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Research Article

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Multivariate risk measures in the non-convex setting

Abstract: The family of admissible positions in a transaction costs model is a random closed set, which is convex in case of proportional transaction costs. However, the convexity fails, e.g., in case of fixed transaction costs or when only a finite number of transfers are possible. The paper presents an approach to measure risks of such positions based on the idea of considering all selections of the portfolio and checking if one of them is acceptable. Properties and basic examples of risk measures of non-convex portfolios are presented.

Keywords: Fixed transaction costs, multivariate risk measure, set-valued risk, selection, random set

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1 Introduction

Multivariate financial positions (portfolios) are usually described by vectors in a Euclidean space. However, if one aims to take into account possible exchanges between the components of the portfolio, it is necessary to consider the whole set of points in space that may be attained from the original position by allowed exchanges. In other words, considering a multiasset portfolio is indispensable from specifying which transactions may be applied to its components. For instance, if all components of the portfolio $C = (C^{(1)}, \dots, C^{(d)})$ represent cash amounts in the same currency and transfers between the components are unrestricted with short-selling permitted, then the attainable positions are all random vectors such that the sum of their components equals the sum of components of C. By allowing disposal of assets (e.g., in the form of consumption), we arrive at the half-space

$$\left\{ \mathbf{x} \in \mathbb{R}^d : \sum_{i=1}^d x^{(i)} \le \sum_{i=1}^d C^{(i)} \right\}.$$

In this case and also in the presence of transaction costs not influenced by C, the attainable positions are points from $C + \mathbf{K}$, where $\mathbf{K}$ is the set of portfolios available at price zero, see [7]. In other situations, possible attainable positions may depend on C in a non-linear way, for instance, when components represent capitals of members of a group and admissible transfers satisfy further restrictions, e.g., requiring that they do not cause insolvency of an otherwise solvent agent, see [3].

In view of the above reasons, it is natural to represent multiasset portfolios as random closed sets. Recall that a *random closed set* $\mathbf{X}$ is a measurable map from a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ to the space of closed sets in $\mathbb{R}^d$ equipped with the σ-algebra generated by the Fell topology. In other words, the measurability of $\mathbf{X}$ means that $\{\omega : \mathbf{X}(\omega) \cap K \ne \emptyset\} \in \mathfrak{F}$ for all compact sets K in $\mathbb{R}^d$, see [11, Section 1.1.1].

A random closed set $\mathbf{X}$ is said to be *lower* if almost all its realizations are lower sets, that is, for almost all ω, $x \in \mathbf{X}(\omega)$ and $y \le x$ coordinatewisely imply that $y \in \mathbf{X}(\omega)$. A random closed set is said to be *convex* if

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almost all its realizations are convex. If $\mathbf{X}$ is a random closed set, then its closed convex hull $\overline{\mathrm{conv}}(\mathbf{X})$ is also a random closed set, see [11, Theorem 1.3.25].

For $p \in [1, \infty]$, denote by $\boldsymbol{L}^p(\mathbf{X})$ the family of p-integrable (essentially bounded if $p = \infty$) random vectors ξ such that $\xi \in \mathbf{X}$ a.s.; such random vectors are called *p-integrable selections* of $\mathbf{X}$. Furthermore, $\boldsymbol{L}^0(\mathbf{X})$ is the family of all selections of $\mathbf{X}$; this family is not empty if $\mathbf{X}$ is a.s. non-empty, see [11, Theorem 1.4.1]. A random closed set $\mathbf{X}$ is called *p-integrable* if it admits at least one p-integrable selection; it is called *p-integrably bounded* if

$$\|\mathbf{X}\| = \sup\{\|x\| : x \in \mathbf{X}\}$$

is a p-integrable random variable for $p \in [1, \infty)$. The random closed set $\mathbf{X}$ is said to be essentially bounded if $\|\mathbf{X}\|$ is a.s. bounded by a constant.

If $\mathbf{X}$ is integrable (that is, 1-integrable), its *selection expectation* is defined by

$$\mathbf{EX} = \mathrm{cl}\{\mathbf{E}\xi : \xi \in \boldsymbol{L}^1(\mathbf{X})\},$$

where $\mathrm{cl}(\cdot)$ denotes the topological closure in $\mathbb{R}^d$. The closed *Minkowski sum*

$$\mathbf{X} + \mathbf{Y} = \mathrm{cl}\{x + y : x \in \mathbf{X}, \, y \in \mathbf{Y}\}$$

of two random closed sets $\mathbf{X}$ and $\mathbf{Y}$ is also a random closed set. Note that

$$-\mathbf{X} = \{-x : x \in \mathbf{X}\}$$

denotes the reflection of $\mathbf{X}$ with respect to the origin; this is not the inverse operation to the addition. We refer to [11] for further material concerning random closed sets.

The paper is organised as follows. In Section 2 we introduce the *selection risk measure* of possibly non-convex random lower closed sets, thereby generalising the setting of [3] and [12]. Due to the non-convexity, it is not possible to assess the risk by working with half-spaces containing the portfolio, as it is the case in [4, 5]. In Section 3 we discuss two basic set-valued risk measures, one based on considering the fixed points of set-valued portfolio, the other is given by the selection expectation of $-\mathbf{X}$. These two cases correspond to taking the negative essential infimum and the negative expectation as the underlying numerical risk measures. Section 4 explores the cases when the selection risk measure takes convex values and is law invariant. The important case of fixed transaction costs is considered in Section 5. Finally, Section 6 deals with the case of only a finite set of admissible transactions.

2 Selection risk measure of non-convex portfolios

2.1 Definition

Fix $p \in \{0\} \cup [1, \infty]$ and a vector $\mathbf{r}(\xi) = (r_1(\xi^{(1)}), \dots, r_d(\xi^{(d)}))$ of monetary $\boldsymbol{L}^p$-risk measures applied to components of a p-integrable random vector $\xi = (\xi^{(1)}, \dots, \xi^{(d)})$. We refer to [1] and [2] for the facts concerning risk measures for random variables. Assume that $\mathbf{r}(0) = 0$ and that all components of $\mathbf{r}$ are finite on p-integrable random variables. When saying that $\mathbf{r}$ is coherent or convex, we mean that all its components are coherent or convex. The convexity or coherency properties will be imposed only when necessary and will be explicitly mentioned.

In many cases below, we consider the following basic numerical risk measures:

1. The negative essential infimum $r(\xi) = -\operatorname{ess\,inf} \xi$, which is an $\boldsymbol{L}^\infty$-risk measure.
2. The negative expectation $r(\xi) = -\mathbf{E}\xi$, an $\boldsymbol{L}^1$-risk measure.
3. The average value-at-risk (or expected shortfall in the non-atomic case)

$$r(\xi) = -\frac{1}{\alpha} \int_0^\alpha F_\xi^{-1}(t) \, dt$$

at level $\alpha \in (0, 1]$ for $\xi \in \boldsymbol{L}^1(\mathbb{R})$, where F_ξ is the cumulative distribution function of ξ and F_ξ^{-1} is the quantile function.

4. The distortion risk measure

$$r(\xi) = -\int_0^1 F_\xi^{-1}(t)\, d\tilde{g}(t) \tag{2.1}$$

for $\xi \in L^p(\mathbb{R})$, where $g\colon [0, 1] \mapsto [0, 1]$ is a (concave) distortion function, $\tilde{g}(t) = 1 - g(1 - t)$ is the dual distortion function, and g is chosen to ensure that the integral is finite.

The *selection risk measure* of a p-integrable lower random closed set $\mathbf{X}$ is defined as

$$R(\mathbf{X}) = \mathrm{cl} \bigcup_{\xi \in L^p(\mathbf{X})} (\mathbf{r}(\xi) + \mathbb{R}_+^d), \tag{2.2}$$

where the union is taken over all p-integrable selections of $\mathbf{X}$. Thus, $x \in R(\mathbf{X})$ if and only if $\liminf \mathbf{r}(\xi_n) \le x$ for $\xi_n \in L^p(\mathbf{X})$, $n \ge 1$. The inequalities between vectors are always coordinatewise and the lower limit is also taken coordinatewisely. The selection risk measure takes values being upper sets, and (2.2) can be seen as the primal representation of $R(\mathbf{X})$. A dual representation is not feasible without imposing convexity on $\mathbf{X}$.

A random set $\mathbf{X}$ is said to be acceptable if $0 \in R(\mathbf{X})$. In other words, $\mathbf{X}$ is acceptable if $\mathbf{X}$ contains a sequence of selections whose risk converges to zero. The monetary property of $\mathbf{r}$ yields that $R(\mathbf{X})$ is the set of all $x \in \mathbb{R}^d$ such that $\mathbf{X} + x$ is acceptable, that is,

$$R(\mathbf{X}) = \{x : R(\mathbf{X} + x) \ni 0\}.$$

2.2 Properties of the selection risk measure

The selection risk measure was introduced in [12] for convex $\mathbf{X}$ and coherent $\mathbf{r}$. Some of its properties for non-convex $\mathbf{X}$ and general monetary $\mathbf{r}$ are easy-to-show replica of those known in the convex coherent setting adopted in [12].

Theorem 2.1. *The selection risk measure satisfies the following properties for p-integrable random lower closed sets $\mathbf{X}$ and $\mathbf{Y}$:*
(i) *Monotonicity, that is, $R(\mathbf{X}) \subseteq R(\mathbf{Y})$ if $\mathbf{X} \subseteq \mathbf{Y}$ a.s.*
(ii) *Cash-invariance, that is, $R(\mathbf{X} + a) = R(\mathbf{X}) - a$ for all deterministic $a \in \mathbb{R}^d$.*
(iii) *If $\mathbf{r}$ is homogeneous, then R is homogeneous, that is, $R(c\mathbf{X}) = cR(\mathbf{X})$ for all deterministic $c > 0$.*
(iv) *If $\mathbf{r}$ is convex, then R is convex, that is,*

$$R(\lambda \mathbf{X} + (1 - \lambda)\mathbf{Y}) \supseteq \lambda R(\mathbf{X}) + (1 - \lambda)R(\mathbf{Y}) \tag{2.3}$$

for all deterministic $\lambda \in [0, 1]$.

Proof. We prove only the convexity, the rest is straightforward. All elements of the set on the right-hand side of (2.3) are coordinatewisely larger than or equal to

$$\liminf(\lambda \mathbf{r}(\xi_n) + (1 - \lambda)\mathbf{r}(\eta_n))$$

for $\xi_n \in L^p(\mathbf{X})$ and $\eta_n \in L^p(\mathbf{Y})$, $n \ge 1$. Then it suffices to note that this convex combination of risks of ξ and η dominates $\mathbf{r}(\lambda \xi_n + (1 - \lambda)\eta_n)$, which is an element of the left-hand side of (2.3). □

The monotonicity property of $\mathbf{r}$ yields that $R(C + \mathbb{R}_-^d) = \mathbf{r}(C) + \mathbb{R}_+^d$ for $C \in L^p(\mathbb{R}^d)$. The selection risk measure is said to be *coherent* if it is homogeneous and convex; this is the case if $\mathbf{r}$ has all coherent components. If $\mathbf{r}$ is coherent, C is a p-integrable random vector, and $\mathbf{X}$ is a p-integrable random lower closed set, then

$$R(C + \mathbf{X}) \supseteq \mathbf{r}(C) + R(\mathbf{X}). \tag{2.4}$$

This is easily seen from (2.3), by choosing $\lambda = 1/2$, $\mathbf{Y} = C + \mathbb{R}_-^d$, and using the homogeneity of $\mathbf{r}$. Note that the equality in (2.4) is not guaranteed even if $\mathbf{X}$ is a deterministic set. Still, in this case, it provides a useful acceptability condition: $C + \mathbf{X}$ is acceptable if $\mathbf{r}(C) + R(\mathbf{X}) \ni 0$.

A general set-valued function (not necessarily constructed using selections) defined for p-integrable random sets is said to be monotonic, cash-invariant, homogeneous or convex if it satisfies the corresponding properties from Theorem 2.1. The set-valued (selection) risk measure is called *law invariant* if its values on identically distributed random sets coincide.

2.3 Choice of selections

The definition of the selection risk measure involves taking union over all p-integrable selections of $\mathbf{X}$. This family may be very rich even for simple random closed sets. In the following, we discuss general approaches suitable to reduce the family of selections needed to determine the selection risk measure.

With a lower closed set F, we associate the set $\partial^+ F$ of its *Pareto optimal* points, that is, the set of points $x \in F$ such that $y \geq x$ for $y \in F$ is only possible if $y = x$. If $\mathbf{X}$ is a random lower closed convex set, then the set $\partial^+ \mathbf{X}$ of Pareto optimal points of $\mathbf{X}$ is a random closed set, see [3, Lemma 3.1]. In the non-convex case, the cited result establishes that $\partial^+ \mathbf{X}$ is graph measurable, so that its closure $\mathrm{cl}\, \partial^+ \mathbf{X}$ is a random closed set, see [10, Proposition 2.6]. If $\partial^+ \mathbf{X}$ is closed and p-integrable, then it is possible to reduce the union in (2.2) to selections of $\partial^+ \mathbf{X}$.

A lower random closed set $\mathbf{X}$ is said to be *quasi-bounded* if $\partial^+ \mathbf{X}$ is essentially bounded; $\mathbf{X}$ is p-integrably quasi-bounded if $\|\partial^+ \mathbf{X}\|$ is p-integrable.

Consider

$$\mathbf{X} = F_1 \cup \cdots \cup F_m, \tag{2.5}$$

where $F_1, \ldots, F_m$ are deterministic lower *convex closed cones*. For the following result, assume that $\mathbf{r}$ is convex law invariant, and the probability space is non-atomic. In this case, $\mathbf{r}$ satisfies the dilatation monotonicity property, that is, $\mathbf{r}(\xi)$ dominates coordinatewisely the risk of a conditional expectation of ξ, see [2, Corollary 4.59] and [9].

Proposition 2.2. *If $\mathbf{X}$ is the deterministic set given by* (2.5)*, then it is possible to reduce the union in* (2.2) *to selections $\xi = \sum_{i=1}^m x_i \mathbf{1}_{A_i}$ for deterministic $x_i \in F_i$, $i = 1, \ldots, m$, and partitions $\mathcal{A} = \{A_1, \ldots, A_m\}$ of the probability space.*

Proof. Consider $\xi = \sum \eta_i \mathbf{1}_{A_i}$ for $\eta_i \in L^p(F_i)$, $i = 1, \ldots, m$. By the dilatation monotonicity, $\mathbf{r}(\xi)$ dominates the risk of the conditional expectation of ξ given $\mathcal{A}$. Thus, it is possible to replace η_i by its conditional expectation, which is also a point in F_i. $\square$

In the convex setting, if $\mathbf{X}$ is the sum of C and a convex closed set F, then the union in (2.2) can be reduced to the selections that are measurable with respect to the σ-algebra generated by C.

3 Fixed points and the expectation

For a random closed set $\mathbf{X}$,

$$F_{\mathbf{X}} = \{x : \mathbf{P}\{x \in \mathbf{X}\} = 1\}$$

denotes the set of its *fixed points*. The set $F_{\mathbf{X}}$ is a lower closed set if $\mathbf{X}$ is a lower closed set, it is convex if $\mathbf{X}$ is convex.

Proposition 3.1. *Let $\mathbf{X}$ be a p-integrable random lower closed set. For the selection risk measure generated by any monetary risk measure $\mathbf{r}$, we have*

$$-F_{\mathbf{X}} \subseteq \mathsf{R}(\mathbf{X}). \tag{3.1}$$

If all components of $\mathbf{r}$ are the negative of the essential infimum, then $\mathsf{R}(\mathbf{X})$ equals the set of fixed points of $-\mathbf{X}$.

Proof. By taking constant selections $\xi = x \in F_{\mathbf{X}}$ in (2.2) and using the fact that $\mathbf{r}(x) = -x$, we see that (3.1) holds.

If $L^\infty(\mathbf{X}) \neq \emptyset$, then $F_\mathbf{X} \neq \emptyset$, since $\mathbf{X}$ is a lower set. Choosing $\mathbf{r}$ with all components being negative of the essential infima, it is easily seen that $\mathbf{X}$ is acceptable if it has a selection with all a.s. non-negative components. In this case, $0 \in \mathbf{X}$ a.s., whence $0 \in F_\mathbf{X}$. Note also that $F_{-\mathbf{X}} = -F_\mathbf{X}$. □

The set of fixed points is a coherent selection risk measure, which is law invariant and not necessarily convex-valued.

Example 3.2. The convex hull of $F_\mathbf{X}$ is a (possibly, strict) subset of the set of fixed points of $\overline{\mathrm{conv}}(\mathbf{X})$. Let $\mathbf{X}$ be a random closed set in $\mathbb{R}^2$ which equally likely takes values $\{(-a, a), (a, -a)\} + \mathbb{R}^2_-$ and $\{(-b, b), (b, -b)\} + \mathbb{R}^2_-$ for $0 < a < b$. Then $F_\mathbf{X} = \{(-b, a), (a, -b)\} + \mathbb{R}^2_-$, while the set of fixed points of $\overline{\mathrm{conv}}(\mathbf{X})$ is the sum of the segment with end-points $(-a, a), (a, -a)$ and $\mathbb{R}^2_-$.

Example 3.3. The set of fixed points appears also in the following context. Let $\Omega = \{\omega_1, \dots, \omega_n\}$ be a finite probability space, and let all components of $\mathbf{r}$ be the average value-at-risk at level $\alpha \leq \mathbf{P}(\{\omega_i\})$, $i = 1, \dots, n$. Then

$$R(\mathbf{X}) = -F_\mathbf{X} = -\bigcap_{i=1}^n \mathbf{X}(\omega_i).$$

Indeed, since $\mathbf{P}(\{\omega_i\}) \geq \alpha$ for all i, we have $r(\xi^{(j)}) = -\min\{\xi^{(j)}(\omega_i), i = 1, \dots, n\}$ for any $\xi \in L^1(\mathbf{X})$. Because each $\mathbf{X}(\omega_i)$ is a lower set, we have $-\mathbf{r}(\xi) \in \mathbf{X}(\omega_i)$ for all i. To show the reverse inclusion, assume that $x \in F_\mathbf{X}$. Then $\xi = x$ is a deterministic selection of $\mathbf{X}$, whence $-x = \mathbf{r}(\xi) \in R(\mathbf{X})$. This also follows from Proposition 3.1.

If $p = 1$ and $\mathbf{r}(\xi) = -\mathbf{E}\xi$ is the negative expectation of ξ, then $R(\mathbf{X})$ becomes the selection expectation of $(-\mathbf{X})$. Note that $R(\mathbf{X}) = -\mathbf{E}\mathbf{X}$ is a coherent selection risk measure, which is law invariant on convex random sets, but may be not law invariant on non-convex ones. Indeed, if the non-convex deterministic set F is considered a random closed set defined on the trivial probability space, then $\mathbf{E}F = F$, while $\mathbf{E}F = \overline{\mathrm{conv}}(F)$ if the underlying probability space is non-atomic, see [11, Theorem 2.1.26].

It might be tempting to define a set-valued risk measure by taking intersection of expected random sets with respect to varying probability measures. This would correspond to the construction of a convex function by taking the supremum of linear ones. However, taking expectation results in convex values for the risk measure if the probability space is non-atomic; otherwise, it depends on the atomic structure of the space. Furthermore, even in the convex setting, such a construction might not correspond to the existence of an acceptable selection from $\mathbf{X}$, as the following remark shows.

Remark 3.4. For any family $\mathcal{Z} \subset L^q(\mathbb{R}^d_+)$ such that $\mathbf{E}\zeta = 1$ for all $\zeta \in \mathcal{Z}$, define

$$R_\mathcal{Z}(\mathbf{X}) = \bigcap_{\zeta \in \mathcal{Z}} \mathbf{E}(-\zeta\mathbf{X}),$$

where $\zeta\mathbf{X} = \{\zeta x : x \in \mathbf{X}\}$. Note that we use vector notation, e.g., $\mathbf{E}\zeta = 1$ means that all components of ζ have mean 1, and

$$\zeta\xi = (\zeta^{(1)}\xi^{(1)}, \dots, \zeta^{(d)}\xi^{(d)})$$

is the coordinatewise product of ζ and ξ. The so-defined $R_\mathcal{Z}(\cdot)$ satisfies all properties from Theorem 2.1. However, $R_\mathcal{Z}$ in general is not a selection risk measure. Indeed, by letting $\mathbf{X} = \xi + \mathbb{R}^d_-$, we see that the corresponding coherent vector-valued risk measure is given by

$$\mathbf{r}(\xi) = \sup_{\zeta \in \mathcal{Z}} \mathbf{E}(-\zeta\xi), \quad \xi \in L^p(\mathbb{R}^d).$$

Assume that $\partial^+\mathbf{X}$ is p-integrably bounded, so that $\mathbf{E}(\zeta\mathbf{X})$ is closed for all $\zeta \in L^q$. Then $0 \in R_\mathcal{Z}(\mathbf{X})$ if and only if $0 \in \mathbf{E}(-\zeta\mathbf{X})$ for all $\zeta \in \mathcal{Z}$, equivalently, for each $\zeta \in \mathcal{Z}$ there is $\xi_\zeta \in L^p(\partial^+\mathbf{X})$ such that $\mathbf{E}(-\zeta\xi_\zeta) \leq 0$. Since these selections ξ_ζ may be different for different ζ, we cannot infer that $\mathbf{X}$ is acceptable with respect to a selection risk measure. Indeed, the acceptability of $\mathbf{X}$ requires the existence of a *single* selection $\xi \in L^p(\mathbf{X})$ such that $\mathbf{E}(-\zeta\xi) \leq 0$ for all ζ. Thus, $R_\mathcal{Z}$ is an example of a coherent set-valued risk measure, which, however, is not necessarily a selection one. The acceptability of $\mathbf{X}$ under $R_\mathcal{Z}$ does not guarantee the existence of an acceptable selection of $\mathbf{X}$. Furthermore, this risk measure does not distinguish between $\mathbf{X}$ and its convex hull.

4 Convexity and law invariance

The monotonicity property yields that $R(\mathbf{X})$ is a subset of $R(\overline{\mathrm{conv}}(\mathbf{X}))$. It is well known that the selection expectation of an integrable random closed set is convex if the underlying probability space is non-atomic, see [11, Theorem 2.1.26]. This result follows from Lyapunov's theorem on ranges of vector-valued measures. The same holds for selection risk measures of convex random sets, if the underlying risk measure $\mathbf{r}$ is convex, see [12, Theorem 3.4]. This is however not the case for non-convex arguments, see Example 3.2 and Section 5.2.

Still, in some cases, $R(\mathbf{X})$ is convex even for non-convex $\mathbf{X}$. Assume that $p \in [1, \infty]$, and the components of $\mathbf{r} = (r_1, \ldots, r_d)$ are $\sigma(\mathbf{L}^p, \mathbf{L}^q)$-lower semicontinuous convex risk measures, so that

$$r_i(\xi) = \sup_{\zeta \in \mathbf{L}^q(\mathbb{R}_+), \mathbf{E}\zeta = 1} (\mathbf{E}(-\zeta\xi) - \alpha_i(\zeta)), \quad \xi \in \mathbf{L}^p(\mathbb{R}), \ i = 1, \ldots, d, \tag{4.1}$$

where $\alpha_i : \mathbf{L}^q(\mathbb{R}_+) \to (-\infty, \infty]$, $i = 1, \ldots, d$, are the penalty functions corresponding to the components of $\mathbf{r}$. The following result generalises Lyapunov's theorem in the sublinear setting, see also [13].

Theorem 4.1. *Let* $(\Omega, \mathfrak{F}, \mathbf{P})$ *be a non-atomic probability space, and let the components of $\mathbf{r}$ admit representation* (4.1)*, with $\alpha_1(\zeta), \ldots, \alpha_d(\zeta)$ being all infinite unless ζ belongs to a finite family from $\mathbf{L}^q(\mathbb{R})$. Then $R(\mathbf{X})$ is convex.*

Proof. We need to show that for two selections $\xi', \xi'' \in \mathbf{L}^p(\mathbf{X})$ and $\lambda \in [0, 1]$, there is $\xi \in \mathbf{L}^p(\mathbf{X})$ such that $\mathbf{r}(\xi) \leq \lambda\mathbf{r}(\xi') + (1 - \lambda)\mathbf{r}(\xi'')$. In view of the convexity of $\mathbf{r}$, it suffices to ensure that

$$\mathbf{r}(\xi) \leq \mathbf{r}(\lambda\xi' + (1 - \lambda)\xi'').$$

Assume that all components of $\alpha(\zeta)$ are infinite for ζ outside a finite set $\mathcal{Z} = \{\zeta_1, \ldots, \zeta_m\}$. Consider the mapping which assigns to each measurable subset $A \subseteq \Omega$ the vector

$$\upsilon(A) = \big(\mathbf{E}(-\mathbf{1}_A\zeta_1\xi'), \ldots, \mathbf{E}(-\mathbf{1}_A\zeta_m\xi'), \mathbf{E}(-\mathbf{1}_A\zeta_1\xi''), \ldots, \mathbf{E}(-\mathbf{1}_A\zeta_m\xi'')\big) \in \mathbb{R}^{2md}.$$

It is easily verified that this map is a vector-valued measure. By Lyapunov's theorem, its image is closed convex, hence there is a measurable subset $A \subseteq \Omega$ such that

$$\upsilon(A) = \lambda\upsilon(\Omega) + (1 - \lambda)\upsilon(\emptyset) = \lambda\upsilon(\Omega).$$

Then $\mathbf{E}(-\mathbf{1}_A\zeta_i\xi') = \lambda\mathbf{E}(-\zeta_i\xi')$ and $\mathbf{E}(-\mathbf{1}_A\zeta_i\xi'') = \lambda\mathbf{E}(-\zeta_i\xi'')$ for all i. Hence,

$$\mathbf{E}(-\lambda\zeta_i\xi' - (1 - \lambda)\zeta_i\xi'') = \mathbf{E}(-\zeta_i(\xi'' + \mathbf{1}_A(\xi' - \xi''))) = \mathbf{E}(-\zeta_i\xi),$$

where $\xi = \xi'\mathbf{1}_A + \xi''\mathbf{1}_{A^c}$ is a selection of $\mathbf{X}$. Therefore,

$$\mathbf{E}(-\zeta_i\xi) - \alpha(\zeta_i) = \mathbf{E}(-\zeta_i(\lambda\xi' + (1 - \lambda)\xi'')) - \alpha(\zeta_i) \leq \mathbf{r}(\lambda\xi' + (1 - \lambda)\xi'')$$

for all i, so ξ is indeed the required selection. $\qquad\square$

Remark 4.2. For a *deterministic* lower closed set F, the selection risk measure $R(F)$ is not always equal to $(-F)$. For instance, this is not the case in the framework of Theorem 4.1, or in the context of fixed transaction costs in Section 5.2. The set F is said to be $\mathbf{r}$-convex (or risk-convex for $\mathbf{r}$), if with any $x_1, x_2 \in F$ and any $A \subseteq \Omega$, we also have $-\mathbf{r}(\mathbf{1}_A x_1 + \mathbf{1}_{A^c} x_2) \in F$. Then F is $\mathbf{r}$-convex if and only if $R(F) = -F$. It is easy to see that the intersection of risk convex sets is also risk convex. If $\mathbf{r}$ is the negative expectation and the probability space is non-atomic, the risk convexity corresponds to the usual notion of convexity. If $\mathbf{r}$ is the negative essential infimum, each lower set is risk convex.

Remark 4.3. Consider $\mathbf{X} = \{\xi, \eta\} + \mathbb{R}^d_-$. Then $R(\mathbf{X})$ is convex if and only if, for each $t \in (0, 1)$, there exists $A \in \mathfrak{F}$ such that

$$t\mathbf{r}(\xi) + (1 - t)\mathbf{r}(\eta) \geq \mathbf{r}(\xi\mathbf{1}_A + \eta\mathbf{1}_{A^c}).$$

The families of selections of random sets are not necessarily law invariant, i.e., they can differ for two random sets having the same distribution, see [11, Section 1.4.1]. This could result in the selection risk measure R not being law invariant. Still, the law invariance of $\mathbf{r}$ yields the law invariance of the selection risk measure for convex $\mathbf{X}$, see [12]. Below we consider the case of a possibly non-convex $\mathbf{X}$.

The risk measure $\mathbf{r}$ is said to be *Lebesgue continuous* if it is continuous on a.s. convergent uniformly p-integrably bounded sequences of random vectors.

Theorem 4.4. *Assume that the probability space is non-atomic and that $\mathbf{r}$ is a Lebesgue continuous risk measure. Then the selection risk measure $R(\mathbf{X})$ is law invariant on p-integrably quasi-bounded portfolios.*

Proof. Let $\mathbf{X}$ and $\mathbf{X}'$ share the same distribution, so that the corresponding closures $\mathbf{Y} = \mathrm{cl}\,\partial^+\mathbf{X}$ and $\mathbf{Y}' = \mathrm{cl}\,\partial^+\mathbf{X}'$ of their Pareto optimal points are p-integrably bounded and share the same distribution. By the Lebesgue property and the p-integrable boundedness of $\mathbf{Y}$ and $\mathbf{Y}'$, it is possible to take the union in (2.2) over p-integrable selections of $\mathbf{Y}$ and $\mathbf{Y}'$, respectively.

Let $x \in \mathbf{r}(\xi) + \mathbb{R}_+^d$ for $\xi \in L^p(\mathbf{Y})$. Since the weak closures of $L^0(\mathbf{Y})$ and $L^0(\mathbf{Y}')$ coincide (see [11, Theorem 1.4.3]), there is a sequence $\eta_n \in L^p(\mathbf{Y}')$ converging weakly to ξ. Then $\|\eta_n\| \leq \|\mathbf{Y}'\|$, and the latter random variable is integrable. Thus, $\{\eta_n, n \geq 1\}$ is relatively compact in $L^1(\mathbb{R}^d)$. By passing to a subsequence, it is possible to assume that $\eta_{n_k} \to \xi$ almost surely.

The Lebesgue continuity property yields that $\mathbf{r}(\eta_{n_k}) \to \mathbf{r}(\xi)$. Thus, $\mathbf{r}(\xi) \in R(\mathbf{Y}')$, since the latter set is closed. Finally, $x \in R(\mathbf{X}')$, since the latter set is upper. □

It is known that each L^p-risk measure with finite values and $p \in [1, \infty)$ is Lebesgue continuous, see [8]. For $p = \infty$, [6, Theorems 2.4, 5.2] provide equivalent formulations of the Lebesgue continuity property for convex risk measures. We give below another criterion.

Proposition 4.5. *Assume that $\mathbf{r}$ is a coherent L^∞-risk measure such that*

$$\mathbf{r}(\xi) = \sup_{\zeta \in \mathcal{Z}} \mathbf{E}(-\zeta\xi),$$

where $\mathcal{Z}$ is a uniformly integrable subset of $L^1(\mathbb{R}_+^d)$. Then $\mathbf{r}$ is Lebesgue continuous.

Proof. Assume that $\xi_n \to \xi$ a.s. and $\|\xi_n\| \leq c$ a.s. for all n and $c > 0$. By Egorov's theorem, for each $\varepsilon > 0$, there is an event A of probability at most ε such that $\xi_n \to \xi$ uniformly on the complement A^c of A.

Using the fact that the absolute value of the difference of two suprema is bounded by the suprema of the absolute values of the differences, we have

$$\|\mathbf{r}(\xi_n) - \mathbf{r}(\xi)\| \leq \sup_{\zeta \in \mathcal{Z}}\|\mathbf{E}(-\zeta(\xi_n - \xi)\| \leq \sup_{\zeta \in \mathcal{Z}} \mathbf{E}\|\zeta\| \sup_{\omega \notin A}\|\xi_n(\omega) - \xi(\omega)\| + 2c\|\mathbf{E}(-\zeta\mathbf{1}_A)\|.$$

The first term on the right-hand side converges to zero by the uniform convergence on A^c, while the second converges to zero by the uniform integrability of $\mathcal{Z}$. □

5 Fixed transaction costs

5.1 Bounds on the selection risk measure

Assume that the components of C represent the same currency and transfers are not restricted, but whenever a transfer is made, a fixed cost $\varkappa > 0$ is incurred. If C is the capital position, then the corresponding set of attainable positions is given by $\mathbf{X} = C + I_\varkappa$ with the non-convex set

$$I_\varkappa = \mathbb{R}_-^d \cup H_{-\varkappa}$$

of portfolios available at price zero, where

$$H_t = \left\{ x \in \mathbb{R}^d : \sum_{i=1}^d x_i \leq t \right\}, \quad t \in \mathbb{R}.$$

The following bounds for the selection risk measure of $C + I_\varkappa$ are straightforward.

Proposition 5.1. *We have*

$$(\mathbf{r}(C) - I_\varkappa) \cup R(C + H_{-\varkappa}) \ \subseteq\ R(C + I_\varkappa) \ \subseteq\ R(C + H_0). \tag{5.1}$$

Proof. The first inclusion follows from the fact that $C + x$ is a selection of $C + I_\varkappa$ for all deterministic $x \in I_\varkappa$ and that $H_{-\varkappa} \subset I_\varkappa$. The second inclusion holds, since $I_\varkappa \subset H_0$. □

Example 5.2. The inclusion on the left-hand side of (5.1) can be strict. Let $d = 2$, and let C be $(-1, 1)$ with probability α and $(0, 0)$ otherwise. For any $0 \leq \beta \leq \alpha$, we can define a selection $\xi \in I_\varkappa$ such that $C + \xi$ equals $(-\varkappa, 0)$ with probability β, $(-1, 1)$ with probability $\alpha - \beta$, and $(0, 0)$ with probability $1 - \alpha$. Taking the risk measure of such selections shows that $R(C + I_\varkappa)$ contains all points on the segments with end-points $(1, 0)$ and $(\varkappa, 0)$.

The following result provides rather simple bounds on the selection risk measure in case of fixed transaction costs.

Proposition 5.3. (i) *If $\varkappa_1 \leq \varkappa_2$ and $C_1 \geq C_2$ componentwisely, then*

$$R(C_1 + I_{\varkappa_1}) \supseteq R(C_2 + I_{\varkappa_2}).$$

(ii) *If $\mathbf{r}$ is subadditive, then*
$$R(C_1 + C_2 + I_\varkappa) \supseteq R(C_1 + I_{\varkappa_1}) + R(C_2 + I_{\varkappa_2})$$

whenever $\varkappa \leq \min(\varkappa_1, \varkappa_2)$.

Proof. (i) Note that $I_{\varkappa_1} \supseteq I_{\varkappa_2}$ for $\varkappa_1 \leq \varkappa_2$, and

$$C_1 + I_{\varkappa_1} \supseteq C_1 + I_{\varkappa_2} \supseteq C_2 + I_{\varkappa_2}.$$

(ii) follows from $I_{\varkappa_1} + I_{\varkappa_2} \subseteq I_\varkappa$ and the monotonicity of the selection risk measure. □

The following result identifies the selection risk measure of $C + H_t$ in some cases in terms of the risk of the total payoff:

$$D = C^{(1)} + \cdots + C^{(d)}.$$

If $\mathbf{r}$ is coherent with all identical components, it is easy to see that $C + H_t$ is acceptable if and only if D is acceptable. The following result concerns the case, when all but one components of $\mathbf{r}$ are identical.

Proposition 5.4. (i) *If all the components of $\mathbf{r}$ are identical convex risk measures r, then*

$$R(C + H_t) = -H_{t-dr(D/d)}.$$

(ii) *If one of the components of $\mathbf{r}$ is the negative essential infimum and all others are identical convex risk measures r, then*
$$R(C + H_t) = -H_{t-(d-1)r(\frac{D}{d-1})}.$$

(iii) *If one of components of $\mathbf{r}$ is the negative expectation and all others are identical convex risk measures r such that $r(\xi) \geq -\mathbf{E}\xi$ for all $\xi \in \mathbf{L}^1(\mathbb{R})$, then*

$$R(C + H_t) = -H_{t-\mathbf{E}D}.$$

Proof. By cash-invariance, it is possible to asssume that $t = 0$. The statement (i) is shown in [3, Theorem 5.1].

(ii) Assume that the first component of $\mathbf{r}$ is the negative essential infimum. Note that $0 \in R(C + H_0)$ if and only if there is a selection ξ such that $\sum_{i=1}^d \xi^{(i)} \leq 0$, $C^{(1)} + \xi^{(1)} \geq 0$ a.s. and $r(C^{(i)} + \xi^{(i)}) \leq 0$ for $i = 2, \ldots, d$. By convexity and monotonicity of r,

$$r\left(\frac{D}{d-1}\right) = r\left(\frac{1}{d-1}\sum_{i=1}^d C^{(i)}\right)$$
$$\leq r\left(\frac{1}{d-1}\sum_{i=2}^d (C^{(i)} + \xi^{(i)})\right) \leq \sum_{i=2}^d \frac{1}{d-1} r(C^{(i)} + \xi^{(i)}).$$

Hence, if $r(C^{(i)} + \xi^{(i)}) \leq 0$ for all $i = 2, \ldots, d$, then $0 \in -H_{-(d-1)r(\frac{D}{d-1})}$. On the other hand, if $r(\frac{D}{d-1}) \leq 0$, then letting $\xi^{(1)} = -C^{(1)}$ and $\xi^{(i)} = -C^{(i)} + D/(d-1)$, $i = 2, \ldots, d$, yields a selection ξ of $C + H_0$ such that $\mathbf{r}(C + \xi) \leq 0$.

(iii) If $0 \in \mathsf{R}(C + H_0)$, then there is ξ such that $\mathbf{E}(C^{(1)} + \xi^{(1)}) \geq 0$ and $r(C^{(i)} + \xi^{(i)}) \leq 0$, $i = 2, \dots, d$. Denote $\eta = D - C^{(1)} - \xi^{(1)}$. Since $\sum_{i=2}^d \xi^{(i)} \leq -\xi^{(1)}$, we have

$$\sum_{i=2}^d (C^{(i)} + \xi^{(i)}) = D - C^{(1)} + \sum_{i=2}^d \xi^{(i)} \leq \eta.$$

Thus,

$$r(\eta/(d-1)) \leq \frac{1}{d-1} r\left(\sum_{i=2}^d (C^{(i)} + \xi^{(i)}) \right) \leq \frac{1}{d-1} \sum_{i=2}^d r(C^{(i)} + \xi^{(i)}) \leq 0. \tag{5.2}$$

Note that $\mathbf{E}(C^{(1)} + \xi^{(1)}) \geq 0$ is equivalent to $\mathbf{E}\eta \leq \mathbf{E}D$. Inequality (5.2) yields that $-\mathbf{E}\eta \leq r(\eta) \leq 0$. Therefore, $0 \leq \mathbf{E}\eta \leq \mathbf{E}D$, as desired.

If $\mathbf{E}D \geq 0$, define a selection of $C + H_0$ by letting $\xi^{(1)} = -C^{(1)} + D$ and $\xi^{(i)} = -C^{(i)}$, $i = 2, \dots, d$. Then $\mathbf{E}(C^{(1)} + \xi^{(1)}) \geq 0$ and $C^{(i)} + \xi^{(i)} = 0$ for $i = 2, \dots, d$, whence $0 \in \mathsf{R}(C + H_0)$. $\square$

5.2 Fixed transaction costs in case $C = 0$

If $C = 0$, then the portfolio $\mathbf{X} = C + I_{\varkappa} = I_{\varkappa}$ is deterministic. However, in the non-convex case, $\mathsf{R}(I_{\varkappa})$ may be a strict superset of $(-I_{\varkappa})$. For instance, this happens in the context of Theorem 4.1 when $\mathsf{R}(I_{\varkappa}) = -H_0 = \overline{\mathrm{conv}}(-I_{\varkappa})$.

In the following assume that $\mathbf{r}$ is a coherent risk measure and $d = 2$. By Proposition 2.2, it suffices to consider selections $\xi = (x, y)\mathbf{1}_A$ satisfying $x + y = -\varkappa$ with $(x, y) \notin \mathbb{R}_-^2$. If $x \geq 0$ and so $y \leq 0$, then

$$\mathbf{r}(\xi) = (x r_1(\mathbf{1}_A), -y r_2(-\mathbf{1}_A)).$$

If $x < 0$, then

$$\mathbf{r}(\xi) = (-x r_1(-\mathbf{1}_A), y r_2(\mathbf{1}_A)).$$

Thus, the risk of $I_{\varkappa}$ is determined by the set

$$B_{\mathbf{r}} = \{(r_1(\mathbf{1}_A), r_2(-\mathbf{1}_A)) : A \in \mathfrak{F}\},$$

where $\mathbf{P}(A) = \beta$ varies between 0 and 1. Then

$$\mathsf{R}(\mathbf{X}) = \bigcup_{t \geq 0, (b^{(1)}, b^{(2)}) \in B_{\mathbf{r}}} \{(t b^{(1)}, (-\varkappa - t)b^{(2)}), (t b^{(2)}, (t - \varkappa)b^{(1)})\} + \mathbb{R}_+^2. \tag{5.3}$$

Example 5.5. Let $d = 2$, and let both components of $\mathbf{r} = (r, r)$ be the average value-at-risk at level α. If $\mathbf{P}(A) = \beta$, then

$$(r(\mathbf{1}_A), r(-\mathbf{1}_A)) = \begin{cases} (0, \beta/\alpha), & \beta \leq \min(\alpha, 1-\alpha), \\ (0, 1), & \alpha < \beta \leq 1 - \alpha, \ \alpha \leq 1/2, \\ (-1 + (1-\beta)/\alpha, \beta/\alpha), & 1 - \alpha < \beta \leq \alpha, \ \alpha > 1/2, \\ (-1 + (1-\beta)/\alpha, 1), & \max(\alpha, 1-\alpha) < \beta \leq 1. \end{cases}$$

Thus, if $\alpha \leq 1/2$, then $B_{\mathbf{r}}$ is the union of two segments $[(0, 0), (0, 1)]$ and $[(0, 1), (-1, 1)]$ and it does not depend on α. In this case, (5.3) yields that $\mathsf{R}(I_{\varkappa}) = -I_{\varkappa}$.

Assume now that $\alpha > 1/2$. Then $B_{\mathbf{r}}$ is the line that joins the points $(0, 0)$, $(0, 1/\alpha - 1)$, $(1/\alpha - 2, 1)$ and $(-1, 1)$. Only the middle segment differs from the case $\alpha \leq 1/2$. If $t > 0$, then the points

$$\{(t b^{(1)}, (t - \varkappa)b^{(2)}) : (b^{(1)}, b^{(2)}) \in B_{\mathbf{r}}\}$$

constitute the segment with the end-points $(0, (t - \varkappa)(1/\alpha - 1))$ and $(t(1/\alpha - 2), t - \varkappa)$. A calculation of the lower envelope of these segments yields that

$$\mathsf{R}(I_{\varkappa}) = \{(-x, y) : x \geq 0, \ y \geq \min(\varkappa + x, (\sqrt{x} + \sqrt{\varkappa(1/\alpha - 1)})^2)\}$$

$$\cup \{(x, -y) : y \geq 0, \ x \geq \min(\varkappa + y, (\sqrt{y} + \sqrt{\varkappa(1/\alpha - 1)})^2)\}.$$

Figure 1 shows the risk of $I_{\varkappa}$ for $\alpha = 0.75$. This set increases as α grows and becomes $\overline{\mathrm{conv}}(-I_{\varkappa})$ if $\alpha = 1$.
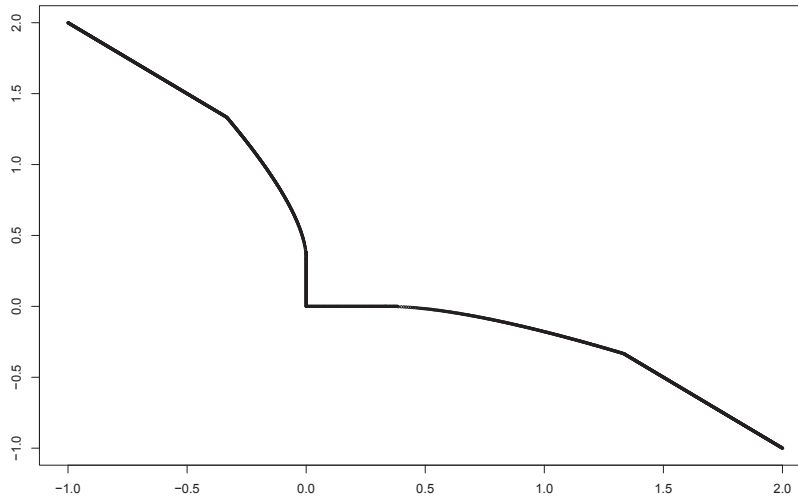
Figure 1: The lower left boundary of the set $R(I_\varkappa)$ for $\varkappa = 1$ and $\alpha = 0.75$.

6 Finite sets of admissible transactions

We consider another special case when the selection risk measure of a non-convex set can be calculated explicitly. Assume that possible transactions are restricted to belong to a finite deterministic set M in $\mathbb{R}^d$, that is,

$$\mathbf{X} = C + M + \mathbb{R}^d_-.$$

Let $\mathbf{r}$ have all components r being the distortion risk measure (2.1) with distortion function g. Since the analytical calculation of $R(\mathbf{X})$ is not feasible, it is possible to use (2.4) to arrive at the bound

$$R(C + M + \mathbb{R}^d_-) \supseteq \mathbf{r}(C) + R(M + \mathbb{R}^d_-).$$

In the following we determine the last term on the right-hand side in dimension $d = 2$.

Example 6.1. Consider the case of a two-point set M. By translating, it is always possible to assume that $0 \in M$. If M consists of two points $(0, 0)$ and (x, y) with $xy < 0$, then $R(\mathbf{X})$ is determined by the set of values $\mathbf{r}((x, y)\mathbf{1}_A)$ for all $A \in \mathfrak{F}$. Without loss of generality, assume that $x > 0$ and $y < 0$. Since $r(\mathbf{1}_A) = -g(\beta)$ and $r(-\mathbf{1}_A) = 1 - g(1 - \beta) = \tilde{g}(\beta)$ if $\mathbf{P}(A) = \beta$, we have

$$R(M + \mathbb{R}^2_-) = \bigcup_{\beta \in [0,1]} \left(-g(\beta)x, \ (g(1 - \beta) - 1)y \right) + \mathbb{R}^2_+.$$

Example 6.2. Let $M = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ consist of three points, and assume that $x_1 < x_2 = 0 < x_3$ and $y_1 > y_2 = 0 > y_3$. In this case, possible selections can be either two-points-selections of two of these three points (in this case the risk is calculated as in Example 6.1), and three point selection attaining all three points with positive probabilities $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$. The risk of the three-point selection can be directly calculated, so that

$$R(M + \mathbb{R}^2_-) = \bigcup_{\substack{\alpha_1 + \alpha_3 \leq 1 \\ \alpha_1, \alpha_3 \geq 0}} \left(-x_1 \tilde{g}(\alpha_1) - x_3 g(\alpha_3), \ -y_1 g(\alpha_1) - y_3 \tilde{g}(\alpha_3) \right) + \mathbb{R}^2_+.$$

References

[1]   F. Delbaen, *Monetary Utility Functions*, Osaka University Press, Osaka, 2012.
[2]   H. Föllmer and A. Schied, *Stochastic Finance. An Introduction in Discrete Time*, 2nd ed., De Gruyter, Berlin, 2004.

[3] A. Haier, I. Molchanov and M. Schmutz, Intragroup transfers, intragroup diversification and their risk assessment, *Ann. Finance* **12** (2016), 363–392.

[4] A. H. Hamel and F. Heyde, Duality for set-valued measures of risk, *SIAM J. Financial Math.* **1** (2010), 66–95.

[5] A. H. Hamel, F. Heyde and B. Rudloff, Set-valued risk measures for conical market models, *Math. Financ. Economics* **5** (2011), 1–28.

[6] E. Jouini, W. Schachermayer and N. Touzi, Law invariant risk measures have the Fatou property, *Adv. Math. Econ.* **9** (2006), 49–71.

[7] Y. M. Kabanov and M. Safarian, *Markets with Transaction Costs. Mathematical Theory*, Springer, Berlin, 2009.

[8] M. Kaina and L. Rüschendorf, On convex risk measures on L^p-spaces, *Math. Methods Oper. Res.* **69** (2009), 475–495.

[9] J. Leitner, Balayage monotonous risk measures, *Int. J. Theor. Appl. Finance* **7** (2004), 887–900.

[10] E. Lépinette and I. Molchanov, Conditional cores and conditional convex hulls of random sets, *J. Math. Anal. Appl.* (2019), DOI 10.1016/j.jmaa.2019.05.010.

[11] I. Molchanov, *Theory of Random Sets*, 2nd ed., Springer, London, 2017.

[12] I. Molchanov and I. Cascos, Multivariate risk measures: A constructive approach based on selections, *Math. Finance* **26** (2016), 867–900.

[13] N. Sagara, A Lyapunov-type theorem for nonadditive vector measures, in: *Modeling Decisions for Artificial Intelligence—MDAI 2009*, Springer, Berlin (2009), 72–80.