Electron. Commun. Probab. **24** (2019), no. 17, 1–8.
https://doi.org/10.1214/19-ECP214
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Conditions for the finiteness of the moments of the volume of level sets

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Abstract

Let $X(t)$ be a Gaussian random field $\mathbb{R}^d \to \mathbb{R}$. Using the notion of $(d-1)$-*integral geometric measures*, we establish a relation between (a) the volume of level sets, and (b) the number of crossings of the restriction of the random field to a line. Using this relation we prove the equivalence between the finiteness of the expectation and the finiteness of the second spectral moment matrix. Sufficient conditions for finiteness of higher moments are also established.

Keywords: Gaussian fields; nodal sets; Crofton formula; Kac-Rice formula.
AMS MSC 2010: 60G15; 60G60; 60D05; 20B15.
Submitted to ECP on November 1, 2018, final version accepted on January 25, 2019.
Supersedes arXiv:1811.01787.

1 Introduction

Let $X(t)$ be a centered, stationary, Gaussian random field

$$X : \Omega \times \mathbb{R}^d \to \mathbb{R},$$

with continuous sample paths. By a scaling argument, and without loss of generality, we may assume that $X(t)$ is centered with variance 1. On the other hand, for a given $u \in \mathbb{R}$, let us consider the level set restricted to some compact set $K \subset \mathbb{R}^d$

$$C_{u,K} := \{t \in K \ : X(t) = u\}. \tag{1.1}$$

If the sample paths of $X(t)$ are almost surely (a.s.) differentiable and if a. s. there exist no point t such that $X(t) = u, \nabla X(t) = 0$ (where $\nabla X(t)$ is X's gradient), then by the implicit function theorem, $C_{u,K}$ is almost surely a manifold and its $(d-1)$-volume is well defined and coincides with its $(d-1)$-Hausdorff measure, namely, $\mathcal{H}_{d-1}(C_{u,K})$. Under some non-degeneracy hypothesis, the Kac-Rice formula (KRF) gives an expression for the moments of this measure, see Azaïs-Wschebor[2]. If we consider the expectation, the compactness of the set K and the KRF imply that the first moment is finite, so we

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already have a sufficient condition of finiteness (but as we will see, the latter is not necessary). For higher moments the KRF provides a multiple integral, the integrand of which is degenerate on the diagonal so the study of finiteness is not straightforward.

When $d = 1$, $C_{u,K}$ is a.s. a set of points and its measure is just the number of points. We have several result on finiteness of moments, see Sections 3 and 4. They all use at some stage the Intermediate Value Theorem. Unfortunately these methods are completely inoperative in higher dimensions. Here, we appeal to integral geometry in order to establish dimension-independent necessary and sufficient conditions of almost sure finiteness of level set volumes that boil down to one-dimensional results.

In Section 2 we recall the definition of the $(d-1)$-dimensional *integral-geometric measure*, which is defined as the integral of the number of points over a family of lines.

Our three main results follow

- In Section 3 we establish the equivalence between (a) the finiteness of the expectation of the $(d-1)$-dimensional *integral-geometric measure* of the level set and (b) the finiteness of the second spectral moment matrix. This result gives a simpler presentation and shorter proof of the results of Wschebor [7] which uses De Gorgi perimeters.
- In Section 4 we give sufficient conditions for finiteness of the second moment (Theorem 4.1) using the Geman condition (see [5]).
- In the same section, we prove finiteness of all moments (Theorem 4.4), under some conditions, when the sample paths are smooth.

2 Integral geometric measure, Crofton formula

Let B be a Borel set in $\mathbb{R}^d$. Following Morgan[6] (and also Federer [4]) we define the $(d-1)$-*integral geometric measure* of B by

$$\mathcal{I}_{d-1}(B) := c_{d-1} \int_{v\in S^{d-1}} \left(\int_{y\in v^\perp} \#\left\{B \cap \ell_{v,y}\right\} d\mathcal{H}_{d-1}(y) \right) dS^{d-1}(v) \tag{2.1}$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$ with its induced Riemannian measure, and $\ell_{v,y}$ is the affine linear space $\{y + tv : t \in \mathbb{R}\}$. The constant can be easily computed, using the Crofton formula below and considering the particular case of the sphere, yielding,

$$c_{d-1} = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{(d-1)/2}}.$$

The integrand in (2.1) is measurable (see for example Morgan[6, page 13]), and since it is non-negative, the integral is always well defined, finite or infinite.

In particular, if B is $(d-1)$-rectifiable, then Crofton's formula [6] p. 31 yields

$$\mathcal{H}_{d-1}(B) = \mathcal{I}_{d-1}(B), \tag{2.2}$$

where $\mathcal{H}_{d-1}$ is the $(d-1)$-Hausdorff measure.

3 Characterisation for the finiteness of the expected volume of the level set.

The spectral measure F of $X(\cdot)$ is a symmetric measure with mass one: it is a probability measure.

Let Λ_2 be the second spectral moment matrix defined by

$$(\Lambda_2)_{ij} := \int_{\mathbb{R}^d} \lambda_i \lambda_j dF(\lambda).$$

This matrix may be finite or infinite, infinite meaning by convention that at least one entry is infinite.

When Λ_2 is finite, i.e. all its entries are finite, it is easy to prove that $X(\cdot)$ is differentiable in quadratic mean. If in addition the sample paths are almost surely differentiable (which is a little stronger) and if a.s. there exist no point t such that $X(t) = u, \nabla X(t) = 0$, we have:

- the level set $C_{u,K}$ is almost surely a submanifold of codimension 1, and its Riemannian volume can be defined and coincides with its $(d-1)$-Hausdorff measure, namely, $\mathcal{H}_{d-1}(C_{u,K})$;
- the KRF (see Adler-Taylor [1] or Azaïs-Wschebor[2]) and (2.2) implies that

$$\begin{aligned}\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K})) &= \mathbb{E}(\mathcal{H}_{d-1}(C_{u,K})) = \mathcal{L}_d(K)\mathbb{E}(\|X'(0)\|)\frac{e^{-u^2/2}}{\sqrt{2\pi}} \\ &= \mathcal{L}_d(K)\mathcal{F}(\Lambda_2)e^{-u^2/2}, \end{aligned} \tag{3.1}$$

  where $\mathcal{I}_{d-1}(C_{u,K})$ is the the $(d-1)$-dimensional *integral-geometric measure* defined above, $\mathcal{L}_d$ is the Lebesgue measure on $\mathbb{R}^d$ and

$$\mathcal{F}(\Lambda_2) = \frac{1}{(2\pi)^{(d+1)/2}} \int_{z\in\mathbb{R}^d} (z^\top \Lambda_2 z)^{1/2} e^{-\|z\|^2/2} d\mathcal{L}_d(z). \tag{3.2}$$

  The second equality in (3.1) is the true Kac-Rice formula, the third is due to classical integration.

When Λ_2 is infinite, i.e. at least one of its entries is infinite, we need to extend the definition of $\mathcal{F}(\Lambda_2)$ by setting it to $+\infty$. So we consider the following relation:

$$\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K})) = \mathcal{L}_d(K)\mathcal{F}(\Lambda_2)e^{-u^2/2}. \tag{3.3}$$

Note that its terms on both hand sides are now always well defined, finite or infinite.

The goal of this section is to prove that in a broad sense this formula is always true:

- Whenever Λ_2 is finite, of course the RHS of (3.1) is finite, but so is the LHS as well and equality holds true.
- If Λ_2 is infinite then both sides of (3.1) are infinite.

Such a kind of property is known since the work of Cramér-Leadbetter [3] for $d = 1$ and from the work of Wschebor [7] for $d > 1$. Our proof uses Cramer-Leadbetter's result and generalised Crofton's formula.
We first recall a result due to Cramér-Leadbetter, main result of Section 10.3 of [3]. The expected number of crossings $N_u([0,T]$ of a stationary processes with any level u on an interval $[0,T]$ is finite if and only if $\lambda_2 < \infty$, where λ_2 denotes the second spectral moment.

In case λ_2 is finite we have furthermore

$$\mathbb{E}(N_u([0,T]) = \frac{T}{\pi}\sqrt{\lambda_2}e^{-u^2/2}.$$

This result is based on polygonal approximation and intermediate values theorem, so it heavily relies on one-dimensional settings.

We now turn to our first main result.

Theorem 3.1. *Let $X(t)$ be a centered, stationary random field $X : \mathbb{R}^d \to \mathbb{R}$, with continuous sample paths. Then, for $u \in \mathbb{R}$, K compact set of $\mathbb{R}^d$ and $C_{u,K}$ defined by* (1.1) *we have equivalence between:*

- $\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K})) < \infty$,
- *The second spectral moment matrix* Λ_2 *is finite.*

In such a case we have

$$\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K})) = \mathcal{L}_d(K)\mathcal{F}(\Lambda_2)e^{-u^2/2}$$

Proof. Since X is almost surely continuous, then $C_{u,K}$ is a Borel set on $\mathbb{R}^d$ a.s., and therefore its integral geometric measure is well defined. By Fubini theorem we get that

$$\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K})) = c_{d-1}\int_{v\in S^{d-1}}\left(\int_{y\in v^\perp}\mathbb{E}(\#\{C_{u,K}\cap \ell_{v,y}\})\,d\mathcal{H}_{d-1}(y)\right)\,dS^{d-1}(v)$$

As a matter of fact, because of stationarity of the process and by Cramér-Leadbetter applied to the process $t\mapsto X(y+tv)$, we get

$$\mathbb{E}(\#\{C_{u,K}\cap \ell_{v,y}\}) = \mathcal{H}_1(K\cap \ell_{v,y})\sqrt{v^\top\Lambda_2 v}\,\frac{1}{\pi}e^{-u^2/2}.$$

Then, $\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K}))$ is equal to

$$e^{-u^2/2}\frac{c_{d-1}}{\pi}\cdot\int_{v\in S^{d-1}}\sqrt{v^\top\Lambda_2 v}\,\left(\int_{y\in v^\perp}\mathcal{H}_1(K\cap\ell_{v,y})\,d\mathcal{H}_{d-1}(y)\right)\,dS^{d-1}(v)$$

and by Fubini, we obtain that

$$\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K})) = \mathcal{L}_d(K)e^{-u^2/2}\frac{c_{d-1}}{\pi}\int_{S^{d-1}}\sqrt{v^\top\Lambda_2 v}\,dS^{d-1}(v). \tag{3.4}$$

Integrating in polar coordinates the expression $\mathcal{F}(\Lambda_2)$, given in (3.2), we obtain

$$\mathcal{F}(\Lambda_2) = \frac{1}{(2\pi)^{(d+1)/2}}\int_0^{+\infty}\rho^d e^{-\rho^2/2}\,d\rho\int_{v\in S^{d-1}}(v^\top\Lambda_2 v)^{1/2}dS^{d-1}(v).$$

Furthermore, making the change of variable $u=\rho^2/2$, it is straighforward to conclude that

$$\mathcal{F}(\Lambda_2) = \frac{c_{d-1}}{\pi}\int_{S^{d-1}}\sqrt{v^\top\Lambda_2 v}\,dS^{d-1}(v), \tag{3.5}$$

and therefore from (3.4) yields

$$\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K})) = \mathcal{L}_d(K)e^{-u^2/2}\mathcal{F}(\Lambda_2). \tag{3.6}$$

We consider the two following cases.

- When Λ_2 is finite the integral on the RHS of (3.6) is finite and therefore we get the desired result in this case.
- When Λ_2 is infinite, we define the linear subspace

  $$G(\Lambda_2) := \{v\in\mathbb{R}^d : v^\top\Lambda_2 v < +\infty\}.$$

  We prove that $G(\Lambda_2)$ is of dimension strictly smaller than d. Let $v_1,\dots,v_{d_0}$ be a maximal set of linearly independent vectors of $G(\Lambda_2)$. Then by standard linear algebra:

  – the space $\mathrm{span}(v_1,\dots,v_{d_0})$ generated by $v_1,\dots,v_{d_0}$ is in $G(\Lambda_2)$. This implies that $d_0<d$,
  – for every $v\notin\mathrm{span}(v_1,\dots,v_{d_0})$: $v^\top\Lambda_2 v = +\infty$ (unless $v_1,\dots,v_{d_0}$ is not maximal),
  – this implies that $G(\Lambda_2)=\mathrm{span}(v_1,\dots,v_{d_0})$.

  In conclusion the integrand in (3.6) is almost everywhere infinite so the integral is infinite and by consequence the expectation of the integral geometric measure is infinite. □

4 Finitness of k-moments of the volume of the level set

Using Formula (2.1) it is possible to obtain sufficient conditions under which the random variable $\mathcal{I}_{d-1}(B)$ has finite moments. To illustrate this we will first consider the second moment. Thus we have the following.

Theorem 4.1. *Assume that*

- *The second spectral moment matrix Λ_2 is non-degenerate.*
- *There exists $\delta > 0$ such that the spectral mesure F satisfies*

$$\int_{\mathbb{R}^d} ||\lambda||^{2+\delta} dF(\lambda) < \infty.$$

Then we have

$$\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K}))^2 < \infty.$$

Remark 4.2. Let us point out that under the assumption that $\int_{\mathbb{R}^d} ||\lambda||^{2+\delta} dF(\lambda)$ is finite, the Kolmogorov-Chentsov criterion implies that the field X has a.s. $\mathcal{C}^1$ sample paths. Thus, the Riemannian volume of $C_{u,K}$ can be defined, and by (2.2), coincides with its $(d-1)$-Hausdorff measure. Thus the following equality takes place
$\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K}))^2 = \mathbb{E}(\mathcal{H}_{d-1}(C_{u,K}))^2$.

Proof. Without loss of generality we assume that $\delta < 2$. Let r be the covariance function of X, and let us first consider the field restricted to the line $\ell_{y,v} : y + tv, t \in \mathbb{R}$: $\tilde{X}_{y,v}(t) = X(y + tv)$. Its covariance function is given by

$$r_v(t) = \mathbb{E}[X(y + tv)X(y)] = r(tv).$$

Note that because of stationarity it does not depend on y.

It is sufficient to prove the assertion of the theorem for a set K being a centred ball B_a with sufficiently small diameter a. In that case note that the integral in the right-hand side of (2.1), for $B = C_{u,K}$, is finite since the regularity of the sample path. Since the second spectral moment matrix is finite and non degenerate

$$\int_{\mathbb{R}^d} \langle \lambda, v\rangle^2 dF(\lambda),$$

is bounded below and above. On the other hand, using a monotone convergence argument, as b tends to infinity

$$\int_{\mathbb{R}^d \setminus B_b} \langle \lambda, v\rangle^2 dF(\lambda) \le \int_{\mathbb{R}^d \setminus B_b} ||\lambda||^2 dF(\lambda) \to 0. \tag{4.1}$$

So it is easy to conclude that for b sufficiently large, for any $v \in S^{d-1}$

$$\int_{B_b} \langle \lambda, v\rangle^2 dF(\lambda) > 1/2 \int_{\mathbb{R}^d} \langle \lambda, v\rangle^2 dF(\lambda). \tag{4.2}$$

In the rest of the paper $\mathbf{C}$ will denote some unimportant constant, its value may change from an occurence to another.

Applying the Jensen inequality (with respect to the integral) yields

$$\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K}))^2 \le \mathbf{C}\, c_{d-1}^2 \int_{v \in S^{d-1}} \int_{y \in v^\perp} \mathbb{E}(\#\{C_{u,K} \cap \ell_{y,v}\})^2 \, d\mathcal{H}_{d-1}(y) \, dS^{d-1}(v).$$

As already remarked, the integral is over a bounded domain and it is sufficient to prove that the integrand is uniformly bounded.

Remark also that $B_a \cap \ell_{y,v}$ is always a centred interval with length $2c$ less that $2a$. Consequently,

$$\mathbb{E}(\#\{C_{u,K} \cap \ell_{y,v}\})^2 \leq \mathbb{E}(\#\{C_{u,K} \cap \ell_{0,v}\})^2.$$

It remains to prove that

$$\mathbb{E}(\#\{C_{u,K} \cap \ell_{0,v}\})^2$$

is uniformly bounded.

In fact, because of the Rolle theorem, if U_u is the number of up-crossings of the level u on the line $\ell_{0,v}$, then

$$\#\{C_{u,K} \cap \ell_{0,v}\} \leq 2U_u + 1.$$

So it is sufficient to bound the second moment of U_u and even, because we have proved in the previous section that the first moment is uniformly bounded, it is sufficient to bound the second factorial moment. Since the variance has been assumed to be 1 and Λ_2 is non-degenerate, the Kac-Rice formula applies and yields

$$\begin{aligned}
&\mathbb{E}(U_u(U_u-1)) \\
&\leq \int_{-a}^{a}\int_{-a}^{a} \mathbb{E}\big(X'^{+}(s)X'^{+}(t)\big|X(s)=X(t)=u\big)\frac{1}{2\pi}\frac{1}{\sqrt{1-r_v^2(s-t)}}dsdt \\
&= \mathbf{C}\int_0^{2a}(2a-\tau)\mathbb{E}\big(X'^{+}(0)X'^{+}(\tau)\big|X(0)=X(\tau)=u\big)\frac{1}{\sqrt{1-r_v^2(\tau)}}d\tau,
\end{aligned}$$

where X stands for $\tilde{X}_{0,v}$.

By a standard regression formula, see for example [2] page 99,

$$\mathbb{E}(X'^{+}(0)\big|X(0)=X(\tau)=u) = -\mathbb{E}(X'^{+}(\tau)\big|X(0)=X(\tau)=u) = \frac{-r_v'(\tau)u}{1+r_v(\tau)}.$$

Also,

$$\begin{aligned}
\sigma_v^2(\tau) :&= \mathrm{Var}(X'(0)|X(0)=X(\tau)=u) \\
&= \mathrm{Var}(X'(\tau)|X(0)=X(\tau)=u) = \frac{\lambda_{2,v}(1-r_v(\tau)) - r_v'^2(\tau)}{1-r_v^2(\tau)}.
\end{aligned}$$

Set $\theta_v(\tau) := r_v(\tau) - 1 + \lambda_{2,v}\tau^2/2$, using the inequality $z^+t^+ \leq (z+t)^2/4$ and the fact that $\theta_v(\tau)$, $\theta_v'(\tau)$, $\theta_v''(\tau)$ are non-negative, we get

$$\mathbb{E}(U_u(U_u-1)) \leq \mathbf{C}a\int_0^{2a} 2\lambda_{2,v}\tau\theta_v'(\tau)\big(1-r_v^2(\tau)\big)^{-3/2}.$$

Now, there exists a constant $\mathbf{C_0}$ such that

$$0 < w < 1 \text{ implies that } 1-\cos(w) \geq \mathbf{C_0}w^2.$$

This implies in turn that for $\tau < 1/b$ where b has been defined in (4.2)

$$\begin{aligned}
1 - r_v(\tau) &= \int_0^{+\infty}(1-\cos(\lambda\tau))\,dF_v(\lambda) \\
&\geq \int_0^{1/\tau}(1-\cos(\lambda\tau))\,dF_v(\lambda) \\
&\geq \mathbf{C_0}\tau^2\int_0^{1/\tau}\lambda^2 dF_v(\lambda) \geq \frac{1}{2}\mathbf{C_0}\tau^2\int_0^{+\infty}\lambda^2 dF_v(\lambda) \geq \mathbf{C}\tau^2,
\end{aligned}$$

where F_v is the spectral measure along the line $\ell_{0,v}$ (for convenience it is on $(0, +\infty)$). The penultimate inequality uses (4.2), the last inequality is due to the fact that Λ_2 is non-degenerate.

On the other hand it is direct to prove that $1 - r_v(\tau) \leq \lambda_{2,v}\tau^2$ and, by compactness, the quantity $\lambda_{2,v}$ is bounded as a function of v giving that $1 + r_v(\tau) \geq 1$ as soon as the radius a of the ball is sufficiently small. This yields

$$1 - r_v^2(\tau) \geq \mathbf{C}\tau^2.$$

As a consequence

$$\mathbb{E}(U_u(U_u - 1)) \leq \mathbf{C}\lambda_{2,v} a \int_0^{2a} 2\frac{\theta_v'(\tau)}{\tau^2} d\tau. \tag{4.3}$$

The integrand in (4.3) can be bounded because

$$\int_0^\infty \lambda^{2+\delta} dF_v(\lambda) = \int_{\mathbb{R}^d} \langle v, \lambda \rangle^{2+\delta} dF(\lambda) \leq I(\delta) := \int_{\mathbb{R}^d} ||\lambda||^{2+\delta} dF(\lambda) < \infty.$$

We have

$$\frac{\theta_v'(\tau)}{\tau^2} = \tau^{-2} \int_0^\infty (\tau\lambda_2 - \lambda \sin(\lambda\tau)) dF_v(\lambda).$$

Define

$$R(u) := (u - \sin(u)).$$

Its behaviour at zero and at infinity implies that for every δ, $0 < \delta < 2$, there exist a constant $\mathcal{C}_\delta$ such that

$$0 \leq R(u) \leq \mathcal{C}_\delta u^{1+\delta}.$$

This implies that

$$\begin{aligned}
\left|\frac{\theta_v'(\tau)}{\tau^2}\right| &\leq \tau^{-2} \int_0^\infty \lambda |R(\lambda\tau)| dF_v(\lambda) \\
&\leq \mathcal{C}_\delta \tau^{-2} \int_0^\infty \lambda (\lambda\tau)^{1+\delta} dF_v(\lambda) \\
&\leq \mathcal{C}_\delta \tau^{\delta - 1} \int_0^\infty \lambda^{2+\delta} dF_v(\lambda),
\end{aligned}$$

resulting in the convergence of the integral in (4.3), uniformly in v. □

Next, we consider a Gaussian field having C^∞ sample paths. This is for instance the case of Gaussian random trigonometric polynomials in several variables or the random plane wave model [8]. A result of Nualart-Wschebor, quoted as Theorem 3.6 in the book [2], can be used for obtaining that all the moments of the random variable $\mathcal{I}_{d-1}(C_{u,K})$ are finite. The background result is the following:

Proposition 4.3. *Consider a real Gaussian process Y over $\mathbb{R}$ satisfying $\mathrm{Var}(Y(t)) > \kappa$ for all $t \in I$, a compact interval of $\mathbb{R}$, and some $\kappa > 0$. Then for all $u \in \mathbb{R}$, and $m, p \in \mathbb{N}$ such that $p > 2m$, it holds*

$$\mathbb{E}[(N_u)^m] \leq C_{p,m}\big[1 + C + \mathbb{E}\big(\|Y^{(p+1)}\|_\infty\big)\big] \tag{4.4}$$

where N_u is the number of points $t \in I$ such that $\chi(t) = u$, $C_{p,m}$ is a constant depending only on p, m and the length of the interval I, and C is a bound for the density of $Y(t)$.

Let us assume now that the field X has C^∞ sample paths and we assume that the variance is bounded below. As in the proof on Theorem 4.1 the process $\tilde{X}_{y,v}(t) = X(y+tv)$ is a real process but now with C^∞ trajectories. Chose $p = 2m+1$ then from Proposition 4.3, for every m,

$$\mathbb{E}(\#\{C_{u,K} \cap \ell_{v,y}\})^m < C_{p,m}\big[1 + C + \mathbb{E}\big(\|X_{y,v}^{(2m+2)}\|_\infty\big)\big].$$

It is an easy consequence of the Borel-Sudakov-Tsirelson inequality that $\mathbb{E}\big(\|X_{y,v}^{(2m+2)}\|_\infty\big)$ is finite. An argument of continuity shows that it is uniformly bounded. A further application of Jensen's inequality gives our third main result

Theorem 4.4. *Let $X(t)$ be a real valued Gaussian random field on $\mathbb{R}^d$ with C^∞ sample paths and with variance bounded below. Then for every integer m and every compact set K,*

$$\mathbb{E}(\mathcal{I}_{d-1}(C_{u,K}))^m < \infty.$$

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Acknowledgments. This work has been partially supported by the French National Research Agency (ANR) through project PEPITO (no ANR-14-CE23-0011).
Partially supported by Agencia Nacional de Investigación e Innovación (ANII), Uruguay, and by CSIC I+D proyect 409.
David Ginsbourger would like to thank Andrew Stuart for a stimulating discussion in Cambridge that has incented the present collaboration, and in turn to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme "Uncertainty quantification for complex systems: theory and methodologies" (supported by EPSRC grant no EP/K032208/1).