.*

Proof. Take the following pullbacks

$$\begin{tikzcd}
d * e \arrow[r, two heads, "j"] \arrow[d, two heads, "i"] & b * f \arrow[r, "h"] \arrow[d, two heads, "e"] & a \arrow[d, two heads, "f"] \\
f * a \arrow[r, two heads, "d"] \arrow[d, "c"] & g * g \arrow[r, "b"] \arrow[d, "a"] & b \arrow[d, "g"] \\
a \arrow[r, two heads, "f"] & b \arrow[r, "g"] & c
\end{tikzcd}$$

Because all small squares are pullbacks, the big one is as well and we know that $d * e \cong (g \circ f) * (g \circ f)$. The morphism h is then given by[2] $d \circ i =_m e \circ j$. As composition of two epis this is epi. $\square$

Proposition A.3.12 ([28]). *If the morphism $f : a \overset{m}{\to} b$ in some finitely complete category $\mathcal{C}$ only factors trivially ($f =_m e \circ m$ with m mono implies that m is iso) then it is strong.*

Proof. We know that f is epi from proposition A.3.10. Let $g : b \overset{m}{\to} c$, $h : b \overset{m}{\to} d$ be any morphisms and $n : c \overset{m}{\to} d$ be a mono such that

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[d, "h"] \\
c \arrow[r, tail, "n"] & d
\end{tikzcd}$$

commutes. Then we can construct the pullback $h^*(n)$ of h along n and get a unique morphism $p : a \overset{m}{\to} n * h$ such that the following diagram commutes.

$$\begin{tikzcd}
a \arrow[rr, "f"] \arrow[dd, "g"] \arrow[dr, dotted, "p"] & & b \arrow[dd, "h"] \\
& n * h \arrow[ur, tail, "h^*(n)"] \arrow[dl, "n^*(h)"] & \\
c \arrow[rr, tail, "n"] & & d
\end{tikzcd}$$

Note that $h^*(n)$ is mono because monos are always preserved under pullback. But then we have a factorization of f and we get to apply proposition A.3.9 and get some $p : b \overset{m}{\to} n * h$ which is an inverse to $h^*(n)$. If we write $k :\equiv n^*(h) \circ p : b \overset{m}{\to} c$ then k is the required unique morphism to show that f is strong.

$$h \circ f =_m n \circ k \circ f =_m n \circ n^*(h) \circ p.$$

---

[2] Technically it should be composed with the isomorphism for $d * e \cong (g \circ f) * (g \circ f)$.

Because n is mono we get from the second and the last term that

$$k \circ f =_m n^*(h) \circ p =_m g.$$

The other triangle follows from being a pullback. Because n is mono k is automatically unique. ☐

Proposition A.3.13 ([8]). *Let $\mathcal{C}$ be a finitely complete category such that every morphism can be factored into a strong epi and a mono and the pullback of a strong epi along any morphism is a strong epi. Then an epi is strong if and only if it is regular.*

Proof. Let $f : a \overset{m}{\to} b$ be strong and u, v be the kernel-pair of f and let $g : a \overset{m}{\to} c$ be any morphism such that $g \circ u =_m g \circ v$. We need to show that there is a unique morphism $w : b \overset{m}{\to} c$ such that $g =_m w \circ f$. Consider

$$\begin{tikzcd}
 & a \arrow[dl, "g"'] \arrow[d, dotted, "h"] \arrow[dr, "f"] & \\
c & c \times b \arrow[l, "p_0"] \arrow[r, "p_1"'] & b
\end{tikzcd}$$

We factor h into $p \circ i$ where p is a strong epi and i is mono. We will show that $p_1 \circ i$ is an iso and that w is given by $p_0 \circ i \circ (p_1 \circ i)^{-1}$.

Consider the diagram below where all squares are pullbacks.

$$\begin{tikzcd}
t * q \arrow[r, "n"] \arrow[d, "m"] \arrow[rr, bend left, "v"] \arrow[dd, bend right, "u"'] & s * p \arrow[r, "y"] \arrow[d, "q"] & a \arrow[d, "p"'] \arrow[dd, bend left, "f"] \\
p * r \arrow[r, "t"] \arrow[d, "x"] & p_1 \circ i * p_1 \circ i \arrow[r, "s"] \arrow[d, "r"] & im \arrow[d, "p_1 \circ i"'] \\
a \arrow[r, "p"] \arrow[rr, bend right, "f"'] & im \arrow[r, "p_1 \circ i"] & b
\end{tikzcd}$$

Because strong epis are pullback stable, and big squares built from pullbacks are pullbacks, we get that $t * q \cong f * f$ and in fact that the morphisms u, v, t, q, n, m are[3] strong epi. Furthermore with proposition A.3.3 we also get that $p_1 \circ i$ and x and y are strong and by pullback r and s which is all morphisms in the diagram. By Commutativity, we have

$$p_1 \circ i \circ r \circ t \circ m =_m p_1 \circ i \circ s \circ t \circ m$$

and we can calculate

$$\begin{aligned}
p_0 \circ i \circ r \circ t \circ m &=_m p_0 \circ i \circ p \circ x \circ m \\
&=_m p_0 \circ h \circ x \circ m \\
&=_m p_0 \circ h \circ u \\
&=_m g \circ u \\
&=_m g \circ v \\
&=_m p_0 \circ h \circ v \\
&=_m p_0 \circ h \circ y \circ n \\
&=_m p_0 \circ i \circ p \circ y \circ n \\
&=_m p_0 \circ i \circ s \circ q \circ n \\
&=_m p_0 \circ i \circ s \circ t \circ m.
\end{aligned}$$

This gets us a cone over the product $c \times b$.

$$\begin{tikzcd}
 & t * q \arrow[dl, "p_0 \circ i \circ r \circ t \circ m"'] \arrow[d, "i \circ s \circ t \circ m"] \arrow[dr, "p_1 \circ i \circ r \circ t \circ m"] & \\
c & c \times b \arrow[l, "p_0"] \arrow[r, "p_1"'] & b
\end{tikzcd}$$

And so $i \circ r \circ t \circ m =_m i \circ s \circ t \circ m$. Since i is mono we get $r \circ t \circ m =_m s \circ t \circ m$ and because t and m are epi, we have $s =_m r$. By proposition A.3.5 this means that $p_1 \circ i$ is mono. But we already know it's also a strong epi and hence with proposition A.3.2 it is an iso and we write $(p_1 \circ i)^{-1}$ for the

---

[3] More correctly, u, v should be written as u', v' since they are precomposed with the iso (say $o : f * f \cong t * q$) to $t * q$ such that $u =_m u' \circ o$. Of course they are still strong epi.

inverse. Now we set $w :\equiv p_0 \circ i \circ (p_1 \circ i)^{-1}$ and calculate

$$\begin{aligned}
w \circ f &=_m p_0 \circ i \circ (p_1 \circ i)^{-1} \circ f \\
&=_m p_0 \circ i \circ (p_1 \circ i)^{-1} \circ p_1 \circ i \circ p \\
&=_m p_0 \circ i \circ p \\
&=_m p_0 \circ h \\
&=_m g
\end{aligned}$$

which is unique since f is an epi. With this we have shown that f is the coequalizer of u and v which means that f is regular. ☐

Corollary A.3.14. *If a Category is regular using the second definition, it is regular using the first one.*

Proof. Propositions A.3.9 and A.3.12 tell us that the factorization of any morphism into the smallest subobject is a factorization into a strong epimorphism followed by a monomorphism. Proposition A.3.13 says that pullback-stable strong epi-mono factorizations imply that epis are strong iff they are regular. Finally, we can apply proposition A.3.7 to show this gives a regular category. ☐

Proposition A.3.15. *If a Category is regular using the first definition, it is regular using the second one.*

Proof. Using the alternative characterization of regular categories from proposition A.3.7 and proposition A.3.4 to relate regular epis to strong one, we need to construct a pullback-stable regular epi-mono factorization.

(a) Let $f : a \overset{m}{\to} b$ be any morphism. We factor it into the coequalizer of the kernel-pair u, v which we call $coeq(u, v) : f * f \overset{m}{\to} i$ and the morphism $m : i \overset{m}{\to} b$ resulting from the universal property of coequalizers. By proposition A.3.8 m is mono and in fact the smallest subobject through which f factors. The required morphism into other epi-mono factorizations is provided by the coequalizer.

(b) We still have to check for stability:

We know that pullbacks along any morphism (if they exist) are functorial, so we can pullback just m and $coeq(u,v)$ and get something isomorphic to the pullback of f. In fact this is a factorization of the pullback of f. Monomorphisms are always stable under pullback and the stability of epimorphisms follows from regularity in the sense of the first definition. The regular epi has a kernel-pair which is the same as the pullback of f and so provides morphisms into other epi-mono factorizations. ☐

A.4. A Direct Construction of $\Pi_f^{\mathbf{ECB}}(p)$

The fact that it is possible to construct the right adjoint to the pullback-functor on slice categories of **ECB** of any morphism $f : x \overset{m}{\to} y$,

$$\mathbf{ECB}\diagup x : \underset{\Pi_f}{\overset{f^*}{\leftrightarrows}} : \mathbf{ECB}\diagup y$$

without having to use the join axiom, might be somewhat surprising at first sight. But it's a direct consequence of how elementary comprehension and the definedness relation interact. Of course the elements of Π_f end up as the usual partial section from some preimage-class of f into the domain of any argument-morphism to $\Pi_f(\cdot)$. The reason we don't need to explicitly mention the preimage-classes dependent on an element in the codomain when we want to make sure $s \dot{\in} \|\Pi_f(p)\|$, is that we can just ask for s to be defined on any relevant preimage and sending these elements to $dom(p)$ (which is not dependent on preimage on any $x_0 \dot{\in} \|f^{-1}\{y_0\}\|$) in a way which is the identity when composed with p.

Of course, all this does, is directly mimicking the usual construction in the category of sets. Namely

$$\Pi_f(p) :\equiv \sum_{x_0 : x} \prod_{y_0 : f^{-1}\{x_0\}} p^{-1}\{y_0\}.$$

What we have to account for is functional extensionality of operations on elements of $f^{-1}\{x_0\}$, and filtering of operations which are not functions. A more involved application of this construction for categorical universes in Explicit Mathematics including a proof that it is isomorphic to the construction given below at each fixed x_0, is given in section 4.1 proposition 4.1.14.

Definition A.4.1. We define Π_f (The right adjoint to the pullback functor associated to f) explicitly without relying on the join axiom. For readability we will abbreviate X, XR, Y, YR, A, AR as $\overline{X}$.

$$\begin{aligned}
PI[u, f, p, \overline{X}] :\equiv{}& \exists y_0, q(u = \langle y_0, q\rangle \wedge y_0 \in Y \wedge \\
&(\forall x \in X)(\overline{f}x \sim_{YR} y_0 \\
&\qquad \to (qx{\downarrow} \wedge qx \in A \wedge \overline{p}(qx) \sim_{XR} x)) \\
&(\forall x_0, x_1 \in X)((\overline{f}x \sim_{YR} y_0 \wedge x_0 \sim_{XR} x_1) \\
&\qquad \to (qx_0 \sim_{AR} qx_1))) \\
pi(f,p) :\equiv{}& t_{PI}(f, p, \|dom(f)\|, dom(f)_{\mathrm{E}}, \\
&\qquad \|cod(f)\|, cod(f)_{\mathrm{E}}, \|dom(p)\|, dom(p)_{\mathrm{E}}) \\
PIE[u, f, p, \overline{X}, PI] :\equiv{}& \exists y_0, y_1, q_0, q_1(u = \langle\langle y_0, q_0\rangle, \langle y_1, q_1\rangle\rangle \\
&\wedge \langle y_0, q_0\rangle \in PI \wedge \langle y_1, q_1\rangle \in PI \\
&\wedge y_0 \sim_{YR} y_1 \\
&\wedge (\forall x \in X)(\overline{f}x \sim_{YR} y_0 \to q_0x \sim_{AR} q_1x) \\
pie(f,p) :\equiv{}& t_{PIE}(f, p, \|dom(f)\|, dom(f)_{\mathrm{E}}, \\
&\qquad \|cod(f)\|, cod(f)_{\mathrm{E}}, \\
&\qquad \|dom(p)\|, dom(p)_{\mathrm{E}}, pi(f,p)) \\
\Pi_f^{\mathbf{ECB}}(p) :\equiv{}& \langle pi(f,p), pie(f,p)\rangle
\end{aligned}$$

The morphism (sometimes also named $\Pi_f^{\mathbf{ECB}}(p)$) to make this an object in the slice category over $cod(f)$ is just the regular projection to the first component.

$$pr : \Pi_f^{\mathbf{ECB}}(p) \overset{m}{\to} cod(f). \qquad \diamond$$

Proposition A.4.2. *The construction given in definition A.4.1 satisfies the required universal property.*

Proof. For $f : a \overset{m}{\to} b$ and $l : \underline{l} \overset{m}{\to} a$ We have to show there is the following natural isomorphism between homsets[4].

Let the following be two functors:

$$\begin{aligned}
\langle z_o, z_m\rangle &: ((\mathbf{ECB}_u/b)^{op}) \times (\mathbf{ECB}_u/a) \to \mathbf{ECB}\\
z_o(k,l) &:\equiv (\mathbf{ECB}_u/b)(k, \Pi_f(l))\\
z_m(\eta,\xi) &:\equiv \langle z_o(dom(\eta), dom(\xi)), z_o(cod(\eta), cod(\xi)),\\
&\qquad \lambda h.(\Pi_f)_o(\xi) \circ h \circ \eta^{op}\rangle\\
\langle w_o, w_m\rangle &: (\mathbf{ECB}_u/b)^{op} \times \mathbf{ECB}_u/a \to \mathbf{ECB}\\
w_o(k,l) &:\equiv (\mathbf{ECB}_u/a)(f^*(k), l)\\
w_m(\eta,\xi) &:\equiv \langle w_o(dom(\eta), dom(\xi)), w_o(cod(\eta), cod(\xi)),\\
&\qquad \lambda g.\xi \circ g \circ (f^*)_m(\eta^{op})\rangle
\end{aligned}$$

A bit more readable this would be:

$$\begin{aligned}
z_m(\eta : k \overset{m}{\rightarrow} w, \xi : l \overset{m}{\rightarrow} v) &:\equiv \lambda h.(\Pi_f)_m(\xi) \circ h \circ \eta^{op}\\
&: z_o(k,l) \overset{m}{\rightarrow} z_o(w,v)\\
w_m(\eta : k \overset{m}{\rightarrow} w, \xi : l \overset{m}{\rightarrow} v) &:\equiv \lambda g.\xi \circ g \circ (f^*)_m(\eta^{op})\\
&: w_o(k,l) \overset{m}{\rightarrow} w_o(w,v)
\end{aligned}$$

To state the natural transformations φ and ψ we first introduce an abbreviation for better readability.

Notation A.4.3. We write $\dddot{f}$ for $\overline{\overline{f}}$, which we need to access the underlying operation of f if f is a morphism in the slice category. ◇

φ and ψ are then given by

$$\begin{aligned}
\varphi(k,l) &:\equiv \langle z_o(k,l), w_o(k,l), \lambda e.\langle f^*k, l, \lambda z.(\pi_1(\dddot{e}(\pi_1 z)))(\pi_0 z)\rangle\rangle\\
\psi(k,l) &:\equiv \langle w_o(k,l), z_o(k,l), \lambda h.\langle k, \Pi_f(l), \lambda k_0.\langle \overline{k} k_0, \lambda x.\dddot{h}\langle x, k_0\rangle\rangle\rangle\rangle.
\end{aligned}$$

---

[4] This is in the slice category $\mathbf{ECB}/a$. In particular, we have *objects* $f^*w : f * w \overset{m}{\rightarrow} a$.

to see that this makes sense we calculate $\overline{\varphi(w,p)\circ\psi(w,p)}(h)$ and the reverse

$$
\begin{aligned}
&\langle f * w, p, \lambda z.(\pi_1 \\
&\qquad \overline{(\langle w, \Pi_f^{\mathbf{ECB}}(p), \lambda w_0.\langle \overline{w} w_0, \lambda x_0.\, \dddot{h}\langle x_0, w_0\rangle\rangle\rangle}(\pi_1 z))(\pi_0 z)\rangle \\
&:\equiv \langle f * w, p, \lambda z.(\pi_1 \\
&\qquad \overline{(\langle w, \Pi_f^{\mathbf{ECB}}(p), \lambda w_0.\langle \overline{w} w_0, \lambda x_0.\, \dddot{h}\langle x_0, w_0\rangle\rangle\rangle}(\pi_1 z))(\pi_0 z)\rangle \\
&=_m \langle f * w, p, \lambda z.(\pi_1(\langle \overline{w}(\pi_1 z), \lambda x_0.\, \dddot{h}\langle x_0, (\pi_1 z)\rangle\rangle))(\pi_0 z)\rangle \\
&=_m \langle f * w, p, \lambda z.(\lambda x_0.\, \dddot{h}\langle x_0, (\pi_1 z)\rangle)(\pi_0 z)\rangle \\
&=_m \langle f * w, p, \lambda z.\, \dddot{h}\langle (\pi_0 z), (\pi_1 z)\rangle\rangle \\
&=_m \langle f * w, p, \lambda z.\, \dddot{h}\, z\rangle \\
&=_m h
\end{aligned}
$$

And the other direction

$$
\begin{aligned}
&\overline{\psi(w,p)\circ\varphi(w,p)}(h) \\
&:\equiv \langle w, \Pi_f^{\mathbf{ECB}}(p), \lambda w_0.\langle \overline{w} w_0, \\
&\qquad \lambda x_0.\overline{\langle f * w, p, \lambda z.(\pi_1(\dddot{h}(\pi_1 z))(\pi_0 z)\rangle}\langle x_0, w_0\rangle\rangle\rangle \\
&:\equiv \langle w, \Pi_f^{\mathbf{ECB}}(p), \lambda w_0.\langle \overline{w} w_0, \\
&\qquad \lambda x_0.\overline{\langle f * w, p, \lambda z.(\pi_1(\dddot{h}(\pi_1 z))(\pi_0 z)\rangle}\langle x_0, w_0\rangle\rangle\rangle \\
&=_m \langle w, \Pi_f^{\mathbf{ECB}}(p), \lambda w_0.\langle \overline{w} w_0, \lambda x_0.\lambda z.(\pi_1(\dddot{h}(\pi_1 z))(\pi_0 z)\langle x_0, w_0\rangle\rangle\rangle \\
&=_m \langle w, \Pi_f^{\mathbf{ECB}}(p), \lambda w_0.\langle \overline{w} w_0, \lambda x_0.(\lambda z.(\pi_1(\dddot{h}(\pi_1 z))(\pi_0 z))\langle x_0, w_0\rangle\rangle\rangle \\
&=_m \langle w, \Pi_f^{\mathbf{ECB}}(p), \lambda w_0.\langle \overline{w} w_0, \lambda x_0.(\pi_1(\dddot{h}(\pi_1\langle x_0, w_0\rangle))(\pi_0\langle x_0, w_0\rangle))\rangle\rangle \\
&=_m \langle w, \Pi_f^{\mathbf{ECB}}(p), \lambda w_0.\langle \overline{w} w_0, \lambda x_0.(\pi_1(\dddot{h}\, w_0) x_0)\rangle\rangle \\
&=_m \langle w, \Pi_f^{\mathbf{ECB}}(p), \lambda w_0.\langle \overline{w} w_0, \pi_1(\dddot{h}\, w_0)\rangle\rangle \\
&=_m \langle w, \Pi_f^{\mathbf{ECB}}(p), \lambda w_0.\langle \pi_0(\dddot{h}\, w_0), \pi_1(\dddot{h}\, w_0)\rangle\rangle \qquad\qquad (*) \\
&=_m \langle w, \Pi_f^{\mathbf{ECB}}(p), \lambda w_0.(\dddot{h}\, w_0)\rangle \\
&=_m h
\end{aligned}
$$

The step $(*)$ needs some explanation but, as we will see, just follows from the well-definedness of the isomorphism.

First consider $\overline{\varphi(w,p)}(h)$. We have $h : f^*w \overset{m}{\to} p$ and so $\dddot{\overline{h}}\langle a_0, w_0\rangle \mathrel{\dot\in} dom(p)$ for $\langle a_0, w_0\rangle$ in the pullback of f and w. Additionally this gives us $a_0 \sim_a \pi_0\langle a_0, w_0\rangle \sim_a \overline{(f^*w)}\langle a_0, w_0\rangle \sim_a \overline{p}(\dddot{\overline{h}}\langle a_0, w_0\rangle)$. From this we can see that $\lambda a_0.\, \dddot{\overline{h}}\langle a_0, w_0\rangle$ (the second component of $\overline{\psi(w,p)}(h)$) induces a morphism $f^{-1}\{\overline{w}w_0\} \overset{m}{\to} dom(p)$ by

$$\begin{aligned} \langle a_0, w_0\rangle \mathrel{\dot\in} dom(f^*w) &\leftrightarrow \overline{f}a_0 \sim_a \overline{w}w_0 \\ &\leftrightarrow a_0 \mathrel{\dot\in} f^{-1}\{\overline{w}w_0\} \end{aligned}$$

given $a_0 \mathrel{\dot\in} a$ and $w_0 \mathrel{\dot\in} w$. But this means that $\langle \overline{w}w_0, \lambda a_0.\, \dddot{\overline{h}}\langle a_0, w_0\rangle\rangle$ really is an element of $\Pi_f^{\mathbf{ECB}}(p)$. The fact, that this is a function follows because we can substitute in preimages (lemma 4.1.8) and because h is already a function.

For the other direction consider $w_0 \mathrel{\dot\in} w$. We have $\dddot{\overline{h}}\, w_0 \sim_{\Pi_f^{\mathbf{ECB}}(p)} \langle b_0, q\rangle$ and $q : f^{-1}\{b_0\} \overset{m}{\to} dom(p) \wedge (\forall a_0 \mathrel{\dot\in} f^{-1}\{b_0\})(\overline{p}(qa_0) \sim_a a_0)$. Because h is a morphism, we also have $b_0 \sim_b \overline{w}w_0$. So given a pair $\langle a_0, w_0\rangle$ from the pullback, we have

$$\pi_0(\dddot{\overline{h}}\, w_0) \sim_b \overline{(pr \circ \overline{h})}(w_0) \sim_b \overline{w}(w_0)$$

and so

$$\begin{aligned} &\langle a_0, w_0\rangle \mathrel{\dot\in} dom(f^*w) \\ &\leftrightarrow \overline{f}a_0 \sim_b \overline{w}(w_0) \\ &\leftrightarrow \overline{f}a_0 \sim_b \pi_0(\dddot{\overline{h}}\, w_0) \\ &\leftrightarrow a_0 \mathrel{\dot\in} f^{-1}\{\pi_0(\dddot{\overline{h}}\, w_0)\} \end{aligned}$$

Hence, $(\pi_1(\dddot{\overline{h}}\, w_0)a_0)$ is defined and gives an element in $dom(p)$ as required. That this is a function follows again because $\pi_1(\dddot{\overline{h}}\, w_0)a_0$ is actually just the composition of functions. (We can argue that π_1 induces the second projection on $\Pi_f^{\mathbf{ECB}}(p)$, which is of course not really well-typed when considered in a system like MLTT. $\Pi_f^{\mathbf{ECB}}(p)$ is basically

$$\sum_{b_0:b}(PartialSectionOn(f^{-1}\{b_0\},p))$$

the dependent sum of partial sections s of p.

$$\begin{tikzcd} & dom(p) \arrow[d, "p"] \\ f^{-1}\{b_0\} \arrow[ur, "s"] \arrow[r, hook] & dom(f) \end{tikzcd}$$

Finally, the step $(*)$ follows from the equation above stating that

$$\overline{w}(w_0) \sim_b \pi_0(\overset{-\,-\,-}{h}\, w_0).$$

The last step is showing naturality. We will write $\mathbf{ECB}(a,b)$ for the implicit Bishop Set $hom_{\mathbf{ECB}}(a,b)$ and its induced hom-functors.

naturality:

$$\begin{tikzcd} z_o(k,l) = (\mathbf{ECB}_u\diagup b)(k,\Pi_f(l)) \arrow[r, leftrightarrow, "\varphi(k,l)", "\psi(k,l)"'] \arrow[d, "z_m(\eta,\xi)"'] & w_o(k,l) = (\mathbf{ECB}_u\diagup a)(f^*(k),l) \arrow[d, "w_m(\eta,\xi)"] \\ z_o(w,v) = (\mathbf{ECB}_u\diagup b)(w,\Pi_f(v)) \arrow[r, leftrightarrow, "\varphi(w,v)", "\psi(w,v)"'] & = (\mathbf{ECB}_u\diagup a)(f^*(w),v) \end{tikzcd}$$

For $f : a \overset{m}{\to} b$ we have:

$$\Pi_f^{\mathbf{ECB}}{}_m(\xi) :\equiv \langle \Pi_f^{\mathbf{ECB}}(l), \Pi_f^{\mathbf{ECB}}(v), \lambda k.\langle \pi_0 k, \lambda z.\, \overset{-\,-}{\xi}((\pi_1 k)z)\rangle\rangle.$$

$$\langle (f^*)_o, (f^*)_m\rangle : \mathbf{ECB}\diagup b \to \mathbf{ECB}\diagup a$$

$$(f^*)_o(q) :\equiv f^*q : f * q \overset{m}{\to} a$$

$$(f^*)_m(\zeta : q \overset{m}{\to} v) :\equiv \langle f^*q, f^*v, f^*[\xi]\rangle.$$

where $f^*[\zeta]$ is the unique morphism in $\mathbf{ECB}$ from the cone given by f^*q and $\zeta \circ q^*f$ over the pullback $f * v$.

$$\begin{tikzcd}
f * q \arrow[rr, "q^* f"] \arrow[dr, dotted, "f^*[\zeta]"] \arrow[dd, "f^* q"'] & & \underline{q} \arrow[dr, "\zeta"] \arrow[dd, "q"] & \\
& f * v \arrow[rr, "v^* f"', crossing over] \arrow[dl, "f^* v"] & & \underline{v} \arrow[dl, "v"] \\
a \arrow[rr, "f"'] & & b &
\end{tikzcd}$$

We have the following, commuting diagram, where $g \dot{\in} w_o(k,l)$ and $e \dot{\in} z_o(k,l)$ are morphisms in the slice-categories and $\underline{k}$ indicates $dom(k)$.

$$\begin{tikzcd}
f * w \arrow[rr, "w^* f"] \arrow[dr, dotted, "f^*[\eta^{op}]"] \arrow[dd, "f^* w"'] & & \underline{w} \arrow[d, "\eta^{op}"] \arrow[ddddl, bend right, "w"'] \arrow[dddd, bend left=60, "\overline{z_m(\eta,\xi)}(e)"] \\
& f * k \arrow[r, "k^* f"] \arrow[dl, "f^* k"'] \arrow[d, "g"] & \underline{k} \arrow[d, "e"] \arrow[dddl, "k"] \\
a \arrow[dd, "f"'] & \underline{l} \arrow[l, "l"] \arrow[d, "\xi"] & \underline{\Pi_f(l)} \arrow[dd, "\Pi_{f_m}(\xi)"] \arrow[ddl, bend left, "\Pi_f(l)"] \\
& \underline{v} \arrow[ul, "v"] & \\
b \arrow[r, equal] & b & \underline{\Pi_f(v)} \arrow[l, "\Pi_f(v)"]
\end{tikzcd}$$

This means in particular that, when considered as morphisms in $\mathbf{ECB}_u$, we have

$$\begin{aligned}
\overline{z_m(\eta,\xi)}(e) \circ w^* f &=_m \Pi_{f_m}(\xi) \circ e \circ \eta^{op} \circ w^* f \\
&=_m \Pi_{f_m}(\xi) \circ e \circ k^* f \circ f^*[\eta^{op}]
\end{aligned}$$

Note that all pullback-morphisms are represented by simple projections π_i. That means, if we translate this to check naturality, we get

Let $e : k \overset{m}{\to} \Pi_f(l)$:

$$\begin{aligned}
&\overline{w_m(\eta,\xi)\circ\varphi(k,l)e}\\
&\sim \overline{w_m(\eta,\xi)}(\overline{\langle z_o(k,l), w_o(k,l), \lambda e.\langle f^*k, l, \lambda z.(\pi_1(\overline{e}(\pi_1 z)))(\pi_0 z)\rangle\rangle}e)\\
&\sim \overline{w_m(\eta,\xi)}(\langle f^*k, l, \lambda z.(\pi_1(\overline{e}(\pi_1 z)))(\pi_0 z)\rangle)\\
&\sim (\lambda g.\xi\circ g\circ(f^*)_m(\eta^{op}))(\langle f^*k, l, \lambda z.(\pi_1(\overline{e}(\pi_1 z)))(\pi_0 z)\rangle)\\
&\sim \xi\circ(\langle f^*k, l, \lambda z.(\pi_1(\overline{e}(\pi_1 z)))(\pi_0 z)\rangle)\circ(f^*)_m(\eta^{op})\\
&\sim \xi\circ(\langle f^*k, l, \lambda z.(\pi_1(\overline{e}(\pi_1 z)))(\pi_0 z)\rangle)\circ(f^*[\eta^{op}] : f^*w \overset{m}{\to} f^*k)\\
&\sim \langle f^*k, v, \lambda z.\,\overline{\xi}((\pi_1(\overline{e\circ k^* f}(z)))(\overline{f^*k}\, z))\rangle\circ(f^*[\eta^{op}] : f^*w \overset{m}{\to} f^*k)\\
&\sim \langle f^*k, v, \lambda z.((\pi_1(\overline{\Pi_{f_m}(\xi)\circ e\circ k^* f}(z)))(\overline{f^*k}\, z))\rangle\circ f^*[\eta^{op}]\\
&\sim \langle f^*w, v, \lambda z.((\pi_1(\overline{\Pi_{f_m}(\xi)\circ e\circ k^* f\circ f^*[\eta^{op}]}(z)))(\overline{f^*k\circ f^*[\eta^{op}]}\, z))\rangle\\
&\sim \langle f^*w, v, \lambda z.((\pi_1(\overline{\Pi_{f_m}(\xi)\circ e\circ k^* f\circ f^*[\eta^{op}]}(z)))(\overline{f^*w}z))\rangle\\
&\sim \langle f^*w, v, \lambda z.(\pi_1(\overline{(\Pi_f)_m(\xi)\circ e\circ\eta^{op}\circ w^* f}(z)))(\overline{f^*w}z)\rangle\\
&\sim \langle f^*w, v, \lambda z.(\pi_1((\overline{(\Pi_f)_m(\xi)\circ e\circ\eta^{op}})(\pi_1 z)))(\pi_0 z)\rangle\\
&\sim \overline{\langle z_o(w,v), w_o(w,v), \lambda e.\langle f^*w, v, \lambda z.(\pi_1(\overline{e}(\pi_1 z)))(\pi_0 z)\rangle\rangle}\\
&\qquad\quad ((\Pi_f)_m(\xi)\circ e\circ\eta^{op})\\
&\sim \overline{\varphi(w,v)}((\lambda h.(\Pi_f)_m(\xi)\circ h\circ\eta^{op})e)\\
&\sim \overline{\varphi(w,v)}((\Pi_f)_m(\xi)\circ e\circ\eta^{op})\\
&\overline{\varphi(w,v)\circ z_m(\eta,\xi)e}
\end{aligned}$$

Note that, strictly speaking we have used the dashed overline not quite correctly. At some point we should have switched to the slice category over b and consider the objects as compositions with f. But like this it is more readable. ☐

Corollary A.4.4. *Dependent products yield exponentials in any slice.*

Proof. The product object $h \times d$ in $\mathbf{ECB}_u / cod(h)$ is given by

$$h\circ(h^*d) =_m \Sigma_h(h^*d) : h * d \overset{m}{\to} cod(h).$$

$$\begin{array}{c}
h \times d \xrightarrow{m} f \\ \hline
\Sigma_h(h^* d) \xrightarrow{m} f \\ \hline
h^* d \xrightarrow{m} h^* f \\ \hline
d \xrightarrow{m} \Pi_h(h^* f)
\end{array}$$

for Σ_h as in definition 3.2.10. And so we have $f^h \cong \Pi_h(h^* f)$. □

Since we have restricted ourselves above to locally small versions of **ECB**, we can now apply corollary 5.1.9 of the Yoneda Lemma, to do the usual proof that exponential objects are preserved by pullback which implies that the Beck-Chevalley condition holds.

Proposition A.4.5. *The pullback-functor* $(f^*) : \mathbf{ECB}_u/b \to \mathbf{ECB}_u/a$ *induced by* $f : a \xrightarrow{m} b$ *preserves exponential objects and hence the ccc structure.*

Proof. Let $g : c \xrightarrow{m} b$ and any $l : \underline{l} \xrightarrow{m} c$ and $k : \underline{k} \xrightarrow{m} b$. We show $f^* \circ \Pi_g \cong \Pi_{f^* g} \circ (g^* f)^*$.

$$\begin{array}{cc}
k \xrightarrow{m} f^*(\Pi_g(l)) & \text{over } b \\ \hline
\Sigma_f k \xrightarrow{m} \Pi_g(l) & \text{over } a \\ \hline
g^*(\Sigma_f k) \xrightarrow{m} l & \text{over } c \\ \hline
g^*(f \circ k) \xrightarrow{m} l & \text{over } c \\ \hline
g^* f \circ ((f^* g)^* k) \xrightarrow{m} l & \text{over } c \\ \hline
\Sigma_{g^* f}(f^* g)^* k) \xrightarrow{m} l & \text{over } c \\ \hline
(f^* g)^* k \xrightarrow{m} (g^* f)^* l & \text{over } a \\ \hline
k \xrightarrow{m} \Pi_{f^* g}((g^* f)^* l) & \text{over } b
\end{array}$$

By corollary 5.1.9, we get $f^*(\Pi_g(l)) \cong \Pi_{f^*g}((g^*f)^*l)$. If we now have $u : \underline{u} \overset{m}{\to} b$ and $v : \underline{v} \overset{m}{\to} b$, we can apply this to exponentials:

$$\begin{aligned} f^*(u^v) &=_m f^*(\Pi_v(v^*u)) \cong \Pi_{f^*v}((v^*f)^*(v^*u)) \\ &\cong \Pi_{f^*v}(f^*[v^*u]) \cong \Pi_{f^*v}((f^*v)^*(f^*u)) =_m (f^*v)^{(f^*u)} \end{aligned}$$

Both the isomorphisms $g^*(f \circ k) \cong g^*f \circ ((f^*g)^k)$ and $(v^*f)^*(v^*u) \cong f^*[v^*u] \cong (f^*v)^*(f^*u)$ follow directly from pullback pasting-lemmas (in the second case from the pullback cube.) The fact that pullback also preserves products and the terminal object, and hence the whole ccc-structure, follows from RAPL (corollary 5.1.12) $\square$

A.5. Proofs for EC

Proposition A.5.1. *Let* $f, f' : b \xrightarrow{m} c$ *and* $g, g' : a \xrightarrow{m} b$ *be morphisms in* $\mathbf{EC}$. *The relation* $=^{\mathbf{EC}}_{m}$ *defined as pointwise equality is an equivalence relation and the composition* $\circ^{\mathbf{EC}}$ *is compatible with it.*

Proof. Reflexivity, symmetry and transitivity of $=^{\mathbf{EC}}_{m}$ is defined as pointwise equality which gets its properties from the axioms in definition 1.0.5 of the theory **EM**. As an example we consider transitivity let $A[u, v, w] :\equiv (u = v \wedge v = w \rightarrow u = w)$. A particular instance of the equality axioms then gives $u = u \wedge u = v \wedge v = w \wedge A[u, u, v] \rightarrow A[u, v, w]$. The other properties are similar.

For composition suppose

$$\begin{aligned} &g =^{\mathbf{EC}}_{m} g', \\ &f =^{\mathbf{EC}}_{m} f'. \end{aligned}$$

Before checking that $f \circ g =^{\mathbf{EC}}_{m} f' \circ g'$ holds, note that η-conversion is part of $=^{\mathbf{EC}}_{m}$:

$$f =^{\mathbf{EC}}_{m} \langle b, c, \lambda x.\overline{f}x \rangle \tag{η}$$

this is because

$$\begin{aligned} (\lambda x.\overline{f}x)z &\simeq (\mathrm{s}(\lambda x.\overline{f})(\lambda x.x))z \\ &\simeq (\mathrm{s}(\mathrm{k}\overline{f})(\mathrm{skk}))z \\ &\simeq (\mathrm{s}(\mathrm{k}\overline{f})(\mathrm{skk}))z \\ &\simeq ((\mathrm{k}\overline{f})z)((\mathrm{skk})z) \\ &\simeq \overline{f}((\mathrm{skk})z) \\ &\simeq \overline{f}((\mathrm{k}z)(\mathrm{k}z)) \\ &\simeq \overline{f}z \end{aligned}$$

and for all elements of the domain of f we have $(\overline{f}z)\!\downarrow$.

Then for all $w \dot{\in} a$ we have

$$\begin{aligned}(\lambda z.\overline{f}(\overline{g}z))w &= \overline{f}(\overline{g}w) \\ &= \overline{f'}(\overline{g}w) \\ &= \overline{f'}(\overline{g'}w) \\ &= (\lambda z.\overline{f'}(\overline{g'}z))w\end{aligned}$$

where for the first equality we have $(\overline{g}w)\downarrow \wedge (\overline{g}w) \dot{\in} b$ and (η), for the second and third we use $f =_m^{\mathbf{EC}} f'$ and $g =_m^{\mathbf{EC}} g'$, and last one is again by (η) because everything is defined.

And so get

$$\begin{aligned}f \circ g &=_m \langle a, c, \lambda z.\overline{f}(\overline{g}z)\rangle \\ &=_m \langle a, c, \lambda z.\overline{f'}(\overline{g'}z)\rangle \\ &=_m f' \circ g'.\end{aligned}$$ □

Lemma A.5.2. *There are two operations on N which are extensionally equal but for which it is impossible to prove equality on terms.*

Proof. Let $f :\equiv \lambda x.0$ and $g :\equiv \lambda x.(\lambda y.y)0$.

For all natural numbers n we have $fn = 0 = gn$. But if we expand the definition of lambda terms we get

$$\begin{aligned}f &= \lambda x.0 = \mathrm{k}0 \\ g &= \lambda x.((\lambda y.y)0) = \mathrm{s}(\lambda x.(\lambda y.y))(\lambda x.0) \\ &= \mathrm{s}(\mathrm{k}(\mathrm{skk}))(\mathrm{k}0)\end{aligned}$$

Note how both s and k only have two (resp. one) arguments supplied at all occurrences. the term model these terms do not reduce any further and hence are not the same which implies that

$$\mathbf{EM} \nvdash f = g.$$ □

Proposition A.5.3. *For all $Ob_{\mathbf{EC}}(a)$ and $f, g : \mathbb{1} \xrightarrow{m} a$ equality is characterized by*

$$f =_m^{\mathbf{EC}} g \leftrightarrow \overline{f}(*) = \overline{g}(*)$$

Proof. By expansion of the definition of $=^{\mathbf{EC}}_{m}$ and $x \mathrel{\dot\in} \mathbb{1} \leftrightarrow x = *$. ☐

Lemma A.5.4. *Non-provability of the existence of a term c such that*

$$(\forall f, g \in (N \to N))(c(f) = c(g) \leftrightarrow (\forall x \in N)(fx = gx)).$$

Proof sketch. Let $f^{\neq 0} :\equiv \lambda x.d_N 10(fx)0$ be an operator. For any operation f which is total on the natural numbers, this means for $fx = 0$ that $f^{\neq 0}x = d_N 10(fx)0 = d_N 1000 = 1$ and for $fx \neq 0$ that $f^{\neq 0}x = 0$. This provides a zero-searching predicate

$$Z[f] :\equiv c(f^{\neq 0}) \neq c(\lambda x.0).$$

But that means we have for total f, that $Z[f] \leftrightarrow \neg(\forall x \in N)(fx \neq 0)$ from which we can conclude

$$f \in (N \to N) \to (Z[f] \leftrightarrow (\exists n \in N)(fx = 0))$$

As there is clearly no Turing machine which can check arbitrary total recursive functions for the existence of zeros and rejecting all those which have none, this shows that such an operation c can not exist in Kleene's first model and hence it's impossible to prove in **EM**. ☐

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Symbols

Erklärung

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....
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