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THE VARIABLE-ORDER DISCONTINUOUS GALERKIN TIME STEPPING SCHEME FOR PARABOLIC EVOLUTION PROBLEMS IS UNIFORMLY L^∞-STABLE*

LARS SCHMUTZ† AND THOMAS P. WIHLER†

Abstract. In this paper we investigate the L^∞-stability of fully discrete approximations of abstract linear parabolic partial differential equations (PDEs). The method under consideration is based on an hp-type discontinuous Galerkin time stepping scheme in combination with general conforming Galerkin discretizations in space. Our main result shows that the global-in-time maximum norm of the discrete solution is bounded by the data of the PDE, with a constant that is robust with respect to the discretization parameters (in particular, it is uniformly bounded with respect to the local time steps and approximation orders).

Key words. discontinuous Galerkin time stepping, Galerkin discretizations, parabolic evolution problems, stability, hp-methods

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1. Introduction.

Let $\mathbb{H}$ and $\mathbb{X}$ be two (real) Hilbert spaces, equipped with the inner products $(\cdot,\cdot)_{\mathbb{H}}$ and $(\cdot,\cdot)_{\mathbb{X}}$, respectively, as well as with the corresponding induced norms $\|\cdot\|_{\mathbb{H}}$ and $\|\cdot\|_{\mathbb{X}}$. The respective dual spaces are denoted by $\mathbb{H}^\star$ and $\mathbb{X}^\star$. Suppose that $\mathbb{X}$ is densely embedded in $\mathbb{H}$, and consider the Gelfand triple

$$\mathbb{X} \hookrightarrow \mathbb{H} \cong \mathbb{H}^\star \hookrightarrow \mathbb{X}^\star. \tag{1.1}$$

In this paper, based on a variable-order discontinuous Galerkin (dG) time stepping method in conjunction with a conforming Galerkin approximation in space, we will study the stability of the fully discrete numerical approximation of the linear parabolic problem

$$\begin{aligned} u'(t) + \mathsf{A}u(t) &= f(t), \qquad t \in (0,T], \\ u(0) &= u_0. \end{aligned} \tag{1.2}$$

Here, $\mathsf{A} : \mathbb{X} \to \mathbb{X}^\star$ is a linear, self-adjoint, and time-independent elliptic operator that is coercive and bounded in the sense that there are two constants $\alpha_{1.3}, \beta_{1.3} > 0$ such that

$$\begin{aligned} \langle \mathsf{A}v, v\rangle_{\mathbb{X}^\star\times\mathbb{X}} &\geq \alpha_{1.3} \|v\|_{\mathbb{X}}^2 \qquad \forall v \in \mathbb{X}, \\ \left|\langle \mathsf{A}v, w\rangle_{\mathbb{X}^\star\times\mathbb{X}}\right| &\leq \beta_{1.3} \|v\|_{\mathbb{X}} \|w\|_{\mathbb{X}} \qquad \forall v, w \in \mathbb{X}. \end{aligned} \tag{1.3}$$

Furthermore, we let $f \in \mathrm{L}^2((0,T);\mathbb{H})$ and $u_0 \in \mathbb{H}$ be a given source term and prescribed initial value, respectively. Applying standard notation for Sobolev and

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†Mathematisches Institut, Universität Bern, CH-3012 Bern, Switzerland (lars.schmutz@math.unibe.ch, thomas.wihler@math.unibe.ch).

Bochner spaces (cf., e.g., [19, section 1.5]), a classical weak formulation of (1.2) is to find $u \in \mathrm{L}^2((0,T);\mathbb{X}) \cap \mathrm{W}^{1,2}((0,T);\mathbb{X}^\star)$ such that, for every $v \in \mathbb{X}$, it holds that

$$\begin{aligned} \langle u', v\rangle_{\mathbb{X}^\star\times\mathbb{X}} + \langle \mathsf{A}u, v\rangle_{\mathbb{X}^\star\times\mathbb{X}} &= (f(t), v)_{\mathbb{H}}\,, \qquad t\in(0,T], \\ u(0) &= u_0. \end{aligned} \tag{1.4}$$

Here, we signify the duality pairing in $\mathbb{X}^\star \times \mathbb{X}$ by $\langle u, v\rangle_{\mathbb{X}^\star\times\mathbb{X}}$; incidentally, this dual product can be seen as an extension of the inner product in $\mathbb{H}$, i.e.,

$$(u,v)_H = \langle u, v\rangle_{\mathbb{X}^\star\times\mathbb{X}} \qquad \forall u\in\mathbb{H}, v\in\mathbb{X}; \tag{1.5}$$

see, e.g., [19, section 7.2]. Recalling the continuous embedding

$$\mathrm{L}^2(0,T;\mathbb{X}) \cap \mathrm{W}^{1,2}(0,T;\mathbb{X}^\star) \hookrightarrow \mathrm{C}^0(0,T;\mathbb{H})$$

(cf., e.g., [19, Lemma 7.3]), we conclude that the solution of (1.4) is continuous in time, i.e., $u \in \mathrm{C}^0(0,T;\mathbb{H})$. Furthermore, the following stability estimate holds:

$$\|u\|_{\mathrm{L}^2(0,T;\mathbb{X})} + \|u'\|_{\mathrm{L}^2(0,T;\mathbb{X}^\star)} + \|u\|_{\mathrm{C}^0(0,T;\mathbb{H})} \le C\left(\|u_0\|_{\mathbb{H}} + \|f\|_{\mathrm{L}^2(0,T;\mathbb{H})}\right); \tag{1.6}$$

see, e.g., [19, Theorem 8.9].

In the context of parabolic partial differential equations (PDEs), the dG time stepping methodology was introduced a few decades ago in [12]. Since then a lot of research has been conducted on this subject: we point the reader to the classical works [4, 5, 6, 7, 8, 14, 25], as well as to the more recent articles [1, 2, 3, 13, 15, 16], where a novel reconstruction technique for the purpose of a posteriori error estimation has been proposed and analyzed. While these articles mainly focus on low-order temporal Galerkin discretizations of fixed degree, the use of *hp*-type dG methods was proposed in [21, 22]. The *hp*-framework permits us to employ locally different time step sizes and arbitrary variations of the local approximation orders, and, thereby, to attain high algebraic or even exponential rates of convergence in time. This feature is particularly powerful if local singularities appear (for instance, in the form of a parabolic time layer due to incompatible initial data) [22, 23, 27] or if highly nonlocal [17, 18] or high-dimensional [26] problems need to be solved.

The present paper centers on the stability of fully discrete *hp*-version dG time discretizations of abstract linear parabolic problems. More precisely, given the solution, u, of (1.2), and its *hp*-dG approximation, U, our goal is to argue that the stability estimate (1.6) holds true also on the discrete level. Indeed, using standard energy arguments, it is fairly straightforward to show that U is bounded with respect to the $\mathrm{L}^2(\mathbb{X})$-norm; indeed, this essentially follows from [22, eq. (2.18)] and the boundedness of the duality pairing. In addition, applying a suitable reconstruction $\widehat{U}$ of U (see, e.g., [16, section 2.1] or [10, section 3.6]) and applying an inf-sup stability result (cf., e.g., [9]) shows that $\widehat{U}'$ is also stable in the $\mathrm{L}^2(\mathbb{X}^\star)$-norm.

In the current work our goal is to establish the stability of the discrete solution U with respect to the $\mathrm{L}^\infty(\mathbb{H})$-norm. We particularly emphasize deriving an estimate with a (known) constant that is *uniformly bounded* with respect to the discretization parameters (i.e., in particular, the local time step lengths and approximation orders). Since our focus is on a pointwise bound, energy arguments are not appropriate in the discrete context; indeed, this is due to the fact that suitable test functions (such as cut-off functions) typically do *not* belong to the underlying discrete test space.

Furthermore, the application of inverse estimates usually involves constants that scale suboptimally with respect to the local approximation orders and, thereby, lead to nonuniform stability results. For these reasons we will pursue a completely different and novel approach: More precisely, we will first derive a pointwise formulation of the fully discrete scheme (section 2.2) using a lifting operator technique as in [24]; cf. also the temporal reconstruction approach [9, 10, 16]. Then, we analyze the fully discrete parabolic operator and show that its inverse operator is $\mathrm{L}^\infty(\mathbb{H})$-stable (section 4). In order to proceed in this direction, in section 2 we will first investigate the special case where $\mathbb{H} = \mathbb{X} = \mathbb{R}$ in (1.1), and we construct a representation formula (section 3.2) which is composed of two terms: The first term is based on the concept of a dG fundamental solution (section 3.1) and relates to the initial value, u_0, in (1.2). The second term, analogously as in the classical Duhamel principle, is an integral that involves the product of the right-hand side function, f, in (1.2), and an exponentially decaying expression in time. Subsequently, using a spectral decomposition, we will employ the scalar analysis on each time step in order to derive a stability bound for the inverse parabolic operator in the abstract case (Proposition 4.1). Finally, inverting the pointwise form of the dG scheme and applying the previous stability analysis eventually implies the main result of this article, i.e., the uniform $\mathrm{L}^\infty(\mathbb{H})$-stability of the dG method (Theorem 4.4). We emphasize that the discrete Duhamel formula and the stability of the inverse of the fully discrete parabolic dG operator, which will be developed in this work, have strong implications that reach far beyond the scope of the present paper; these include, for instance, the analysis of nonlinear parabolic PDE approximation schemes (see, e.g., [20, sections 5 and 7]).

2. Fully discrete discontinuous Galerkin time stepping.

2.1. Variable-order time partitions and discrete spaces. On an interval $I = [0, T]$, $T > 0$, consider time nodes $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$, which introduce a time partition $\mathcal{M} = \{I_m\}_{m=0}^M$ of I into $M + 1$ time intervals $I_m = (t_{m-1}, t_m]$, $m = 1, \ldots, M$, and $I_0 = \{t_0\}$. The (possibly varying) length $k_m = t_m - t_{m-1}$ of a time interval is called the mth time step. We define the one-sided limits of an $\mathcal{M}$-wise continuous function v at each time node t_m, $0 \le m \le M - 1$, by

$$v_m^+ := \lim_{s \searrow 0} v(t_m + s), \qquad v_m^- := \lim_{s \searrow 0} v(t_m - s),$$

where v_0^- is considered to be a prescribed initial value. Then, the discontinuity jump of v at t_m, $0 \le m \le M - 1$, is defined by $[\![v]\!]_m := v_m^+ - v_m^-$.

Furthermore, to each interval we associate a polynomial degree $r_m \ge 0$, which takes the role of a local approximation order. Moreover, given any (real) Hilbert (sub)space $\mathbb{V} \subset \mathbb{H}$, an integer $r \in \mathbb{N}_0$, and an interval $J \subset \mathbb{R}$, the set

$$\mathbb{P}^r(J; \mathbb{V}) = \left\{ p \in \mathrm{C}^0(\bar{J}; \mathbb{V}) : p(t) = \sum_{i=0}^r v_i t^i,\ v_i \in \mathbb{V} \right\}$$

signifies the space of all polynomials of degree at most r on J with values in $\mathbb{V}$. If $\mathbb{V} = \mathbb{R}$, then we simply write $\mathbb{P}^r(J)$.

A fully discrete framework for (1.4) is based on replacing the Hilbert space $\mathbb{X}$ from (1.1) by finite-dimensional subspaces $\mathbb{X}_m \subset \mathbb{X}$, $n_m := \dim(\mathbb{X}_m) < \infty$, on each interval I_m, $0 \le m \le M$. The $\mathbb{H}$-orthogonal projection from $\mathbb{H}$ to $\mathbb{X}_m$, for $0 \le m \le M$, is given by

$$\pi_m : \mathbb{H} \to \mathbb{X}_m, \qquad v \mapsto \pi_m v : \quad (v - \pi_m v, w)_{\mathbb{H}} = 0 \quad \forall w \in \mathbb{X}_m.$$

Notice the obvious stability property,

$$\|\pi_m v\|_{\mathbb{H}} \leq \|v\|_{\mathbb{H}} \qquad \forall v \in \mathbb{H}. \tag{2.1}$$

Moreover, $\mathsf{A}_m : \mathbb{X} \to \mathbb{X}_m$ denotes the discretization of A defined by

$$(\mathsf{A}_m u, v)_{\mathbb{H}} = \langle \mathsf{A}u, v\rangle_{\mathbb{X}^\star \times \mathbb{X}} \qquad \forall v \in \mathbb{X}_m \tag{2.2}$$

for $1 \leq m \leq M$. Recalling (1.3), we observe that A_m is invertible as an operator from $\mathbb{X}_m$ to $\mathbb{X}_m$.

2.2. Fully discrete dG time stepping. Based on the previous definitions, the *fully discrete dG-in-time/conforming-in-space scheme* for (1.2) is given iteratively as follows: Find $U|_{I_m} \in \mathbb{P}^{r_m}(I_m; \mathbb{X}_m)$ through the weak formulation

$$\begin{aligned} \int_{I_m} (U', V)_{\mathbb{H}} \,\mathsf{d}t + ([\![U]\!]_{m-1}, V_{m-1}^+)_{\mathbb{H}} + \int_{I_m} \langle \mathsf{A}U, V\rangle_{\mathbb{X}^\star \times \mathbb{X}} \,\mathsf{d}t& \\ = \int_{I_m} (f, V)_{\mathbb{H}} \,\mathsf{d}t \qquad \forall V \in \mathbb{P}^{r_m}(I_m; \mathbb{X}_m)& \end{aligned} \tag{2.3}$$

for any $1 \leq m \leq M$. Here, for $m = 1$, we let

$$U_0^- := \pi_0 u_0, \tag{2.4}$$

where $u_0 \in \mathbb{H}$ is the initial value from (1.2), and, thereby, $[\![U]\!]_0 = U_0^+ - \pi_0 u_0$.

In order to write (2.3) in pointwise form, we proceed along the lines of [24]. Specifically, for $1 \leq m \leq M$, and any $z \in \mathbb{X}_m$, we define the (linear) lifting operator

$$\mathsf{L}_m^{r_m} : \mathbb{X}_m \to \mathbb{P}^{r_m}(I_m; \mathbb{X}_m)$$

by

$$\int_{I_m} (\mathsf{L}_m^{r_m}(z), V)_{\mathbb{H}} \,\mathsf{d}t = (z, V(t_{m-1}))_{\mathbb{H}} \qquad \forall V \in \mathbb{P}^{r_m}(I_m; \mathbb{X}_m).$$

Referring to [24, Lemma 6], the explicit representation formula

$$\mathsf{L}_m^{r_m}(z) = \frac{z}{k_m} \sum_{i=0}^{r_m} (-1)^i (2i+1) K_i^m(t) \tag{2.5}$$

holds, where $\{K_i^m\}_{i \geq 0}$ is the family of Legendre polynomials, affinely scaled from $[-1, 1]$ to I_m, such that

$$(-1)^i K_i^m(t_{m-1}) = K_i^m(t_m) = 1, \qquad i \geq 0, \tag{2.6}$$

and

$$\int_{I_m} K_i^m(t) K_j^m(t) \,\mathsf{d}t = \frac{k_m}{2i+1} \delta_{ij} \qquad \forall i, j \in \mathbb{N}_0; \tag{2.7}$$

see [24, section 3.1] for details. For later purposes, we also introduce the endpoint lifting operator

$$\widetilde{\mathsf{L}}_m^{r_m} : \mathbb{X}_m \to \mathbb{P}^{r_m}(I_m; \mathbb{X}_m)$$

by

$$\int_{I_m} \left(\widetilde{\mathsf{L}}_m^{r_m}(z), V\right)_{\mathbb{H}} \, \mathsf{d}t = (z, V(t_m))_{\mathbb{H}} \qquad \forall V \in \mathbb{P}^{r_m}(I_m; \mathbb{X}_m).$$

Using (2.5) and (2.6), we may represent it as

$$\widetilde{\mathsf{L}}_m^{r_m}(z) = \frac{z}{k_m} \sum_{i=0}^{r_m} (-1)^i (2i+1) K_i^m(-t) = \frac{z}{k_m} \sum_{i=0}^{r_m} (2i+1) K_i^m(t). \tag{2.8}$$

For any $w \in \mathrm{L}^2(I_m; \mathbb{H})$ we denote by $\Pi_m^{r_m}(w) \in \mathbb{P}^{r_m}(I_m; \mathbb{X}_m)$ the fully discrete $\mathrm{L}^2(I_m; \mathbb{H})$-projection defined by

$$\int_{I_m} (\Pi_m^{r_m}(w), V)_{\mathbb{H}} \, \mathsf{d}t = \int_{I_m} (w, V)_{\mathbb{H}} \, \mathsf{d}t \qquad \forall V \in \mathbb{P}^{r_m}(I_m; \mathbb{X}_m).$$

Then, employing the spatial projection π_m from (2.1) and the discrete elliptic operator A_m from (2.2), and using the lifting operator $\mathsf{L}_m^{r_m}$, we transform (2.3) into

$$\int_{I_m} (U' + \mathsf{L}_m^{r_m}(\pi_m [\![U]\!]_{m-1}) + \mathsf{A}_m U - \Pi_m^{r_m} f, V)_{\mathbb{H}} \, \mathsf{d}t = 0 \qquad \forall V \in \mathbb{P}^{r_m}(I_m; \mathbb{X}_m).$$

This immediately implies the pointwise form

$$U' + \mathsf{L}_m^{r_m}(\pi_m [\![U]\!]_{m-1}) + \mathsf{A}_m U = \Pi_m^{r_m} f, \qquad t \in I_m. \tag{2.9}$$

Following [11], for $1 \leq m \leq M$, we consider the dG-time operator

$$\chi_m^{r_m} : \mathbb{P}^{r_m}(I_m; \mathbb{X}_m) \to \mathbb{P}^{r_m}(I_m; \mathbb{X}_m),$$

given by

$$\chi_m^{r_m}(U) := U' + \mathsf{L}_m^{r_m}(U_{m-1}^+), \qquad U \in \mathbb{P}^{r_m}(I_m; \mathbb{X}_m). \tag{2.10}$$

Consequently, introducing the operator

$$\Gamma_m^{r_m} : \mathbb{P}^{r_m}(I_m; \mathbb{X}_m) \to \mathbb{P}^{r_m}(I_m; \mathbb{X}_m)$$

given by

$$\Gamma_m^{r_m} := \chi_m^{r_m} + \mathsf{A}_m, \tag{2.11}$$

we can write (2.9) as

$$\Gamma_m^{r_m}(U) = \Pi_m^{r_m} f + \mathsf{L}_m^{r_m}(\pi_m U_{m-1}^-) \tag{2.12}$$

for $1 \leq m \leq M$. Referring to [22, Proposition 2.6], we note that (2.3) is uniquely solvable, and hence the operator $\Gamma_m^{r_m}$ from (2.11) is an isomorphism on $\mathbb{P}^{r_m}(I_m; \mathbb{X}_m)$.

3. Scalar problem. In order to derive a stability analysis for the fully discrete scheme (2.12), we focus first on the case where $\mathbb{H} = \mathbb{X} = \mathbb{R}$. Specifically, for $1 \leq m \leq M$, consider the scalar problem of finding a function $u : I_m \to \mathbb{R}$ such that

$$\begin{aligned} u'(t) + \lambda u(t) &= f(t), \qquad t \in I_m, \\ u(t_{m-1}) &= u_{m-1}. \end{aligned}$$

Here, $\lambda > 0$ is a fixed parameter, $u_{m-1} \in \mathbb{R}$ is a prescribed initial value, and $f : [0,T] \to \mathbb{R}$ is a given source function. The dG time discretization of this problem is formulated in strong form as

$$\Gamma_{\lambda,m}^{r_m}(U) = \Pi_m^{r_m} f + \mathsf{L}_m^{r_m}(u_{m-1}), \qquad t \in I_m, \tag{3.1}$$

where, in this simplified context, $\Pi_m^{r_m} : \mathrm{L}^2(I_m) \to \mathbb{P}^{r_m}(I_m)$ is the L^2-projection onto $\mathbb{P}^{r_m}(I_m)$, and

$$\Gamma_{\lambda,m}^{r_m} : \mathbb{P}^{r_m}(I_m) \to \mathbb{P}^{r_m}(I_m), \qquad \Gamma_{\lambda,m}^{r_m}(v) = \chi_m^{r_m}(v) + \lambda v \tag{3.2}$$

is the scalar version of (2.11). As mentioned earlier, $\Gamma_{\lambda,m}^{r_m}$ is an isomorphism on the space $\mathbb{P}^{r_m}(I_m)$. Hence, applying the inverse operator $(\Gamma_{\lambda,m}^{r_m})^{-1}$ to (3.1), the dG solution U on I_m can be represented as follows:

$$U = (\Gamma_{\lambda,m}^{r_m})^{-1} \left[\mathsf{L}_m^{r_m}(u_{m-1}) + \Pi_m^{r_m} f\right] \quad \text{on } I_m. \tag{3.3}$$

Consequently, the stability of the inverse of $\Gamma_m^{r_m}$ is crucial in our analysis. We will attend to this matter by means of the classical scalar model problem

$$\begin{aligned} \psi'(t) + \lambda\psi(t) &= 0, \qquad t \in I_m, \\ \psi(t_{m-1}) &= 1, \end{aligned} \tag{3.4}$$

with the solution $\psi(t) = e^{-\lambda(t-t_{m-1})}$.

3.1. dG fundamental solution. We denote the dG time stepping approximation of (3.4) by $\psi_\lambda^{r_m} \in \mathbb{P}^{r_m}(I_m)$ and call it the *dG fundamental solution of degree r_m on I_m*. Based on (3.1) and (3.3), with $f \equiv 0$ and $u_{m-1} = 1$, it holds that

$$\Gamma_{\lambda,m}^{r_m}(\psi_\lambda^{r_m}) = \mathsf{L}_m^{r_m}(1) \tag{3.5}$$

and

$$\psi_\lambda^{r_m} = (\Gamma_{\lambda,m}^{r_m})^{-1}(\mathsf{L}_m^{r_m}(1)), \tag{3.6}$$

respectively.

Our goal is to derive an explicit representation formula for $(\Gamma_{\lambda,m}^{r_m})^{-1}$. To this end, we consider the subspace

$$\mathbb{P}_0^{r_m}(I_m) := \{v \in \mathbb{P}^{r_m}(I_m) : v(t_{m-1}) = 0\}$$

as well as its image under $\Gamma_{\lambda,m}^{r_m}$, i.e.,

$$\mathbb{W}_\lambda^{r_m}(I_m) := \Gamma_{\lambda,m}^{r_m}(\mathbb{P}_0^{r_m}(I_m)).$$

LEMMA 3.1. *Let $\lambda \geq 0$. It holds that* $\dim \mathbb{W}_\lambda^{r_m}(I_m) = r_m$, *and we have the direct sum*

$$\mathbb{W}_\lambda^{r_m}(I_m) \oplus \operatorname{span}\{\mathsf{L}_m^{r_m}(1)\} = \mathbb{P}^{r_m}(I_m).$$

Proof. If $\lambda = 0$, then the result simply follows by observing that the derivative operator maps the space $\mathbb{P}^{r_m}(I_m)$ onto $\mathbb{P}^{r_m-1}(I_m)$ if $r_m > 0$, and by noticing that the lifting operator is of exact degree r_m. Hence, let us consider the case $\lambda > 0$. For $i \geq 0$ and $1 \leq m \leq M$, consider the integrated Legendre polynomials

$$Q_i^m(t) := \frac{2}{k_m}\int_{t_{m-1}}^{t} K_i^m(s)\,\mathsf{d}s, \qquad t \in I_m. \tag{3.7}$$

Evidently, the set $\{Q_i^m\}_{i=0}^{r_m-1}$ is a basis of $\mathbb{P}_0^{r_m}(I_m)$. Furthermore, since the polynomial degree of $\Gamma_{\lambda,m}^{r_m}(Q_i^m)$ is exactly $i+1$, for $i \geq 0$, it follows that $\{\Gamma_{\lambda,m}^{r_m}(Q_i^m)\}_{i=0}^{r_m-1}$ forms a basis of $\mathbb{W}_\lambda^{r_m}(I_m)$. It therefore remains to show that the intersection of span$\{\mathsf{L}_m^{r_m}(1)\}$ and $\mathbb{W}_\lambda^{r_m}(I_m)$ is trivial. Take any $w \in \mathbb{W}_\lambda^{r_m}(I_m)$, and choose $v \in \mathbb{P}_0^{r_m}(I_m)$ such that $w = v' + \lambda v = \alpha \mathsf{L}_m^{r_m}(1)$ for some $\alpha \in \mathbb{R}$. Then, testing by v and integrating over I_m yields

$$0 = \alpha v(t_{m-1}) = \alpha \int_{I_m} \mathsf{L}_m^{r_m}(1) v\,\mathsf{d}t = \int_{I_m}(v' + \lambda v)v\,\mathsf{d}t = \frac{1}{2}v(t_m)^2 + \lambda \|v\|_{\mathrm{L}^2(I_m)}^2.$$

Hence, we conclude that $v \equiv 0$, and therefore $w \equiv 0$. ∎

It is interesting and useful for the subsequent analysis to notice that the above setup gives rise to the *dG dual solution of degree r_m on I_m*, which we denote by $\phi_\lambda^{r_m} \in \mathbb{P}^{r_m}(I_m)$. It is defined via the differential equation

$$(\phi_\lambda^{r_m})' - \lambda \phi_\lambda^{r_m} - \widetilde{\mathsf{L}}_m^{r_m}(\phi_\lambda^{r_m}(t_m)) = \mathsf{L}_m^{r_m}(1), \tag{3.8}$$

where the lifting operators $\mathsf{L}_m^{r_m}$ and $\widetilde{\mathsf{L}}_m^{r_m}$ are given in (2.5) and (2.8), respectively, with $\mathbb{X}_m$ being replaced by $\mathbb{R}$.

LEMMA 3.2. *Suppose that $\lambda \geq 0$. There exists exactly one solution of* (3.8) *in $\mathbb{P}^{r_m}(I_m)$, i.e., the dG dual solution $\phi_\lambda^{r_m}$ is well defined in $\mathbb{P}^{r_m}(I_m)$. Furthermore, $\phi_\lambda^{r_m}$ is L^2-orthogonal to $\mathbb{W}_\lambda^{r_m}(I_m)$, i.e.,*

$$\int_{I_m} \phi_\lambda^{r_m} w\,\mathsf{d}t = 0 \qquad \forall w \in \mathbb{W}_\lambda^{r_m}(I_m).$$

Proof. Let us define an operator $\Psi : \mathbb{P}^{r_m}(I_m) \to \mathbb{P}^{r_m}(I_m)$ by

$$\Psi(v) := v' - \lambda v - \widetilde{\mathsf{L}}_m^{r_m}(v(t_m)). \tag{3.9}$$

We show that the kernel of Ψ is trivial, i.e., Ψ is an isomorphism. Suppose that $v \in \mathbb{P}^{r_m}(I_m)$ and $\Psi(v) \equiv 0$. In the case when $\lambda = 0$, this implies that $v' = \widetilde{\mathsf{L}}_m^{r_m}(v(t_m))$. Now, since $\widetilde{\mathsf{L}}_m^{r_m}(v(t_m))$ has degree exactly r_m, unless $v(t_m) = 0$, we conclude that $v' \equiv 0$ as well as $v(t_m) = 0$. This, in turn, leads to $v \equiv 0$. Otherwise, if $\lambda > 0$, we test (3.9) by $v \in \mathbb{P}^{r_m}(I_m)$ and integrate over I_m. Then,

$$\begin{aligned}
0 = \int_{I_m} \Psi(v) v\,\mathsf{d}t &= \int_{I_m}\Big(v' - \lambda v - \widetilde{\mathsf{L}}_m^{r_m}(v(t_m))\Big)\,v\,\mathsf{d}t \\
&= \frac{1}{2}\left(v(t_m)^2 - v(t_{m-1})^2\right) - \lambda \|v\|_{\mathrm{L}^2(I_m)}^2 - v(t_m)^2 \\
&= -\frac{1}{2}(v(t_m)^2 + v(t_{m-1})^2) - \lambda \|v\|_{\mathrm{L}^2(I_m)}^2.
\end{aligned}$$

This immediately results in $v \equiv 0$. Hence, there exists exactly one $\phi_\lambda^{r_m} \in \mathbb{P}^{r_m}(I_m)$ such that $\Psi(\phi_\lambda^{r_m}) = \mathsf{L}_m^{r_m}(1)$.

In order to prove the second assertion, we let $w \in \mathbb{W}_\lambda^{r_m}(I_m)$ and choose $v \in \mathbb{P}_0^{r_m}(I_m)$ such that $w = v' + \lambda v$. Then, integrating by parts, it holds that

$$\begin{aligned}\int_{I_m} \phi_\lambda^{r_m} w \,\mathsf{d}t &= \int_{I_m} \phi_\lambda^{r_m}(v' + \lambda v)\,\mathsf{d}t \\ &= \int_{I_m} (-(\phi_\lambda^{r_m})' + \lambda \phi_\lambda^{r_m}) v \,\mathsf{d}t + \phi_\lambda^{r_m}(t_m) v(t_m) \\ &= \int_{I_m} (-(\phi_\lambda^{r_m})' + \lambda \phi_\lambda^{r_m} + \widetilde{\mathsf{L}}_m^{r_m}(\phi_\lambda^{r_m}(t_m))) v \,\mathsf{d}t.\end{aligned}$$

Invoking (3.8), we obtain

$$\int_{I_m} \phi_\lambda^{r_m} w \,\mathsf{d}t = \int_{I_m} -\mathsf{L}_m^{r_m}(1) v = -v(t_{m-1}) = 0.$$

Therefore, $\phi_\lambda^{r_m}$ is in the orthogonal complement of $\mathbb{W}_\lambda^{r_m}(I_m)$. □

LEMMA 3.3. *Let $\lambda \geq 0$. The initial values of the dG fundamental solution $\psi_\lambda^{r_m}$ and the dG dual solution $\phi_\lambda^{r_m}$ satisfy*

$$\phi_\lambda^{r_m}(t_{m-1}) = -\psi_\lambda^{r_m}(t_{m-1}). \tag{3.10}$$

Proof. Testing (3.8) by $\psi_\lambda^{r_m}$ and integrating over I_m by parts, we obtain

$$\begin{aligned}0 &= \int_{I_m} \Big((\phi_\lambda^{r_m})' - \lambda \phi_\lambda^{r_m} - \widetilde{\mathsf{L}}_m^{r_m}(\phi_\lambda^{r_m}(t_m)) - \mathsf{L}_m^{r_m}(1)\Big) \psi_\lambda^{r_m} \,\mathsf{d}t \\ &= -\int_{I_m} \phi_\lambda^{r_m} (\psi_\lambda^{r_m})' \,\mathsf{d}t + \phi_\lambda^{r_m}(t_m)\psi_\lambda^{r_m}(t_m) - \phi_\lambda^{r_m}(t_{m-1})\psi_\lambda^{r_m}(t_{m-1}) \\ &\quad - \lambda \int_{I_m} \phi_\lambda^{r_m} \psi_\lambda^{r_m} \,\mathsf{d}t - \phi_\lambda^{r_m}(t_m)\psi_\lambda^{r_m}(t_m) - \psi_\lambda^{r_m}(t_{m-1}) \\ &= -\int_{I_m} \phi_\lambda^{r_m} \left\{(\psi_\lambda^{r_m})' + \lambda \psi_\lambda^{r_m} + \mathsf{L}_m^{r_m}(\psi_\lambda^{r_m}(t_{m-1}))\right\} \mathsf{d}t - \psi_\lambda^{r_m}(t_{m-1}).\end{aligned}$$

Implementing the definition (3.5) of the dG fundamental solution yields

$$0 = -\int_{I_m} \phi_\lambda^{r_m} \mathsf{L}_m^{r_m}(1)\,\mathsf{d}t - \psi_\lambda^{r_m}(t_{m-1}) = -\phi_\lambda^{r_m}(t_{m-1}) - \psi_\lambda^{r_m}(t_{m-1}),$$

which is (3.10). □

Our next step is to prove that the dG dual solution takes the value of its maximum norm at t_{m-1}.

PROPOSITION 3.4 (stability of $\phi_\lambda^{r_m}$). *Suppose that $\lambda > 0$. It holds that*

$$\|\phi_\lambda^{r_m}\|_{\mathrm{L}^\infty(I_m)} = |\phi_\lambda^{r_m}(t_{m-1})| \tag{3.11}$$

and

$$-1 < \phi_\lambda^{r_m}(t_{m-1}) < 0. \tag{3.12}$$

The proof of the above proposition, to be presented later, is based on some properties of the Legendre expansion of the dG dual solution. More precisely, write

$$\phi_\lambda^{r_m} = \sum_{i=0}^{r_m} a_i K_i^m, \tag{3.13}$$

with the Legendre polynomials $\{K_i^m\}_{i\geq 0}$ from (2.6) and (2.7).

LEMMA 3.5. *Let $\lambda > 0$. Then, for the coefficients $a_0, \dots, a_{r_m}$ in the Legendre expansion* (3.13)*, the following recursion formulas hold:*

$$a_0 = -\frac{1 + \phi_\lambda^{r_m}(t_{m-1})}{k_m \lambda}, \tag{3.14}$$

$$a_1 = -\frac{3}{\lambda}\left(\frac{2}{k_m} + \lambda\right) a_0, \tag{3.15}$$

$$a_i = (2i+1)\left(\frac{a_{i-2}}{2i-3} - \frac{2a_{i-1}}{k_m \lambda}\right) \quad \text{for } 2 \leq i \leq r_m. \tag{3.16}$$

Furthermore, we have that $a_i \neq 0$ as well as that

$$\operatorname{sign}(a_i) = (-1)^{i+1} \tag{3.17}$$

for any $i = 0, \dots, r_m$.

Proof. We begin by integrating (3.8) over I_m, which yields

$$\lambda \int_{I_m} \phi_\lambda^{r_m}(t)\,\mathsf{d}t = -1 - \phi_\lambda^{r_m}(t_{m-1}).$$

Then, making use of the expansion (3.13) as well as of the fact that

$$\int_{I_m} K_i^m \,\mathsf{d}t = 0 \qquad \forall i \geq 1,$$

we see that $\lambda k_m a_0 = -1 - \phi_\lambda^{r_m}(t_{m-1})$, and hence

$$a_0 = -\frac{1 + \phi_\lambda^{r_m}(t_{m-1})}{k_m \lambda},$$

which proves (3.14). Next, we employ again the integrated Legendre polynomials defined in (3.7) and notice the following properties (see, e.g., [24, eq. (9)]):

$$Q_0^m = K_0^m + K_1^m, \qquad Q_i^m = \frac{1}{2i+1}(K_{i+1}^m - K_{i-1}^m), \quad i \geq 1. \tag{3.18}$$

Due to Lemma 3.2, we note that

$$0 = \int_{I_m} \phi_\lambda^{r_m}(v' + \lambda v)\,\mathsf{d}t \qquad \forall v \in \mathbb{P}_0^{r_m}(I_m).$$

Thus, applying the expansion (3.13) and choosing $v := Q_j^m$, we obtain

$$0 = \sum_{i=0}^{r_m} a_i \int_{I_m} K_i^m \left(\frac{2}{k_m} K_j^m + \lambda Q_j^m\right) \mathsf{d}t, \qquad j = 0, \dots, r_{m-1}.$$

Involving (3.18) and using the orthogonality property (2.7) of the Legendre polynomials, we arrive at

$$\begin{aligned} 0 &= \left(\frac{2}{k_m} + \lambda\right) a_0 + \frac{\lambda}{3} a_1, \\ 0 &= -\frac{\lambda}{2j-1} a_{j-1} + \frac{2}{k_m} a_j + \frac{\lambda}{2j+3} a_{j+1}, \qquad 1 \leq j \leq r_{m-1}. \end{aligned} \tag{3.19}$$

Rewriting these equalities yields the asserted recursion relations (3.15) and (3.16). Here, we note that $a_0 \neq 0$ since otherwise all coefficients would be zero, which, in turn, would lead to $\phi_\lambda^{r_m} \equiv 0$. Moreover, the recursion formulas (3.19) immediately show that the coefficients a_j, $j = 1, \ldots, r_m$, never vanish, and they have alternating signs.

It remains to show the sign alternation property (3.17). To this end, we test (3.8) by the Legendre polynomial $K_{r_m}^m$ and integrate over I_m. Then, observing that $(\phi_\lambda^{r_m})'$ is L^2-orthogonal to $K_{r_m}^m$ (because it has degree $r_m - 1$) and applying the properties (2.6) and (2.7) leads to

$$0 = -\frac{k_m \lambda}{2r_m + 1} a_{r_m} - \phi_\lambda^{r_m}(t_m) - (-1)^{r_m},$$

and therefore,

$$a_{r_m} = -\frac{2r_m + 1}{k_m \lambda} \left(\phi_\lambda^{r_m}(t_m) + (-1)^{r_m} \right). \tag{3.20}$$

Next, we test (3.8) by $\phi_\lambda^{r_m}$ and integrate over I_m. A brief calculation reveals that

$$2\lambda \, \|\phi_\lambda^{r_m}\|_{\mathrm{L}^2(I_m)}^2 + |\phi_\lambda^{r_m}(t_m)|^2 = -\phi_\lambda^{r_m}(t_{m-1})(2 + \phi_\lambda^{r_m}(t_{m-1})). \tag{3.21}$$

Since the left-hand side of (3.21) consists only of nonnegative terms, it follows that $\phi_\lambda^{r_m}(t_{m-1}) \in [-2, 0]$. In addition, we note that $\max_{x \in [-2,0]} [-x(2 + x)] = 1$. Hence, the right-hand side of (3.21) and thereby also the left-hand side are both bounded by 1. This implies, in particular, that $|\phi_\lambda^{r_m}(t_m)| \leq 1$. Therefore, from (3.20) and because $a_{r_m} \neq 0$, we infer that $\mathrm{sign}(a_{r_m}) = (-1)^{r_m + 1}$. Since the sign of the coefficients a_j are alternating, we necessarily arrive at

$$\mathrm{sign}(a_j) = \mathrm{sign}(a_{r_m})(-1)^{r_m - j} = (-1)^{2r_m + 1 - j} = (-1)^{j+1}$$

for $0 \leq j \leq r_m$. □

Proof of Proposition 3.4. We apply the Legendre expansion (3.13) of $\phi_\lambda^{r_m}$. Then, recalling (3.17) and invoking (2.6), we deduce that

$$\phi_\lambda^{r_m}(t_{m-1}) = -\sum_{i=0}^{r_m} |a_i| < 0. \tag{3.22}$$

This is the upper bound in (3.12). In addition, noticing that

$$\|K_i^m\|_{\mathrm{L}^\infty(I_m)} = 1, \tag{3.23}$$

we infer

$$\|\phi_\lambda^{r_m}\|_{\mathrm{L}^\infty(I_m)} \leq \sum_{i=0}^{r_m} |a_i| \, \|K_i^m\|_{\mathrm{L}^\infty(I_m)} = \sum_{i=0}^{r_m} |a_i|. \tag{3.24}$$

Combining (3.22) and (3.24), we arrive at (3.11). Finally, the lower bound in (3.12) follows from the fact that $a_0 < 0$ (cf. (3.17)) and from (3.14). □

The ensuing lemma provides further properties of the dG dual solution which will be crucial in the stability analysis below.

LEMMA 3.6. *For $\lambda > 0$, the coefficient a_{r_m} in the Legendre expansion of $\phi_\lambda^{r_m}$ (cf.* (3.13)*) satisfies the bound*

$$|a_{r_m}| \le \left(1 + \frac{\lambda k_m}{2(2r_m+1)}\right)^{-1}.$$

Proof. We use the formulas for the Legendre coefficients $a_0, \dots, a_{r_m}$ of $\phi_\lambda^{r_m}$ from Lemma 3.5. Specifically, from (3.14) and (3.17) it follows that

$$\lambda k_m |a_0| = 1 + \phi_\lambda^{r_m}(t_{m-1}). \tag{3.25}$$

Moreover, taking moduli in (3.15), we deduce that

$$|a_1| = \frac{3(2+\lambda k_m)}{\lambda k_m}|a_0|. \tag{3.26}$$

In addition, rearranging (3.16), we have

$$a_i = \frac{\lambda k_m}{2}\left(\frac{a_{i-1}}{2i-1} - \frac{a_{i+1}}{2i+3}\right), \qquad 1 \le i \le r_m - 1,$$

which, involving again (3.17), leads to

$$|a_i| = \frac{\lambda k_m}{2}\left(\frac{|a_{i+1}|}{2i+3} - \frac{|a_{i-1}|}{2i-1}\right), \qquad 1 \le i \le r_m - 1. \tag{3.27}$$

Inserting (3.27) into (3.22) implies

$$-\phi_\lambda^{r_m}(t_{m-1}) = |a_0| + |a_{r_m}| + \frac{\lambda k_m}{2}\sum_{i=1}^{r_m-1}\left(\frac{|a_{i+1}|}{2i+3} - \frac{|a_{i-1}|}{2i-1}\right).$$

Observing the telescope sum on the right-hand side results in

$$\begin{aligned} &-\phi_\lambda^{r_m}(t_{m-1}) \\ &= \left(1 - \frac{\lambda k_m}{2}\right)|a_0| - \frac{\lambda k_m}{6}|a_1| + \frac{\lambda k_m}{2(2r_m-1)}|a_{r_m-1}| + \left(1 + \frac{\lambda k_m}{2(2r_m+1)}\right)|a_{r_m}|. \end{aligned}$$

Applying (3.26), we note that

$$-\phi_\lambda^{r_m}(t_{m-1}) = -\lambda k_m|a_0| + \frac{\lambda k_m}{2(2r_m-1)}|a_{r_m-1}| + \left(1 + \frac{\lambda k_m}{2(2r_m+1)}\right)|a_{r_m}|.$$

Making use of (3.25), we arrive at

$$1 = \frac{\lambda k_m}{2(2r_m-1)}|a_{r_m-1}| + \left(1 + \frac{\lambda k_m}{2(2r_m+1)}\right)|a_{r_m}|,$$

which yields the bound

$$1 \ge \left(1 + \frac{\lambda k_m}{2(2r_m+1)}\right)|a_{r_m}|.$$

This completes the proof. ∎

LEMMA 3.7. *Let $\lambda > 0$. For the dG dual solution from (3.8) it holds that*

$$\|\phi_\lambda^{r_m}\|_{L^2(I_m)} \leq \Upsilon_{3.28}(r_m, k_m\lambda)k_m^{1/2},$$

where, for $r \in \mathbb{N}_0$ and $\varrho > 0$, we let

$$\Upsilon_{3.28}(r, \varrho) := \left(\frac{3}{2(2r+1)+\varrho}\right)^{1/2}. \tag{3.28}$$

In particular, $\|\phi_\lambda^{r_m}\|_{L^2(I_m)} \to 0$, as $r_m \to \infty$, uniformly with respect to λ.

Proof. Recalling (3.20), we have that

$$\left|\phi_\lambda^{r_m}(t_m) - (-1)^{r_m+1}\right| = \frac{k_m\lambda}{2r_m+1}|a_{r_m}|. \tag{3.29}$$

Moreover, due to Proposition 3.4 we notice that

$$0 < 1 + \phi_\lambda^{r_m}(t_{m-1}) = 1 - \|\phi_\lambda^{r_m}\|_{L^\infty(I_m)} \leq 1 - |\phi_\lambda^{r_m}(t_m)| \leq \left|(-1)^{r_m+1} - \phi_\lambda^{r_m}(t_m)\right|.$$

Hence,

$$0 < 1 + \phi_\lambda^{r_m}(t_{m-1}) \leq \frac{k_m\lambda}{2r_m+1}|a_{r_m}|. \tag{3.30}$$

From (3.21), we recall that

$$2\lambda\,\|\phi_\lambda^{r_m}\|^2_{L^2(I_m)} = -|\phi_\lambda^{r_m}(t_m)|^2 - \phi_\lambda^{r_m}(t_{m-1})(2 + \phi_\lambda^{r_m}(t_{m-1})). \tag{3.31}$$

We estimate the terms on the right-hand side of the above identity separately. First,

$$\begin{aligned}
|\phi_\lambda^{r_m}(t_m)|^2 &= \left|\phi_\lambda^{r_m}(t_m) - (-1)^{r_m+1}\right|^2 + 2(-1)^{r_m+1}\phi_\lambda^{r_m}(t_m) - 1 \\
&\geq 1 + 2(-1)^{r_m+1}\left(-(-1)^{r_m+1} + \phi_\lambda^{r_m}(t_m)\right) \\
&\geq 1 - 2\left|\phi_\lambda^{r_m}(t_m) - (-1)^{r_m+1}\right|,
\end{aligned}$$

and thus, upon exploiting (3.29),

$$-|\phi_\lambda^{r_m}(t_m)|^2 \leq -1 + \frac{2k_m\lambda}{2r_m+1}|a_{r_m}|.$$

Next, with (3.30), it follows that

$$2 + \phi_\lambda^{r_m}(t_{m-1}) \leq 1 + \frac{k_m\lambda}{2r_m+1}|a_{r_m}|.$$

Inserting these estimates into (3.31) and recalling the fact that $0 < -\phi_\lambda^{r_m}(t_{m-1}) < 1$ (cf. (3.12)), we conclude that

$$2\lambda\,\|\phi_\lambda^{r_m}\|^2_{L^2(I_m)} \leq \frac{3k_m\lambda}{2r_m+1}|a_{r_m}|.$$

Finally, employing Lemma 3.6 results in

$$2\lambda\,\|\phi_\lambda^{r_m}\|^2_{L^2(I_m)} \leq \frac{6k_m\lambda}{2(2r_m+1)+k_m\lambda},$$

and dividing by 2λ completes the proof. ∎

Remark 3.8. For $\lambda = 0$ it is fairly elementary to verify that

$$\phi_0^{r_m} = (-1)^{r_m+1} K_{r_m}^m \tag{3.32}$$

in (3.8), where $K_{r_m}^m$ is the Legendre polynomial of degree r_m on I_m. Therefore, revisiting (2.7), we observe that

$$\|\phi_0^{r_m}\|_{\mathrm{L}^2(I_m)} = \left(\frac{k_m}{2r_m+1}\right)^{1/2},$$

which slightly improves the estimate from Lemma 3.7 above.

The following result is the analogue of Proposition 3.4 for the dG fundamental solution.

PROPOSITION 3.9 (stability of $\psi_\lambda^{r_m}$). *Let $\lambda > 0$ and $1 \le m \le M$. For the dG fundamental solution from* (3.8) *the identities*

$$\|\psi_\lambda^{r_m}\|_{\mathrm{L}^\infty(I_m)} = \psi_\lambda^{r_m}(t_{m-1}) \tag{3.33}$$

and

$$\|(\psi_\lambda^{r_m})'\|_{\mathrm{L}^\infty(I_m)} = -(\psi_\lambda^{r_m})'(t_{m-1}) \tag{3.34}$$

hold true.

Proof. For simplicity of presentation, we suppose that $r_m \ge 4$ (the cases $0 \le r_m \le 3$ can be verified directly). We show (3.34) first. For this purpose, let us expand $(\psi_\lambda^{r_m})'$ in a Legendre series, i.e.,

$$(\psi_\lambda^{r_m})' = \sum_{i=0}^{r_m-1} b_i K_i^m, \tag{3.35}$$

with coefficients $b_0, \dots, b_{r_m-1}$. Recalling (3.7) and using (3.18), for $t \in I_m$, we have

$$\begin{aligned}
\psi_\lambda^{r_m}(t) &= \psi_\lambda^{r_m}(t_{m-1}) + \int_{t_{m-1}}^t (\psi_\lambda^{r_m})' \,\mathsf{d}s \\
&= \psi_\lambda^{r_m}(t_{m-1}) + \frac{k_m}{2} \sum_{i=0}^{r_m-1} b_i Q_i^m(s) \,\mathsf{d}s \\
&= \psi_\lambda^{r_m}(t_{m-1}) + \frac{k_m}{2} \left(b_0(K_0^m + K_1^m) + \sum_{i=1}^{r_m-1} \frac{b_i}{2i+1}(K_{i+1}^m - K_{i-1}^m) \right). \qquad (3.36)
\end{aligned}$$

Note that $K_0^m \equiv 1$. Then, inserting (3.35) and (3.36) into (3.5), using the representation (2.5) of the lifting operator, and comparing coefficients leads to the equations

(3.37)
$$
\begin{aligned}
\lambda\psi_\lambda^{r_m}(t_{m-1}) + \left(1+\frac{\lambda k_m}{2}\right)b_0 - \frac{\lambda k_m}{6}b_1 &= \frac{1}{k_m}\mathsf{e}_{\lambda,m}^+,\\
\frac{\lambda k_m}{2}b_0 + b_1 - \frac{\lambda k_m}{10}b_2 &= -\frac{3}{k_m}\mathsf{e}_{\lambda,m}^+,\\
\frac{\lambda k_m}{2(2i-1)}b_{i-1} + b_i - \frac{\lambda k_m}{2(2i+3)}b_{i+1} &= \frac{1}{k_m}(-1)^i(2i+1)\mathsf{e}_{\lambda,m}^+ \quad (2\le i\le r_m-2),\\
\frac{\lambda k_m}{2(2r_m-3)}b_{r_m-2} + b_{r_m-1} &= \frac{1}{k_m}(-1)^{r_m-1}(2r_m-1)\mathsf{e}_{\lambda,m}^+,\\
\frac{\lambda k_m}{2(2r_m-1)}b_{r_m-1} &= \frac{1}{k_m}(-1)^{r_m}(2r_m+1)\mathsf{e}_{\lambda,m}^+.
\end{aligned}
$$

Here, we denote by $\mathsf{e}_{\lambda,m}^+ = 1-\psi_\lambda^{r_m}(t_{m-1})$ the error between the initial values of ψ from (3.4) and its dG approximation $\psi_\lambda^{r_m}$. In order to show (3.34), we first illustrate that the signs of the coefficients $b_0,\ldots,b_{r_m-1}$ are alternating. We focus on the case where r_m is even. Let us first observe, by (3.10) and (3.12), that

$$0 < \psi_\lambda^{r_m}(t_{m-1}) < 1. \tag{3.38}$$

Rewriting the last equation in (3.37), we have

$$b_{r_m-1} = \frac{(-1)^{r_m}}{2\lambda k_m^2}(4r_m^2-1)\mathsf{e}_{\lambda,m}^+.$$

Using (3.38), we notice that

$$\mathsf{e}_{\lambda,m}^+ > 0, \tag{3.39}$$

and because r_m is even, we arrive at $b_{r_m-1} > 0$. Then, from the second-to-last equation in (3.37), we infer

$$b_{r_m-2} = -\frac{2}{\lambda k_m}(2r_m-3)b_{r_m-1} + \frac{2(-1)^{r_m-1}}{\lambda k_m^2}(2r_m-3)(2r_m-1)\mathsf{e}_{\lambda,m}^+ < 0.$$

Analogously, the third equation in (3.37), with $i = r_m-2$, implies that

$$
\begin{aligned}
b_{r_m-3} = &-\frac{2}{\lambda k_m}(2r_m-5)b_{r_m-2}\\
&+ \frac{2r_m-5}{2r_m-1}b_{r_m-1} + \frac{2(-1)^{r_m-2}}{\lambda k_m^2}(2r_m-5)(2r_m-3)\mathsf{e}_{\lambda,m}^+ > 0.
\end{aligned}
$$

We continue in the same way to conclude that $\operatorname{sign}(b_i) = (-1)^{i+1}$ for $1 \le i < r_m-1$. Finally, applying the second equation in (3.37), it holds that

$$b_0 = \frac{2}{\lambda k_m}\left(-b_1 + \frac{\lambda k_m}{10}b_2 - \frac{3}{k_m}\mathsf{e}_{\lambda,m}^+\right) < 0.$$

Then, from (2.6) and (3.23), we obtain

$$
\begin{aligned}
-(\psi_\lambda^{r_m})'(t_{m-1}) = -\sum_{i=0}^{r_m-1}(-1)^i b_i = \sum_{i=0}^{r_m-1}|b_i| = \sum_{i=0}^{r_m-1}|b_i|\,\|K_i^m\|_{\mathrm{L}^\infty(I_m)}\\
\ge \|(\psi_\lambda^{r_m})'\|_{\mathrm{L}^\infty(I_m)},
\end{aligned}
$$

which gives (3.34). For r_m odd we may proceed similarly.

In order to complete the proof, we show (3.33). To this end, we evaluate (3.5) at $t = t_{m-1}$:

$$(\psi_\lambda^{r_m})'(t_{m-1}) + \lambda\psi_\lambda^{r_m}(t_{m-1}) - \mathsf{e}_{\lambda,m}^+ \mathsf{L}_m^{r_m}(t_{m-1}) = 0.$$

Since the coefficients of the lifting operator $\mathsf{L}_m^{r_m}$ are alternating, and because of property (2.6), it is straightforward to see that $\|\mathsf{L}_m^{r_m}\|_{\mathrm{L}^\infty(I_m)} = \mathsf{L}_m^{r_m}(t_{m-1}) > 0$. Hence, with $\mathsf{e}_{\lambda,m}^+ > 0$ and by means of (3.34), we see that

$$\lambda\psi_\lambda^{r_m}(t_{m-1}) = \|(\psi_\lambda^{r_m})'\|_{\mathrm{L}^\infty(I_m)} + \mathsf{e}_{\lambda,m}^+ \|\mathsf{L}_m^{r_m}\|_{\mathrm{L}^\infty(I_m)}.$$

Thus, in view of (3.5), which implies that

$$\lambda\|\psi_\lambda^{r_m}\|_{\mathrm{L}^\infty(I_m)} \leq \|(\psi_\lambda^{r_m})'\|_{\mathrm{L}^\infty(I_m)} + \mathsf{e}_{\lambda,m}^+ \|\mathsf{L}_m^{r_m}\|_{\mathrm{L}^\infty(I_m)},$$

we conclude that $\psi_\lambda^{r_m}$ takes its maximum at $t = t_{m-1}$. ∎

3.2. Representation formulas. In this section we derive explicit representation formulas for the operator $(\Gamma_{\lambda,m}^{r_m})^{-1}$ defined in (3.2). Observing (3.5) and Lemma 3.1, it is sufficient to investigate how $(\Gamma_{\lambda,m}^{r_m})^{-1}$ acts on $\mathbb{W}_\lambda^{r_m}(I_m)$.

LEMMA 3.10. *Let $w \in \mathbb{W}_\lambda^{r_m}(I_m)$, then it holds that*

$$(\Gamma_{\lambda,m}^{r_m})^{-1}(w) = \int_{t_{m-1}}^t e^{\lambda(s-t)} w(s)\,\mathsf{d}s.$$

Proof. Let $w \in \mathbb{W}_\lambda^{r_m}(I_m)$, and choose $v \in \mathbb{P}_0^{r_m}(I_m)$ with $w = v' + \lambda v$. Then, we have the following equation:

$$w(s) = \Gamma_{\lambda,m}^{r_m}(v)(s) = v'(s) + \lambda v(s) = e^{-\lambda(s-t_{m-1})}\frac{\mathsf{d}}{\mathsf{d}s}\left(e^{\lambda(s-t_{m-1})}v(s)\right), \quad s \in I_m.$$

Hence, it follows that $\frac{\mathsf{d}}{\mathsf{d}s}\left(e^{\lambda(s-t_{m-1})}v(s)\right) = e^{\lambda(s-t_{m-1})}w(s)$. Integrating with respect to s over (t_{m-1}, t) and using $v(t_{m-1}) = 0$, we obtain

$$e^{\lambda(t-t_{m-1})}v(t) = \int_{t_{m-1}}^t e^{\lambda(s-t_{m-1})}w(s)\,\mathsf{d}s,$$

and therefore,

$$(\Gamma_{\lambda,m}^{r_m})^{-1}(w)(t) = v(t) = \int_{t_{m-1}}^t e^{\lambda(s-t)}w(s)\,\mathsf{d}s.$$

This completes the proof. ∎

PROPOSITION 3.11. *For any $w \in \mathbb{P}^{r_m}(I_m)$ it holds that*

$$(\Gamma_{\lambda,m}^{r_m})^{-1}(w) = -e^{-\lambda(t-t_{m-1})}\left(\int_{I_m} w\phi_\lambda^{r_m}\,\mathsf{d}s\right)\eta_{\lambda,m}^{r_m}(t) + \int_{t_{m-1}}^t e^{\lambda(s-t)}w\,\mathsf{d}s,$$

where

$$\eta_{\lambda,m}^{r_m}(t) := 1 - \int_{t_{m-1}}^t e^{\lambda(s-t_{m-1})}\mathsf{L}_m^{r_m}(1)\,\mathsf{d}s. \tag{3.40}$$

Proof. Consider any $w \in \mathbb{P}^{r_m}(I_m)$. Then, Lemma 3.1 implies that there exist $\alpha \in \mathbb{R}$ and $w_0 \in \mathbb{W}_\lambda^{r_m}(I_m)$ such that $w = w_0 + \alpha \mathsf{L}_m^{r_m}(1)$. Hence, applying Lemma 3.10 and recalling (3.5) yields

$$\begin{aligned}(\Gamma_{\lambda,m}^{r_m})^{-1}(w) &= (\Gamma_{\lambda,m}^{r_m})^{-1}(w_0) + \alpha(\Gamma_{\lambda,m}^{r_m})^{-1}(\mathsf{L}_m^{r_m}(1)) \\ &= \int_{t_{m-1}}^t e^{\lambda(s-t)} w_0 \,\mathsf{d}s + \alpha \psi_\lambda^{r_m} \\ &= \int_{t_{m-1}}^t e^{\lambda(s-t)} \left(w - \alpha \mathsf{L}_m^{r_m}(1)\right) \mathsf{d}s + \alpha \psi_\lambda^{r_m} \\ &= \alpha \left(\psi_\lambda^{r_m} - \int_{t_{m-1}}^t e^{\lambda(s-t)} \mathsf{L}_m^{r_m}(1)\,\mathsf{d}s \right) + \int_{t_{m-1}}^t e^{\lambda(s-t)} w \,\mathsf{d}s.\end{aligned}$$

Setting

$$\Theta_{\lambda,m}^{r_m} := \psi_\lambda^{r_m} - \int_{t_{m-1}}^t e^{\lambda(s-t)} \mathsf{L}_m^{r_m}(1)\,\mathsf{d}s$$

and using the fact that $\psi_\lambda^{r_m}$ is the solution of (3.5), an elementary calculation reveals that

$$(\Theta_{\lambda,m}^{r_m})' + \lambda \Theta_{\lambda,m}^{r_m} = -\psi_\lambda^{r_m}(t_{m-1}) \mathsf{L}_m^{r_m}(1).$$

Integrating this identity, we arrive at

$$\Theta_{\lambda,m}^{r_m}(t) = e^{-\lambda(t-t_{m-1})} \psi_\lambda^{r_m}(t_{m-1}) \eta_{\lambda,m}^{r_m}(t).$$

Therefore,

$$(\Gamma_{\lambda,m}^{r_m})^{-1}(w) = \alpha e^{-\lambda(t-t_{m-1})} \psi_\lambda^{r_m}(t_{m-1}) \eta_{\lambda,m}^{r_m}(t) + \int_{t_{m-1}}^t e^{\lambda(s-t)} w \,\mathsf{d}s.$$

In order to determine the value of α, we employ Lemmas 3.2 and 3.3. This yields

$$\int_{I_m} w \phi_\lambda^{r_m} \,\mathsf{d}t = \alpha \int_{I_m} \mathsf{L}_m^{r_m}(1) \phi_\lambda^{r_m} \,\mathsf{d}t = \alpha \phi_\lambda^{r_m}(t_{m-1}) = -\alpha \psi_\lambda^{r_m}(t_{m-1}),$$

which directly leads to the desired formula. ∎

The following lemma gives an interesting interpretation of $\eta_{\lambda,m}^{r_m}$ defined in (3.40). Let us denote by

$$\mathsf{e}_{\lambda,m} := e^{-\lambda(t-t_{m-1})} - \psi_\lambda^{r_m}(t), \qquad t \in I_m, \tag{3.41}$$

the pointwise error between the solution ψ of (3.4) and the dG fundamental solution $\psi_\lambda^{r_m}$ from (3.6).

LEMMA 3.12. *We have the identity*

$$e^{-\lambda(t-t_{m-1})} \eta_{\lambda,m}^{r_m}(t) = \frac{\mathsf{e}_{\lambda,m}(t)}{\mathsf{e}_{\lambda,m}(t_{m-1})}, \qquad t \in I_m.$$

Proof. Due to (3.39), let us first note that the right-hand side in the above identity is well defined. Recalling (3.6) and applying Proposition 3.11 with $w = \mathsf{L}_m^{r_m}(1)$, we note that

$$\begin{aligned}\psi_\lambda^{r_m}(t) &= -e^{-\lambda(t-t_{m-1})}\left(\int_{I_m} \mathsf{L}_m^{r_m}(1)\phi_\lambda^{r_m}\,\mathsf{d}s\right)\eta_{\lambda,m}^{r_m}(t) + \int_{t_{m-1}}^t e^{\lambda(s-t)}\mathsf{L}_m^{r_m}(1)\,\mathsf{d}s \\ &= -\phi_\lambda^{r_m}(t_{m-1})e^{-\lambda(t-t_{m-1})}\eta_{\lambda,m}^{r_m}(t) + \int_{t_{m-1}}^t e^{\lambda(s-t)}\mathsf{L}_m^{r_m}(1)\,\mathsf{d}s.\end{aligned}$$

By virtue of Lemma 3.3, this leads to

$$\psi_\lambda^{r_m}(t) = \psi_\lambda^{r_m}(t_{m-1})e^{-\lambda(t-t_{m-1})} + (1-\psi_\lambda^{r_m}(t_{m-1}))\int_{t_{m-1}}^t e^{\lambda(s-t)}\mathsf{L}_m^{r_m}(1)\,\mathsf{d}s,$$

and thus,

$$\begin{aligned}-\mathsf{e}_{\lambda,m}(t) &= -\mathsf{e}_{\lambda,m}(t_{m-1})e^{-\lambda(t-t_{m-1})} + \mathsf{e}_{\lambda,m}(t_{m-1})\int_{t_{m-1}}^t e^{\lambda(s-t)}\mathsf{L}_m^{r_m}(1)\,\mathsf{d}s \\ &= -\mathsf{e}_{\lambda,m}(t_{m-1})e^{-\lambda(t-t_{m-1})}\eta_{\lambda,m}^{r_m}(t).\end{aligned}$$

This proves the lemma. ∎

Summarizing the above results, we obtain the following representation expression.

Corollary 3.13. *For any $w \in \mathbb{P}^{r_m}(I_m)$, the identity*

$$(\Gamma_{\lambda,m}^{r_m})^{-1}(w) = -\frac{\mathsf{e}_{\lambda,m}(t)}{\mathsf{e}_{\lambda,m}(t_{m-1})}\int_{I_m} w\phi_\lambda^{r_m}(t)\,\mathsf{d}t + \int_{t_{m-1}}^t e^{\lambda(s-t)}w\,\mathsf{d}s \tag{3.42}$$

holds true.

3.3. Stability. We are now in a position to derive stability bounds for $(\Gamma_{\lambda,m}^{r_m})^{-1}$ as well as for the scalar dG time stepping solution from (3.3). In this section, let us suppose that $\lambda > 0$.

Proposition 3.14 (L^∞-L^2-stability of $(\Gamma_{\lambda,m}^{r_m})^{-1}$). *Let $w \in \mathbb{P}^{r_m}(I_m)$, $1 \le m \le M$. Then the stability estimate*

$$\left\|(\Gamma_{\lambda,m}^{r_m})^{-1}(w)\right\|_{\mathrm{L}^\infty(I_m)} \le C_{\lambda,r_m}^{\mathrm{L}^2} k_m^{1/2}\|w\|_{\mathrm{L}^2(I_m)} \tag{3.43}$$

holds, where

$$C_{\lambda,r_m}^{\mathrm{L}^2} := \Upsilon_{3.28}(r_m, k_m\lambda)\left\|\frac{\mathsf{e}_{\lambda,m}}{\mathsf{e}_{\lambda,m}(t_{m-1})}\right\|_{\mathrm{L}^\infty(I_m)} + \min\left(1, (2\lambda k_m)^{-1/2}\right). \tag{3.44}$$

Proof. We separately bound the two terms on the right-hand side of (3.42). By means of the Cauchy–Schwarz inequality and Lemma 3.7, we have

$$\left|\int_{I_m} w\phi_\lambda^{r_m}\,\mathsf{d}t\right| \le \|w\|_{\mathrm{L}^2(I_m)}\|\phi_\lambda^{r_m}\|_{\mathrm{L}^2(I_m)} \le k_m^{1/2}\Upsilon_{3.28}(r_m,k_m\lambda)\|w\|_{\mathrm{L}^2(I_m)}.$$

Therefore, we infer that

$$\left|\frac{\mathsf{e}_{\lambda,m}(t)}{\mathsf{e}_{\lambda,m}(t_{m-1})}\int_{I_m} w\phi_\lambda^{r_m}\,\mathsf{d}t\right| \le k_m^{1/2}\Upsilon_{3.28}(r_m,k_m\lambda)\left\|\frac{\mathsf{e}_{\lambda,m}(t)}{\mathsf{e}_{\lambda,m}(t_{m-1})}\right\|_{\mathrm{L}^\infty(I_m)}\|w\|_{L^2(I_m)}. \tag{3.45}$$

Similarly, it holds that

$$\left\|\int_{t_{m-1}}^{t} e^{\lambda(s-t)}w(s)\,\mathsf{d}s\right\|_{\mathrm{L}^\infty(I_m)} \le \sup_{t\in I_m}\left(\int_{t_{m-1}}^{t} e^{2\lambda(s-t)}\,\mathsf{d}s\right)^{1/2}\|w\|_{\mathrm{L}^2(I_m)} \le \min\left(k_m^{1/2},(2\lambda)^{-1/2}\right)\|w\|_{\mathrm{L}^2(I_m)}\,. \tag{3.46}$$

The two estimates (3.45) and (3.46) immediately imply the asserted result. ∎

Remark 3.15 (L^∞-L^1-stability of $(\Gamma^{r_m}_{\lambda,m})^{-1}$). As in Proposition 3.14 above, for $w\in\mathbb{P}^{r_m}(I_m)$, $1\le m\le M$, we can derive the bound

$$\left\|(\Gamma^{r_m}_{\lambda,m})^{-1}(w)\right\|_{\mathrm{L}^\infty(I_m)} \le C^{\mathrm{L}^1}_{\lambda,r_m}\|w\|_{\mathrm{L}^1(I_m)}\,, \tag{3.47}$$

where

$$C^{\mathrm{L}^1}_{\lambda,r_m} := \left\|\frac{\mathsf{e}_{\lambda,m}}{\mathsf{e}_{\lambda,m}(t_{m-1})}\right\|_{\mathrm{L}^\infty(I_m)} + 1. \tag{3.48}$$

Indeed, to see this, for the first term on the right-hand side of (3.42) we employ Proposition 3.4 to obtain

$$\left|\int_{I_m} w\phi^{r_m}_\lambda\,\mathsf{d}t\right| \le \|w\|_{\mathrm{L}^1(I_m)}\|\phi^{r_m}_\lambda\|_{\mathrm{L}^\infty(I_m)} = \|w\|_{\mathrm{L}^1(I_m)}|\phi^{r_m}_\lambda(t_{m-1})| \le \|w\|_{\mathrm{L}^1(I_m)}\,.$$

Therefore, we obtain the bound

$$\left|\frac{\mathsf{e}_{\lambda,m}(t)}{\mathsf{e}_{\lambda,m}(t_{m-1})}\int_{I_m} w\phi^{r_m}_\lambda\,\mathsf{d}t\right| \le \left\|\frac{\mathsf{e}_{\lambda,m}(t)}{\mathsf{e}_{\lambda,m}(t_{m-1})}\right\|_{\mathrm{L}^\infty(I_m)}\|w\|_{L^1(I_m)}. \tag{3.49}$$

As for the second term, we note that

$$\left\|\int_{t_{m-1}}^{t} e^{\lambda(s-t)}w(s)\,\mathsf{d}s\right\|_{\mathrm{L}^\infty(I_m)} \le \sup_{t\in I_m}\int_{t_{m-1}}^{t} e^{\lambda(s-t)}|w(s)|\,\mathsf{d}s \le \|w\|_{\mathrm{L}^1(I_m)}\,. \tag{3.50}$$

Thence, combining (3.49) and (3.50) gives (3.47).

Remark 3.16. The term $\left\|\mathsf{e}_{\lambda,m}(t_{m-1})^{-1}\mathsf{e}_{\lambda,m}\right\|_{\mathrm{L}^\infty(I_m)}$ arising in the two constants $C^{\mathrm{L}^2}_{\lambda,r_m}$ and $C^{\mathrm{L}^1}_{\lambda,r_m}$ from (3.44) and (3.48), respectively, can be estimated uniformly with respect to the time step k_m and the polynomial degree r_m. In fact, performing an integration by parts in (3.40), we note that

$$\eta^{r_m}_{\lambda,m}(t) = 1 - e^{\lambda(t-t_{m-1})}\rho^{r_m}_m(t) + \lambda\int_{t_{m-1}}^{t} e^{\lambda(s-t_{m-1})}\rho^{r_m}_m(s)\,\mathsf{d}s,$$

where we define

$$\rho^{r_m}_m(t) := \int_{t_{m-1}}^{t}\mathsf{L}^{r_m}_m(1)\,\mathsf{d}s, \qquad t\in I_m.$$

Rearranging terms, we obtain

$$\eta^{r_m}_{\lambda,m}(t) = e^{\lambda(t-t_{m-1})}(1-\rho^{r_m}_m(t)) - \lambda\int_{t_{m-1}}^{t} e^{\lambda(s-t_{m-1})}(1-\rho^{r_m}_m(s))\,\mathsf{d}s, \qquad t\in I_m.$$

Referring to [11, Lemma 1], it holds that

$$\|1-\rho_m^{r_m}\|_{\mathrm{L}^\infty(I_m)} = 1. \tag{3.51}$$

Consequently, we conclude that

$$|\eta_{\lambda,m}^{r_m}(t)| \leq e^{\lambda(t-t_{m-1})} + \lambda \int_{t_{m-1}}^t e^{\lambda(s-t_{m-1})}\,\mathsf{d}s = 2e^{\lambda(t-t_{m-1})} - 1.$$

Recalling Lemma 3.12 results in

$$\left\|\frac{\mathsf{e}_{\lambda,m}}{\mathsf{e}_{\lambda,m}(t_{m-1})}\right\|_{\mathrm{L}^\infty(I_m)} \leq \sup_{t\in I_m}\left(2-e^{-\lambda(t-t_{m-1})}\right) = 2-e^{-\lambda k_m}. \tag{3.52}$$

In particular,

$$\begin{aligned} C_{\lambda,r_m}^{\mathrm{L}^2} &\leq \left(2-e^{-\lambda k_m}\right)\Upsilon_{3.28}(r_m,k_m\lambda) + \min\left(1,(2\lambda k_m)^{-1/2}\right) \\ &\leq 2\left(\frac{3}{2(2r_m+1)+k_m\lambda}\right)^{1/2} + 1 \leq \sqrt{6}+1 \end{aligned} \tag{3.53}$$

in (3.44), and thus $C_{\lambda,r_m}^{\mathrm{L}^2}$ is uniformly bounded with respect to the parameters k_m, r_m, and λ. Incidentally, a considerably more involved analysis in [20, section 4] reveals that it even holds that $\left\|\mathsf{e}_{\lambda,m}(t_{m-1})^{-1}\mathsf{e}_{\lambda,m}\right\|_{\mathrm{L}^\infty(I_m)} = 1$; i.e., the above inequality improves, for example, to $C_{\lambda,r_m}^{\mathrm{L}^2} \leq \sqrt{3/2}+1$.

Remark 3.17. For $\lambda = 0$, recalling (3.32), we see that Proposition 3.11 implies the following representation formula for $(\chi^{r_m})^{-1}$(cf. (2.10)):

$$(\chi_m^{r_m})^{-1}(w) = (-1)^{r_m+1}\left(\int_{I_m} wK_{r_m}^m\,\mathsf{d}t\right)(1-\rho_m^{r_m}(t)) + \int_{t_{m-1}}^t w\,\mathsf{d}s$$

for any $w \in \mathbb{P}^{r_m}(I_m)$. Revisiting (3.51) and denoting by

$$w_{r_m} := \frac{2r_m+1}{k_m}\int_{I_m} wK_{r_m}^m\,\mathsf{d}t$$

the r_mth Legendre coefficient of w (cf. (2.7)) leads to the stability estimate

$$\left\|(\chi_m^{r_m})^{-1}(w)\right\|_{\mathrm{L}^\infty(I_m)} \leq |w_{r_m}| + \|w\|_{\mathrm{L}^1(I_m)}, \qquad w \in \mathbb{P}^{r_m}(I_m),$$

which is an improvement of [11, Proposition 1].

The above Proposition 3.14 immediately implies an $\mathrm{L}^\infty(I_m)$-stability bound for the dG time stepping solution $U \in \mathbb{P}^{r_m}(I_m)$ from (3.3).

THEOREM 3.18 (L^∞-stability of scalar dG solution). *The dG solution $U \in \mathbb{P}^{r_m}(I_m)$ from* (3.3) *satisfies*

$$\|U\|_{\mathrm{L}^\infty(I_m)} \leq |u_{m-1}| + C_{\lambda,r_m}^{\mathrm{L}^2} k_m^{1/2}\|f\|_{\mathrm{L}^2(I_m)},$$

with $C_{\lambda,r_m}^{\mathrm{L}^2}$ from (3.44).

Proof. Employing the triangle inequality to (3.3), together with the linearity of $(\Gamma^{r_m}_{\lambda,m})^{-1}$ and $\mathsf{L}^{r_m}_m$, we have

$$\|U\|_{\mathrm{L}^\infty(I_m)} \leq |u_{m-1}| \left\|(\Gamma^{r_m}_{\lambda,m})^{-1}(\mathsf{L}^{r_m}_m(1))\right\|_{\mathrm{L}^\infty(I_m)} + \left\|(\Gamma^{r_m}_{\lambda,m})^{-1}(\Pi^{r_m}_m f)\right\|_{\mathrm{L}^\infty(I_m)}.$$

Recalling (3.6), it follows that

$$\|U\|_{\mathrm{L}^\infty(I_m)} \leq |u_{m-1}| \, \|\psi^{r_m}_m\|_{\mathrm{L}^\infty(I_m)} + \left\|(\Gamma^{r_m}_{\lambda,m})^{-1}(\Pi^{r_m}_m f)\right\|_{\mathrm{L}^\infty(I_m)}.$$

Using (3.33) and (3.38) and estimating the second term on the right-hand side of the above inequality by means of Proposition 3.14, we deduce that

$$\|U\|_{\mathrm{L}^\infty(I_m)} \leq |u_{m-1}| + C^{\mathrm{L}^2}_{\lambda,r_m} k_m^{1/2} \, \|\Pi^{r_m}_m f\|_{\mathrm{L}^2(I_m)}.$$

The proof now follows from applying the $\mathrm{L}^2(I_m)$-stability of $\Pi^{r_m}_m$. ∎

4. Linear parabolic equations. We now attend to the stability of the fully discrete dG time discretization (2.9) for the linear parabolic evolution problem (1.2). For this purpose, for $1 \leq m \leq M$, we make use of the spectral decomposition of the discrete elliptic operator A_m introduced in (2.2): Since A_m is self-adjoint and positive definite, there exist orthonormal basis functions $\{\varphi_i\}_{i=1}^{n_m} \subset \mathbb{X}_m$, $\mathbb{X}_m = \operatorname{span}\{\varphi_1, \dots, \varphi_{n_m}\}$, which are eigenfunctions of A_m:

$$(\varphi_i, \varphi_j)_{\mathbb{H}} = \delta_{ij}, \qquad \mathsf{A}_m \varphi_i = \lambda_i \varphi_i, \qquad i, j = 1, \dots, n_m. \tag{4.1}$$

Here, for $1 \leq i \leq n_m$, we signify by $\lambda_i > 0$ the (real) eigenvalue corresponding to φ_i. Then, any function $w \in \mathbb{P}^{r_m}(I_m; \mathbb{X}_m)$ can be represented as

$$w(t) = \sum_{i=1}^{n_m} a_i(t) \varphi_i, \tag{4.2}$$

where $a_i \in \mathbb{P}^{r_m}(I_m)$ are time-dependent coefficients, and by Parseval's identity it holds that

$$\|w(t)\|^2_{\mathbb{H}} = \sum_{i=1}^{n_m} a_i(t)^2, \qquad t \in I_m.$$

Furthermore, for the purpose of Remark 4.6 below, we derive a lower bound on the dual norm. Specifically, for $w(t) \neq 0$ as in (4.2), using (1.5), we have

$$\|w(t)\|^2_{\mathbb{X}^\star} = \sup_{0 \neq v \in \mathbb{X}} \frac{\langle w(t), v\rangle^2_{\mathbb{X}^\star \times \mathbb{X}}}{\|v\|^2_{\mathbb{X}}} \geq \frac{\left\langle w(t), \mathsf{A}_m^{-1} w(t)\right\rangle^2_{\mathbb{X}^\star \times \mathbb{X}}}{\|\mathsf{A}_m^{-1} w(t)\|^2_{\mathbb{X}}} = \frac{\left(w(t), \mathsf{A}_m^{-1} w(t)\right)^2_{\mathbb{H}}}{\|\mathsf{A}_m^{-1} w(t)\|^2_{\mathbb{X}}}.$$

Moreover, with the aid of (1.3) and (2.2), we deduce that

$$\alpha_{1.3} \|\mathsf{A}_m^{-1} w(t)\|^2_{\mathbb{X}} \leq \left\langle \mathsf{A}\mathsf{A}_m^{-1} w(t), \mathsf{A}_m^{-1} w(t)\right\rangle_{\mathbb{X}^\star \times \mathbb{X}} = \left(w(t), \mathsf{A}_m^{-1} w(t)\right)_{\mathbb{H}}.$$

Hence,

$$\|w(t)\|^2_{\mathbb{X}^\star} \geq \alpha_{1.3} \left(w(t), \mathsf{A}_m^{-1} w(t)\right)_{\mathbb{H}}.$$

Due to (4.1) we infer that

$$\mathsf{A}_m^{-1} w(t) = \sum_{i=1}^{n_m} a_i(t)\lambda_i^{-1}\varphi_i,$$

and in conclusion,

$$\|w(t)\|_{\mathbb{X}^\star}^2 \geq \alpha_{1.3} \sum_{i=1}^{n_m} a_i(t)^2 \lambda_i^{-1}. \tag{4.3}$$

4.1. Stability of dG solution operator. Following our approach in section 3.3 we now investigate the stability of the inverse of the discrete parabolic operator $\Gamma_m^{r_m}$ from (2.11).

PROPOSITION 4.1. *Given $w \in \mathbb{P}^{r_m}(I_m;\mathbb{X}_m)$, with a spectral representation as in* (4.2)*, we have*

$$(\Gamma_m^{r_m})^{-1}(w) = \sum_{i=1}^{n_m} (\Gamma_{\lambda_i,m}^{r_m})^{-1}(a_i)\varphi_i, \tag{4.4}$$

where $\Gamma_m^{r_m}$ and $\Gamma_{\lambda_i,m}^{r_m}$ are the discrete operators defined in (2.11) *and* (3.2)*, respectively. Moreover, the estimate*

$$\left\|(\Gamma_m^{r_m})^{-1}(w)\right\|_{\mathrm{L}^\infty(I_m;\mathbb{H})} \leq C_m k_m^{1/2} \|w\|_{\mathrm{L}^2(I_m;\mathbb{H})} \tag{4.5}$$

holds true, with

$$C_m := \max_{1\leq i\leq n_m} C_{\lambda_i,r_m}^{\mathrm{L}^2}, \tag{4.6}$$

where $C_{\lambda_i,r_m}^{\mathrm{L}^2}$ is defined in (3.44)*; cf. also* (3.53)*.*

Proof. Let $w \in \mathbb{P}^{r_m}(I_m;\mathbb{X}_m)$. Since $\Gamma_m^{r_m}$ is an isomorphism on $\mathbb{P}^{r_m}(I_m;\mathbb{X}_m)$ there exists a unique $v \in \mathbb{P}^{r_m}(I_m;\mathbb{X}_m)$,

$$v = \sum_{i=1}^{n_m} b_i\varphi_i, \qquad b_i \in \mathbb{P}^{r_m}(I_m), \quad 1 \leq i \leq n_m,$$

such that $w = \Gamma_m^{r_m}(v)$. Equivalently, by linearity of $\Gamma_m^{r_m}$,

$$\begin{aligned} w &= \sum_{i=1}^{n_m} \Gamma_m^{r_m}(b_i\varphi_i) = \sum_{i=1}^{n_m} \chi_m^{r_m}(b_i)\varphi_i + b_i\mathsf{A}_m\varphi_i = \sum_{i=1}^{n_m} \left(\chi_m^{r_m}(b_i) + \lambda_i b_i\right)\varphi_i \\ &= \sum_{i=1}^{n_m} \Gamma_{\lambda_i,m}^{r_m}(b_i)\varphi_i. \end{aligned}$$

Comparing coefficients with (4.2), we infer that $a_i = \Gamma_{\lambda_i,m}^{r_m}(b_i)$, and thus that $b_i = (\Gamma_{\lambda_i,m}^{r_m})^{-1}(a_i)$, for any $i = 1,\ldots,n_m$. Therefore,

$$(\Gamma_m^{r_m})^{-1}(w) = v = \sum_{i=1}^{n_m} (\Gamma_{\lambda_i,m}^{r_m})^{-1}(a_i)\varphi_i,$$

which is (4.4). Now, employing (4.1), we obtain

$$\left\|(\Gamma_m^{r_m})^{-1}(w)\right\|^2_{\mathrm{L}^\infty(I_m;\mathbb{H})} = \sup_{I_m} \sum_{i=1}^{n_m} \left|(\Gamma_{\lambda_i,m}^{r_m})^{-1}(a_i)\right|^2 \leq \sum_{i=1}^{n_m} \left\|(\Gamma_{\lambda_i,m}^{r_m})^{-1}(a_i)\right\|^2_{\mathrm{L}^\infty(I_m)}.$$

Applying Proposition 3.14, we arrive at

$$\left\|(\Gamma_m^{r_m})^{-1}(w)\right\|_{\mathrm{L}^\infty(I_m;\mathbb{H})} \leq C_m k_m^{1/2} \left(\sum_{i=1}^{n_m} \|a_i\|^2_{\mathrm{L}^2(I_m)}\right)^{1/2} = C_m k_m^{1/2} \|w\|_{\mathrm{L}^2(I_m;\mathbb{H})}.$$

Recalling (3.53) completes the proof. ∎

4.2. Stability of homogeneous problem. For $1 \leq m \leq M$, denote by $\mathbf{\Psi}^{r_m} \in \mathbb{P}^{r_m}(I_m;\mathbb{X}_m)$ the solution of the discrete problem

$$\frac{\mathsf{d}}{\mathsf{d}t}\mathbf{\Psi}^{r_m} + \mathsf{A}_m \mathbf{\Psi}^{r_m} + \mathsf{L}_m^{r_m}(\mathbf{\Psi}^{r_m}(t_{m-1})) = \mathsf{L}_m^{r_m}(\pi_m U_{m-1}^-), \tag{4.7}$$

where $U_{m-1}^- \in \mathbb{H}$ is a given value. Note that this is (2.9) with $f \equiv 0$.

LEMMA 4.2. *Let $\mathbf{\Psi}^{r_m} \in \mathbb{P}^{r_m}(I_m;\mathbb{X}_m)$ be the solution of* (4.7). *Then, we have the stability estimate* $\|\mathbf{\Psi}^{r_m}\|_{\mathrm{L}^\infty(I_m;\mathbb{H})} \leq \left\|U_{m-1}^-\right\|_{\mathbb{H}}$.

Proof. We use the spectral decomposition $\pi_m U_{m-1}^- = \sum_{j=1}^{n_m} a_j\varphi_j$, with constant coefficients $a_1,\ldots,a_{n_m}$. Furthermore, exploiting the representation of the lifting operator from (2.5) and involving (3.5), it holds that

$$\mathsf{L}_m^{r_m}(\pi_m U_{m-1}^-) = \sum_{j=1}^{n_m} a_j \mathsf{L}_m^{r_m}(1)\varphi_j = \sum_{j=1}^{n_m} a_j \Gamma_{\lambda_j,m}^{r_m}(\psi_{\lambda_j}^{r_m})\varphi_j,$$

where we slightly abuse notation by denoting the lifting operator on $\mathbb{X}_m$ and on $\mathbb{R}$ in the same way. Hence, by virtue of (2.12), with $f \equiv 0$, and due to (4.4), we observe that

$$\mathbf{\Psi}^{r_m} = (\Gamma_m^{r_m})^{-1}(\mathsf{L}_m^{r_m}(\pi_m U_{m-1}^-)) = \sum_{j=1}^{n_m} a_j\varphi_j\psi_{\lambda_j}^{r_m}. \tag{4.8}$$

Using orthogonality and applying (3.33) and (3.38), leads to

$$\|\mathbf{\Psi}^{r_m}\|^2_{\mathrm{L}^\infty(I_m;\mathbb{H})} \leq \sum_{j=1}^{n_m} a_j^2 \left\|\psi_{\lambda_j}^{r_m}\right\|^2_{\mathrm{L}^\infty(I_m)} \leq \sum_{j=1}^{n_m} a_j^2 = \|\pi_m U_{m-1}^-\|^2_{\mathbb{H}}.$$

Finally, applying the stability property (2.1) completes the proof. ∎

Remark 4.3. We notice that $\mathbf{\Psi}^{r_m}$ defined in (4.7) is the fully discrete approximation of the solution of the homogeneous parabolic equation (1.2), with $f \equiv 0$, on the time interval I_m. For $t \in I_m$, the latter can be represented as $\Psi(t) = e^{-\mathsf{A}(t-t_{m-1})}u(t_{m-1})$. Consequently, for $t \in I_m$, the error satisfies the identity

$$\begin{aligned}\Psi(t) - \mathbf{\Psi}^{r_m}(t) &= e^{-\mathsf{A}(t-t_{m-1})}\left(u(t_{m-1}) - \pi_m U_{m-1}^-\right)\\ &\quad + \left(e^{-\mathsf{A}(t-t_{m-1})} - e^{-\mathsf{A}_m(t-t_{m-1})}\right)\pi_m U_{m-1}^-\\ &\quad + \left(e^{-\mathsf{A}_m(t-t_{m-1})}\pi_m U_{m-1}^- - \mathbf{\Psi}^{r_m}(t)\right).\end{aligned} \tag{4.9}$$

Let us briefly discuss the three terms on the right-hand side of the above equality. By stability, the first term in (4.9) may simply be estimated by

$$\sup_{t\in I_m}\left\|e^{-\mathsf{A}(t-t_{m-1})}\left(u(t_{m-1})-\pi_m U_{m-1}^-\right)\right\|_{\mathbb{H}}$$
$$\leq \left\|u(t_{m-1})-\pi_m U_{m-1}^-\right\|_{\mathbb{H}} \leq \left\|u(t_{m-1})-U_{m-1}^-\right\|_{\mathbb{H}} + \left\|U_{m-1}^- - \pi_m U_{m-1}^-\right\|_{\mathbb{H}},$$

which shows that this term is bounded by the error in the previous time step and by a mesh change contribution. Moreover, the second term in (4.9) refers to a Galerkin discretization error in space. Finally, using the spectral decomposition of $\pi_m U_{m-1}^-$ as in the proof of Lemma 4.2, and recalling (4.8), the third term in (4.9) can be written in the form

$$e^{-\mathsf{A}_m(t-t_{m-1})}\pi_m U_{m-1}^- - \mathbf{\Psi}^{r_m}(t) = \sum_{j=1}^{n_m} a_j\varphi_j\left(e^{-\lambda_j(t-t_{m-1})}-\psi_{\lambda_j}^{r_m}\right), \qquad t\in I_m.$$

Thus,

$$\sup_{t\in I_m}\left\|e^{-\mathsf{A}_m(t-t_{m-1})}\pi_m U_{m-1}^- - \mathbf{\Psi}^{r_m}(t)\right\|_{\mathbb{H}}^2 \leq \sum_{j=1}^{n_m} a_j^2\left\|\mathsf{e}_{\lambda_j,m}\right\|_{\mathrm{L}^\infty(I_m)}^2,$$

where the scalar error $\mathsf{e}_{\lambda,m}$ is defined in (3.41). Employing (3.52), we notice that $\left\|\mathsf{e}_{\lambda_j,m}\right\|_{\mathrm{L}^\infty(I_m)} \leq 2|\mathsf{e}_{\lambda_j,m}(t_{m-1})|$ and therefore obtain

$$\sup_{t\in I_m}\left\|e^{-\mathsf{A}_m(t-t_{m-1})}\pi_m U_{m-1}^- - \mathbf{\Psi}^{r_m}(t)\right\|_{\mathbb{H}} \leq 2\left\|\pi_m U_{m-1}^-\right\|_{\mathbb{H}}\sup_j\left|1-\psi_{\lambda_j}^{r_m}(t_{m-1})\right|.$$

In particular, we see that the third term converges spectrally as $r_m\to\infty$.

4.3. Stability of inhomogeneous problem. Let us now turn to the stability of the fully discrete dG discretization (2.3)–(2.4) of the linear parabolic problem (1.2).

THEOREM 4.4 ($\mathrm{L}^\infty(\mathbb{H})$-stability of the dG time stepping method). *For any* $1\leq m\leq M$, *the fully discrete dG time stepping solution* $U\in\prod_{m=1}^M\mathbb{P}^{r_m}(I_m;\mathbb{X}_m)$ *from* (2.3) *fulfills the stability estimate*

$$\|U\|_{\mathrm{L}^\infty((0,t_m);\mathbb{H})} \leq \|\pi_0 u_0\|_{\mathbb{H}} + \gamma_m t_m^{1/2}\|f\|_{\mathrm{L}^2((0,t_m);\mathbb{H})}. \tag{4.10}$$

Here, we let $\gamma_m := \max_{1\leq i\leq m} C_i$, *where, for* $1\leq i\leq M$, *the constant* C_i *is defined in* (4.6).

Proof. For $1\leq i\leq m$, we invert (2.12) to infer the solution formula

$$U|_{I_i} = (\Gamma_i^{r_i})^{-1}(\mathsf{L}_i^{r_i}(\pi_i U_{i-1}^-)) + (\Gamma_i^{r_i})^{-1}(\Pi_i^{r_i}f) = \mathbf{\Psi}^{r_i} + (\Gamma_i^{r_i})^{-1}(\Pi_i^{r_i}f), \tag{4.11}$$

where $\mathbf{\Psi}^{r_i}$ is the solution from (4.8). Then, Lemma 4.2 implies that

$$\|U\|_{\mathrm{L}^\infty(I_i;\mathbb{H})} \leq \|U_{i-1}^-\|_{\mathbb{H}} + \left\|(\Gamma_i^{r_i})^{-1}(\Pi_i^{r_i}f)\right\|_{\mathrm{L}^\infty(I_i;\mathbb{H})}.$$

Furthermore, employing (4.5), together with the $\mathrm{L}^2(I_i;\mathbb{H})$-stability of $\Pi_i^{r_i}$, we have

$$\left\|(\Gamma_i^{r_i})^{-1}(\Pi_i^{r_i}f)\right\|_{\mathrm{L}^\infty(I_i;\mathbb{H})} \leq C_i k_i^{1/2}\|\Pi_i^{r_i}f\|_{\mathrm{L}^2(I_i;\mathbb{H})} \leq C_i k_i^{1/2}\|f\|_{\mathrm{L}^2(I_i;\mathbb{H})}. \tag{4.12}$$

This yields the bound

$$\|U\|_{\mathrm{L}^\infty(I_i;\mathbb{H})} \leq \|U_{i-1}^-\|_{\mathbb{H}} + C_i k_i^{1/2} \|f\|_{\mathrm{L}^2(I_i;\mathbb{H})} . \tag{4.13}$$

Now select $i^\star \in \{1,\ldots,m\}$ such that $\|U\|_{\mathrm{L}^\infty((0,t_m);\mathbb{H})} = \|U\|_{\mathrm{L}^\infty(I_{i^\star};\mathbb{H})}$. Then, with (4.13) it holds that

$$\|U\|_{\mathrm{L}^\infty((0,t_m);\mathbb{H})} \leq \|U_{i^\star-1}^-\|_{\mathbb{H}} + C_{i^\star} k_{i^\star}^{1/2} \|f\|_{\mathrm{L}^2(I_{i^\star};\mathbb{H})} .$$

In order to estimate the first term on the right-hand side of the above inequality, we iterate the bound (4.13), thereby yielding

$$\begin{aligned} \|U\|_{\mathrm{L}^\infty((0,t_m);\mathbb{H})} &\leq \|U\|_{\mathrm{L}^\infty(I_{i^\star-1};\mathbb{H})} + C_{i^\star} k_{i^\star}^{1/2} \|f\|_{\mathrm{L}^2(I_{i^\star};\mathbb{H})} \\ &\leq \|U_{i^\star-2}^-\|_{\mathbb{H}} + \sum_{i=i^\star-1}^{i^\star} C_i k_i^{1/2} \|f\|_{\mathrm{L}^2(I_i;\mathbb{H})} \\ &\;\vdots \\ &\leq \|U_0^-\|_{\mathbb{H}} + \sum_{i=1}^{i^\star} C_i k_i^{1/2} \|f\|_{\mathrm{L}^2(I_i;\mathbb{H})} . \end{aligned} \tag{4.14}$$

Recalling (2.4) and applying the Cauchy–Schwarz inequality, we obtain

$$\|U\|_{\mathrm{L}^\infty((0,t_m);\mathbb{H})} \leq \|\pi_0 u_0\|_{\mathbb{H}} + \gamma_m \left(\sum_{i=1}^m k_i\right)^{1/2} \left(\sum_{i=1}^m \|f\|^2_{\mathrm{L}^2(I_i;\mathbb{H})}\right)^{1/2},$$

and the proof is complete. ∎

Remark 4.5. Recalling (4.6) and exploiting (3.53), we infer a (rough) estimate for γ_m from (4.10), namely $\gamma_m \leq \max_{1\leq i\leq m} C_i \leq \sqrt{6}+1$. In particular, γ_m is uniformly bounded with respect to any discretization parameters.

Remark 4.6. Revisiting (3.43) and invoking (3.53), for any $w \in \mathbb{P}^{r_m}(I_m)$, we deduce the bound

$$\left\|(\Gamma^{r_m}_{\lambda,m})^{-1}(w)\right\|_{\mathrm{L}^\infty(I_m)} \leq \widetilde{C}^{\mathrm{L}^2}_{\lambda,r_m} \lambda^{-1/2} \|w\|_{\mathrm{L}^2(I_m)} , \tag{4.15}$$

where $\widetilde{C}^{\mathrm{L}^2}_{\lambda,r_m} := (2-e^{-\lambda k_m})(k_m\lambda)^{1/2}\Upsilon_{3.28}(r_m,k_m\lambda)+2^{-1/2}$. Similarly as in (3.53) it is immediately verified that $\widetilde{C}^{\mathrm{L}^2}_{\lambda,r_m}$ is uniformly bounded with respect to k_m, r_m, and λ. Now, proceeding as in the proof of Proposition 4.1, and utilizing (4.15), we obtain

$$\left\|(\Gamma^{r_m}_m)^{-1}(w)\right\|^2_{\mathrm{L}^\infty(I_m;\mathbb{H})} \leq \sum_{i=1}^{n_m} \left\|(\Gamma^{r_m}_{\lambda_i,m})^{-1}(a_i)\right\|^2_{\mathrm{L}^\infty(I_m)} \leq \widetilde{C}_m^2 \sum_{i=1}^{n_m} \lambda_i^{-1} \|a_i\|^2_{\mathrm{L}^2(I_m)} ,$$

with $\widetilde{C}_m := \max_{1\leq i\leq n_m} \widetilde{C}^{\mathrm{L}^2}_{\lambda_i,r_m}$. Invoking (4.3), this transforms into

$$\left\|(\Gamma^{r_m}_m)^{-1}(w)\right\|_{\mathrm{L}^\infty(I_m;\mathbb{H})} \leq \widetilde{C}_m \alpha_{1.3}^{-1/2} \|w\|_{\mathrm{L}^2(I_m;\mathbb{X}^\star)}.$$

Then, the bound (4.12) is modified to

$$\left\|(\Gamma^{r_m}_m)^{-1}(\Pi^{r_m}_m f)\right\|_{\mathrm{L}^\infty(I_m;\mathbb{H})} \leq \widetilde{C}_m \alpha_{1.3}^{-1/2} \|\Pi^{r_m}_m f\|_{\mathrm{L}^2(I_m;\mathbb{X}^\star)} \leq \widetilde{C}_m \alpha_{1.3}^{-1/2} \|f\|_{\mathrm{L}^2(I_m;\mathbb{X}^\star)} ,$$

where the projection $\Pi_m^{r_m}$ needs to be extended to $\mathrm{L}^2(I_m;\mathbb{X}^\star)$; cf. [24, eq. (10)]. Thereby, we obtain the following analogue to (4.13):

$$\|U\|_{\mathrm{L}^\infty(I_i;\mathbb{H})} \leq \|U_{i-1}^-\|_{\mathbb{H}} + \widetilde{C}_i\alpha_{1.3}^{-1/2}\,\|f\|_{\mathrm{L}^2(I_i;\mathbb{X}^\star)}\,.$$

Adding as in (4.14), we arrive at

$$\|U\|_{\mathrm{L}^\infty((0,t_m);\mathbb{H})} \leq \|U_0^-\|_{\mathbb{H}} + \alpha_{1.3}^{-1/2}\sum_{i=1}^{m}\widetilde{C}_i\,\|f\|_{\mathrm{L}^2(I_i;\mathbb{X}^\star)}\,.$$

We emphasize that, in contrast to the bound derived in (4.14), there are no local scaling factors $k_i^{1/2}$ in the previous estimate. Consequently, when aiming at a global $\mathrm{L}^2((0,t_m);\mathbb{X}^\star)$ norm of the data f, the application of the Cauchy–Schwarz inequality causes an unfavorable m-dependence (i.e., the number of time steps) in the resulting bound:

$$\|U\|_{\mathrm{L}^\infty((0,t_m);\mathbb{H})} \leq \|\pi_0 u_0\|_{\mathbb{H}} + \widetilde{\gamma}_m m^{1/2}\,\|f\|_{\mathrm{L}^2((0,t_m);\mathbb{X}^\star)}\,; \tag{4.16}$$

here, $\widetilde{\gamma}_m := \max_{1\leq i\leq m}\widetilde{C}_i$. We underline that (4.16) is relevant, for instance, in the spectral context, where $r_m\to\infty$, $m=1,\ldots,M$, on a small number M of (possibly large) time steps.

Remark 4.7. For $t\in I_m$, the solution of the linear parabolic problem (1.2) is given by

$$u(t) = e^{-\mathsf{A}(t-t_{m-1})}u(t_{m-1}) + \int_{t_{m-1}}^{t} e^{-\mathsf{A}(t-s)}f(s)\,\mathsf{d}s.$$

Hence, recalling the solution formula (4.11) for the discrete problem on I_m, we have

$$u(t)-U(t) = \mathfrak{H}(t)+\mathfrak{I}(t), \qquad t\in I_m,$$

where the terms $\mathfrak{H}(t) = e^{-\mathsf{A}(t-t_{m-1})}u(t_{m-1}) - \mathbf{\Psi}^{r_m}(t)$, with $\mathbf{\Psi}^{r_m}$ from (4.8), and

$$\mathfrak{I}(t) = \int_{t_{m-1}}^{t} e^{-\mathsf{A}(t-s)}f(s)\,\mathsf{d}s - (\Gamma_m^{r_m})^{-1}(\Pi_m^{r_m}f)(t)$$

correspond to the homogeneous and inhomogeneous part of the PDE, respectively. Here, to bound the error $\|u-U\|_{\mathrm{L}^\infty(I_m;\mathbb{H})}$, we can employ our previous analysis from Remark 4.3 to control $\|\mathfrak{H}\|_{\mathrm{L}^\infty(I_m)}$. Additionally, in order to estimate $\|\mathfrak{I}\|_{\mathrm{L}^\infty(I_m)}$, let $\Pi_m^{r_m}f = \sum_{i=1}^{n_m} f_i(t)\varphi_i$ be the spectral decomposition of $\Pi_m^{r_m}f$. By Proposition 4.1 and Corollary 3.13 we have that $(\Gamma_m^{r_m})^{-1}(\Pi_m^{r_m}f) = \sum_{i=1}^{n_m}(\Gamma_{\lambda_i,m}^{r_m})^{-1}(f_i)\varphi_i$, and thus

$$\begin{aligned}(\Gamma_m^{r_m})^{-1}&(\Pi_m^{r_m}f)\\ &= \sum_{i=1}^{n_m} -\frac{\mathsf{e}_{\lambda_i,m}(t)}{\mathsf{e}_{\lambda_i,m}(t_{m-1})}\int_{I_m} f_i(s)\phi_{\lambda_i}^{r_m}(s)\,\mathsf{d}s + \int_{t_{m-1}}^{t}\sum_{i=1}^{n_m} e^{-\lambda_i(t-s)}f_i(s)\varphi_i\,\mathsf{d}s\\ &= \sum_{i=1}^{n_m} -\frac{\mathsf{e}_{\lambda_i,m}(t)}{\mathsf{e}_{\lambda_i,m}(t_{m-1})}\int_{I_m} f_i(s)\phi_{\lambda_i}^{r_m}(s)\,\mathsf{d}s + \int_{t_{m-1}}^{t} e^{-\mathsf{A}_m(t-s)}\Pi_m^{r_m}f(s)\,\mathsf{d}s.\end{aligned}$$

Then,

$$\mathfrak{I}(t) = \sum_{i=1}^{n_m} \frac{\mathsf{e}_{\lambda_i,m}(t)}{\mathsf{e}_{\lambda_i,m}(t_{m-1})} \int_{I_m} f_i(s)\phi_{\lambda_i}^{r_m}(s)\,\mathsf{d}s + \int_{t_{m-1}}^{t} e^{-\mathsf{A}(t-s)}\left(f(s) - \Pi_m^{r_m} f(s)\right)\mathsf{d}s$$
$$+ \int_{t_{m-1}}^{t} \left(e^{-\mathsf{A}(t-s)} - e^{-\mathsf{A}_m(t-s)}\right)\Pi_m^{r_m} f(s)\,\mathsf{d}s.$$

We notice that the second integral is a data approximation term (which, with the aid of stability, can be estimated further), and the third integral relates to the spatial Galerkin discretization. Incidentally, the second term in (4.9) and the third integral above add to the semidiscrete error in space; cf. [25, section 6]. Moreover, recalling (3.52), the first term can be estimated by

$$\left|\sum_{i=1}^{n_m} \frac{\mathsf{e}_{\lambda_i,m}(t)}{\mathsf{e}_{\lambda_i,m}(t_{m-1})} \int_{I_m} f_i(s)\phi_{\lambda_i}^{r_m}(s)\,\mathsf{d}s\right| \leq 2\sum_{i=1}^{n_m}\left|\int_{I_m} f_i(s)\phi_{\lambda_i}^{r_m}(s)\,\mathsf{d}s\right|.$$

Even though both sides of the the above inequality are computable, we could proceed as follows by means of the Cauchy–Schwarz inequality (which results in a more pessimistic bound):

$$\sum_{i=1}^{n_m}\left|\int_{I_m} f_i(s)\phi_{\lambda_i}^{r_m}(s)\,\mathsf{d}s\right| \leq \left(\sum_{i=1}^{n_m}\|f_i\|_{\mathrm{L}^2(I_m)}^2\right)^{1/2}\left(\sum_{i=1}^{n_m}\left\|\phi_{\lambda_i}^{r_m}\right\|_{\mathrm{L}^2(I_m)}^2\right)^{1/2}.$$

While the first term on the right-hand side of the above inequality can be bounded by $\|f\|_{\mathrm{L}^2(I_m;\mathbb{H})}$ the second term can be estimated by means of Lemma 3.7.

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