Conformal Dimensions in the Large Charge Sectors at the $O(4)$ Wilson-Fisher Fixed Point

Debasish Banerjee,[1,*] Shailesh Chandrasekharan,[2,†] Domenico Orlando,[3,4,‡] and Susanne Reffert[4,§]
[1]Institut für Physik, Humboldt-Universität zu Berlin, Zum Großen Windkanal 6, D-12489 Berlin, Germany
[2]Department of Physics, Duke University, Durham, North Carolina 27708, USA
[3]INFN, sezione di Torino and ArnoldRegge Center, via Pietro Giuria 1, 10125 Torino, Italy
[4]Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, CH-3012, Bern, Switzerland

(Received 25 March 2019; published 1 August 2019)

We study the $O(4)$ Wilson-Fisher fixed point in $2+1$ dimensions in fixed large-charge sectors identified by products of two spin-j representations (j_L, j_R). Using effective field theory we derive a formula for the conformal dimensions $D(j_L, j_R)$ of the leading operator in terms of two constants, $c_{3/2}$ and $c_{1/2}$, when the sum $j_L + j_R$ is much larger than the difference $|j_L - j_R|$. We compute $D(j_L, j_R)$ when $j_L = j_R$ with Monte Carlo calculations in a discrete formulation of the $O(4)$ lattice field theory, and show excellent agreement with the predicted formula and estimate $c_{3/2} = 1.068(4)$ and $c_{1/2} = 0.083(3)$.

DOI: 10.1103/PhysRevLett.123.051603

Introduction.—Conformal field theory (CFT) holds a central place in the study of quantum field theory (QFT), as it is relevant to both particle physics and condensed matter systems at criticality, and via the gauge-gravity correspondence even to the description of quantum gravity. Generically, CFTs do not contain any small couplings that can be used in a perturbative analysis. However, the conformal symmetry constrains its observables such that we can determine any n-point function using only operator dimensions and three-point function coefficients. While it is possible to treat strongly coupled theories with methods such as the large-N expansion, the ϵ expansion (see [1] for a review), and the conformal bootstrap [2], they are notoriously difficult to access analytically. In simple cases, Monte Carlo (MC) techniques offer a reliable numerical alternative [3,4].

Recently, it has been shown in a series of papers [5–10] that working in a sector of large global charge results in important simplifications and gives us a perturbative handle to study CFTs using effective field theories (EFTs): it is possible to write an effective action as an expansion in terms of a large conserved charge with unknown coefficients. For the Wilson-Fisher point in the three-dimensional $O(N)$ vector model [11], except for two low-energy couplings, all terms are suppressed by inverse powers of the large charge [6]. The approximate physics of the CFT becomes accessible as a function of these two couplings which we label as $c_{3/2}$ and $c_{1/2}$. This suggests a double-pronged approach to CFTs, which involves using the large-charge expansion to determine the effective action, paired with MC calculations to determine the low-energy couplings. For the case of the $O(2)$ Wilson-Fisher CFT, this approach has been successfully implemented recently [12]. In particular, it was shown that the predictions obtained with the two couplings remain very accurate even for low charges.

In this Letter, we explore the viability of this approach for the $O(4)$ Wilson-Fisher CFT, which has qualitatively distinct features from the $O(2)$ model studied earlier. The fact that $O(4)$ symmetry is non-Abelian and that it leads to two conserved global charges j_L and j_R creates novel challenges. The ground state can become spatially inhomogeneous requiring a different analysis in the EFT, and the construction of a worldline-based lattice model becomes necessary to access easily the large-charge sectors. The CFT with $O(4)$ symmetry is also interesting in many subfields of physics. For example, it arises naturally in the study of finite-temperature chiral phase transitions in two-flavor quantum chromodynamics (QCD) with massless quarks [13,14]. It is also of interest in studies of strongly correlated electronic systems at half filling built out of models of interacting electrons with spin [15].

Traditional $O(4)$ lattice models are constructed using classical vectors. Unfortunately, in the study of large charge sectors, using traditional MC methods based on sampling classical vectors leads to severe signal-to-noise ratio problems. While worldline representations can in principle solve these problems [16,17], the presence of an infinite Hilbert space at each lattice site can still lead to algorithmic inefficiencies. Fortunately, a discrete version of the $O(4)$ model with a finite Hilbert space per lattice site is easy to construct [18,19]. In this work we use this discrete formulation to accurately compute the conformal dimensions

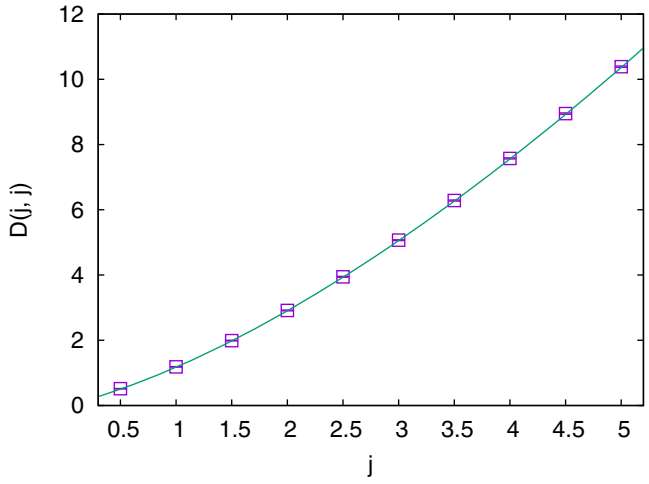

FIG. 1. Plot of $D(j,j)$ as a function of j. The squares represent the data obtained using MC calculations with the lattice model in Eq. (11). The solid line is the large-charge prediction Eq. (5) with $c_{3/2} = 1.068(4)$ and $c_{1/2} = 0.083(3)$.

$D(j_L, j_R)$ (defined below) at the $O(4)$ Wilson-Fisher fixed point, when $j_L = j_R = j$ (see Fig. 1). The large-charge prediction [see Eq. (5)] is an excellent fit to the lattice data even up to the smallest charge giving us $c_{3/2} = 1.068(4)$ and $c_{1/2} = 0.083(3)$.

Large charge predictions.—The EFT approach to the $O(4)$ CFT is based on the construction of an effective action at large charge. Using the fact that $SU_L(2) \times SU_R(2)$ is a double cover of $O(4)$, this action can be put into the form [6,8]

$$S = \int_{\mathbb{R}\times\Sigma} dt d\Sigma \left(\frac{\sqrt{2}}{27 c_{3/2}^2} \|dg\|^3 - \frac{c_{1/2}}{3\sqrt{2} c_{3/2}} R \|dg\| + \ldots \right), \tag{1}$$

where $g(\mathbf{r}, t) \in SU(2)$, $\mathbf{r}$ is the coordinate on Σ, $\|dg\|^2 = \mathrm{Tr}(\partial_\mu g^\dagger \partial^\mu g)$, and $c_{3/2}$ and $c_{1/2}$ are the two leading low-energy couplings referred to earlier. This action is to be understood as an expansion around the fixed-charge ground state. We study the system on a spatial Riemann surface Σ with scalar curvature R. The field g transforms as $g \to V_L g V_R^{-1}$ under the global $SU(2)_L \times SU(2)_R$. The corresponding Noether charges are the two-by-two Hermitian traceless matrices in the $su(2)$ algebra:

$$Q_L = i \int d\Sigma c_J \partial_0 g g^\dagger, \qquad Q_R = i \int d\Sigma c_J \partial_0 g^\dagger g, \tag{2}$$

where $c_J = \sqrt{2}\|dg\|/(9c_{3/2}^2) - c_{1/2} R/(6\sqrt{2} c_{3/2} \|dg\|)$. Their two eigenvalues are $\pm j_L$ and $\pm j_R$. Under the action of $SU(2)_L \times SU(2)_R$ the charges in Eq. (2) transform as

$$Q_L \to V_L Q_L V_L^{-1}, \qquad Q_R \to V_R Q_R V_R^{-1}, \tag{3}$$

but j_L and j_R remain invariant. We will refer to the class of configurations g connected by $SU(2)_L \times SU(2)_R$ transformations as the (j_L, j_R) sector. In the underlying QFT the sectors (j_L, j_R) naturally label the irreducible representation space of $SU(2)_L \times SU(2)_R$; hence the values of j_L and j_R are quantized, i.e., $j_L, j_R \in (1/2)\mathbb{Z}$. Their sum must be integer, $j_L + j_R \in \mathbb{Z}$, because we consider only states that are representations of $O(4)$.

Instead of $Q_{L,R}$ it is more convenient to work with the projections $q_{L,R} = \mathrm{Tr}(Q_{L,R}\sigma_3)/2$. In a fixed (j_L, j_R) sector these projections will take values in the range $-j_{L,R} \le q_{L,R} \le j_{L,R}$. It is natural to identify them with the quantized charges of the states in the representation with highest weights (j_L, j_R). In Ref. [6] it was shown that the minimal energy solutions to the equation of motion (EOM) for the action (1) for fixed values of $q_{L,R}$ are homogeneous in space and arise in sectors with $j_L = j_R = \max(q_L, q_R)$. This leads to the formula for the minimal energy in a fixed (j,j) sector:

$$E(j,j) = \sqrt{\frac{8j^3}{V}} \left(c_{3/2} + c_{1/2} \frac{RV}{4j} + \ldots \right) + \zeta(-1/2|\Sigma), \tag{4}$$

where $\zeta(s|\Sigma)$ is the ζ function for the Laplacian on the surface Σ and represents the contribution of the Casimir energy. Since in a CFT the conformal dimension of an operator is identified with the energy on the unit sphere $\Sigma = S^2$ of the corresponding state, we deduce a formula for the dimension of the lowest operator in the (j,j) sector:

$$D(j,j) = \sqrt{\frac{2j^3}{\pi}} \left[c_{3/2} + c_{1/2} \frac{2\pi}{j} + \mathcal{O}\left(\frac{1}{j^2}\right) \right] + c_0, \tag{5}$$

where $c_0 = \zeta(-1/2|S^2) \approx -0.094$ is a universal constant [20,21].

In this work we generalize this formula to any representation (j_L, j_R). We need to find the minimal-energy solutions admitted by the action Eq. (1) whose charge matrices Q_L and Q_R are diagonal and correspond, by the argument above, to highest-weight states in a representation of $SO(4)$. In order to study the sectors with $j_L \neq j_R$, the analysis in Refs. [6,22,23] suggests that we need to look for inhomogeneous field configurations $g(\mathbf{r}, t)$ of the form

$$g(\mathbf{r}, t) = \begin{pmatrix} \cos(p(\mathbf{r}))e^{i\mu_1 t} & \sin(p(\mathbf{r}))e^{i\mu_2 t} \\ -\sin(p(\mathbf{r}))e^{-i\mu_2 t} & \cos(p(\mathbf{r}))e^{-i\mu_1 t} \end{pmatrix}, \tag{6}$$

where μ_1 and μ_2 are constants parametrizing the action of $SU(2)_L \times SU(2)_R$ on the inhomogeneous configuration encoded by the undetermined function $p(\mathbf{r})$. The homogeneous solution of Ref. [6] with $\mu_1 = \mu_2 = \mathcal{O}(j^{1/2})$ and

$p(\mathbf{r}) = 0$ describes the $j_L = j_R$ sectors. In order to explore solutions with small nonzero values of $|j_L - j_R|/\max(j_L, j_R)$ we may expand the action in a series in $\eta^2 = (\mu_2 - \mu_1)/(\mu_2 + \mu_1)$ which will naturally be small. At leading order in η^2 the EOM is an elliptic sine-Gordon equation

$$2\triangle p(\mathbf{r}) + \Lambda^2 \sin[2p(\mathbf{r})] = 0, \tag{7}$$

where $\triangle$ is the Laplacian on Σ and $\Lambda^2 = \mu_2^2 - \mu_1^2$. The case $\Sigma = T^2$ was already discussed in [22,23]. Here we concentrate on $\Sigma = S^2(r_0)$ in order to calculate operator dimensions. We express the parameters μ_1 and μ_2 as functions of the eigenvalues j_L and j_R and find that the leading (tree-level) contribution to the energy of the solutions to the EOM has the same form as Eq. (4), where now $j = j_m = \max(j_L, j_R)$, plus an extra contribution that captures the inhomogeneity of the solution [i.e., $\nabla p(\mathbf{r}) \neq 0$]:

$$E_\Sigma^{\mathrm{tr}} = \sqrt{\frac{8j_m^3}{V}}\left(c_{3/2} + \frac{c_{1/2}RV}{4j_m} + \int_\Sigma d\Sigma \frac{(\nabla p)^2}{6c_{3/2}j_m} + \ldots\right). \tag{8}$$

As discussed in the Supplemental Material [24], when $\Sigma = S^2(r_0)$, the EOM (7) admits different branches of smooth solutions, parametrized by an integer ℓ which counts the zeros of $p(\mathbf{r})$. The energy is minimal in the first nontrivial branch ($\ell = 1$), where $2 \leq r_0^2\Lambda^2 < 6$. Here the integral of the divergence can be computed numerically in terms of an expansion in $|j_L - j_R|/j_m$ to give

$$\frac{1}{4\pi}\int_{S^2} d\Omega(\nabla p)^2 = \frac{|j_L - j_R|}{j_m} + \lambda_2\left(\frac{|j_L - j_R|}{j_m}\right)^2 + \ldots \tag{9}$$

with $\lambda_2 \approx 0.2455$. This is the leading contribution in the large-charge expansion. There will be in general higher-order corrections suppressed by inverse powers of the large charges due to subleading terms in the tree-level action in Eq. (1) and to quantum corrections.

There is only one term of order $\mathcal{O}(j^0)$: the Casimir energy of the Goldstones resulting from the spontaneous symmetry breaking $SO(3) \times D \times SO(2)^2 \to SO(2) \times D'$ discussed in the Supplemental Material [24]. The two broken generators of the isometries on the sphere only give rise to one Goldstone degree of freedom (d.o.f.). Together with the 2 d.o.f. from the broken internal symmetries, they are arranged into one type-I and one type-II Goldstone field in the notation of [26]. Only the former contributes to the Casimir energy as $E_0 = \zeta(-1/2|S^1)/(2\sqrt{2})$. The zero-point energy is different from the one in the (j, j) sector because the low-energy excitations only propagate in the direction of the unbroken sphere isometry. Once again we can use the state-operator correspondence and obtain the final formula for the conformal dimension of the lowest operator in the representation (j_L, j_R) of $SO(4)$ when $j_L \neq j_R$:

$$D(j_L, j_R) = \sqrt{\frac{2j_m^3}{\pi}}\left[c_{3/2} + c_{1/2}\frac{2\pi}{j_m} + \frac{1}{3c_{3/2}}\left(\frac{|j_L - j_R|}{j_m} + \lambda_2\frac{(j_L - j_R)^2}{j_m^2} + \ldots\right) \times \frac{2\pi}{j_m} + \ldots\right] - \frac{1}{12\sqrt{2}}. \tag{10}$$

As we have stressed, the conformal dimensions only depend on the two Wilsonian couplings $c_{3/2}$ and $c_{1/2}$, which are the same coefficients that appear in Eq. (5) for the $j_L = j_R$ case. We now explain how we determine them using MC methods with our lattice model.

Lattice simulations.—Our lattice model was first introduced in Ref. [18] as a model for pion physics in two-flavor QCD and studied with an efficient MC algorithm. It is constructed using four Grassmann fields $\psi_\alpha(x), \bar{\psi}_\alpha(x), \alpha = 1, 2$ at every three-dimensional periodic cubic lattice site $x = (\mathbf{r}, t)$ of size L in all the directions. If we arrange these four-fields into a 2×2 matrix of the form $g_{\alpha\beta}(x) = \psi_\alpha\bar{\psi}_\beta$ we can write the lattice action as

$$S = -\sum_{\langle xy\rangle}\mathrm{Tr}(g_x g_y) - \frac{U}{2}\sum_x \det(g_x), \tag{11}$$

where $\langle xy\rangle$ are nearest-neighbor bonds. This action is invariant under the $SU(2) \times SU(2)$ transformations $g_x \to V_L g_x V_R^{-1}$ on odd sites and $g_x \to V_R g_x V_L^{-1}$ on even sites. The partition function of the model can be expressed as a sum over configurations where each site either contains a vacuum site or a worldline of an $O(4)$ particle in the vector representation. Thus, each worldline has four possible states that label the eigenvalues $(q_L, q_R) = (\pm 1/2, \pm 1/2)$ of particles that travel through the sites. These can be thought

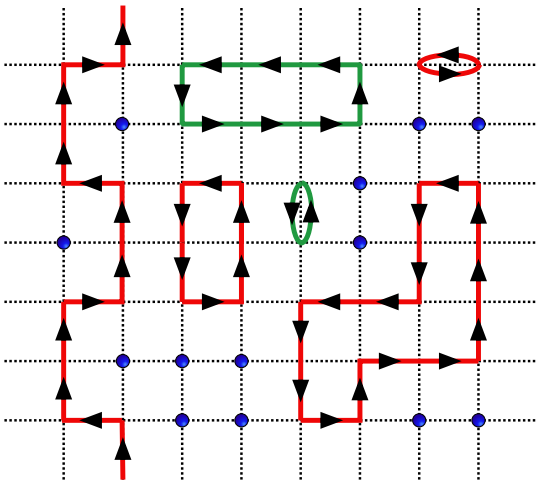

FIG. 2. Illustration of an $O(4)$ worldline configuration in two dimensions. The solid circles represent vacuum sites, each of which have a weight U. All other sites have a single $O(4)$ particle with charge $(q_L, q_R) = (\pm 1/2, \pm 1/2)$ moving in space-time.

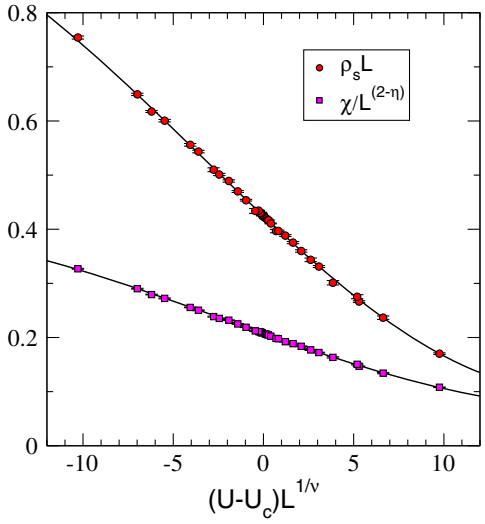

FIG. 3. The critical scaling plots of $\rho_s L$ (circles) and $\chi L^{2-\eta}$ as a function of the scaling variable $(U-U_c)L^{1/\nu}$. The solid lines show the goodness of the combined fit of all the data shown to polynomials to fourth order.

of as oriented loops with two colors (say red and green). An illustration of a configuration is shown in Fig. 2.

The weight of a worldline configuration is given by U^{N_m} where N_m is the number of vacuum sites. As U is tuned, the model undergoes a phase transition between the massive symmetric phase at large values to a phase where the $O(4)$ symmetry is spontaneously broken at small values. Using well-established MC methods [18,19] we first demonstrate that at the critical point we obtain the $O(4)$ Wilson-Fisher CFT by computing the critical exponents ν and η. For this purpose we compute the current susceptibility ρ_s and the order parameter susceptibility χ, details of which can be found in the Supplemental Material [24]. Finite-size scaling theory near a second-order phase transition predicts that $\rho_s L$ and $\chi L^{2-\eta}$ must be simple polynomials of $(U-U_c)L^{1/\nu}$. A combined fit of our data gives $U_c = 1.655394(3)$, $\nu = 0.746(3)$, and $\eta = 0.0353(10)$. In Fig. 3 we plot our data and the fit. These exponents are in excellent agreement with earlier results, $\nu = 0.749(2)$ and $\eta = 0.0365(10)$, obtained from the traditional lattice model [27].

TABLE I. Results for the conformal dimensions $D(j,j)$ up to $j = 5$ computed using worldline MC methods in this work (second and fifth column). We also compare our results with earlier calculations up to $j = 2$ found in [28].

	$D(j,j)$	
j	(this work)	(from [28])
1/2	0.515(3)	0.5180(3)
3/2	1.989(5)	1.9768(10)
5/2	3.945(6)	⋯
7/2	6.284(8)	⋯
9/2	8.949(10)	⋯

	$D(j,j)$	
j	(this work)	(from [28])
1	1.185(4)	1.1855(5)
2	2.915(6)	2.875(5)
3	5.069(7)	⋯
4	7.575(9)	⋯
5	10.386(11)	⋯

Having established that our lattice model indeed reproduces the $O(4)$ CFT when $U = U_c$, we can use the method we developed in Ref. [12] to accurately compute the conformal dimensions $D(j,j)$ at the $O(4)$ CFT. We can create configurations in a specific (j_L, j_R) sector by placing appropriately charged sources and sinks at $t = 0$ and $t = L/2$ respectively. More concretely, sources that create a red loop are assigned the charge $(1/2, 1/2)$ and the sinks that annihilate them are assigned the charge $(-1/2, -1/2)$. Similarly, those that create and annihilate the green loops are assigned charges $(1/2, -1/2)$ and $(-1/2, 1/2)$. Using these fundamental sources we can construct sources and sinks with any charge (q_L, q_R). However, since each site can only have one red or one green source, to create a source with a large charge we distribute the fundamental sources in a local region near the origin (see Supplemental Material [24] for more details). Since the couplings $c_{3/2}$ and $c_{1/2}$ can be computed by fitting the data for $D(j,j)$ to the predicted form in (5), in this work we only study the sector with $j_L = j_R = j, j = 1/2, 1, 3/2, \ldots$. For this purpose we only work with sources and sinks of equal charges by creating $2j$ sources of red loops at $t = 0$ and annihilating them at $t = L/2$. This naturally projects us into the highest-weight representation sector with $j_L = j_R = j$. Let $Z_j(L)$ be the partition function in the presence of these sources and sinks. In Ref. [12] we developed an efficient algorithm to compute the ratio $R_j(L) = Z_j(L)/Z_{j-1/2}(L)$, which is expected to scale as $C/L^{2\Delta(j)}$ for large values of L. By evaluating $R_j(L)$ for various values of j, L and fitting to the expected form we can accurately compute the difference in the conformal dimensions $\Delta(j) = D(j,j) - D(j-1/2, j-1/2)$. From these differences we can also estimate $D(j,j)$, since conformal invariance fixes $D(0,0) = 0$. Our final results are tabulated in Table I up to $j = 5$. As the table shows, our results are also in good agreement with earlier calculations up to $j = 2$ [28]. We first confirm the EFT prediction of $j^{3/2}$ for large values of j, by fitting the conformal dimensions in Table I for $j > 2$ to the form Cj^p and obtain $p = 1.49 \pm 0.02$ with a small χ^2/d.o.f.. Then fitting all the data in Table I to the form in Eq. (5) we obtain $c_{3/2} = 1.068(4)$ and $c_{1/2} = 0.083(3)$ again with a small χ^2/d.o.f. (see Fig. 1).

Conclusions.—In this Letter we provided a new prediction for the anomalous dimensions $D(j_L, j_R)$ [see (10)]

at the $O(4)$ Wilson-Fisher fixed point in terms of the two couplings that appear in the fixed large-charge effective action (1). Our prediction is valid in the limit of large (j_L, j_R) and small $|j_L - j_R|/\max(j_L, j_R)$. We then use a discrete lattice $O(4)$ model to compute the two couplings by fitting the data for $D(j, j)$ to the prediction in Eq. (5) obtained from an earlier work. We also demonstrate that this prediction provides an excellent approximation even at small values of j (see Fig. 1). Our estimate $c_{3/2} = 1.068(4)$ and $c_{1/2} = 0.083(3)$ can be used in (10) to predict $D(j_L, j_R)$ even for $j_L \neq j_R$. While our lattice model can in principle be used to check the validity of these predictions, our method is likely to suffer from signal-to-noise ratio problems when j_L and j_R are sufficiently large and different. Discrete lattice models like ours can in principle also be designed for other non-Abelian symmetry groups, thus allowing us to explore the robustness of the large-charge EFT method for general CFTs. Such extensions are likely to bring new challenges providing a fertile ground for further research.

We wish to thank Andrew Gasbarro, Simeon Hellerman, Francesco Vitali, Masataka Watanabe, Urs Wenger, and Uwe-Jens Wiese for valuable discussions. The material presented here is based upon work supported by the U.S. Department of Energy, Office of Science, Nuclear Physics program under Award No. DE-FG02-05ER41368. D. B. is supported by the German Research Foundation (DFG), Project No. 392051989. D. O. acknowledges partial support by the NCCR 51NF40-141869 "The Mathematics of Physics" (SwissMAP). The work of S. R. is supported by the Swiss National Science Foundation (SNF) under Grant No. PP00P2_157571/1.

*debasish.banerjee@physik.hu-berlin.de
†sch@phy.duke.edu
‡domenico.orlando@to.infn.it
§sreffert@itp.unibe.ch

[1] A. Pelissetto and E. Vicari, Phys. Rep. **368**, 549 (2002).
[2] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, J. High Energy Phys. 12 (2008) 031.
[3] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B **63**, 214503 (2001).
[4] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B **65**, 144520 (2002).
[5] S. Hellerman, D. Orlando, S. Reffert, and M. Watanabe, J. High Energy Phys. 12 (**2015**) 071.
[6] L. Alvarez-Gaume, O. Loukas, D. Orlando, and S. Reffert, J. High Energy Phys. 04 (2017) 059.
[7] A. Monin, D. Pirtskhalava, R. Rattazzi, and F. K. Seibold, J. High Energy Phys. 06 (**2017**) 011.
[8] O. Loukas, D. Orlando, and S. Reffert, J. High Energy Phys. 10 (2017) 085.
[9] O. Loukas, D. Orlando, S. Reffert, and D. Sarkar, Nucl. Phys. **B934**, 437 (2018).
[10] S. Favrod, D. Orlando, and S. Reffert, J. High Energy Phys. 12 (2018) 052.
[11] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972).
[12] D. Banerjee, S. Chandrasekharan, and D. Orlando, Phys. Rev. Lett. **120**, 061603 (2018).
[13] R. D. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338 (1984).
[14] F. Wilczek, Int. J. Mod. Phys. A **07**, 3911 (1992); **07**, 6951(E) (1992).
[15] C. N. Yang and S. Zhang, Mod. Phys. Lett. B **04**, 759 (1990).
[16] U. Wolff, Proc. Sci., LATTICE2010 (**2010**) 020 [arXiv:1009.0657].
[17] C. Gattringer, D. Gschl, and C. Marchis, Phys. Lett. B **778**, 435 (2018).
[18] D. J. Cecile and S. Chandrasekharan, Phys. Rev. D **77**, 014506 (2008).
[19] S. Chandrasekharan and A. Li, J. High Energy Phys. 12 (2010) 021.
[20] E. Elizalde, Lect. Notes Phys. **855**, 1 (2012); Lect. Notes Phys., M: Monogr. **35**, 1 (1995).
[21] A. De La Fuente, J. High Energy Phys. 08 (2018) 041.
[22] S. Hellerman, N. Kobayashi, S. Maeda, and M. Watanabe, arXiv:1705.05825.
[23] S. Hellerman, N. Kobayashi, S. Maeda, and M. Watanabe, arXiv:1804.06495.
[24] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.051603, which includes [25], for details of the solutions of the equations and the numerical algorithm employed.
[25] I. Low and A. V. Manohar, Phys. Rev. Lett. **88**, 101602 (2002).
[26] H. B. Nielsen and S. Chadha, Nucl. Phys. **B105**, 445 (1976).
[27] M. Hasenbusch, J. Phys. A **34**, 8221 (2001).
[28] M. Hasenbusch and E. Vicari, Phys. Rev. B **84**, 125136 (2011).