Published for SISSA by Springer

Received: *July 17, 2019*
Accepted: *August 14, 2019*
Published: *September 5, 2019*

Non-abelian Toda theory on AdS_2 and $AdS_2/CFT_2^{1/2}$ duality

Matteo Beccaria,[a] Hongliang Jiang[b] and Arkady A. Tseytlin[c,1]

[a] *Dipartimento di Matematica e Fisica Ennio De Giorgi, Università del Salento & INFN, Via Arnesano, 73100 Lecce, Italy*

[b] *Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland*

[c] *Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.*

E-mail: matteo.beccaria@le.infn.it, jiang@itp.unibe.ch, tseytlin@imperial.ac.uk

Abstract: It was recently observed that boundary correlators of the elementary scalar field of the Liouville theory on AdS_2 background are the same (up to a non-trivial proportionality coefficient) as the correlators of the chiral stress tensor of the Liouville CFT on the complex plane restricted to the real line. The same relation generalizes to the conformal abelian Toda theory: boundary correlators of Toda scalars on AdS_2 are directly related to the correlation functions of the chiral $\mathcal{W}$-symmetry generators in the Toda CFT and thus are essentially controlled by the underlying infinite-dimensional symmetry. These may be viewed as examples of AdS_2/CFT_1 duality where the CFT_1 is the chiral half of a 2d CFT; we shall refer to this as $AdS_2/CFT_2^{1/2}$. In this paper we demonstrate that this duality applies also to the non-abelian Toda theory containing a Liouville scalar coupled to a 2d σ-model originating from the $SL(2,\mathbb{R})/U(1)$ gauged WZW model. Here the Liouville scalar is again dual to the chiral stress tensor T while the other two scalars are dual to the parafermionic operators $V^\pm$ of the non-abelian Toda CFT. We explicitly check the duality at the next-to-leading order in the large central charge expansion by matching the chiral CFT correlators of (T, V^+, V^-) (computed using a free field representation) with the boundary correlators of the three Toda scalars given by the tree-level and one-loop Witten diagrams in AdS_2.

Keywords: AdS-CFT Correspondence, Conformal and W Symmetry

ArXiv ePrint: 1907.01357

[1] Also at the Institute for Theoretical and Mathematical Physics, Moscow University and Lebedev Institute, Moscow.

https://doi.org/10.1007/JHEP09(2019)036

JHEP09(2019)036

Contents

1 Introduction and summary

Quantum field theories in rigid AdS_2 background (studied from different perspectives, e.g., in [1–6]) were recently discussed in the context of AdS_5/CFT_4: a conformal "defect" model describing correlators of operators inserted on a straight or circular Wilson line [7–10] is represented at strong coupling by an effective 2d σ-model in AdS_2 background that follows from the $AdS_5 \times S^5$ superstring action expanded near the corresponding minimal surface. To find strong coupling corrections to Wilson line correlators requires computing loop corrections in AdS_2 [8, 10]; this is, in general, a challenging problem (cf. e.g. [11–14] and refs. there).

As the theory in AdS_2 originating [8] from the $AdS_5 \times S^5$ superstring action should be quantum scale-invariant (having no 2d UV divergences), one may hope to learn some important lessons by first investigating simpler examples of Lagrangian conformal 2d field theories (like Liouville or Toda) defined on rigid AdS_2 background. Having conformal (Weyl) invariance, one may expect the bulk correlators in curved conformally flat space like AdS_2 to be directly related to the correlators in flat space. A novel feature in non-compact AdS space is that while the elementary fields (like scalars of Toda theory) are not good conformal fields in flat space, their boundary correlators in AdS_2 (which are the observables in AdS) are well defined and thus are of interest.

Somewhat surprisingly, they happen to be directly related [15–17] to the correlators of chiral (holomorphic) primary operators in the 2d CFT defined by the same action in flat space. For example, the boundary correlators of the Liouville scalar in AdS_2 have the same form as the correlators of the holomorphic stress tensor on a plane and thus are essentially controlled by the underlying Virasoro symmetry. Thus here the chiral half of 2d CFT may be identified with the effective 1d CFT dual to the conformal field theory in AdS_2. We shall refer to this relation as the $AdS_2/CFT_2^{1/2}$ duality.[1]

Examples of models exhibiting $AdS_2/CFT_2^{1/2}$ duality are conformal abelian Toda field theories with exponential potentials associated with a finite Lie algebra $\mathfrak{g}$. This correspondence was first noticed at the level of the classical AdS_2 theory (or large c CFT) in [15] for the $\mathfrak{g} = A_1$ Liouville theory and also in the particular rank-2 examples of $\mathfrak{g} = A_2$ and B_2. The case of $\mathfrak{g} = A_n$ was discussed in [17]. The generalization to AdS_2 loop level (or subleading $1/c$ corrections) was presented for the Liouville theory in [16] where also the exact expression for the map between the AdS_2 scalar and CFT stress tensor correlators was found. Below we shall extend the Liouville theory loop-level results of [16] to the case of the A_2 Toda theory.

One reason why the duality between the elementary scalars in AdS_2 ($ds^2 = \frac{d\mathsf{z}^2+d\mathsf{t}^2}{\mathsf{z}^2}$) and chiral CFT operators on the plane is possible is that expanded near the minimum of its potential the AdS_2 action describes massive ($m^2 = \Delta(\Delta-1)$) scalar fields φ_Δ that should correspond to the dimension $\Delta = 2, 3, \ldots$ operators V_Δ at the boundary (with $V_2 \equiv T$). Another is that the Weyl symmetry of the AdS_2 theory suggests enhancement of the global 1d conformal $SL(2,\mathbb{R})$ symmetry to the Virasoro symmetry. Then the boundary correlators of φ_Δ (with $\mathsf{z} \to 0$ asymptotics $\mathsf{z}^\Delta \Phi_\Delta$) that we shall denote as $\langle\!\langle \Phi_{\Delta_1}(\mathsf{t}_1) \ldots \Phi_{\Delta_k}(\mathsf{t}_k) \rangle\!\rangle$ may be related (on symmetry grounds) to the chiral CFT correlators $\langle\!\langle V_{\Delta_1}(z_1) \ldots V_{\Delta_k}(z_k) \rangle\!\rangle$ restricted to the boundary of half-plane ($z = \mathsf{t} + iy \to \mathsf{t}$). A non-trivial question is a mechanism that determines the proportionality coefficients in this duality relation (i.e., symbolically, κ_Δ in $\Phi_\Delta \to \kappa_\Delta V_\Delta$). Using the Weyl invariance of the AdS_2 theory one may map it to the flat-space theory on the upper half-plane and then try to relate the half-plane boundary asymptotics of the scalar fields to the CFT operators using their free-field (quantum Miura) representation. The free fields are, in general, related to the elementary bulk scalars by a non-linear differential (quantum) Bäcklund transformation, but this relation should effectively simplify in the boundary limit. Details of this remain to be understood.

[1] This is thus an example of AdS_2/CFT_1 where the boundary 1d conformal symmetry is enhanced from $SL(2,\mathbb{R})$ to the chiral half of the 2d Virasoro symmetry. This happens when a Weyl-covariant theory is put on a rigid AdS_2 background. While superficially similar, this is to be distinguished from what happens when one considers quantum gravity in AdS_2 [18] where the bulk diffeomorphism symmetry implies an enlarged asymptotic global symmetry corresponding to 1d reparametrizations or Virasoro symmetry in the boundary theory [19, 20] (which may be spontaneously broken [21–24]).

JHEP09(2019)036

The main focus of the present paper will be on the demonstration of the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality in the case of the non-abelian conformal Toda theory of [25]. Here the derivative part of the Lagrangian is no longer free but is described by an effective σ-model originating from the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ gauged WZW model [26, 27]. This theory comes closer to the string-theory related AdS_2 model in [8] (containing derivative interactions) and turns out to be much more non-trivial than the abelian Toda one.

Below we shall first review (in section 1.1) the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality in the case of the Liouville theory and then summarize the analogous statements for the abelian Toda theory in section 1.2. A summary of our results for the non-abelian Toda theory will be given in section 1.3.

Section 2 will contain details of new one-loop AdS_2 boundary correlator computations in the A_2 abelian Toda model providing solid check of the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality in this case. Section 3 will be devoted to a systematic discussion of the duality in the non-abelian Toda theory. In appendix A we will present a heuristic proposal for the all-order proportionality coefficient between the boundary correlators of the second scalar of the A_2 abelian Toda theory in AdS_2 and correlators of the dual spin 3 chiral generator in the CFT. In appendix B we will collect useful relations that are used to compute some AdS_2 integrals. Appendices C and D will contain details of the one-loop AdS_2 calculations of the coefficients in the three-point boundary correlators that are required to check the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality in the A_2 abelian and non-abelian Toda theories.

1.1 Liouville theory

The action of the Liouville theory defined on a 2d space with curvature R is [28, 29]

$$\mathcal{S} = \frac{1}{4\pi}\int d^2x\,\sqrt{g}\,\left[(\partial\varphi)^2 + \mu^2\,e^{2\,b\,\varphi} + Q\,R\,\varphi\right], \qquad\qquad Q = b^{-1} + b\,. \tag{1.1}$$

For the above value of Q this model is Weyl-covariant with central charge

$$c = 1 + 6\,Q^2 = b^{-2} + 13 + b^2\,. \tag{1.2}$$

Considering the unit-radius Euclidean AdS_2 background with $x = (\mathsf{t},\mathsf{z})$ and the Poincaré plane metric

$$ds^2 = \frac{1}{\mathsf{z}^2}(d\mathsf{t}^2 + d\mathsf{z}^2)\,, \qquad\qquad R = -2\,, \tag{1.3}$$

the Liouville field φ can be expanded near its constant vacuum expectation value,

$$\varphi = \varphi_0 + \zeta\,, \qquad\qquad \varphi_0 = \frac{1}{2b}\log\frac{Q}{b\mu^2}\,, \tag{1.4}$$

and then the fluctuation ζ has classical mass $m^2 = 2$. Perturbation theory in the AdS_2 bulk was studied previously in [2, 5, 30]. One may also compute [16] the boundary correlators of ζ (relevant from the usual AdS/CFT point of view) by assuming the Dirichlet boundary conditions for ζ at the boundary line $\mathsf{z} = 0$, i.e.

$$\zeta(\mathsf{t},\mathsf{z})\big|_{\mathsf{z}\to 0} = \mathsf{z}^2\Phi(\mathsf{t}) + \dots\,. \tag{1.5}$$

The 1d field $\Phi(\mathsf{t})$ is associated to a boundary conformal operator with the scaling dimension $\Delta = 2$ (here $m^2 = \Delta(\Delta-1) = 2$). Then the boundary correlators are defined as

$$\langle\!\langle \prod_{i=1}^{N} \Phi(\mathsf{t}_i) \rangle\!\rangle \stackrel{\rm def}{=} \lim_{\mathsf{z}_i\to 0} \langle \prod_{i=1}^{N} \mathsf{z}_i^{-2}\, \zeta(\mathsf{t}_i,\mathsf{z}_i)\rangle. \tag{1.6}$$

As was noticed in [15], the tree level ($b\to 0$) Witten diagrams computing (1.6) in perturbation theory match the leading large c limit of the correlators of the chiral part of stress tensor $T(z)$ of the flat-space CFT (the generator of the Virasoro algebra with the central charge c given by (1.2))

$$\langle\!\langle \prod_{i=1}^{N} \Phi(\mathsf{t}_i) \rangle\!\rangle = \kappa^N \langle \prod_{i=1}^{N} T(z_i)\rangle\Big|_{z_i\to \mathsf{t}_i}, \tag{1.7}$$

where $\kappa = \kappa(b)$ is the constant in the identification $\Phi(\mathsf{t}) \to \kappa\, T(\mathsf{t})$. Here the r.h.s. may be viewed as a chiral stress tensor correlator restricted to the real-line boundary of the half-plane $z_i = \mathsf{t}_i + i y_i \to \mathsf{t}_i$. The relation (1.7) was demonstrated also to hold [16] at the one-loop in AdS_2 (i.e. subleading order in large c expansion). Furthermore, it was argued in [16] using boundary CFT considerations that the all-order expression for $\kappa(b)$ is given by

$$\kappa = -\frac{4Q}{c} = -\frac{4\,b(1+b^2)}{(3+2\,b^2)(2+3\,b^2)} = -\frac{2}{3}\,b + \frac{7}{9}\,b^3 + \cdots. \tag{1.8}$$

Using that

$$\langle T(z_1)T(z_2)\rangle = \frac{c}{2z_{12}^4}, \qquad \langle T(z_1)T(z_2)T(z_3)\rangle = \frac{c}{z_{12}^2 z_{23}^2 z_{31}^2}, \tag{1.9}$$

the duality relation (1.7) means that the coefficients in the perturbative expansion of the two-point and three-point boundary correlators (with the structure controlled by the conformal symmetry)[2]

$$\langle\!\langle \Phi(\mathsf{t}_1)\,\Phi(\mathsf{t}_2)\,\rangle\!\rangle = \frac{C_{22}}{\mathsf{t}_{12}^4}, \qquad \langle\!\langle \Phi(\mathsf{t}_1)\,\Phi(\mathsf{t}_2)\,\Phi(\mathsf{t}_3)\,\rangle\!\rangle = \frac{C_{222}}{\mathsf{t}_{12}^2\,\mathsf{t}_{13}^2\,\mathsf{t}_{23}^2}, \tag{1.10}$$

should be given by

$$\begin{aligned} C_{22} &= \kappa^2\,\frac{c}{2} = \frac{8\,(1+b^2)^2}{(3+2b^2)(2+4b^2)} = \frac{4}{3} - \frac{2}{9}\,b^2 + \frac{13}{27}\,b^4 + \cdots, \\ C_{222} &= \kappa^3\,c = -\frac{64\,b\,(1+b^2)^3}{(3+2b^2)^2(2+3b^2)^2} = -\frac{16}{9}\,b + \frac{64}{27}\,b^3 - \frac{100}{27}\,b^5 + \cdots. \end{aligned} \tag{1.11}$$

These expansions were checked directly (for the leading tree and one-loop terms) in [16]. The four-point boundary correlators may be decomposed into the disconnected and connected parts

$$\langle\!\langle \Phi(\mathsf{t}_1)\,\cdots\,\Phi(\mathsf{t}_4)\rangle\!\rangle = \langle\!\langle \Phi(\mathsf{t}_1)\,\cdots\,\Phi(\mathsf{t}_4)\rangle\!\rangle_{\rm disc} + \langle\!\langle \Phi(\mathsf{t}_1)\,\cdots\,\Phi(\mathsf{t}_4)\rangle\!\rangle_{\rm conn}. \tag{1.12}$$

[2]Indices of coefficients indicate dimensions of operators involved.

Since the four-point function for the chiral part of the stress tensor T is controlled by the Virasoro symmetry

$$\begin{aligned}\langle T(z_1)\cdots T(z_4)\rangle = \frac{c^2}{4}\left(\frac{1}{z_{12}^4\,z_{34}^4}+\frac{1}{z_{13}^4\,z_{24}^4}+\frac{1}{z_{14}^4\,z_{23}^4}\right)\\ + c\left(\frac{1}{z_{12}^2\,z_{23}^2\,z_{34}^2\,z_{14}^2}+\frac{1}{z_{13}^2\,z_{24}^2\,z_{14}^2\,z_{23}^2}+\frac{1}{z_{12}^2\,z_{24}^2\,z_{34}^2\,z_{13}^2}\right)\end{aligned} \tag{1.13}$$

the relation (1.7) implies that

$$\begin{aligned}\langle\!\langle\Phi(\mathrm{t}_1)\cdots\Phi(\mathrm{t}_4)\rangle\!\rangle_{\text{disc}} &= (C_{22})^2\left(\frac{1}{\mathrm{t}_{12}^2\,\mathrm{t}_{34}^2}+\frac{1}{\mathrm{t}_{13}^2\,\mathrm{t}_{24}^2}+\frac{1}{\mathrm{t}_{14}^2\,\mathrm{t}_{23}^2}\right),\\ \langle\!\langle\Phi(\mathrm{t}_1)\cdots\Phi(\mathrm{t}_4)\rangle\!\rangle_{\text{conn}} &= C_{2222}\left(\frac{1}{\mathrm{t}_{12}^2\,\mathrm{t}_{23}^2\,\mathrm{t}_{34}^2\,\mathrm{t}_{14}^2}+\frac{1}{\mathrm{t}_{13}^2\,\mathrm{t}_{24}^2\,\mathrm{t}_{14}^2\,\mathrm{t}_{23}^2}+\frac{1}{\mathrm{t}_{12}^2\,\mathrm{t}_{24}^2\,\mathrm{t}_{34}^2\,\mathrm{t}_{13}^2}\right),\end{aligned} \tag{1.14}$$

where according to (1.8)

$$C_{2222} = \kappa^4\,c = \frac{256\,b^2\,(1+b^2)^4}{(3+2b^2)^3(2+3b^2)^3} = \frac{32}{27}\,b^2 - \frac{80}{27}\,b^4+\cdots. \tag{1.15}$$

This was also verified in [16] (using numerical computation for one-loop integrals appearing in the computation of the four-point boundary correlator (1.14)).

To conclude, the relation (1.7) found in the Liouville model in AdS_2 provides the simplest example of the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality. The fact that the bulk theory is conformal implies that the structure of the boundary correlators is essentially fixed by the Virasoro symmetry.[3] While this duality may be viewed as being essentially kinematical, the non-trivial expression for κ in (1.8) receiving corrections from all orders in the small b expansion provides an important constraint on how the higher-loop Witten AdS_2 diagrams are to be evaluated in order to maintain the underlying infinite-dimensional Virasoro symmetry.

1.2 Abelian Toda theory

The generalization to abelian Toda theory for A_2 and B_2 algebras was discussed in [15] and for A_n algebras in [17]. A novel feature is that in addition to the Liouville field here the Lagrangian contains other scalar fields that are massive in AdS_2 and are dual to the chiral generators of the $\mathcal{W}_n$ symmetry of the Toda theory.[4]

In section 2 of this paper we will extend the discussion of the corresponding $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality to the one-loop level on the example of the A_2 theory with main results summarized below.

The action of the A_2 abelian Toda theory in curved 2d background contains in addition to the Liouville field φ another scalar ψ, i.e. is given by the following generalization of (1.1)

$$\mathcal{S} = \frac{1}{4\pi}\int d^2x\,\sqrt{g}\,\big[(\partial\varphi)^2+(\partial\psi)^2+\mu\,e^{2b\varphi}\cosh(2\sqrt{3}\,b\,\psi)+Q\,R\,\varphi\big], \tag{1.16}$$

$$Q = b^{-1}+4\,b, \qquad\qquad c = 2+6\,Q^2. \tag{1.17}$$

[3]In particular, the four-point boundary correlators of operators that have protected dimension here do not contain logarithms of 1d cross ratio, etc.

[4]Toda field theories associated with a finite (non-affine) Lie algebra $\mathfrak{g}$ of rank n (see, e.g., [31–33]) may be formulated in curved space, see, for instance, [34].

This theory is Weyl-covariant at the quantum level: the required value of Q in (1.17) can be determined, e.g., by viewing (1.16) as a string model in a linear dilaton and tachyon backgrounds and solving the corresponding tachyon β-function equation [35, 36].

Considering the AdS$_2$ background (1.3) and expanding near the constant vacuum value for φ as in the Liouville theory (with φ_0 given again by (1.4)) one can develop the perturbation theory in small b, i.e. in powers of the fluctuation fields $\zeta \equiv \varphi - \varphi_0$ and ψ. These happen to have masses $m_\zeta^2 = 2$ and $m_\psi^2 = 6$ corresponding (according to $m^2 = \Delta(\Delta-1)$) to the dual operator dimensions $\Delta_\zeta = 2$ and $\Delta_\psi = 3$. Let us label the boundary fields as Φ and Φ_3 (cf. (1.5))[5]

$$\zeta(\mathsf{t},\mathsf{z})\big|_{\mathsf{z}\to 0} = \mathsf{z}^2\Phi(\mathsf{t}) + \dots\,, \qquad\qquad \psi(\mathsf{t},\mathsf{z})\big|_{\mathsf{z}\to 0} = \mathsf{z}^3\Phi_3(\mathsf{t}) + \dots\,. \tag{1.18}$$

In this case the dual CFT is isomorphic to the chiral sector of the $\mathcal{W}_3$ extension of the Virasoro algebra [37, 38], with Φ dual to the dimension 2 stress tensor T and Φ_3 to the dimension (or spin) 3 generator V_3 of the $\mathcal{W}_3$ symmetry algebra. Defining the boundary correlators as in (1.6)

$$\langle\!\langle \prod_{i=1}^{N}\Phi(\mathsf{t}_i)\prod_{j=1}^{M}\Phi_3(\mathsf{t}'_j)\rangle\!\rangle \equiv \lim_{\mathsf{z}_i,\mathsf{z}'_j\to 0}\langle \prod_{i=1}^{N}\mathsf{z}_i^{-2}\zeta(\mathsf{t}_i,\mathsf{z}_i)\prod_{j=1}^{M}\mathsf{z}_j'^{-3}\psi(\mathsf{t}'_j,\mathsf{z}'_j)\rangle, \tag{1.19}$$

the expected correspondence is expressed by a generalization of (1.7), i.e.

$$\langle\!\langle \prod_{i=1}^{N}\Phi(\mathsf{t}_i)\prod_{j=1}^{M}\Phi_3(\mathsf{t}'_j)\rangle\!\rangle = \kappa^N\,\kappa_3^M\,\langle \prod_{i=1}^{N}T(z_i)\prod_{j=1}^{M}V_3(z'_j)\rangle\Big|_{z_i,z'_j\to \mathsf{t}_i,\mathsf{t}'_j}. \tag{1.20}$$

Here κ has the same form as in the Liouville theory (cf. (1.8)) and the coefficient κ_3 in the duality between Φ_3 and V_3 turns out to be a non-trivial function of b

$$\kappa = -\frac{4\,Q}{c} = -\frac{2}{3}\,b + \frac{26}{9}\,b^3 + \cdots\,, \qquad \kappa_3 = \frac{24\,Q^2}{c\,\sqrt{5c+22}} = \frac{2\sqrt{2}}{\sqrt{15}}\left(b - \frac{73}{15}\,b^3 + \cdots\right). \tag{1.21}$$

We shall verify (1.20) and (1.21) by the one-loop AdS$_2$ computations in section 2 and present an argument for the above expression for κ_3 in appendix A.

The CFT 2- and 3-point functions are constrained by the conformal invariance to have the form (we adopt the standard normalization for the spin 3 generator V_3)[6]

$$\begin{aligned}
\langle T(z_1)T(z_2)\rangle &= \frac{c}{2\,z_{12}^4}, & \langle T(z_1)T(z_2)T(z_3)\rangle &= \frac{c}{z_{12}^2 z_{13}^2 z_{23}^2},\\
\langle V_3(z_1)V_3(z_2)\rangle &= \frac{c}{3\,z_{12}^6}, & \langle T(z_1)V_3(z_2)V_3(z_3)\rangle &= \frac{c}{z_{12}^2 z_{13}^2 z_{23}^4}\,.
\end{aligned} \tag{1.22}$$

[5]The index of Φ_3 indicates the dimension of the associated operator (we omit index 2 on the Liouville field Φ).

[6]In general, given a primary field V_Δ with the dimension Δ and the two-point function $\langle V_\Delta(z_1)\,V_\Delta(z_2)\rangle = \frac{C}{z_{12}^{2\Delta}}$ the conformal symmetry Ward identity implies that $\langle T(z_1)V_\Delta(z_2)V_\Delta(z_3)\rangle = \frac{C\,\Delta}{z_{12}^2\,z_{13}^2\,z_{23}^{2\Delta-2}}$. This is consistent with the values of the coefficients in (1.22) (note that in the $\langle TTT\rangle$ case the central term in the OPE of $T(z)T(0)$ does not contribute to the three-point function as $\langle T\rangle = 0$).

The corresponding boundary correlators can be parametrized as in (1.10)

$$\langle\!\langle \Phi(\mathrm{t}_1)\Phi(\mathrm{t}_2)\rangle\!\rangle = \frac{C_{22}}{\mathrm{t}_{12}^4}, \qquad \langle\!\langle \Phi(\mathrm{t}_1)\Phi(\mathrm{t}_2)\Phi(\mathrm{t}_3)\rangle\!\rangle = \frac{C_{222}}{\mathrm{t}_{12}^2\mathrm{t}_{13}^2\mathrm{t}_{23}^2},$$
$$\langle\!\langle \Phi_3(\mathrm{t}_1)\Phi_3(\mathrm{t}_2)\rangle\!\rangle = \frac{C_{33}}{\mathrm{t}_{12}^6}, \qquad \langle\!\langle \Phi(\mathrm{t}_1)\Phi_3(\mathrm{t}_2)\Phi_3(\mathrm{t}_3)\rangle\!\rangle = \frac{C_{233}}{\mathrm{t}_{12}^2\mathrm{t}_{13}^2\mathrm{t}_{23}^4}. \tag{1.23}$$

We shall explicitly check the relations implied by (1.20), (1.21)

$$C_{22} = \frac{c}{2}\,\kappa^2 = \frac{4}{3} - \frac{4}{9}\,b^2 + \cdots, \qquad C_{33} = \frac{c}{3}\,\kappa_3^2 = \frac{16}{15} - \frac{112}{75}\,b^2 + \cdots,$$
$$C_{233} = c\,\kappa\,\kappa_3^2 = -\frac{32}{15}\,b + \frac{2752}{225}\,b^3 + \cdots, \qquad C_{222} = c\,\kappa^3 = -\frac{16}{9}\,b + \frac{224}{27}\,b^3 + \cdots, \tag{1.24}$$

by the one-loop computations in AdS_2 in section 2.

The non-vanishing four-point correlators $\langle\!\langle\Phi\Phi\Phi\Phi\rangle\!\rangle$, $\langle\!\langle\Phi\Phi\Phi_3\Phi_3\rangle\!\rangle$, and $\langle\!\langle\Phi_3\Phi_3\Phi_3\Phi_3\rangle\!\rangle$ were considered at the tree level in [15] confirming the expected $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ relations. While no conceptual difficulties are expected at the one-loop level where the computation is similar to the one in the Liouville theory in [16] here will not discuss it because of technical limitations.[7]

Similar results should hold also for higher rank abelian Toda models where there are more scalar fields with different masses corresponding to the higher-spin generators of the underlying $\mathcal{W}$-algebra symmetry. One implication of the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality (1.20) is that it can then be used to compute the higher-loop AdS_2 boundary correlators for the Toda theory using purely $\mathcal{W}$-algebraic methods.

1.3 Non-abelian Toda theory

In the abelian Toda theory the chiral CFT operators (conserved currents) have protected dimensions so their correlators are constrained by $\mathcal{W}$ symmetry modulo overall normalizations and thus the corresponding $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality is a direct generalization of the Liouville theory case. The story becomes more intricate in the case of the conformal non-abelian Toda (NAT) theory [25] (see also [39, 40]) that we shall discuss in detail in section 3 below. Here we will summarize the main results.

Keeping only the leading order (one-loop) terms in the corresponding Weyl-invariant target space metric and dilaton (cf. (3.8)) the NAT action is the following generalization of the Liouville (1.1) or abelian Toda (1.16) actions

$$\mathcal{S} = \frac{1}{4\pi}\int d^2x\,\sqrt{g}\Big[(\partial\varphi)^2 + (\partial r)^2 + \tanh^2(br)\,(\partial y)^2 \tag{1.25}$$
$$+\,\mu^2\,e^{2b\varphi}\cosh(2br)\,+R\,\Big(Q\varphi - \log\cosh(br)\Big)\Big], \qquad Q = b^{-1} + 3b.$$

[7] The coefficients C_{233} and C_{222} in the three-point functions in (1.23) are expressed in terms of finite AdS_2 integrals. Most of them may be computed analytically, but a few have to be evaluated numerically because the available analytic tools turn out not to be sufficient. In the case of the coefficients in the four-point correlators the number of diagrams to be evaluated numerically is larger and the expected numerical accuracy is too low to be reliable.

Here in addition to the Liouville field φ with linear dilaton term we have the (r, y) sector with the kinetic term originating from the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ gauged WZW model and the dilaton term $R \log\cosh(br)$ required for the Weyl invariance of this σ-model [27]. The expression for Q is fixed by the condition of quantum Weyl invariance of the potential term, i.e. by satisfaction of the tachyon equation (3.10) [35, 41]. The (exact) relations between the constant b, the WZW level k and the total central charge are (cf. (3.9), (3.17))

$$b = \frac{1}{\sqrt{k-2}}\,, \qquad c = 3 + 6Q^2 + 6b^2 = 6b^{-2} + 39 + b^2\,, \tag{1.26}$$

so that the small b expansion is the same as the large k or large c expansion.

Expanding (1.25) around the minimum of the potential for φ on the AdS_2 background as in (1.4), one finds the action for 3 massive fluctuation fields ζ and ξ_1, ξ_2 (related to r, y as in (3.13) with $a = b$). They have the same mass $m^2 = 2$ (to leading order in b), i.e. should be dual to the boundary operators with the classical dimension 2. We shall denote the corresponding boundary fields as Φ and $\Phi^\pm$ (cf. (1.5), (1.18))

$$\zeta(\mathsf{t},\mathsf{z})\big|_{\mathsf{z}\to 0} = \mathsf{z}^2\Phi(\mathsf{t}) + \ldots\,, \qquad \xi^\pm(\mathsf{t},\mathsf{z})\big|_{\mathsf{z}\to 0} = \mathsf{z}^{\Delta_V}\Phi^\pm(\mathsf{t}) + \ldots\,, \qquad \xi^\pm \equiv \frac{1}{\sqrt{2}}(\xi_1 \pm i\xi_2)\,, \tag{1.27}$$

where $\Delta_V = 2 + \ldots$ (anticipating possible anomalous dimension). At the classical level, the NAT model in flat space has three conserved holomorphic currents with dimension 2: the stress tensor $T_{\rm cl}$ and a $\mathrm{U}(1)$ doublet of "parafermions" $V^\pm_{\rm cl}$ [39] (generalizing the classical parafermions of the gWZW model [26]). A natural suggestion is that the corresponding quantum operators T and $V^\pm$ should be related, respectively, to the AdS_2 boundary fields Φ and $\Phi^\pm$. To check the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality in this case we will compute the AdS_2 boundary correlators using Witten diagrams and compare them to the chiral CFT correlators of T and $V^\pm$.

The CFT correlators can be found using an explicit free field representation for T and parafermions $V^\pm$ that we shall present in section 3.2. A novel feature compared to the abelian Toda theory is that the dimension of the primaries $V^\pm$ is not protected, i.e. they have a non-zero anomalous dimension

$$\Delta_V = 2 + \gamma_V\,, \qquad \gamma_V = \frac{1}{k} = \frac{b^2}{1+2b^2}\,. \tag{1.28}$$

Defining the boundary correlators as in (1.6), (1.19)[8]

$$\langle\!\langle \prod_{i=1}^{N} \Phi(\mathsf{t}_i) \prod_{j^\pm=1}^{M^\pm} \Phi^\pm(\mathsf{t}^\pm_j) \rangle\!\rangle \equiv \lim_{\mathsf{z}_i, \mathsf{z}^\pm_j \to 0} \langle \prod_{i=1}^{N} \mathsf{z}_i^{-2}\zeta(\mathsf{t}_i,\mathsf{z}_i) \prod_{j^\pm=1}^{M^\pm} (\mathsf{z}^\pm_j)^{-\Delta_V}\xi^\pm(\mathsf{t}^\pm_j, \mathsf{z}^\pm_j)\rangle\,, \tag{1.29}$$

the statement of the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality should be (cf. (1.7), (1.20))

$$\langle\!\langle \prod_{i=1}^{N} \Phi(\mathsf{t}_i) \prod_{j^\pm=1}^{M^\pm} \Phi^\pm(\mathsf{t}^\pm_j) \rangle\!\rangle = \kappa^N\, \kappa_\pm^{M^+ + M^-}\, \langle \prod_{i=1}^{N} T(z_i) \prod_{j^\pm=1}^{M^\pm} V^\pm(z^\pm_j)\rangle\Big|_{z_i, z^\pm_j \to \mathsf{t}_i, \mathsf{t}^\pm_j}, \tag{1.30}$$

where κ and $\kappa_+ = \kappa_-$ are the coefficients in the correspondence $\Phi \to \kappa T$, $\Phi^\pm \to \kappa_\pm V^\pm$.

[8] Here we use a short-cut notation: the $\pm$ products are, of course, independent.

As the dimension 2 of the stress tensor is protected, its normalization is universal: $\langle T(z_1)T(z_2)\rangle = \frac{c}{2z_{12}^4}$. As a result, as in the Liouville (1.8) or the abelian Toda theory (1.21), the coefficient κ has the scheme-independent expression in terms of Q and c

$$\kappa = -\frac{4Q}{c} = -\frac{4\,b\,(1+3\,b^2)}{3\,(1+4\,b^2)(2+5\,b^2)} = -\frac{2}{3}\,b + \frac{7}{3}\,b^3 + \cdots\,. \tag{1.31}$$

At the same time, as $V^\pm$ has the non-zero anomalous dimension (1.28), the coefficient C^{+-} in the corresponding 2-point function

$$\langle V^+(z_1)V^-(z_2)\rangle = \frac{C^{+-}}{z_{12}^{2\Delta_V}}\,, \tag{1.32}$$

is scheme-dependent (it, in general, contains a factor of $\Lambda^{-2\Delta_V}$ where Λ is a renormalization scale). We shall assume a scheme in which C^{+-} is given by

$$C^{+-} = \frac{c}{\Delta_V} = \frac{3k(k+2)}{k-2}\,, \tag{1.33}$$

where k is the WZW level related to b by (1.26). In this scheme our proposed exact expression for $\kappa_\pm$ is (cf. (1.28))

$$\kappa_\pm = \frac{2b}{3(1+4b^2)}\,2^{\gamma_V} = \frac{2}{3}b - \frac{2}{3}\big(4-\log 2\big)\,b^3 + \cdots\,. \tag{1.34}$$

Representing the AdS_2 boundary two-point correlators as

$$\langle\!\langle \Phi(\mathsf{t}_1)\,\Phi(\mathsf{t}_2)\rangle\!\rangle = \frac{C_{22}}{\mathsf{t}_{12}^4}, \qquad \langle\!\langle \Phi^+(\mathsf{t}_1)\,\Phi^-(\mathsf{t}_2)\rangle\!\rangle = \frac{C_{+-}}{(\mathsf{t}_{12}^2)^{\Delta_V}}, \tag{1.35}$$

the correspondence (1.30) implies the following relations

$$C_{22} = \kappa^2\,\frac{c}{2} = \frac{8\,(1+3b^2)^2}{3\,(1+4b^2)(2+5b^2)} = \frac{4}{3} - \frac{2}{3}\,b^2 + \cdots\,, \tag{1.36}$$

$$C_{+-} = \kappa_\pm^2\,C^{+-} = \kappa_\pm^2\,\frac{3\,(1+2b^2)(1+4b^2)}{b^2} = \frac{4}{3} + \frac{8}{3}\,(\log 2 - 1)\,b^2 + \cdots\,. \tag{1.37}$$

The conformal invariance (Virasoro algebra) fixes the three-point functions of the chiral primary operators to have the following form (after using the Ward identity in footnote 6 and (1.32), (1.33))

$$\langle T(z_1)T(z_2)T(z_3)\rangle = \frac{c}{z_{12}^2\,z_{13}^2\,z_{23}^2}\,, \qquad \langle T(z_1)V^+(z_2)V^-(z_3)\rangle = \frac{c}{z_{12}^2\,z_{13}^2\,z_{23}^{2\Delta_V-2}}\,. \tag{1.38}$$

With the AdS_2 boundary three-point correlators written as

$$\langle\!\langle \Phi(\mathsf{t}_1)\Phi(\mathsf{t}_2)\Phi(\mathsf{t}_3)\rangle\!\rangle = \frac{C_{222}}{\mathsf{t}_{12}^2\,\mathsf{t}_{13}^2\,\mathsf{t}_{23}^2}\,, \qquad \langle\!\langle \Phi(\mathsf{t}_1)\Phi^+(\mathsf{t}_2)\Phi^-(\mathsf{t}_3)\rangle\!\rangle = \frac{C_{2+-}}{\mathsf{t}_{12}^2\,\mathsf{t}_{13}^2\,\mathsf{t}_{23}^{2\Delta_V-2}}\,, \tag{1.39}$$

the duality (1.30) then rests on the validity of the following relations

$$C_{222} = \kappa^3\, c = -\frac{64\, b\,(1+3b)^3}{9\,(1+4b^2)^2(2+5b^2)^2} = -\frac{16}{9}\, b + \frac{64}{9}\, b^3 + \cdots\,,$$
$$C_{2+-} = \kappa\, \kappa_\pm^2\, c = -C_{+-}\, \frac{4b\,(1+3b^2)}{3\,(1+2b^2)(1+4b^2)} = -\frac{16}{9}\, b + \frac{16}{9}\Big(5 - 2\,\log 2\Big)\, b^3 + \cdots\,. \tag{1.40}$$

We shall verify these predictions in section 3 by the tree and one-loop Witten diagram calculations of (1.39) starting with the AdS_2 action (1.25) in expanded in powers of b. This provides strong evidence for the consistency of the duality (1.30).

We will also compute (at tree level) the non-vanishing four-point correlators: $\langle\!\langle \Phi(\mathrm{t}_1)\Phi(\mathrm{t}_2)\Phi(\mathrm{t}_3)\Phi(\mathrm{t}_4)\rangle\!\rangle$ (which has the same form as in (1.12), (1.14), cf. (1.13)) and also

$$\langle\!\langle \Phi(\mathrm{t}_1)\Phi(\mathrm{t}_2)\Phi^+(\mathrm{t}_3)\Phi^-(\mathrm{t}_4)\rangle\!\rangle, \qquad \langle\!\langle \Phi^+(\mathrm{t}_1)\Phi^-(\mathrm{t}_2)\Phi^+(\mathrm{t}_3)\Phi^-(\mathrm{t}_4)\rangle\!\rangle\,. \tag{1.41}$$

The expressions for the latter are only partially constrained by the conformal invariance. Nevertheless, we will demonstrate that their dependence on the conformally invariant cross ratio, the structure of kinematical singularities and the conformal block expansions are in full agreement with the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ relations (1.30). It would be interesting to evaluate the four-point functions in (1.41) also at the one-loop level, but like in the abelian Toda theory, this would require developing more efficient computational tools (see footnote 7).

2 A_2 abelian Toda theory

Here we will discuss loop corrections in the A_2 Toda theory in AdS_2 with the action (1.16) with the aim to test the duality relation (1.20). The exact expressions for the coefficients in (1.21) can be argued for by adapting the conformal Ward identity method used in the Liouville theory case in [16] as explained in appendix A.

2.1 Perturbation theory

As discussed in detail in [16], the perturbation theory of an abelian Toda theory in AdS_2 may be set up in two natural ways that are equivalent at the level of the expressions for the boundary correlators. In the first approach, proposed in the context of the Liouville theory in [5] (and thus named ZZ in [16]), one starts with the Toda action in flat space and expands around a non-trivial solution corresponding to an effective AdS_2 geometry. In the second (AdS) approach the theory is put on AdS_2 background from the very beginning and the perturbation theory is developed around the constant minimum of the effective potential that includes the curvature coupling term. Technically, the difference between the two schemes happens to be in the treatment of the short distance singularity of the Toda field propagator or the value of $g_\Delta(z,z)$. In the ZZ scheme, this is a non-trivial quantity, while in the AdS scheme it is set to zero. Equivalence between the two approaches is possible thanks to special identities that ensure a compensation between the tadpole contributions (present in the ZZ and absent in the AdS case) and the different effective couplings in the two schemes.[9]

[9]In the Liouville theory, the basic identity is in eq. (A.9) of [16]. Completely similar results can be proved in the A_2 Toda theory; in general, this should be a consequence of the equations of motion.

Here we shall adopt the AdS scheme which is simpler as one can simply ignore all tadpole contributions. Thus we start with the action (1.16) on the AdS_2 background (1.3) and expand φ near its vacuum value as in (1.4)

$$\mathcal{S} = \frac{1}{2\pi}\int d\mathsf{t}\,d\mathsf{z}\,\left[\frac{1}{2}(\partial_a\zeta)^2+\frac{1}{2}(\partial_a\psi)^2+\frac{Q}{2b\,\mathsf{z}^2}\,e^{2b\zeta}\Big(\cosh(2\sqrt{3}b\psi)-2b\zeta-1\Big)\right]. \tag{2.1}$$

Here the interaction terms in (2.1) are multiplied by the common factor $\frac{Q}{b}=\frac{1}{b^2}(1+4b^2)$ (see (1.17)). The values of the masses $m_\zeta^2=2$, $m_\psi^2=6$ correspond to the dimensions of the boundary operators being $\Delta_\zeta=2$ and $\Delta_\psi=3$.

In general, the AdS_2 propagator for a free scalar field $\varphi_\Delta(x)=\varphi_\Delta(\mathsf{t},\mathsf{z})$ with mass $m^2=\Delta(\Delta-1)$ and kinetic term normalized as in (2.1) is $\langle\varphi_\Delta(x)\varphi_\Delta(x')\rangle_{\rm free}=2\,\pi\,G_\Delta(x,x')$ where

$$\begin{aligned}
G_\Delta &= \frac{\mathcal{C}_\Delta}{(4u)^\Delta}\,{}_2F_1\left(\Delta,\Delta,2\Delta,-\frac{4}{u}\right),\\
\mathcal{C}_\Delta &= \frac{\Gamma(\Delta)}{2\sqrt{\pi}\Gamma(\Delta+1/2)},\\
u(x,x') &= \frac{(\mathsf{z}-\mathsf{z}')^2+(\mathsf{t}-\mathsf{t}')^2}{4\mathsf{z}\mathsf{z}'}.
\end{aligned} \tag{2.2}$$

In particular, for the free propagators for the fields ζ and ψ are[10]

$$\begin{aligned}
&g(x,x')\equiv\langle\zeta(x)\zeta(x')\rangle_{\rm free}=\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet=-\frac{1}{2}\Big(\frac{1+\eta}{1-\eta}\log\eta+2\Big), \qquad \eta\equiv\eta(x,x')=\frac{u(x,x')}{1+u(x,x')},\\
&h(x,x')\equiv\langle\psi(x)\psi(x')\rangle_{\rm free}=\bullet\text{- - - - - -}\bullet=-\frac{3(1-\eta^2)+(\eta^2+4\eta+1)\log\eta}{2(\eta-1)^2}.
\end{aligned} \tag{2.3}$$

As was mentioned above, in the AdS approach we shall use a particular AdS_2 covariant UV regularization in which

$$g(x,x)=0, \qquad\qquad h(x,x)=0\,, \tag{2.4}$$

and thus may ignore all tadpole contributions.

The explicit computations of the AdS_2 loop integrals will be often performed by changing the coordinates from the Poincaré half plane $x=(\mathsf{t},\mathsf{z})$ to the unit disk $|z|<1$ as follows

$$w\equiv\mathsf{t}+i\mathsf{z}=-i\frac{z+1}{z-1}, \qquad z=\frac{w-i}{w+i}, \qquad ds^2=-\frac{4\,dw\,d\bar w}{(w-\bar w)^2}=\frac{4\,dzd\bar z}{(1-z\bar z)^2}\,. \tag{2.5}$$

Then the action (2.1) becomes ($z=x_1+ix_2$, $d^2z=dx_1dx_2$)

$$\mathcal{S}=\frac{1}{2\pi}\int d^2z\left[\frac{1}{2}(\partial_a\zeta)^2+\frac{1}{2}(\partial_a\psi)^2+\frac{2Q}{b\,(1-z\bar z)^2}e^{2b\zeta}\Big(\cosh(2\sqrt{3}b\psi)-2b\zeta-1\Big)\right]. \tag{2.6}$$

The mass and interaction terms in (2.6) have the following explicit expansion in b

$$\begin{aligned}
&\int \mathsf{d}^2z\,\frac{Q}{b}\left(e^{2b\zeta}\cosh(2\sqrt{3}b\psi)-2b\zeta-1\right)=\int\mathsf{d}^2z\,(1+4b^2)\Big(2\zeta^2+6\psi^2\\
&\qquad\qquad\qquad\qquad +12b\,\zeta\psi^2+\frac{4}{3}b\,\zeta^3+12b^2\,\zeta^2\psi^2+6b^2\,\psi^4+\frac{2}{3}b^2\,\zeta^4+\cdots\Big),\\
&\mathsf{d}^2z\equiv\frac{d^2z}{\pi(1-z\bar z)}\,.
\end{aligned} \tag{2.7}$$

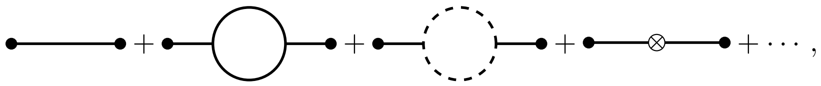
[10]In the following, we shall often omit the arguments in $u\equiv u(x,x')$ and $\eta\equiv\eta(x,x')$.

2.2 Two-point functions

Let us now discuss quantum corrections to the two-point functions of the ζ and ψ fields, first in the bulk, and then in the boundary limit defined as in (1.18), (1.19). We shall use the disk coordinates (2.5), (2.6) and apply the results of appendix B.

Bulk two-point function of ζ and its boundary limit $\langle\!\langle \Phi\Phi \rangle\!\rangle$

The non-vanishing one-loop contributions to the two-point function of the ζ field are given by[11]

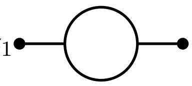
$\langle \zeta(z_1)\zeta(z_2)\rangle =$  (2.8)

$g(z_1,z_2)$ $\Sigma_{\zeta\zeta}(z_1,z_2)$ $\Sigma_{\psi\psi}(z_1,z_2)$ $\Sigma^{(\zeta)}_{\text{ins.}}(z_1,z_2)$

where

$$\Sigma_{\zeta\zeta} = \frac{(-8b)^2}{2} D_{\zeta\zeta}(z_1,z_2), \qquad \Sigma_{\psi\psi} = \frac{(-24b)^2}{2} D_{\psi\psi}(z_1,z_2), \qquad \Sigma^{(\zeta)}_{\text{ins.}} = -16\, b^2\, \widehat{\Sigma}_\zeta(z_1,z_2). \tag{2.9}$$

Here, $D_{\zeta\zeta}$ is the bubble diagram with virtual ζ fields, $D_{\psi\psi}$ is the similar diagram with virtual ψ fields. The additional order b^2 contribution $\Sigma_{\text{ins.}}$ is associated with the insertion of the vertex $b^2\zeta^2$ coming from the $1+4b^2$ factor in (2.7). Let us give the expressions for the various contributions. The ζ loop correction $D_{\zeta\zeta}$ is

$D_{\zeta\zeta}(z_1,z_2) = z_1$  $z_2 = \int \mathsf{d}^2z'\, g(z_1,z')B(z',z_2)\,,$ (2.10)

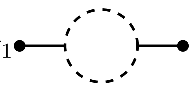
where

$$B(z_1,z_2) = \int \mathsf{d}^2z' \big[g(z_1,z')\big]^2 g(z',z_2) = \frac{1}{8} - \frac{\eta\log^2\eta}{8(1-\eta)^2}\,. \tag{2.11}$$

As a result,

$$D_{\zeta\zeta} = \frac{1}{576}\left[15 + \frac{\pi^2(\eta+1)}{\eta-1}\right] - \frac{\eta\log\eta}{48\,(\eta-1)} + \frac{\eta^2\log^2\eta}{64\,(\eta-1)^2} + \frac{\log(1-\eta)}{48}\left[1 - \frac{(\eta+1)\log\eta}{\eta-1}\right] - \frac{(\eta+1)\mathrm{Li}_2(\eta)}{96(\eta-1)}. \tag{2.12}$$

Similarly, the ψ loop contribution $D_{\psi\psi}$ reads

$D_{\psi\psi}(z_1,z_2) = z_1$  $z_2 = \int \mathsf{d}^2z'\, g(z_1,z')C(z',z_2),$ (2.13)

where

$$C(z_1,z_2) = \int \mathsf{d}^2z' [h(z_1,z')]^2 g(z',z_2) = \frac{\eta^2-6\eta+1}{16(\eta-1)^2} - \frac{\eta\big[2-2\eta^2+(\eta^2+1)\log\eta\big]\log\eta}{8(\eta-1)^4}\,. \tag{2.14}$$

[11] Since we assume the vanishing of simple tadpoles (2.4) only connected one-loop diagrams will contribute (for a general discussion of tadpole contributions see [16]).

Thus we obtain

$$D_{\psi\psi}(z_1,z_2)=\frac{\pi^2(\eta^2-1)+12\eta^2-20\eta+12}{576(\eta-1)^2}-\frac{\eta(\eta^2-\eta+2)\log\eta}{144(\eta-1)^3}+\frac{\eta^2(5\eta^2-10\eta+9)\log^2\eta}{576(\eta-1)^4}$$
$$+\frac{\log(1-\eta)}{72}\left[\frac{1}{2}-\frac{(\eta+1)\log\eta}{\eta-1}\right]-\frac{(\eta+1)\mathrm{Li}_2(\eta)}{96(\eta-1)}. \tag{2.15}$$

Finally, the insertion diagram in (2.9) is determined by

$$\widehat{\Sigma}_\zeta=\int \mathsf{d}^2z'g(z_1,z')g(z',z_2)$$
$$=-\frac{\eta\log\eta}{6(\eta-1)}+\frac{\log(1-\eta)}{6}\left[1+\frac{(\eta+1)\log\eta}{2(\eta-1)}\right]+\frac{(\eta+1)\mathrm{Li}_2(\eta)}{6(\eta-1)}-\frac{1}{6}-\frac{\pi^2(\eta+1)}{36(\eta-1)}. \tag{2.16}$$

Then (2.8), (2.9) give the one-loop bulk two-point function. Going back to the Poincaré plane parametrization (cf. (2.2)) let us define the boundary correlator on the real line as in (1.18), (1.19). As a result, we get

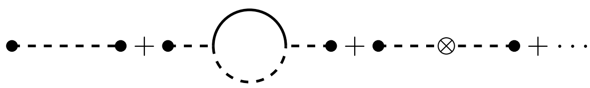
$$\langle\!\langle \Phi(\mathsf{t}_1)\Phi(\mathsf{t}_2)\rangle\!\rangle=\lim_{\mathsf{z}_1,\mathsf{z}_2\to 0}\mathsf{z}_1^{-2}\,\mathsf{z}_2^{-2}\langle\zeta(\mathsf{t}_1,\mathsf{z}_1)\zeta(\mathsf{t}_2,\mathsf{z}_2)\rangle=\frac{1}{\mathsf{t}_{12}^4}\left(\frac{4}{3}-\frac{4}{9}b^2+\cdots\right). \tag{2.17}$$

This expression is in agreement with the expected value of C_{22} in (1.23), (1.24).

Bulk two-point function of ψ and its boundary limit $\langle\!\langle\Phi_3\Phi_3\rangle\!\rangle$

The corresponding two-point function reads

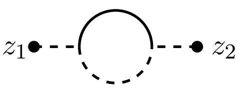
$$\langle\psi(z_1)\psi(z_2)\rangle_{\text{conn}}= \tag{2.18}$$



$h(z_1,z_2)$ $\quad\Sigma_{\zeta\psi}(z_1,z_2)\quad$ $\Sigma^{(\psi)}_{\text{ins.}}(z_1,z_2)$

The non-trivial diagram is $\Sigma_{\zeta\psi}(z_1,z_2)=(-24b)^2D_{\zeta\psi}(z_1,z_2)$, while the insertion contribution is $\Sigma^{(\psi)}_{\text{ins.}}(z_1,z_2)=-48\,b^2\,\widehat{\Sigma}_\psi(z_1,z_2)$. The bubble is given by

$$D_{\zeta\psi}(z_1,z_2)=$$



$$=\int \mathsf{d}^2z'h(z_1,z')B_\psi(z',z_2)\,. \tag{2.19}$$

Here

$$B_\psi(z_1,z_2)=\int \mathsf{d}^2z'h(z_1,z')g(z',z_1)h(z',z_2)=\frac{1+\eta}{16(1-\eta)}-\frac{\eta\log\eta}{8(\eta-1)^2}+\frac{\eta(1+\eta)\log^2\eta}{8(\eta-1)^3}\,. \tag{2.20}$$

As a result

$$D_{\zeta\psi}(z_1,z_2)=-\frac{15\left(\eta^2-1\right)+\pi^2(\eta(\eta+4)+1)}{2880(\eta-1)^2}-\frac{\eta^2(\eta+3)\log^2\eta}{192(\eta-1)^3}+\frac{\eta(2\eta+1)\log\eta}{160(\eta-1)^2}$$
$$+\frac{1}{80}\log(1-\eta)\left[-\frac{\eta+1}{\eta-1}+\frac{(\eta(\eta+4)+1)\log\eta}{2(\eta-1)^2}\right]+\frac{(\eta(\eta+4)+1)\mathrm{Li}_2(\eta)}{480(\eta-1)^2}\,. \tag{2.21}$$

The insertion contribution in (2.18) is proportional to

$$
\begin{aligned}
\widehat{\Sigma}_\psi &= \int \mathsf{d}^2 z'\, h(z_1, z')h(z', z_2) \\
&= \frac{15\left(\eta^2-1\right)+2\pi^2(\eta(\eta+4)+1)}{120(\eta-1)^2}+\frac{\eta(3\eta+4)\log(\eta)}{20(\eta-1)^2} \\
&\quad +\log(1-\eta)\left[-\frac{3(\eta+1)}{20(\eta-1)}-\frac{(\eta(\eta+4)+1)\log(\eta)}{20(\eta-1)^2}\right]-\frac{(\eta(\eta+4)+1)\mathrm{Li}_2(\eta)}{10(\eta-1)^2}\,.
\end{aligned}
\tag{2.22}
$$

Defining again the boundary correlator according to (1.18), (1.19) we finish with

$$
\langle\!\langle \Phi_3(\mathsf{t}_1)\Phi_3(\mathsf{t}_2)\rangle\!\rangle = \lim_{\mathsf{z}_1,\mathsf{z}_2\to 0} \mathsf{z}_1^{-3}\,\mathsf{z}_2^{-3}\langle\psi(\mathsf{t}_1,\mathsf{z}_1)\psi(\mathsf{t}_2,\mathsf{z}_2)\rangle = \frac{1}{\mathsf{t}_{12}^6}\left(\frac{16}{15}-\frac{112}{75}b^2+\cdots\right), \tag{2.23}
$$

which is in agreement with the expression for C_{33} in (1.23), (1.24).

2.3 Three-point functions

We can also check the relations (1.30) for the three-point functions

$$
\langle\!\langle \Phi\Phi\Phi\rangle\!\rangle = \kappa^3\,\langle TTT\rangle, \qquad\qquad \langle\!\langle \Phi\Phi_3\Phi_3\rangle\!\rangle = \kappa\,\kappa_3^2\,\langle TV_3V_3\rangle \tag{2.24}
$$

by reproducing the one-loop perturbative expansions of the coefficients C_{222} and C_{233} in (1.24). The relevant one-loop diagrams can be computed using the disc parametrization of AdS_2. The only type of bulk diagram that cannot be computed analytically is the one with a triangle loop but it can be evaluated numerically with good precision well below one percent level. As a result, we confirmed the duality predictions in (1.24). The details can be found in appendix C.

3 Non-abelian Toda theory

Another non-trivial example of $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality is provided by the non-abelian Toda (NAT) theory of [25] (see also [39, 40]). This theory may be viewed as a special case of a conformal model with a 3-dimensional target space with the kinetic term given by a scalar plus a σ-model originating from the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ gauged WZW model [26, 27] supplemented by a potential analogous to the abelian Toda model one.

We shall discuss the quantum conformal properties of the NAT model, in particular, give a free field representation for its 3 conserved currents: the stress tensor T and a pair of "parafermions" $V^\pm$. This will allow us to compute their CFT correlators. One novel feature will be that in contrast to the $\mathcal{W}$ symmetry generators in the abelian Toda case here $V^\pm$ will have a non-trivial anomalous dimension (1.28). This will allow us to compare the CFT correlators with the boundary correlators of the corresponding elementary fields in the NAT Lagrangian in AdS_2 (1.25). The boundary correlators given by the Witten diagrams will be computed at the tree and one-loop level. As as result, we will verify the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ correspondence (1.30).

3.1 Definition of the model on AdS_2 background

Let us start with a σ-model like theory on a curved 2d background

$$\mathcal{S} = \frac{1}{4\pi}\int d^2z\,\sqrt{g}\,\mathcal{L}\,, \qquad \mathcal{L} = G_{\mu\nu}(x)\,\partial_a x^\mu \partial^a x^\nu + \mathrm{T}(x) + R\,\mathit{\Phi}(x)\,. \tag{3.1}$$

Here $\mu,\nu = 1,\ldots,D$, R is 2d curvature, $\mathit{\Phi}$ is a "dilaton" and T is a "tachyon". The standard Weyl-invariance conditions of decoupling of the conformal factor of the 2d metric are [42, 43]

$$R_{\mu\nu} + 2\,\nabla_\mu\nabla_\nu\mathit{\Phi} + \ldots = 0\,, \qquad -\tfrac{1}{2}\nabla^2\mathit{\Phi} + (\partial\mathit{\Phi})^2 + \cdots = \tfrac{c-D}{6}\,, \tag{3.2}$$

$$-\tfrac{1}{2}\nabla^2\mathrm{T} + G^{\mu\nu}\partial_\mu\mathit{\Phi}\partial_\nu\mathrm{T} - 2\,\mathrm{T} + \ldots = 0\,. \tag{3.3}$$

Here we presented only the leading one-loop terms in the corresponding Weyl-anomaly coefficients and ignored $\mathcal{O}(\mathrm{T}^2)$ terms in the β-functions corresponding to non-perturbative divergences [35, 36] that will not be relevant for the model discussed below (with T given by a sum of exponentials with non-constant products).

The $D = 3$ model (3.1) (with $x^\mu = (\varphi, r, y)$) we will be interested in will be

$$\mathcal{L} = (\partial\varphi)^2 + (\partial r)^2 + G(r)(\partial y)^2 + \mathrm{T}(\varphi, r) + R\,\mathit{\Phi}(\varphi, r)\,, \tag{3.4}$$

with $G, \mathit{\Phi}, \mathrm{T}$ given by a particular solution of the leading-order equations (3.2), (3.3)

$$G(r) = \tanh^2(ar)\,, \qquad \mathit{\Phi} = Q\,\varphi - \log\cosh(ar), \tag{3.5}$$

$$\mathrm{T} = \mu^2\,e^{2b\varphi}\,\cosh(2ar)\,. \tag{3.6}$$

Here the constants a, b and Q are related by

$$1 + b^2 - b\,Q + 2a^2 = 0\,, \qquad \text{i.e.} \qquad Q = b^{-1}(1+2a^2) + b\,, \qquad c = 3 + 6Q^2 + 6a^2\,. \tag{3.7}$$

The case of $a = 0$ brings us back to the Liouville theory (plus two extra free fields).

In general, the leading-order metric coefficient G and the dilaton $\mathit{\Phi}$ in (3.5) require a modification in order to satisfy the Weyl-invariance equations (3.2) at the 2-loop level [44]. Including the 3- and 4-loop terms in the β-functions requires further modifications [45] (whose form depends, in general, on a renormalization scheme [46]). Recognizing that (3.5) is the "classical" background for the $SL(2,\mathbb{R})/U(1)$ gWZW model corresponding to a coset CFT allows one to determine the exact metric and dilaton [47]

$$G(r) = \frac{\tanh^2(ar)}{1-\frac{2}{k}\tanh^2(ar)}\,, \qquad \mathit{\Phi} = Q\,\varphi - \log\cosh(ar) - \tfrac{1}{4}\log\left[1-\tfrac{2}{k}\tanh^2(ar)\right], \tag{3.8}$$

$$c = 3+6Q^2+6a^2 = 6Q^2 + \frac{3k}{k-2}\,, \qquad a = \frac{1}{\sqrt{k-2}}\,. \tag{3.9}$$

Here k is the level of the gWZW theory so that the total central charge is the sum of the one for the linear dilaton φ-theory and the $SL(2,\mathbb{R})/U(1)$ coset theory: $c = (1+6Q^2)+(\frac{3k}{k-2}-1)$. The exact background (3.8) was first found [47] by identifying the "point-particle" probe equation ($L_0\mathrm{T} = 2\,\mathrm{T}$) in the coset CFT with the leading-order tachyon equation (3.3).[12] This corresponds to the choice of a "CFT scheme" in which the tachyon equation is not modified by quantum (or α') corrections [46], i.e. its exact form is the leading-order one in (3.3) or

$$-\frac{1}{\sqrt{G}e^{-2\Phi}}\partial_\mu(\sqrt{G}e^{-2\Phi}\,G^{\mu\nu}\partial_\nu)\mathrm{T} - 4\,\mathrm{T} = 0\,. \tag{3.10}$$

As the potential T in (3.4) is assumed not to depend on the isometric y coordinate and since for the exact background (3.8) one gets no corrections to the combination $\sqrt{G}e^{-2\Phi} = e^{-2Q\varphi}\sinh(ar)\cosh(ar)$ we conclude that the corresponding exact solution for T is still given by (3.6) with b related to a and Q by (3.7).

Below we will only consider the leading order terms in loop expansion, i.e. will ignore finite quantum counterterms coming from $1/k$ expansion of (3.8) (which are, in general, required for the preservation of the conformal invariance and also integrability [51] beyond the 1-loop order). We shall thus start with the following Lagrangian (3.4) on a unit-radius AdS_2 background ($R = -2$)

$$\mathcal{L} = (\partial\varphi)^2 + (\partial r)^2 + \tanh^2(ar)(\partial y)^2 + \mu^2 e^{2b\varphi}\cosh(2ar) \; - 2Q\varphi + 2\log\cosh(ar)\,. \tag{3.11}$$

Here the last two terms come from the dilaton coupling in (3.4). The coupling constants a and b are a priori independent with Q expressed in terms of them by (3.7). As in the Liouville or abelian Toda model the presence of an extra "dilaton" term in the action on the AdS_2 background implies the existence of a constant extremum of the resulting potential (see (1.4)): $\varphi_0 = \frac{1}{2b}\log\frac{Q}{b\mu^2}$, $r = 0$. The perturbation theory near this vacuum is then described by ($\varphi = \varphi_0 + \zeta$)

$$\mathcal{L} = (\partial\zeta)^2 + (\partial r)^2 + \tanh^2(ar)(\partial y)^2 + \tfrac{Q}{b}\left[e^{2b\zeta}\cosh(2ar) - 2b\zeta\right] + 2\log\cosh(ar)\,. \tag{3.12}$$

Introducing the new coordinates $(r, y) \to (\xi_1, \xi_2)$ (with y-isometry becoming rotation in the ξ_i plane)

$$\xi_1 = a^{-1}\sinh(ar)\,\cos(ay), \qquad \xi_2 = a^{-1}\sinh(ar)\,\sin(ay)\,, \qquad \xi^2 \equiv \xi_1^2 + \xi_2^2\,, \tag{3.13}$$

the Lagrangian (3.12) takes the following SO(2) invariant form

$$\mathcal{L} = (\partial\zeta)^2 + \frac{(\partial\xi_1)^2 + (\partial\xi_2)^2}{1 + a^2\,\xi^2} + \tfrac{Q}{b}\left[e^{2b\zeta}(1 + 2a^2\,\xi^2) - 2b\zeta\right] + \log(1 + a^2\xi^2)\,. \tag{3.14}$$

Expanding to quadratic order in the fluctuation fields (ζ, ξ_1, ξ_2) we find that their masses are

$$m_\zeta^2 = 2\,b\,Q = 2\,(1 + 2a^2 + b^2)\,, \qquad m_{\xi_{1,2}}^2 = \frac{a^2}{b}\,(b + 2Q) = \frac{2a^2}{b^2}(1 + 2a^2 + \tfrac{3}{2}b^2)\,. \tag{3.15}$$

We observe that in the special case when

$$a = b\,, \tag{3.16}$$

[12] The same background (3.8) can be found also from the quantum effective action for the gWZW model [48–50].

all masses have the same value $m^2 = 2$ at leading order in expansion in the coupling b. With (3.16) we get from (3.7) (see also (1.25), (1.26))

$$Q = b^{-1} + 3b\,, \qquad\qquad c = 3 + 6Q^2 + 6b^2 = 6b^{-2} + 39 + b^2\,. \tag{3.17}$$

The case of (3.11) with (3.16) corresponds to the NAT model of [25] with the exact conformal background given by (3.6), (3.8) and $b = \frac{1}{\sqrt{k-2}}$.[13]

This is the model we shall study below. It can be also obtained by a Hamiltonian reduction [52–54] of the $B_2 = \mathrm{SO}(5)$ WZW model over a nilpotent subgroup [25]. It should be possible to derive it directly from a gauged WZW model obtained by "null" gauging of a solvable subgroup (similarly to how that was done for the abelian Toda models in [55]). This should give a σ-model with a 5d target space, containing, in addition to the first three terms in (3.11), also $\Delta\mathcal{L} = [e^{2b(\varphi+r)} + e^{2b(\varphi-r)}]^{-1}\partial_+ u \partial_- v$. Solving for the two extra fields u and v as in [55] will then reproduce the $e^{2b\varphi}\cosh(2br)$ potential in (3.11).

Our starting point in the computation of the AdS_2 boundary correlators will thus be the Lagrangian (3.14) with $a = b$ expanded to quartic order in the fluctuation fields or to b^2 order in the coupling[14]

$$\begin{aligned}\mathcal{L} = {}& \tfrac{1}{2}(\partial\zeta)^2 + \tfrac{1}{2}(\partial\xi_i)^2 + (1+3b^2)\,\zeta^2 + (1+\tfrac{7}{2}\,b^2)\,\xi^2 \\ & + b(1+3b^2)\big(2\,\zeta\,\xi^2 + \tfrac{2}{3}\,\zeta^3\big) + b^2\big[2\,\zeta^2\,\xi^2 + \tfrac{1}{3}\,\zeta^4 - \tfrac{1}{2}\,\xi^2\,(\partial\xi_i)^2\big] + \mathcal{O}(b^4)\,.\end{aligned} \tag{3.18}$$

Note that the expansion of higher order terms in $\frac{1}{k} = \frac{b^2}{1+2b^2}$ in the exact background (3.8) which we ignored in (3.12) produces only higher order $\mathcal{O}(b^4)$ terms in (3.18) that will not contribute to the computations performed in this paper.

3.2 Underlying flat-space CFT

Following [25], the classical integrable structure of the NAT model was elucidated in [39]. It was shown to admit three conserved holomorphic (plus anti-holomorphic) quantities: the stress tensor T and the two currents $V^{\pm}$ with the same classical dimension 2. Using a non-trivial change of variables they may be written in terms of free fields. After quantization T and $V^{\pm}$ should represent primary operators in the underlying CFT. The chiral half of this CFT should then play the role of the boundary CFT in the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ correspondence.

Since our application of the free-field construction to the quantum NAT model appears to be new, let us start with recalling first the free-field representation for the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ gWZW model (see, e.g., [56–58]), i.e. for the (r, y) subsector of (3.11) with no tachyon coupling. The classical conserved quantities in this case are

$$T_{0,\rm cl} = -\frac{1}{2}\,(\partial\varphi_1)^2 - \frac{1}{2}\,(\partial\varphi_2)^2, \qquad\qquad U^{\pm}_{cl} = \frac{1}{\sqrt{2}}\big(\partial\varphi_1 \pm i\,\partial\varphi_2\big)\,e^{\pm i\,\frac{\sqrt{2}}{\sqrt{k}}\,\varphi_2}, \tag{3.19}$$

[13] Let us mention that the misleading claim in [40] that the leading-order model of [25] defined by the metric and tachyon in (3.5), (3.6) is not conformally invariant was due to ignoring the contribution of the dilaton coupling in (3.5) (see also [41]).

[14] Here we redefine $\mathcal{L}$ by $\frac{1}{2}$ to restore the standard normalization of the fields, i.e. the action is now given by $\mathcal{S} = \frac{1}{2\pi}\int d^2z\,\sqrt{g}\,\mathcal{L}$.

where k is the level. The quantization is performed by assuming that φ_i ($i = 1, 2$) are two free quantum fields with the standard OPE

$$\partial\varphi_i(z)\,\partial\varphi_j(0) \sim -\frac{1}{z^2}\delta_{ij}\,. \tag{3.20}$$

The conserved stress tensor and the parafermions $U^\pm$ of the quantum gWZW model are given by the following generalizations of the classical quantities in (3.19) (see, e.g., [58])

$$T_{_0} = -\frac{1}{2} : (\partial\varphi_1)^2 : -\frac{1}{2} : (\partial\varphi_2)^2 : -\frac{1}{\sqrt{2}\,\sqrt{k-2}}\,\partial^2\varphi_1\,, \tag{3.21}$$

$$U^\pm = \frac{1}{\sqrt{2}} : \left(\sqrt{\tfrac{k-2}{k}}\,\partial\varphi_1 \pm i\,\partial\varphi_2\right) e^{\pm i\,\frac{\sqrt{2}}{\sqrt{k}}\,\varphi_2} : . \tag{3.22}$$

This $T_{_0}$ obeys the Virasoro algebra with the central charge of the gWZW theory

$$c_0 = \frac{3k}{k-2} - 1\,, \tag{3.23}$$

while $U^\pm$ turn out to be the primary fields with dimension $\Delta_{_U}$, i.e.

$$T_{_0}(z)\,U^\pm(0) \sim \frac{\Delta_{_U}}{z^2}\,U^\pm(0) + \frac{1}{z}\,\partial U^\pm(0) + \ldots\,, \qquad\qquad \Delta_{_U} = 1 + \frac{1}{k}\,. \tag{3.24}$$

The operators U^+ and U^- have a non-trivial OPE

$$U^+(z)\,U^-(0) \sim \frac{1}{z^{2\Delta_{_U}}}\left[-1 - \frac{2\Delta_{_U}}{c_0}\,T_{_0}(0) + \ldots\right]. \tag{3.25}$$

Going back to the NAT theory, this free-field construction may be generalized by adding another free field φ_3 (again obeying (3.20)). Then the analogs of the classical conserved quantities $T_{_\mathrm{cl}}$ and $V^\pm_{_\mathrm{cl}}$ found in [39] are proposed to be (cf. (3.21), (3.22))

$$\begin{aligned} T &= T_{_0} - \frac{1}{2} : (\partial\varphi_3)^2 : -\frac{1}{\sqrt{2}} Q\,\partial^2\varphi_3 \\ &= -\frac{1}{2}\sum_{i=1}^{3} : (\partial\varphi_i)^2 : -\frac{1}{\sqrt{2}\sqrt{k-2}}\,\partial^2\varphi_1 - \frac{1}{\sqrt{2}} Q\,\partial^2\varphi_3\,, \end{aligned} \tag{3.26}$$

$$V^\pm = \big(\partial\varphi_3 - p\,\partial\big)U^\pm = :\big(\partial\varphi_3 - p\,\partial\big)\,\frac{1}{\sqrt{2}}\left(\sqrt{\tfrac{k-2}{k}}\,\partial\varphi_1 \pm i\,\partial\varphi_2\right) e^{\pm\frac{i\sqrt{2}}{\sqrt{k}}\,\varphi_2} :, \tag{3.27}$$

where Q and p are constants to be fixed as functions of the level k. Note that the classical dimension of $V^\pm(z)$ in (3.27) differs by 1 from that of $U^\pm$ in (3.22). The stress tensor T in (3.26) obeys the Virasoro algebra with the central charge (cf. (3.9))

$$c = c_0 + 1 + 6Q^2 = \frac{3k}{k-2} + 6Q^2\,. \tag{3.28}$$

Requiring that this matches the NAT central charge in (3.17) (where $b = \frac{1}{\sqrt{k-2}}$) gives the same value of Q as in (3.17), i.e.

$$Q = b^{-1} + 3b = \frac{k+1}{\sqrt{k-2}}\,. \tag{3.29}$$

Computing the OPE of $T(z)V^{\pm}(0)$ (generalizing the one in (3.24)) one finds that the leading singularity is $\mathcal{O}(z^{-3})$ with the residue $\sim (\sqrt{2}\,k - 2\sqrt{k-2}\,p)$, i.e. the necessary condition for $V^{\pm}$ to be primary is

$$p = \frac{k}{\sqrt{2}\,\sqrt{k-2}}\,. \tag{3.30}$$

This turns out to be also the sufficient condition, i.e. assuming (3.30) we find that $V^{\pm}$ are primary with the same dimension $\Delta_{_V} = \Delta_{_U} + 1 = 2 + \frac{1}{k}$ (cf. (3.24)), i.e.

$$T(z)\,V^{\pm}(0) \sim \frac{\Delta_{_V}}{z^2}\,V^{\pm}(0) + \frac{1}{z}\,\partial V^{\pm}(0) + \dots\,, \qquad\qquad \Delta_{_V} = 2 + \frac{1}{k}\,. \tag{3.31}$$

The OPE for V^+ and V^- is found to be (cf. (3.25), (1.32))

$$\begin{aligned} V^+(z)\,V^-(0) &\sim \frac{C^{+-}}{z^{2\Delta_{_V}}}\left[1 + z^2\,\frac{2\Delta_{_V}}{c}\,T + z^3\,\frac{\Delta_{_V}}{c}\,\partial T + z^4\,\Omega_4 + \dots\right], \\ C^{+-} &= \frac{3k(k+2)}{k-2}\,, \end{aligned} \tag{3.32}$$

where c is the NAT central charge in (3.28). Ω_4 is a dimension 4 operator that may be decomposed in a dimension 4 descendent of the identity plus a primary of dimension 4.[15]

Computing the OPE of $V^+(z)V^+(0)$ we find that there are no poles in z and first regular term is

$$V^+(z)\,V^+(0) \sim z^{2/k}\,\Omega_0(0) + \dots\,, \tag{3.33}$$

where the operator Ω_0 has somewhat complicated expression (that will not be needed below).[16] One can check that Ω_0 is a primary field with dimension

$$\Delta_{_{\Omega_0}} = 4 + \frac{4}{k}\,. \tag{3.34}$$

3.3 Two-point and three-point functions

To check the $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ duality (1.30) let us start with two-point functions. The coefficient C_{22} in the boundary correlator of ζ fields in (1.35) can be computed as in the Liouville theory [16], but noting that now we can have also the fields $\xi_{1,2}$ propagating in the loop. Taking into account the coefficients of the interaction terms in the Lagrangian (3.18) and the fact that to leading order ξ_i has the same mass as ζ, the one-loop result is obtained multiplying by 3 the one-loop correction in the Liouville theory

$$C_{22} = \frac{4}{3}\left(1 - 3\times\frac{1}{6}b^2\right) + \dots = \frac{4}{3} - \frac{2}{3}\,b^2 + \cdots\,, \tag{3.35}$$

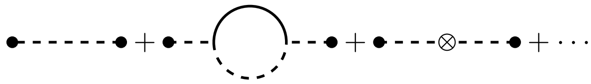
which is indeed in agreement with (1.36).

[15]Note that in the pure $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ gWZW model (without extra Liouville φ field and potential) the dimension 3 operator in the corresponding OPE (3.25) is not simply ∂T, but the spin 3 generator of the $\mathcal{W}_3$ extension of the Virasoro algebra (see, for instance, [58, 59]). In the NAT case the first non-trivial correction in (3.32) appears at order z^4 in the square brackets. It would be interesting to explore if this is an indication of an underlying $\mathcal{W}_4$ symmetry (cf. [60]).

[16]It is the normal ordered product of several monomials in φ_1, φ_2, φ_3 with total of 4 derivatives times $\exp\left(\frac{2\,i\,\sqrt{2}}{\sqrt{k}}\,\varphi_2\right)$.

The correlator of the ξ_i fields can also be found by a simple modification of the previous calculations (cf. (2.8), (2.18)):

$$\langle \xi_i(z_1)\xi_j(z_2)\rangle_{\text{conn}} = \delta_{ij}\hat{g}(z_1,z_2)\,,$$

$$\hat{g} =$$  (3.36)

$$= g(z_1,z_2) + (-8b)^2 D_{\zeta\zeta}(z_1,z_2) + \frac{7}{2}\cdot(-4)b^2\,\widehat{\Sigma}_\zeta(z_1,z_2) + \ldots,$$

where $D_{\zeta\zeta}$ and $\widehat{\Sigma}_\zeta$ are given in (2.12) and (2.16). In the limit when one point is sent to the boundary, $\mathsf{z}_2 \to 0$, the expression for (3.36) gets simplified to

$$\begin{aligned}\langle \xi^+(\mathsf{z}_1,\mathsf{t}_1)\xi^-(\mathsf{z}_2,\mathsf{t}_2)\rangle_{\text{conn}} &= \hat{g}(\mathsf{z}_1,\mathsf{t}_1;\mathsf{z}_2,\mathsf{t}_2)\\ &= \frac{4}{3}\left[\frac{\mathsf{z}_1\mathsf{z}_2}{\mathsf{z}_1^2+\mathsf{t}_{12}^2} + \mathcal{O}\left(\mathsf{z}_2^2\right)\right]^{2+b^2}\left[1+2b^2(\log 2-1)\right] + \mathcal{O}(b^4),\end{aligned} \tag{3.37}$$

where we kept only the leading order term in $\mathsf{z}_2 \to 0$ and defined $\xi^\pm = \frac{1}{\sqrt{2}}(\xi_1 \pm i\xi_2)$ as in (1.27). The above expression is consistent with the field ξ being dual to an operator with dimension

$$\Delta_V = 2+\gamma_V\,, \qquad\qquad \gamma_V = b^2 + \mathcal{O}(b^4)\,, \tag{3.38}$$

in agreement with (1.28) obtained from the CFT. Then the 1-loop corrected bulk-to-boundary propagator turns out to be

$$\begin{aligned}\hat{g}(\mathsf{t}';\mathsf{t},\mathsf{z}) &= \lim_{\mathsf{z}'\to 0}\frac{1}{\mathsf{z}'^{\Delta_V}}\langle \xi^+(\mathsf{z},\mathsf{t})\xi^-(\mathsf{z}',\mathsf{t}')\rangle_{\text{conn}}\\ &= \frac{4}{3}\left(\frac{\mathsf{z}}{\mathsf{z}^2+(\mathsf{t}-\mathsf{t}')^2}\right)^{\Delta_V}\left[1+2b^2(\log 2-1)\right] + \mathcal{O}(b^4)\,.\end{aligned} \tag{3.39}$$

Setting both legs to the boundary $\mathsf{z},\mathsf{z}' \to 0$, we get the boundary limit of this correlator (cf. (1.27), (1.29)) with the coefficient in (1.35) being

$$C_{+-} = \frac{4}{3}\Big[1+2b^2(\log 2-1)+\cdots\Big]\,. \tag{3.40}$$

This is also in agreement with (1.37) where we substituted (3.32) and the 1-loop expression for $\kappa_\pm$ in (1.34).

The exact form of $\kappa_\pm$ proposed in (1.34) is motivated by the free-field representation (3.27) of $V^\pm$ and the OPE (3.32). Following the argument used to determine κ_3 of the abelian Toda theory (1.21) in appendix A, one may conjecture that

$$\kappa_\pm = 2\sqrt{2}\,\frac{p}{C^{+-}}\,2^{1/k} = \frac{2^{1+1/k}\sqrt{k-2}}{3\,(k+2)}\,, \tag{3.41}$$

where the factor $2^{1/k}$ takes into account the anomalous dimension (3.31) of $V^\pm$ and the overall coefficient is fixed by matching with known tree level value $\kappa_\pm = \frac{2}{3}b+\cdots$. Written in terms of $b = \frac{1}{\sqrt{k-2}}$, eq. (3.41) becomes the same as (1.34) (see also (1.28)).

JHEP09(2019)036

Next, let us consider the three-point correlator in (1.29), i.e. $\langle\!\langle \Phi\Phi\Phi \rangle\!\rangle \sim \langle TTT \rangle$. Again, we can use the result in the Liouville theory since the virtual $\xi_{1,2}$ fields behave as copies of the Liouville field, taking into account the values of the coupling coefficients in (3.18). Thus we find for C_{222} in (1.39)

$$C_{222} = -\frac{16b}{9} + 3 \times \frac{64b^3}{27} + \cdots = -\frac{16}{9}\,b + \frac{64}{9}\,b^3 + \cdots . \tag{3.42}$$

This agrees with the expected expression in (1.40).

Turning to the second three-point function $\langle\!\langle \Phi\Phi^+\Phi^- \rangle\!\rangle \sim \langle TV^+V^- \rangle$, using (3.40), the prediction for its coefficient in (1.39) is given in (1.40), i.e.

$$C_{2+-}(b) = -\frac{16}{9}\,b + \frac{16}{9}\Big(5 - 2\,\log 2\Big)\,b^3 + \cdots . \tag{3.43}$$

We have confirmed it by a detailed computation described in appendix D.

3.4 Four-point functions

The structure of $\langle TTTT \rangle$ is fully determined by the conformal symmetry (1.13). The $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ correspondence (1.30) then predicts that the corresponding boundary correlator should be given by (1.14), (1.15). These predictions can be checked in the same way as in the Liouville theory, taking into account the multiplicity of virtual exchanges of ζ and ξ_i. Indeed, for the coefficient of the connected part in (1.14) we have in the Liouville and in the non-abelian Toda cases

$$C_{2222} = \kappa^4 c = \frac{256\,Q^4}{c^3} = \begin{cases} \frac{32}{27}\,b^2 - \frac{80}{27}\,b^4 + \frac{496}{81}\,b^6 + \cdots , & \text{Liouville} \\ \frac{32}{27}\,b^2 - 3 \times \frac{80}{27}\,b^4 + \frac{464}{9}\,b^6 + \cdots , & \text{NAT} \end{cases} \tag{3.44}$$

where we used the model-dependent specific values of Q and c (see (1.1), (1.2), (1.25), (1.26)). As expected (cf. (3.40), (3.42)), the one-loop b^4 correction in the NAT model is simply three times that of the Liouville theory. There is no such simple relation at higher orders in b.

The expressions for $\langle TTV^+V^- \rangle$ and $\langle V^+V^-V^+V^- \rangle$ related to the boundary correlators in (1.41) are not fully fixed by the conformal symmetry and thus represent a particular interest. In the first case the dual four-point correlator $\langle\!\langle \Phi\Phi\Phi^+\Phi^- \rangle\!\rangle$ has a disconnected contribution that is just $\langle\!\langle \Phi\Phi \rangle\!\rangle\langle\!\langle \Phi^+\Phi^- \rangle\!\rangle$ and it then obeys $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ as a consequence of the relation between the two-point functions. In the connected part, at the tree $\mathcal{O}(b^2)$ level, the fields ξ_i play again the same role as the Liouville field ζ and the matching can be easily checked.

The story is more complicated for the remaining non-vanishing correlator $\langle\!\langle \Phi^+\Phi^-\Phi^+\Phi^- \rangle\!\rangle$ which according to (1.30) should be proportional to the CFT correlator $\langle V^+V^-V^+V^- \rangle$ restricted to the real line. The latter is non-trivial because of the non-locality in the OPE of VV in (3.32) and (3.33). To compute this correlator one should use the free field representation (3.27) perturbatively in small b or large k.

AdS_2 boundary correlator $\langle\!\langle \Phi^+\Phi^-\Phi^+\Phi^-\rangle\!\rangle$

The boundary correlators in AdS_2 have, in general, the same form as correlators in 1d CFT. If we have four primary operators $\mathcal{O}$ in 1d CFT of the same dimension Δ, their correlator is constrained by the global SO(2, 1) conformal invariance to take the following form

$$\langle\!\langle \mathcal{O}_1(\mathsf{t}_1)\,\mathcal{O}_2(\mathsf{t}_2)\,\mathcal{O}_3(\mathsf{t}_3)\,\mathcal{O}_4(\mathsf{t}_4)\rangle\!\rangle = \frac{1}{(\mathsf{t}_{12}\,\mathsf{t}_{34})^{2\Delta}}\,G(\chi)\,, \qquad \chi = \frac{\mathsf{t}_{12}\,\mathsf{t}_{34}}{\mathsf{t}_{13}\,\mathsf{t}_{24}}\,. \tag{3.45}$$

Here χ is 1d conformally invariant cross-ratio. The function $G(\chi)$ in (3.45) admits the s-channel expansion (see, e.g., [61])

$$G(\chi) = \sum_h c_h\,\mathsf{F}_h(\chi)\,, \qquad \mathsf{F}_h \equiv \chi^h\ {}_2F_1(h,h,2h,\chi)\,, \tag{3.46}$$

where h labels the conformal dimension of the fields appearing in the OPE $\mathcal{O}_1\mathcal{O}_2 = [\mathcal{O}_h] + [\mathcal{O}_{h'}] + \cdots$, and the coefficients c_h may be expressed in terms of the coefficients in the 2-point and 3-point functions of $\mathcal{O}_1, \mathcal{O}_2$ and the exchanged field.

Let us consider the AdS_2 boundary correlator of for fields ξ_i using the notation Φ_i for their boundary values as in (1.27), (1.29). According to (3.45) the result should read

$$\langle\!\langle \Phi_{i_1}(\mathsf{t}_1)\,\Phi_{i_2}(\mathsf{t}_2)\,\Phi_{i_3}(\mathsf{t}_3)\,\Phi_{i_4}(\mathsf{t}_4)\rangle\!\rangle = \frac{1}{(\mathsf{t}_{12}\mathsf{t}_{34})^{2\Delta_V}}G_{i_1i_2i_3i_4}(\chi)\,, \qquad \Delta_V = 2+b^2+\ldots\,, \tag{3.47}$$

where in general we should have $\Delta_V = 2+\frac{b^2}{1+2b^2}$ (see (1.28), (3.31)). At order $\mathcal{O}(b^2)$ we have trivial disconnected contributions where the two-point function $\langle\!\langle \Phi_i\Phi_j\rangle\!\rangle$ includes propagator loop corrections. These contributions automatically respect the structure in (3.45) and also are consistent with $\mathrm{AdS}_2/\mathrm{CFT}_2^{1/2}$ as discussed in section 3.3 above. Indeed, the disconnected part can be written in the generalized free field form

$$G^{\rm disc}_{i_1i_2i_3i_4}(\chi) = C^2_{+-}\left[\delta_{i_1i_2}\delta_{i_3i_4} + \chi^{2\Delta_V}\,\delta_{i_1i_3}\delta_{i_2i_4} + \left(\frac{\chi}{1-\chi}\right)^{2\Delta_V}\,\delta_{i_1i_4}\delta_{i_2i_3}\right], \tag{3.48}$$

where (cf. (1.37))

$$C_{+-} = \mathcal{C} + \mathcal{O}(b^2)\,, \qquad \mathcal{C} = 2\pi\frac{2}{3\pi} = \frac{4}{3} \tag{3.49}$$

is the coefficient in the two-point function of Φ_i in (1.35). The disconnected $\mathcal{O}(b^2)$ contributions can be found from the tree-level disconnected diagrams where the pairs of Φ_i fields are connected by a tree propagator and the b^2 correction comes from the expansion of the prefactor in (3.45) depending on the coupling b because of the non-zero anomalous dimension in Δ_V.

The disconnected contribution can be represented as (cf. (3.49))

$$\begin{aligned} G^{\rm disc}_{i_1i_2i_3i_4}(\chi) &= \mathcal{C}^2\,(\mathsf{t}_{12}\mathsf{t}_{34})^{2(2+b^2+\cdots)}\left[\frac{\delta_{i_1i_2}\delta_{i_3i_4}}{(\mathsf{t}_{12}\mathsf{t}_{34})^{2(2+b^2+\cdots)}} + 2\text{ crossed terms} + \text{self-energy loops}\right] \\ &= G^{(0)\,\rm disc}_{i_1i_2i_3i_4}(\chi) + \left[G^{(1)\,\rm disc}_{i_1i_2i_3i_4}(\chi) + G^{(1)\,\rm self}_{i_1i_2i_3i_4}(\chi)\right] + \mathcal{O}(b^4). \end{aligned} \tag{3.50}$$

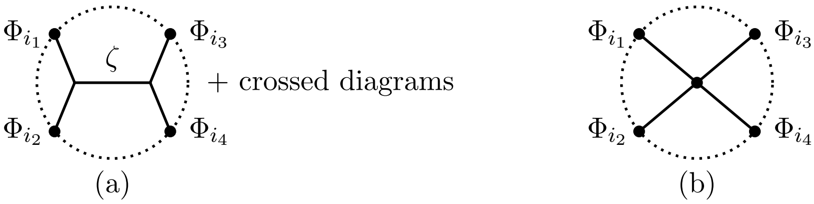


Figure 1. Connected diagrams of order $\mathcal{O}(b^2)$ contributing to $\langle\!\langle \Phi_{i_1}(\mathsf{t}_1)\cdots\Phi_{i_4}(\mathsf{t}_4)\rangle\!\rangle$.

In particular,

$$\begin{aligned}
G^{(0)\,\mathrm{disc}}_{i_1i_2i_3i_4}(\chi) &= \mathcal{C}^2\left[\delta_{i_1i_2}\delta_{i_3i_4}+\chi^4\,\delta_{i_1i_3}\delta_{i_2i_4}+\frac{\chi^4}{(1-\chi)^4}\delta_{i_1i_4}\delta_{i_2i_3}\right],\\
G^{(1)\,\mathrm{disc}}_{i_1i_2i_3i_4}(\chi) &= b^2\,\mathcal{C}^2\,\chi^4\left[2\log\chi\,\delta_{i_1i_3}\delta_{i_2i_4}+2\frac{\log\chi-\log(1-\chi)}{(1-\chi)^4}\delta_{i_1i_4}\delta_{i_2i_3}\right],\\
G^{(1)\,\mathrm{self}}_{i_1i_2i_3i_4}(\chi) &= 2\,(2\log 2-2)\,b^2\,G^{(0)\,\mathrm{disc}}_{i_1i_2i_3i_4}(\chi)\,.
\end{aligned}\tag{3.51}$$

There are also connected $\mathcal{O}(b^2)$ tree diagrams of the two types (see figure 1): the exchange diagrams of the form (a) and the contact diagram (b) where the internal vertex is the derivative-dependent $\xi^2(\partial\xi)^2$ σ-model interaction in (3.18), i.e. the leading-order connected contribution is

$$G^{\mathrm{conn}}_{i_1i_2i_3i_4} = G^{(1)\,\mathrm{exch}}_{i_1i_2i_3i_4}(\chi)+G^{(1)\,\mathrm{cont}}_{i_1i_2i_3i_4}(\chi)+\cdots\,. \tag{3.52}$$

The contribution of the exchange diagrams in figure 1 (a) sums up to

$$G^{(1)\,\mathrm{exch}}_{i_1i_2i_3i_4}(\chi) = \frac{1}{2\pi}\,16\,b^2\,\mathcal{C}^4\left[\delta_{i_1i_2}\delta_{i_3i_4}\frac{D_{1122}}{4\mathsf{t}_{12}^2}+\delta_{i_1i_3}\delta_{i_2i_4}\frac{D_{1212}}{4\mathsf{t}_{13}^2}+\delta_{i_1i_4}\delta_{i_2i_3}\frac{D_{1122}}{4\mathsf{t}_{12}^2}\right], \tag{3.53}$$

where the D-functions are defined by the AdS_2 integral [62–64]

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(\mathsf{t}_1,\mathsf{t}_2,\mathsf{t}_3,\mathsf{t}_4) = \int\frac{d\mathsf{t}\,d\mathsf{z}}{\mathsf{z}^2}\prod_{i=1}^{4}\left[\frac{\mathsf{z}}{\mathsf{z}^2+(\mathsf{t}-\mathsf{t}_i)^2}\right]^{\Delta_i}. \tag{3.54}$$

Using the known explicit expressions for the D-functions in (3.53) we obtain

$$\begin{aligned}
G^{(1)\,\mathrm{exch}}_{i_1i_2i_3i_4}(\chi) = \frac{b^2}{4}\,\mathcal{C}^4\Bigg\{&\delta_{i_1i_2}\delta_{i_3i_4}\left[\frac{\chi^4\log\chi}{(\chi-1)^2}-\frac{\chi^2}{\chi-1}-(\chi+2)\;\chi\log(1-\chi)\right]\\
&+\delta_{i_1i_4}\delta_{i_2,i_3}\left[-\frac{(\chi-3)\chi^4\log\chi}{(\chi-1)^3}+\frac{\chi^3}{(\chi-1)^2}+\chi^2\;\log(1-\chi)\right]\\
&+\delta_{i_1i_3}\delta_{i_2i_4}\left[\frac{(2\chi-3)\chi^4\log\chi}{(\chi-1)^2}+\frac{\chi^3}{\chi-1}-(2\chi+1)\chi^2\log(1-\chi)\right]\Bigg\}.
\end{aligned}\tag{3.55}$$

The contact diagram in figure 1 (b) evaluates to

$$
\begin{aligned}
G^{(1)\,\mathrm{cont}}_{i_1i_2i_3i_4}(\chi) = \frac{b^2}{4}\mathcal{C}^4 \Bigg\{ &\delta_{i_1i_4}\delta_{i_2i_3}\bigg[-\frac{(2\chi-3)(3\chi-2)(3\chi^2+2\chi+3)\chi}{4(\chi-1)^2} \\
&-\frac{1}{2}(3\chi^2-4\chi+3)(3\chi^2+4\chi+3)\log(1-\chi)-\frac{(3\chi-5)(3\chi^2-4\chi+3)\chi^4\log\chi}{2(1-\chi)^3}\bigg] \\
&+\delta_{i_1i_3}\delta_{i_2i_4}\bigg[-\frac{(\chi-3)(\chi+2)(8\chi^2-8\chi+3)\chi}{4(1-\chi)^3}+\frac{1}{2}(2\chi+3)(2\chi^2-2\chi+3)\log(1-\chi) \\
&-\frac{(2\chi-5)(2\chi^2-2\chi+3)\chi^4\log\chi}{2(1-\chi)^4}\bigg]+\delta_{i_1i_2}\delta_{i_3i_4}\bigg[-\frac{(2\chi+1)(3\chi-1)(3\chi^2-8\chi+8)\chi^2}{4(1-\chi)^3} \\
&+\frac{1}{2}(3\chi+2)(3\chi^2-2\chi+2)\chi\log(1-\chi)-\frac{(3\chi^2-10\chi+10)(3\chi^2-2\chi+2)\chi^4\log\chi}{2(1-\chi)^4}\bigg]\Bigg\}.
\end{aligned}
\tag{3.56}
$$

All the three of the above contributions (disconnected (3.51), exchange (3.53) and contact (3.56)) are separately crossing invariant. In particular, if we consider $\langle\!\langle \Phi^+\Phi^-\Phi^+\Phi^-\rangle\!\rangle$ where $\Phi^\pm = \frac{1}{\sqrt{2}}(\Phi_1 \pm i\Phi_2)$ correspond to bulk fields $\xi^\pm$, we can check the $1 \leftrightarrow 3$ crossing invariance relation for each of the contributions

$$
G_{+-+-}(\chi) = \left(\frac{\chi}{1-\chi}\right)^{2(2+b^2+\cdots)} G_{+-+-}(1-\chi)\,. \tag{3.57}
$$

The total $\mathcal{O}(b^2)$ term in $G = G^{\mathrm{disc}} + G^{\mathrm{conn}}$ turns out to be

$$
\begin{aligned}
G_{+-+-}(\chi) = C^2_{+-}\left[1+\frac{\chi^4}{(1-\chi)^4}\right] + b^2\bigg[&-\frac{16\,\chi\,(6-19\chi+19\chi^2)}{27\,(1-\chi)^3} \\
&-\frac{32}{9}\left[1+\frac{\chi^4}{(1-\chi)^4}\right]\log(1-\chi)\bigg]+\cdots.
\end{aligned}
\tag{3.58}
$$

Notice the cancellation of the $\log\chi$ terms which is a non-trivial fact (cf. [8, 10]). This is in agreement with the OPE (3.32) where the first exchanged primary has dimension 4 with no anomalous contribution.

Another interesting case is the four-point function $\langle\!\langle \Phi^+\Phi^+\Phi^-\Phi^-\rangle\!\rangle$. Its coefficient function G_{++--} is related to the above one in (3.58) by the $2 \leftrightarrow 3$ crossing transformation

$$
G_{++--}(\chi) = \chi^{2(2+b^2+\cdots)}\, G_{+-+-}(\chi^{-1})\,. \tag{3.59}
$$

From (3.55) one finds

$$
\begin{aligned}
G_{++--}(\chi) = C^2_{+-}\left[\chi^4+\frac{\chi^4}{(1-\chi)^4}\right]+b^2\bigg\{&\frac{16(6\chi^2-19\chi+19)\chi^4}{27(1-\chi)^3} \\
&-\frac{32}{9}\left[\chi^4+\frac{\chi^4}{(1-\chi)^4}\right][\log(1-\chi)-2\log\chi]\bigg\}+\mathcal{O}(b^4).
\end{aligned}
\tag{3.60}
$$

Expanding in conformal blocks (cf. (3.46))

$$
G_{++--}(\chi) = \sum_{n=4}^{\infty}(a_n+b^2 d_n+\cdots)\,\mathsf{F}_{h_n}(\chi)\,, \qquad h_n = n+b^2\gamma_n+\cdots, \tag{3.61}
$$

we find non-zero contributions only with even values of index n. The leading order coefficients a_{2n} are

$$a_{2n} = \frac{\sqrt{\pi}\, 4^{3-2n}\,(2n-1)(2n-3)(n-1)\Gamma(2n+3)}{81\,\Gamma(2n-\frac{1}{2})}\,, \tag{3.62}$$

while the first few values of d_{2n} are

$$d_4 = \frac{304}{27}, \qquad d_6 = \frac{5056}{729}, \qquad d_8 = \frac{219320}{61347}, \qquad \dots . \tag{3.63}$$

Remarkably, all anomalous dimensions γ_n of the exchanged operators are equal, i.e.

$$\gamma_n = 4\,. \tag{3.64}$$

This follows from the fact that the $b^2 \log\chi$ term in (3.60) is proportional to the tree level disconnected contribution. Again, this feature has a simple explanation on CFT side — from the point of view of the OPE of V^+V^+. All operators in this OPE have a common exponential factor of the form $\exp(i\frac{2\sqrt{2}}{\sqrt{k}}\varphi_2)$ that is just the squared exponential in V^+ in (3.27) and this is the origin of the common anomalous contribution $\frac{1}{2}(\frac{2\sqrt{2}}{\sqrt{k}})^2 = \frac{4}{k}$. In particular, this is true for $n=4$, in agreement with (3.34), i.e. $h_4 = \Delta_{\Omega_0} = 4+\frac{4}{k} = 4+4b^2+\cdots$.

CFT correlator $\langle V^+V^-V^+V^-\rangle$

We can now test $\text{AdS}_2/\text{CFT}_2^{1/2}$ by comparing the above result for $\langle\!\langle \Phi^+\Phi^-\Phi^+\Phi^-\rangle\!\rangle$ in (3.47), (3.60) with the CFT correlator $\langle V^+V^-V^+V^-\rangle$. The latter can be found using the free field representation (3.27) and expanding in large k. This reduces to a purely Wick contraction calculation since after the $1/k$ expansion the exponential operators in (3.27) become a sum of polynomial local operators. Representing the large k expansion as

$$\langle V^+V^-V^+V^-\rangle = k^2\,\mathcal{G}_0(\chi) + k\,\mathcal{G}_1(\chi) + \cdots\,, \tag{3.65}$$

we find

$$\begin{aligned}
\mathcal{G}_0 &= 9\left[1+\frac{\chi^4}{(1-\chi)^4}\right],\\
\mathcal{G}_1 &= \frac{3\,(24-102\chi+169\chi^2-134\chi^3+67\chi^4)}{(1-\chi)^4} - 18\left[1+\frac{\chi^4}{(1-\chi)^4}\right]\log(1-\chi)\,.
\end{aligned} \tag{3.66}$$

Notice that the logarithmic term in (3.66) arises from the contraction of two φ_2 fields with no derivatives which appear from the large k expansion of the exponential in (3.26). As a result, the coefficient of this logarithmic term is proportional to $\mathcal{G}_0$.

To satisfy the $\text{AdS}_2/\text{CFT}_2^{1/2}$ relation (1.30) we should have

$$C_{+-}^2\,\mathsf{G}_0 + b^2\mathsf{G}_1 + \cdots = \kappa_\pm^2(k^2\mathcal{G}_0 + k\mathcal{G}_1 + \cdots)\,, \tag{3.67}$$

where from (3.55)

$$\begin{aligned}
\mathsf{G}_0 &= 1+\frac{\chi^4}{(1-\chi)^4}\,,\\
\mathsf{G}_1 &= -\frac{16\,\chi\,(6-19\chi+19\chi^2)}{27\,(1-\chi)^3} - \frac{32}{9}\left[1+\frac{\chi^4}{(1-\chi)^4}\right]\log(1-\chi)\,.
\end{aligned} \tag{3.68}$$

Expanding (3.67) in small $b = \frac{1}{k} + \ldots$ using the values of $\kappa_\pm$ and $C_{+-}(b)$ from (1.34) and (1.37) gives

$$\mathcal{G}_0 = 9\,\mathsf{G}_0, \qquad \mathcal{G}_1 = \tfrac{81}{16}\,\mathsf{G}_1 + 72\,\mathsf{G}_0\,. \tag{3.69}$$

Comparing (3.66) with (3.68) we find that these relations are indeed satisfied.

Acknowledgments

We would like to thank S. Giombi, E. Perlmutter and V. Rosenhaus for useful discussions of related questions. HJ was supported by Swiss National Science Foundation. AAT was supported by the STFC grant ST/P000762/1.

A Expression for κ_3 in A_2 Abelian Toda theory

To suggest an exact expression for $\kappa_3(b)$ in (1.21) we shall use similar considerations as in the Liouville case in [16]. We shall start with a free field realization of the spin 2 and spin 3 $\mathcal{W}_3$ symmetry generators (T, V_3) based on two real free bosons ϕ_1,ϕ_2 with normalization (different by $\sqrt{2}$ from the one in (3.20))

$$\phi_i(z)\phi_j(0) \sim -2\,\delta_{ij}\,\log z\,. \tag{A.1}$$

Explicitly (conformal normal ordering is understood in all composite operators) [65]

$$T = -\frac{1}{4}\,(\partial\phi_1)^2 - \frac{1}{4}\,(\partial\phi_2)^2 + i\,\alpha_0\,\partial^2\phi_1, \tag{A.2}$$

$$V_3 = \frac{\beta}{12\,i}\,\Big[(\partial\phi_2)^3 - 3\,(\partial\phi_1)^2\,\partial\phi_2 + 3\,i\,\alpha_0\,\partial^2\phi_1\,\partial\phi_2 + 9\,i\,\alpha_0\,\partial\phi_1\partial^2\phi_2 + 6\,\alpha_0^2\,\partial^3\phi_2\Big], \tag{A.3}$$

where (cf. (1.17))

$$\alpha_0 = \sqrt{\frac{2-c}{24}} = -\frac{i}{2}\,Q, \qquad \beta = -\frac{4}{\sqrt{5c+22}}\,, \qquad c = 2 + 6Q^2\,. \tag{A.4}$$

Naively, if we could ignore the contribution of the potential term in the flat-space Toda action (1.16) to its stress tensor (or assume that the stress tensor becomes traceless at the quantum level) we could formally identify the fields in (A.2) as $(\phi_1, \phi_2) = 2\,(\varphi, \psi)$.[17]

In the semiclassical limit starting with (1.16) one may formally eliminate the conformal factor of the AdS$_2$ metric by redefining the Liouville field as $\varphi(t, \mathsf{z}) \to \phi(t, \mathsf{z}) + Q\log\mathsf{z}$, i.e. transform the action into the flat-space one. This leads to the Toda CFT for the field (ϕ, ψ) defined on a flat upper half-plane ($w = \mathsf{t} + i\,\mathsf{z}$, $\mathsf{z} > 0$) with the stress tensor $T(w) = -(\partial_w\phi)^2 - (\partial_w\psi)^2 + Q\,\partial_w^2\phi$ (where $\partial_w = \frac{1}{2}(\partial_\mathsf{t} - i\,\partial_\mathsf{z})$). The field ϕ has then the boundary asymptotics $\phi(\mathsf{t}, \mathsf{z})\big|_{\mathsf{z}\to 0} = \mathsf{z}^2\,\Phi(\mathsf{t}) - Q\log\mathsf{z} + \ldots$. Taking the boundary limit in T ($w \to \mathsf{t}$) gives $T(\mathsf{t}) = -\frac{3}{2}\,Q\,\Phi(t) + \mathcal{O}(\mathsf{z}^2)$. This is precisely the operator relation which is required for the validity of the expression for κ in (1.8), i.e. $\kappa(b) = -\frac{2}{3}\,b + \cdots$. Identifying

[17] In fact, the transformation between the fields in the Liouville or Toda action and the free fields involves a non-trivial Bäcklund transformation [66].

$(\phi_1, \phi_2) = 2\,(\varphi, \psi)$ we get in the boundary limit $\mathsf{z} \to 0$: $\phi_1 \to 2\,\mathsf{z}^2\Phi - 2\,Q\,\log \mathsf{z}$, $\phi_2 \to 2\,\mathsf{z}^3\Phi_3$. The boundary limit of the product $V_3\phi_2$ is then

$$V_3(w)\phi_2(0)\Big|_{\mathsf{z}\to 0} \to 2V_3(\mathsf{t})\,\mathsf{z}^3\Phi_3(0) = 2\,\mathsf{z}^3\,\kappa_3\;V_3(\mathsf{t})V_3(0)\,, \tag{A.5}$$

where we replaced $\Phi_3 \to \kappa_3 V_3$ as it should be in the correlation functions in (1.20). Hence, using that $V_3(w)V_3(0) \sim \frac{c}{3}w^{-6} + \cdots$ we can write κ_3 in terms of the coefficient K in the leading singularity in the OPE (A.5)

$$\kappa_3 = \frac{3}{2\,c}\,K\,, \qquad\qquad V_3(w)\phi_2(0)\Big|_{\mathsf{z}\to 0} \sim \mathsf{z}^3\frac{K}{\mathsf{t}^6} + \cdots\,. \tag{A.6}$$

A straightforward computation using the free-field representation (A.3) gives

$$V_3(w)\phi_2(0) \sim \frac{2\,i\,\beta\,\alpha_0^2}{w^3} + \frac{3}{2}\frac{\beta\,\alpha_0\,\partial\phi_1(0)}{w^2} + \frac{\beta}{w}\left[-\frac{i}{12}\mathcal{X}(0) + \frac{3}{2}\alpha_0\partial^2\phi_1(0)\right]\,, \tag{A.7}$$

where $\mathcal{X} = 6\,(\partial\phi_1)^2 - 6\,(\partial\phi_2)^2 - 6\,i\,\alpha_0\,\partial^2\phi_1$. Then contributions to (A.6) come from the first $\sim w^{-3}$ term and also terms originating from the $-2Q\log\mathsf{z}$ piece in ϕ_1, i.e.

$$\partial\phi_1 = \frac{i\,Q}{\mathsf{z}} + \cdots\,, \qquad\qquad -\frac{i}{12}\mathcal{X} + \frac{3}{2}\alpha_0\partial^2\phi_1 = \frac{3\,i\,Q^2}{4\mathsf{z}^2} + \cdots\,. \tag{A.8}$$

To find the latter one has to take into account the upper half plane mirror poles in the OPE (see, for instance, [67]) and expand near the boundary $\mathsf{z} \to 0$ according to[18]

$$\begin{aligned} \frac{1}{(\mathsf{t}-i\,\mathsf{z})^3} - \frac{1}{(\mathsf{t}+i\,\mathsf{z})^3} &= \cdots - \frac{20\,i\,\mathsf{z}^2}{\mathsf{t}^6} + \cdots\,,\\ \frac{1}{\mathsf{z}}\left[\frac{1}{(\mathsf{t}-i\,\mathsf{z})^2} + \frac{1}{(\mathsf{t}+i\,\mathsf{z})^2}\right] &= \cdots + \frac{10\,\mathsf{z}^2}{\mathsf{t}^6} + \cdots\,,\\ \frac{1}{\mathsf{z}^2}\left[\frac{1}{(\mathsf{t}-i\,\mathsf{z})} - \frac{1}{(\mathsf{t}+i\,\mathsf{z})}\right] &= \cdots + \frac{2\,i\,\mathsf{z}^2}{\mathsf{t}^6} + \cdots\,. \end{aligned} \tag{A.9}$$

This gives in total the following expression for the coefficient K in the OPE in (A.6)

$$K = 2\,i\,\beta\,\alpha_0^2 \times (-20\,i) + \frac{3}{2}\,\beta\,\alpha_0 \times i\,Q \times 10\,i + \beta \times \tfrac{3i}{4}Q^2 \times 2i = -4Q^2\,\beta\,. \tag{A.10}$$

As a result,

$$\kappa_3 = \frac{3}{2\,c}\,K = -\frac{6\,Q^2\,\beta}{c} = \frac{24\,Q^2}{c\,\sqrt{5c+22}}\,, \tag{A.11}$$

which is the expression given in (1.21).

[18]The Dirichlet boundary conditions for the free fields ϕ_1 and ϕ_2 require a non trivial gluing map for the odd spin chiral generators, i.e. the non-trivial reflection relation $V_3(z) = -\overline{V}_3(\overline{z})$ for the odd spin field V_3. The same happens in the familiar example of the spin 1 current $J(z) = i\,\partial\phi$ when ϕ has a Dirichlet boundary condition, see, e.g., section 4.1 of [68].

JHEP09(2019)036

B Fourier expansion of the $\Delta = 3$ propagator in AdS_2

A useful tool to compute the AdS_2 integrals in disk parametrization (2.5) is the Fourier representation of the $\Delta = 2, 3$ propagators (2.3). As illustrated in [69], see also appendix A of [16], the simplest diagrams that occur in the calculations of this paper are multiple AdS_2 integrals that depend on a number of fixed points and on internal points that are integrated in SU(1, 1) covariant way. Using a Fourier representation, one first integrates the relative angles of involved points and performs radial integration as the last step. The coefficients of the Fourier expansion in the $\Delta = 2$ case of $g(x, x')$ are given in [69]. Here we need the generalization to the $\Delta = 3$ case, i.e. the propagator $h(x, x')$ in (2.3). Using the disk coordinates ($x \equiv z$, $x' \equiv z'$) we have η in (2.3) given by $\eta = \left|\frac{z-z'}{1-z\bar{z}'}\right|^2$. The Fourier expansion of $h(z, z')$ then reads

$$h(z, z') = \sum_{n=0}^{\infty} h_n(|z|^2, |z'|^2)\, \cos(n\,\vartheta(z, z')), \tag{B.1}$$

$$h_n(x, y) = \theta(x-y)\, c_n(y)\, d_n(x) + \theta(y-x)\, c_n(x)\, d_n(y)\,, \tag{B.2}$$

where $\vartheta(z, z')$ is the angle between the disc points z and z'. The coefficients $c_n(x)$ are given by (for any $n \geq 0$)

$$c_n(x) = \frac{x^{n/2}}{(1-x)^2}\left[1 - \frac{2\,(n-2)}{n+1}\, x + \frac{(n-1)(n-2)}{(n+2)(n+1)}\, x^2\right]. \tag{B.3}$$

The coefficients $d_n(x)$ take the following special values for $n = 0, 1, 2$

$$\begin{aligned} d_0(x) &= \frac{3(x+1)}{2(x-1)} - \frac{(x^2+4x+1)\log x}{2(x-1)^2}, \\ d_1(x) &= \frac{-x^2-10x-1}{(x-1)\sqrt{x}} + \frac{6\sqrt{x}(x+1)\log x}{(x-1)^2}, \\ d_2(x) &= \frac{x^3-7x^2-7x+1}{2x-2x^2} - \frac{6x\log x}{(x-1)^2}, \end{aligned} \tag{B.4}$$

while for all $n \geq 3$

$$\begin{aligned} d_n(x) = &\frac{1}{n(n-1)(n-2)\, x^{n/2}\,(1-x)^2}\Big[-(n-2)(n-1)x^{n+2} + 2(n-2)(n+2)x^{n+1} \\ &-(n+1)(n+2)x^n + (n+1)(n+2)x^2 - 2(n-2)(n+2)x + (n-2)(n-1)\Big]. \end{aligned} \tag{B.5}$$

C Perturbative calculation of C_{222} and C_{233} in A_2 abelian Toda theory

To compute three-point boundary correlators in (2.24) it is useful first to recall the form of the tree level bulk-to-boundary scalar propagator in AdS_2 in disc parametrization. On the Poincaré half plane, assuming Dirichlet boundary conditions, for generic mass or Δ ($m^2 = \Delta(\Delta - 1)$) we may define (here $w = (\mathsf{t}, \mathsf{z})$)

$$g_\Delta(\mathsf{t}, w') = \mathsf{z}^{-\Delta}\, g_\Delta(w, w')\Big|_{\mathsf{z}\to 0}, \tag{C.1}$$

where for the $\Delta = 2, 3$ fields ζ and ψ of the A_2 abelian Toda theory (cf. (2.3))

$$g(\mathsf{t}, w') \equiv g_2(\mathsf{t}, w') = \frac{4}{3}\left[\frac{\mathsf{z}'}{\mathsf{z}'^2 + (\mathsf{t}-\mathsf{t}')^2}\right]^2, \qquad h(\mathsf{t}, w') \equiv g_3(\mathsf{t}, w') = \frac{16}{15}\left[\frac{\mathsf{z}'}{\mathsf{z}'^2 + (\mathsf{t}-\mathsf{t}')^2}\right]^3. \tag{C.2}$$

Under the map (2.5) the disk boundary $|z| = 1$ is mapped to the real axis $\mathsf{z} = 0$ of Poincaré plane as

$$\mathsf{t}(\theta) = -i\frac{e^{i\theta}+1}{e^{i\theta}-1} = -\cot\frac{\theta}{2}, \qquad\qquad z = r\, e^{i\theta}. \tag{C.3}$$

Then the bulk-to-boundary propagators (C.2) become

$$g(\theta, z') = \frac{4}{3}\frac{\sin^4\frac{\theta}{2}\,(1-|z'|)^2}{|e^{i\theta}-z'|^4}, \qquad\qquad h(\theta, z') = \frac{16}{15}\frac{\sin^6\frac{\theta}{2}\,(1-|z'|)^3}{|e^{i\theta}-z'|^6}. \tag{C.4}$$

In general, for three fields of dimensions Δ_k we will have

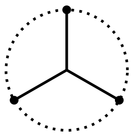
$$\langle\!\langle \Phi_{\Delta_1}\Phi_{\Delta_2}\Phi_{\Delta_3}\rangle\!\rangle = C_{\Delta_1\Delta_2\Delta_3}\,\mathcal{K}_{\Delta_1\Delta_2\Delta_3}(\theta_1,\theta_2,\theta_3), \tag{C.5}$$
$$\mathcal{K}_{\Delta_1\Delta_2\Delta_3}(\theta_1,\theta_2,\theta_3) = |\mathsf{t}(\theta_1)-\mathsf{t}(\theta_2)|^{\Delta_3-\Delta_1-\Delta_2}\,|\mathsf{t}(\theta_2)-\mathsf{t}(\theta_3)|^{\Delta_1-\Delta_2-\Delta_3}\,|\mathsf{t}(\theta_3)-\mathsf{t}(\theta_1)|^{\Delta_2-\Delta_1-\Delta_3}.$$

Let us now consider the Witten diagrams contributing the two non-vanishing three-point boundary correlators $\langle\!\langle\Phi\Phi\Phi\rangle\!\rangle$ and $\langle\!\langle\Phi\Phi_3\Phi_3\rangle\!\rangle$.

Coefficient C_{222} in $\langle\!\langle\Phi\Phi\Phi\rangle\!\rangle$

Let us start with computing the coefficient C_{222} of the contributions proportional to $\mathcal{K} \equiv \mathcal{K}_{222}$ in (C.5). The expression for $\langle\!\langle\Phi\Phi\Phi\rangle\!\rangle$ will differ, in general, from the one in the Liouville theory because here the second field ψ may also appear in virtual exchanges.

Tree level diagram. At leading order in b there is a single diagram

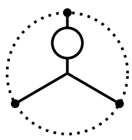
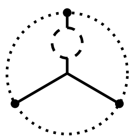
$C^{(0)}_{222}\, b =$  $= \frac{1}{\mathcal{K}}(-8b)\int d^2w\, g(\mathsf{t}_1, w)g(\mathsf{t}_2, w)g(\mathsf{t}_3, w) = \frac{2}{9}(-8b) = -\frac{16}{9}b.$ (C.6)

This is a special case of the general expression

$$\begin{aligned}&\frac{1}{\mathcal{K}}\int d^2w\, g_{\Delta_1}(\mathsf{t}_1, w)g_{\Delta_2}(\mathsf{t}_2, w)g_{\Delta_3}(\mathsf{t}_3, w)\\ &= \frac{\pi}{8}\frac{\Gamma(\frac{\Delta_1+\Delta_2-\Delta_3}{2})\Gamma(\frac{\Delta_1+\Delta_3-\Delta_2}{2})\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_1+\Delta_2+\Delta_3-1}{2})}{\Gamma(\frac{1}{2}+\Delta_1)\Gamma(\frac{1}{2}+\Delta_2)\Gamma(\frac{1}{2}+\Delta_3)},\end{aligned} \tag{C.7}$$

giving $\frac{2}{9}$ for $\Delta_1 = \Delta_2 = \Delta_3 = 2$.

Diagrams with dressed propagators. These diagrams give

$C^{(1)}_{222,1}\, b^3 =$  $+$  $+ \quad 2$ permutations. (C.8)

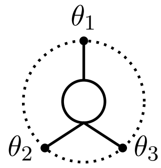
Their contribution is

$$C^{(1)}_{222,1} = 3\left(-\frac{1}{3}\right)C^{(0)}_{222} = -C^{(0)}_{222}. \tag{C.9}$$

JHEP09(2019)036

Diagrams with ζ loop. The contribution of the diagrams

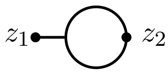
$$C^{(1)}_{222,2} b^3 =$$



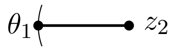
$$+ \quad 2 \text{ permutations}, \tag{C.10}$$

may be found by taking the boundary limit of (the propagator on the r.h.s. is bulk-to-boundary one)

$$\lim_{z_1 \to 0} \frac{1}{z_1^2}$$



$$= \frac{1}{8}$$



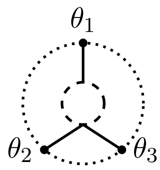
$$. \tag{C.11}$$

This gives

$$C^{(1)}_{222,2} b^3 = \frac{1}{8} (-8b) (-16b^2) \frac{1}{2} C^{(0)}_{222} b \left(-\frac{1}{8b}\right) \times 3 = -3\, C^{(0)}_{222}\, b^3. \tag{C.12}$$

Diagrams with ψ loop. Similarly, the diagrams

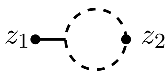
$$C^{(1)}_{222,3} b^3 =$$



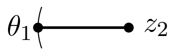
$$+ \quad 2 \text{ permutations}\,, \tag{C.13}$$

may be computed from

$$\lim_{z_1 \to 0} \frac{1}{z_1^2}$$



$$= \frac{1}{24}$$



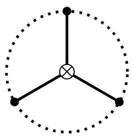
$$, \tag{C.14}$$

and give

$$C^{(1)}_{222,3} b^3 = \frac{1}{24} \big(-12b\, 2!\big) \big(-12b^2\, 2!\, 2!\big) \frac{1}{2} C^{(0)}_{222} b \left(-\frac{1}{8b}\right) \times 3 = -9\, C^{(0)}_{222}\, b^3. \tag{C.15}$$

Diagram with vertex insertion. According to the structure of the interaction Lagrangian in (2.7), at order b^3 we need to include also a special tree diagram

$$C^{(1)}_{222,4}\, b^3 =$$



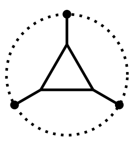
$$, \tag{C.16}$$

where the insertion represents the b^3 vertex appearing due to the overall factor $1 + 4b^2$ in (2.7). This gives (cf. (C.6))

$$C^{(1)}_{222,4} = 4\, C^{(0)}_{222}. \tag{C.17}$$

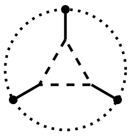
Triangle diagram with ζ loop. This non-trivial diagram is the same as in the Liouville theory and thus can be found from [16]:

$$C^{(1)}_{222,5} = b^{-3}$$



$$= \frac{7}{6} C^{(0)}_{222} = -\frac{56}{27}. \tag{C.18}$$

Triangle diagrams with ψ loop. This diagram is given by the following finite disk integral

$$C^{(1)}_{222,6}\, b^3 =$$  $$\text{(C.19)}$$

$$= \frac{(-24b)^3}{\mathcal{K}} \int \mathsf{d}^2 w_1 \mathsf{d}^2 w_2 \mathsf{d}^2 w_3\, g(\mathsf{t}_1, w_1) g(\mathsf{t}_2, w_2) g(\mathsf{t}_3, w_3) h(w_1, w_2) h(w_1, w_3) h(w_2, w_3).$$

Its numerical calculation by the same method as discussed in [16] gives[19]

$$C^{(1)}_{222,6} = -5.619 \pm 0.011. \tag{C.20}$$

Total coefficient of three-point correlator. The duality prediction (1.24) amounts to

$$\frac{C^{(1)}_{222}}{C^{(0)}_{222}} \equiv \sum_{a=1}^{6} \frac{C^{(1)}_{222,a}}{C^{(0)}_{222}} = -\frac{14}{3}. \tag{C.21}$$

Thus, we should have the following exact value of the contribution $C^{(1)}_{222,6}$ in (C.20)

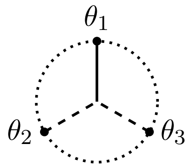
$$C^{(1)}_{222,6} = C^{(0)}_{222} \left(-\frac{14}{3} + 1 - 4 + 3 + 9 - \frac{7}{6} \right) = \frac{19}{6} C^{(0)}_{222} = -\frac{152}{27} = -5.\overline{629}. \tag{C.22}$$

Comparing this to (C.20), a relative error is just 0.2%, thus confirming the validity of (1.24).

Coefficient C_{233} in $\langle\langle \Phi \Phi_3 \Phi_3 \rangle\rangle$

Let us now check the prediction (1.24) for C_{233}. Here we shall use the notation $\mathcal{K} \equiv \mathcal{K}_{233}$ (see (C.5)).

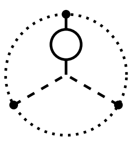
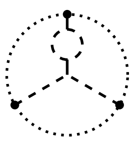
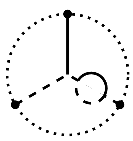
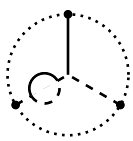
Tree level diagram. The leading order $\mathcal{O}(b)$ contribution is given by

$$C^{(0)}_{233} b =$$ 

$$= \frac{1}{\mathcal{K}} (-24b) \int \mathsf{d}^2 w\, g(\mathsf{t}_1, w) h(\mathsf{t}_2, w) h(\mathsf{t}_3, w) = \frac{4}{45} (-24b)$$

$$= -\frac{32}{15} b\,. \tag{C.23}$$

Diagrams with dressed propagators. The diagrams that are obtained from the tree level diagram by adding loop to one of the propagators are

$$C^{(1)}_{233,1}\, b^3 =$$  $+$  $+$  $+$  . $$\text{(C.24)}$$

[19]In principle, one could try to use an analytic approach based on an exact integral representation for the triangle diagram. One such option is the split-representation discussed, e.g., in [14]. However, if one is interested in the final number and not just in special analytical features (like residues) such representation does not appear to be very useful.

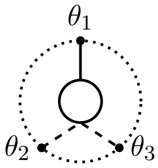
JHEP09(2019)036

Their contribution is

$$C^{(1)}_{233,1} = \left[\left(-\frac{1}{6} - \frac{1}{6}\right) + 2\cdot\left(-\frac{7}{5}\right)\right] C^{(0)}_{233} = -\frac{47}{15}\, C^{(0)}_{233}. \tag{C.25}$$

Diagram with ζ loop. Taking into account (C.11) we find

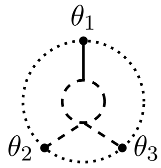
$$C^{(1)}_{233,2} b^3 = \tag{C.26}$$



$$= \frac{1}{8}\left(-\frac{4b}{3}\,3!\right)\left(-12b^2\,2!\,2!\right)\frac{1}{2}\,C^{(0)}_{233} b\left(-\frac{1}{24b}\right) = -C^{(0)}_{233}\, b^3. \tag{C.27}$$

Diagram with ψ loop. In a similar way, the diagram

$$C^{(1)}_{233,3}\, b^3 = \quad , \tag{C.28}$$

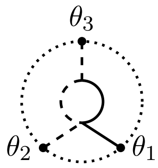


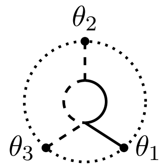
may be computed by using (C.14) giving

$$C^{(1)}_{233,3}\, b^3 = \frac{1}{24}\left(-12b\,2!\right)\left(-6b^2\,4!\right)\frac{1}{2}\,C^{(0)}_{233}\, b\left(-\frac{1}{24b}\right) = -3\,C^{(0)}_{233}\, b^3. \tag{C.29}$$

Diagrams with mixed $\zeta\psi$ loop. The two diagrams

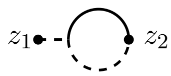
$$C^{(1)}_{233,4}\, b^3 = \quad + \quad , \tag{C.30}$$





can be computed using by the relation

$$\lim_{z_1\to 0} \frac{1}{z_1^3} \quad = \frac{1}{12} \quad , \tag{C.31}$$

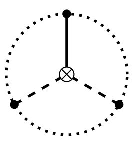


leading to

$$C^{(1)}_{233,4} = \frac{1}{12}\left(-12b\,2!\right)\left(-12b^2\,2!\,2!\right)C(0)_{233} b\left(-\frac{1}{24b}\right)\times 2 = -8\,C^{(0)}_{233}\, b^3. \tag{C.32}$$

Diagram with vertex insertion. The diagram

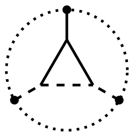
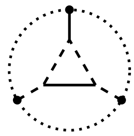
$$C^{(1)}_{233,5}\, b^3 = \quad , \tag{C.33}$$



comes from the b^3 cubic vertex in the action (2.7), giving

$$C^{(1)}_{233,5} = 4\,C^{(0)}_{233}\,. \tag{C.34}$$

Triangle loop diagrams. The two most complicated diagrams

$$C^{(1)}_{233,6}\, b^3 =$$  $$+$$  , (C.35)

correspond to

$$\begin{aligned} &C^{(1)}_{233,6}\, b^3 = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{C.36}) \\ &\frac{1}{\mathcal{K}}(-8b)\,(-24b)^2 \int \mathsf{d}^2 w_1 \mathsf{d}^2 w_2 \mathsf{d}^2 w_3\, g(\mathsf{t}_1, w_1) h(\mathsf{t}_2, w_2) h(\mathsf{t}_3, w_3) g(w_1, w_2) g(w_1, w_3) h(w_2, w_3) \\ &+ \frac{1}{\mathcal{K}}(-24b)^3 \int \mathsf{d}^2 w_1 \mathsf{d}^2 w_2 \mathsf{d}^2 w_3\, g(\mathsf{t}_1, w_1) h(\mathsf{t}_2, w_2) h(\mathsf{t}_3, w_3) h(w_1, w_2) h(w_1, w_3) g(w_2, w_3)\,. \end{aligned}$$

These integrals require a numerical evaluation following the same method as in [16] giving

$$C^{(1)}_{233,6} = -11.47 \pm 0.03. \tag{C.37}$$

Total coefficient in three-point correlator. From the expected result for C_{233} in (1.24) we should have

$$\frac{C^{(1)}_{233}}{C^{(0)}_{233}} \equiv \sum_{a=1}^{6} \frac{C^{(1)}_{233,a}}{C^{(0)}_{233}} = -\frac{86}{15}. \tag{C.38}$$

This corresponds to the following prediction for the numerical coefficient $C^{(1)}_{233,6}$

$$C^{(1)}_{233,6} = C^{(0)}_{233}\left(-\frac{86}{15} + \frac{47}{15} + 1 + 3 + 8 - 4\right) = \frac{27}{5} C^{(0)}_{233} = -\frac{288}{25} = -11.52\ldots\,. \tag{C.39}$$

Compared with the numerical value in (C.37), we find good agreement with a relative error of about 0.4%. This strongly supports the validity of (1.24).

D Perturbative calculation of C_{2+-} in non-abelian Toda theory

Here we collect the details of the calculation of the coefficient C_{2+-} in the boundary correlator in (1.39) leading to the result in (1.40), (3.43). The aim is to reproduce the expansion

$$\langle\!\langle \Phi^+(\mathsf{t}_1)\Phi^-(\mathsf{t}_2)\Phi(\mathsf{t}_3)\rangle\!\rangle = \frac{\mathsf{C}^{(0)}\, b}{(\mathsf{t}_1-\mathsf{t}_2)^2(\mathsf{t}_3-\mathsf{t}_1)^2(\mathsf{t}_3-\mathsf{t}_2)^2}\left[1 + b^2\left(\mathsf{C}^{(1)} + \mathsf{C}^{(1)}_{\log}\log|\mathsf{t}_{12}|\right) + \cdots\right], \tag{D.1}$$

where

$$\mathsf{C}^{(0)} = -\frac{16}{9}, \qquad \mathsf{C}^{(1)} = 2\log 2 - 5, \qquad \mathsf{C}^{(1)}_{\log} = -2\,. \tag{D.2}$$

We shall consider in turn various classes of loop diagrams contributing to (D.1) starting with the AdS_2 action (3.18). The coefficient $\mathsf{C}^{(1)}_{\log}$ will get contributions only from diagrams with dressed ξ propagators while $\mathsf{C}^{(1)}$ will receive several types of contributions.

Diagrams with dressed ξ propagator

The diagrams with dressed ξ propagators can be obtained from the three-point tree-level contact diagrams by replacing one of the bulk-to-boundary ξ-propagator by its loop corrected version $g^{\text{1-loop}}(\mathsf{t},w)$ in (3.39). Explicitly it can be computed as follows

$$
\begin{aligned}
&(-8b)\int d^2w\, g^{\text{1-loop}}(\mathsf{t}_1,w)\; g^{\text{1-loop}}(\mathsf{t}_2,w)\; g(\mathsf{t}_3,w)\\
&=(-8b)\left[1+b^2(2\log 2-2)\right]^2\frac{1}{4\pi}\left(\frac{4}{3}\right)^3\int\frac{d\mathsf{t}d\mathsf{z}}{\mathsf{z}^2}\left(\frac{\mathsf{z}}{\mathsf{z}^2+(\mathsf{t}-\mathsf{t}_1)^2}\right)^{2+b^2}\left(\frac{\mathsf{z}}{\mathsf{z}^2+(\mathsf{t}-\mathsf{t}_2)^2}\right)^{2+b^2}\\
&\quad\times\left(\frac{\mathsf{z}}{\mathsf{z}^2+(\mathsf{t}-\mathsf{t}_3)^2}\right)^2\\
&=-\frac{16b}{9}\frac{1}{|\mathsf{t}_{12}|^2|\mathsf{t}_{23}|^2|\mathsf{t}_{31}|^2}\left[1+b^2\left(2\log 2-\frac{10}{3}\right)-2b^2\log|\mathsf{t}_{12}|\right]+\mathcal{O}\left(b^4\right),
\end{aligned}
\tag{D.3}
$$

where we used the relation (C.7) written in the form

$$
\int\frac{d\mathsf{t}d\mathsf{z}}{\mathsf{z}^2}\prod_{i=1}^{3}\left(\frac{\mathsf{z}}{\mathsf{z}^2+(\mathsf{t}-\mathsf{t}_i)^2}\right)^{\Delta_i}=\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{\Delta_1+\Delta_2-\Delta_3}{2})\Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2})\Gamma(\frac{\Delta_1+\Delta_3-\Delta_2}{2})\Gamma(\frac{\Delta_1+\Delta_2+\Delta_3-1}{2})}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)|\mathsf{t}_{12}|^{\Delta_1+\Delta_2-\Delta_3}|\mathsf{t}_{23}|^{\Delta_2+\Delta_3-\Delta_1}|\mathsf{t}_{31}|^{\Delta_3+\Delta_1-\Delta_2}}.
\tag{D.4}
$$

Comparing with (D.1) we see that the overall coefficient $\mathsf{C}^{(0)}$ and the logarithmic coefficient $\mathsf{C}^{(1)}_{\log}$ already match (D.2) while the coefficient $\mathsf{C}^{(1)}$ gets the following contribution

$$
\mathsf{C}^{(1)}_1=2\log 2-\frac{10}{3}.
\tag{D.5}
$$

Diagrams with non-derivative vertices

There is a set of simple diagrams which do not involve the derivative interaction vertex $\xi^2(\partial\xi)^2$ in (3.18) and thus do not require new calculations: their contributions can be obtained from previous results in the Liouville or abelian A_2 Toda theory by adjusting combinatorial coefficients (below we indicate this by using the symbol "$\times$"). These are:

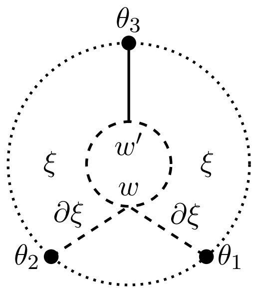
1. Diagrams with dressed ζ propagator:
$$
\mathsf{C}^{(1)}_2=-\frac{1}{6}\times 3=-\frac{1}{2}.
\tag{D.6}
$$
2. Diagram with extra cubic vertex from the factor $\frac{Q}{b}=\frac{1}{b^2}(1+3\,b^2)$ in (3.14):
$$
\mathsf{C}^{(1)}_3=3\times 1=3.
\tag{D.7}
$$
3. Triangle diagrams with two possible fields in the loop:
$$
\mathsf{C}^{(1)}_4=2\times\frac{7}{6}=\frac{7}{3}.
\tag{D.8}
$$
4. Diagrams with a cubic and a non-derivative quartic vertex: (i) for only ζ in the loop (one cubic ζ^3 and one quartic $\zeta^2\xi^2$ vertex)
$$
\mathsf{C}^{(1)}_5=-1,
\tag{D.9}
$$
and with both ζ and ξ in the loop (one cubic $\zeta\xi^2$ and one quartic $\zeta^2\xi^2$ vertices) we get
$$
\mathsf{C}^{(1)}_6=2\times(-2)=-4.
\tag{D.10}
$$

Diagrams with derivative vertex $\xi^2(\partial\xi)^2$

This are the diagrams involve the σ-model derivative interaction $\xi^2(\partial\xi)^2$ in (3.18). There are different types of them depending on which leg (bulk-to-bulk or bulk-to-boundary) is acted on by the two derivatives.

Type I. The first relevant diagram is

$$\mathsf{C}_7^{(1)}\,\mathsf{C}^{(0)} b^3 = \tag{D.11}$$



To evaluate it we find it convenient to use the identity in eq. (4.10) of [8], i.e.

$$\begin{aligned} &g^{ab}\partial_a \widetilde{K}_{\Delta_1}(\mathsf{z},\mathsf{t};\mathsf{t}_1)\,\partial_b \widetilde{K}_{\Delta_2}(\mathsf{z},\mathsf{t};\mathsf{t}_2) \\ &\quad = \Delta_1\Delta_2\Big[\widetilde{K}_{\Delta_1}(\mathsf{z},\mathsf{t};\mathsf{t}_1)\widetilde{K}_{\Delta_2}(\mathsf{z},\mathsf{t};\mathsf{t}_2) - 2\mathsf{t}_{12}^2\widetilde{K}_{\Delta_1+1}(\mathsf{z},\mathsf{t};\mathsf{t}_1)\widetilde{K}_{\Delta_2+1}(\mathsf{z},\mathsf{t};\mathsf{t}_2)\Big], \end{aligned} \tag{D.12}$$

where $g^{ab} = \mathsf{z}^2\delta^{ab}$ is the AdS_2 metric and $\widetilde{K}_\Delta(\mathsf{z},\mathsf{t},\mathsf{t}') = \left(\frac{\mathsf{z}}{\mathsf{z}^2+(\mathsf{t}-\mathsf{t}')^2}\right)^\Delta$. This allows us to transform the expression for the above diagram into another one with $\Delta = 3$ external propagators and only non-derivative interactions. Using that (here $\mathcal{C}_2 = \frac{2}{3\pi}$ as in (2.2))

$$\begin{aligned} &\int \mathsf{d}^2w\,\partial^a g(w,\mathsf{t}_1)\partial_a g(w,\mathsf{t}_2) g(w,\mathsf{t}_3) = (2\pi\mathcal{C}_2)^3 \int \mathsf{d}^2 w\,\partial^a \widetilde{K}_2(w,\mathsf{t}_1)\partial_a\widetilde{K}_2(w,\mathsf{t}_2)\widetilde{K}_2(w,\mathsf{t}_3) \\ &= (2\pi\mathcal{C}_2)^3 \int \mathsf{d}^2 w\,4\Big(\widetilde{K}_2(w,\mathsf{t}_1)\widetilde{K}_2(w',\mathsf{t}_2) - 2\mathsf{t}_{12}^2\widetilde{K}_3(w,\mathsf{t}_1)\widetilde{K}_3(w,\mathsf{t}_2)\Big)\widetilde{K}_2(w,\mathsf{t}_3) \\ &= 4(2\pi\mathcal{C}_2)^3\left(\frac{3}{32\,|\mathsf{t}_{12}|^2|\mathsf{t}_{23}|^2|\mathsf{t}_{31}|^2} - 2\mathsf{t}_{12}^2\frac{15}{256\,|\mathsf{t}_{12}|^4|\mathsf{t}_{23}|^2\mathsf{t}_{31}|^2}\right) = -\frac{2}{9}\,\frac{1}{|\mathsf{t}_{12}|^2|\mathsf{t}_{23}|^2|\mathsf{t}_{31}|^2}, \end{aligned} \tag{D.13}$$

this diagram evaluates to (cf. (2.11), (C.11))

$$\begin{aligned} &\int \mathsf{d}^2 w\;\partial^a g(w,\mathsf{t}_1)\partial_a g(w,\mathsf{t}_2)\;\frac{1}{\mathsf{z}_3^2}B(w,w_3) \overset{\mathsf{z}_3\to 0}{=} \frac{1}{8}\int \mathsf{d}^2 w\;\partial^a g(w,\mathsf{t}_1)\partial_a g(w,\mathsf{t}_2) g(w,\mathsf{t}_3) \\ &\quad = \frac{1}{8}\left(-\frac{2}{9}\right)\,\frac{1}{|\mathsf{t}_{12}|^2|\mathsf{t}_{23}|^2\mathsf{t}_{31}|^2} = -\frac{1}{36}\,\frac{1}{|\mathsf{t}_{12}|^2|\mathsf{t}_{23}|^2\mathsf{t}_{31}|^2}\,. \end{aligned} \tag{D.14}$$

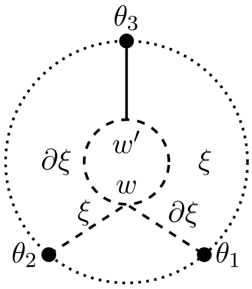
As a result, its contribution to $\mathsf{C}^{(1)}$ is given by

$$\mathsf{C}_7^{(1)}\,\mathsf{C}^{(0)} b^3 = (-8b)\,(b^2\times 2\times 2)\,2\times\frac{1}{2}\left(-\frac{1}{36}\right) = -\frac{1}{2}\mathsf{C}^{(0)}b^3 \quad\to\quad \mathsf{C}_7^{(1)} = -\frac{1}{2}\,, \tag{D.15}$$

where the factor of 2 comes from the two species of the fields in the loop and $\frac{1}{2}$ is a symmetry factor.

Type II. If one derivative from $\xi^2(\partial\xi)^2$ is on a bulk propagator while the other is on a bulk-to-boundary propagator, we get the diagram

$$\mathsf{C}_8^{(1)}\,\mathsf{C}^{(0)} b^3 = \tag{D.16}$$



It can be computed in terms of the function defined in (2.11)

$$B(w, w_3) = \int \mathsf{d}^2 w' \, \left[g(w, w')\right]^2 g(w', w_3). \tag{D.17}$$

Taking a derivative over w^a gives

$$\partial_a B(w, w_3) = 2 \int \mathsf{d}^2 w' \, \partial_a g(w, w') g(w, w') g(w', w_3). \tag{D.18}$$

On the other hand, sending w_3 to the boundary, we have

$$\lim_{\mathsf{z}_3 \to 0} \frac{1}{\mathsf{z}_3^2} B(w, w_3) = \frac{1}{8} g(w, \mathsf{t}_3)\,, \tag{D.19}$$

and thus

$$\int \mathsf{d}^2 w' \, \partial_a g(w, w') g(w, w') g(w', w_3) = \frac{1}{16} \partial_a g(w, \mathsf{t}_3)\,. \tag{D.20}$$

These simple manipulations allow us to write the contribution of the above diagram as

$$\begin{aligned} &\int \mathsf{d}^2 w' \, \mathsf{d}^2 w \; \partial^a g(w, \mathsf{t}_1) \partial_a g(w, w') g(w, w') g(w', w_3) g(w, \mathsf{t}_2) \\ &\quad = \frac{1}{16} \int \mathsf{d}^2 w \; \partial^a g(w, \mathsf{t}_1) \partial_a g(w', w_3) g(w, \mathsf{t}_2). \end{aligned} \tag{D.21}$$

This is the same expression as for the diagram of Type I (cf. (D.14)) and thus we get

$$\frac{1}{16} \left(-\frac{2}{9}\right) \frac{1}{|\mathsf{t}_{12}|^2 |\mathsf{t}_{23}|^2 \mathsf{t}_{31}|^2} = -\frac{1}{72} \frac{1}{|\mathsf{t}_{12}|^2 \mathsf{t}_{23}|^2 |\mathsf{t}_{31}|^2}\,. \tag{D.22}$$

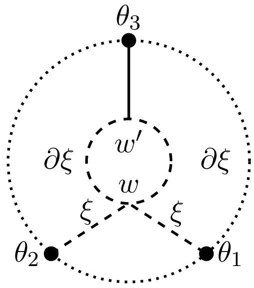
As a result

$$\mathsf{C}_8^{(1)}\,\mathsf{C}^{(0)}\; b^3 = (-8b)\,(b^2 \times 2 \times 2) \times 2 \left(-\frac{1}{72}\right) = -\frac{1}{2} \mathsf{C}^{(0)} b^3 \quad \to \quad \mathsf{C}_8^{(1)} = -\frac{1}{2}\,, \tag{D.23}$$

where the factor 2 comes from the two possibilities of assigning the derivative (on either leg 1 or 2).

Type III. When both derivatives from the σ-model vertex are on the bulk propagators we obtain

$$\mathsf{C}_9^{(1)}\,\mathsf{C}^{(0)} b^3 = \tag{D.24}$$



The diagram requires the calculation of

$$B_{\partial\partial}(w, w_3) = \int \mathsf{d}^2 w'\; \partial^a g(w,w')\partial_a g(w,w') g(w',w_3). \tag{D.25}$$

Taking another derivative of (D.18) gives ($\Box \equiv \frac{1}{\sqrt{g}}\partial^a\partial_a$)

$$\Box B(w,w_3) = 2\int \mathsf{d}^2 w'\; \Box g(w,w') g(w,w') g(w',w_3) + 2\, B_{\partial\partial}(w,w_3). \tag{D.26}$$

The $m^2 = 2$ scalar propagator in AdS_2 satisfies

$$\Box g(w,w') = 2 g(w,w') - \frac{1}{\sqrt{g}}\,\delta^{(2)}(w,w')\,. \tag{D.27}$$

Using this in (D.26) the δ-function gives a contact term proportional to the propagator at coincident points $g(w,w)$ which we set to zero in the AdS scheme (consistent with the AdS_2 symmetry). Thus

$$B_{\partial\partial}(w,w_3) = \frac{1}{2}\Box B(w,w_3) - 2B(w,w_3). \tag{D.28}$$

To compute $B_{\partial\partial}$ as a function of the η-invariant in (2.3) we observe that

$$\Box B = \eta(\eta-1)^2 \partial_\eta\partial_\eta B + (\eta-1)^2\partial_\eta B\,, \tag{D.29}$$

where we used the relations $\Box\eta = \eta(\eta-1)^2$ and $\partial^a\eta\partial_a\eta = (\eta-1)^2$, that follow from the definition of η in (2.3). Sending w_3 to the boundary, we obtain

$$\lim_{\mathsf{z}_3\to 0}\frac{1}{\mathsf{z}_3^2} B_{\partial\partial}(w,w_3) = -\frac{1}{8} g(w,\mathsf{t}_3). \tag{D.30}$$

Finally, the above diagram thus gives

$$-\frac{1}{8}\frac{2}{9}\frac{1}{|\mathsf{t}_{12}|^2 \mathsf{t}_{23}|^2\mathsf{t}_{31}|^2} = -\frac{1}{36}\frac{1}{|\mathsf{t}_{12}|^2\mathsf{t}_{23}|^2\mathsf{t}_{31}|^2}, \tag{D.31}$$

which results in the following contribution to the coefficient $\mathsf{C}^{(1)}$

$$\mathsf{C}_9^{(1)}\,\mathsf{C}^{(0)}\,b^3 = (-8b)\,(b^2\times 2\times 2)\times 2\times\frac{1}{2}\left(-\frac{1}{36}\right) = -\frac{1}{2}\mathsf{C}^{(0)}\,b^3 \quad\to\quad \mathsf{C}_9^{(1)} = -\frac{1}{2}, \tag{D.32}$$

where the factor of 2 accounts for the two possible fields in the loop and $\frac{1}{2}$ is the symmetry factor.

JHEP09(2019)036

Total result for $\mathsf{C}^{(1)}$

Summing up all the 9 contributions given above in (D.5), (D.6), (D.7), (D.8), (D.9), (D.10), (D.15), (D.23), (D.32) we get for the total value of the coefficient $\mathsf{C}^{(1)}$ in (D.1)

$$\mathsf{C}^{(1)} = \sum_{i=1}^{9} \mathsf{C}_i^{(1)} = 2\log 2 - \frac{10}{3} - \frac{1}{2} + 3 + \frac{7}{3} - 1 - 4 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = 2\,\log 2 - 5\,, \tag{D.33}$$

which is in agreement with (D.2).

References

[1] E. D'Hoker and R. Jackiw, *Space translation breaking and compactification in the Liouville theory*, *Phys. Rev. Lett.* **50** (1983) 1719 [INSPIRE].

[2] E. D'Hoker, D.Z. Freedman and R. Jackiw, SO(2, 1) *Invariant Quantization of the Liouville Theory*, *Phys. Rev.* **D 28** (1983) 2583 [INSPIRE].

[3] T. Inami and H. Ooguri, *Dynamical breakdown of supersymmetry in two-dimensional Anti de Sitter space*, *Nucl. Phys.* **B 273** (1986) 487 [INSPIRE].

[4] C.G. Callan Jr. and F. Wilczek, *Iinfrared behaviour at negative curvature*, *Nucl. Phys.* **B 340** (1990) 366 [INSPIRE].

[5] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152 [INSPIRE].

[6] D. Carmi, L. Di Pietro and S. Komatsu, *A Study of Quantum Field Theories in AdS at Finite Coupling*, *JHEP* **01** (2019) 200 [arXiv:1810.04185] [INSPIRE].

[7] N. Drukker and S. Kawamoto, *Small deformations of supersymmetric Wilson loops and open spin-chains*, *JHEP* **07** (2006) 024 [hep-th/0604124] [INSPIRE].

[8] S. Giombi, R. Roiban and A.A. Tseytlin, *Half-BPS Wilson loop and AdS_2/CFT_1*, *Nucl. Phys.* **B 922** (2017) 499 [arXiv:1706.00756] [INSPIRE].

[9] M. Beccaria and A.A. Tseytlin, *On non-supersymmetric generalizations of the Wilson-Maldacena loops in $\mathcal{N}=4$ SYM*, *Nucl. Phys.* **B 934** (2018) 466 [arXiv:1804.02179] [INSPIRE].

[10] M. Beccaria, S. Giombi and A.A. Tseytlin, *Correlators on non-supersymmetric Wilson line in $\mathcal{N}=4$ SYM and AdS_2/CFT_1*, *JHEP* **05** (2019) 122 [arXiv:1903.04365] [INSPIRE].

[11] S. Giombi, C. Sleight and M. Taronna, *Spinning AdS Loop Diagrams: Two Point Functions*, *JHEP* **06** (2018) 030 [arXiv:1708.08404] [INSPIRE].

[12] I. Bertan, I. Sachs and E.D. Skvortsov, *Quantum ϕ^4 Theory in AdS_4 and its CFT Dual*, *JHEP* **02** (2019) 099 [arXiv:1810.00907] [INSPIRE].

[13] E.Y. Yuan, *Simplicity in AdS Perturbative Dynamics*, arXiv:1801.07283 [INSPIRE].

[14] J. Liu, E. Perlmutter, V. Rosenhaus and D. Simmons-Duffin, *d-dimensional SYK, AdS Loops, and 6j Symbols*, *JHEP* **03** (2019) 052 [arXiv:1808.00612] [INSPIRE].

JHEP09(2019)036

[15] H. Ouyang, *Holographic four-point functions in Toda field theories in AdS_2*, *JHEP* **04** (2019) 159 [arXiv:1902.10536] [INSPIRE].

[16] M. Beccaria and A.A. Tseytlin, *On boundary correlators in Liouville theory on AdS_2*, *JHEP* **07** (2019) 008 [arXiv:1904.12753] [INSPIRE].

[17] M. Beccaria and G. Landolfi, *Toda theory in AdS_2 and $\mathcal{W}A_n$-algebra structure of boundary correlators*, arXiv:1906.06485 [INSPIRE].

[18] A. Strominger, *AdS_2 quantum gravity and string theory*, *JHEP* **01** (1999) 007 [hep-th/9809027] [INSPIRE].

[19] M. Hotta, *Asymptotic isometry and two-dimensional anti-de Sitter gravity*, gr-qc/9809035 [INSPIRE].

[20] M. Cadoni and S. Mignemi, *Asymptotic symmetries of AdS_2 and conformal group in $d = 1$*, *Nucl. Phys.* **B 557** (1999) 165 [hep-th/9902040] [INSPIRE].

[21] A. Almheiri and J. Polchinski, *Models of AdS_2 backreaction and holography*, *JHEP* **11** (2015) 014 [arXiv:1402.6334] [INSPIRE].

[22] K. Jensen, *Chaos in AdS_2 Holography*, *Phys. Rev. Lett.* **117** (2016) 111601 [arXiv:1605.06098] [INSPIRE].

[23] J. Maldacena, D. Stanford and Z. Yang, *Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space*, *PTEP* **2016** (2016) 12C104 [arXiv:1606.01857] [INSPIRE].

[24] J. Engelsöy, T.G. Mertens and H. Verlinde, *An investigation of AdS_2 backreaction and holography*, *JHEP* **07** (2016) 139 [arXiv:1606.03438] [INSPIRE].

[25] J.-L. Gervais and M.V. Savelev, *Black holes from nonAbelian Toda theories*, *Phys. Lett.* **B 286** (1992) 271 [hep-th/9203039] [INSPIRE].

[26] K. Bardakci, M.J. Crescimanno and E. Rabinovici, *Parafermions From Coset Models*, *Nucl. Phys.* **B 344** (1990) 344 [INSPIRE].

[27] E. Witten, *On string theory and black holes*, *Phys. Rev.* **D 44** (1991) 314 [INSPIRE].

[28] A.M. Polyakov, *Quantum Geometry of Bosonic Strings*, *Phys. Lett.* **B 103** (1981) 207 [INSPIRE].

[29] Y. Nakayama, *Liouville field theory: A decade after the revolution*, *Int. J. Mod. Phys.* **A 19** (2004) 2771 [hep-th/0402009] [INSPIRE].

[30] P. Menotti and E. Tonni, *Standard and geometric approaches to quantum Liouville theory on the pseudosphere*, *Nucl. Phys.* **B 707** (2005) 321 [hep-th/0406014] [INSPIRE].

[31] A.N. Leznov and M.V. Saveliev, *Representation of zero curvature for the system of nonlinear partial differential equations $\chi_{\alpha,z\bar{z}} = \exp(k\chi)_\alpha$ and its integrability*, *Lett. Math. Phys.* **3** (1979) 489 [INSPIRE].

[32] A. Bilal and J.-L. Gervais, *Systematic Construction of Conformal Theories with Higher Spin Virasoro Symmetries*, *Nucl. Phys.* **B 318** (1989) 579 [INSPIRE].

[33] T.J. Hollowood and P. Mansfield, *Quantum Group Structure of Quantum Toda Conformal Field Theories. 1.*, *Nucl. Phys.* **B 330** (1990) 720 [INSPIRE].

[34] V.A. Fateev and A.V. Litvinov, *Correlation functions in conformal Toda field theory. I.*, *JHEP* **11** (2007) 002 [arXiv:0709.3806] [INSPIRE].

[35] A.A. Tseytlin, *On the Structure of the Renormalization Group Beta Functions in a Class of Two-dimensional Models*, *Phys. Lett.* **B 241** (1990) 233 [INSPIRE].

[36] M.T. Grisaru, A. Lerda, S. Penati and D. Zanon, *Renormalization Group Flows in Generalized Toda Field Theories*, *Nucl. Phys.* **B 346** (1990) 264 [INSPIRE].

[37] A.B. Zamolodchikov, *Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory*, *Theor. Math. Phys.* **65** (1985) 1205 [INSPIRE].

[38] P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, *Phys. Rept.* **223** (1993) 183 [hep-th/9210010] [INSPIRE].

[39] A. Bilal, *NonAbelian Toda theory: A completely integrable model for strings on a black hole background*, *Nucl. Phys.* **B 422** (1994) 258 [hep-th/9312108] [INSPIRE].

[40] A. Bilal, *Consistent string backgrounds and completely integrable 2-D field theories*, *Nucl. Phys. Proc. Suppl.* **45A** (1996) 105 [hep-th/9508062] [INSPIRE].

[41] I. Jack, D.R.T. Jones and J. Panvel, *Quantum nonAbelian Toda field theories*, *Int. J. Mod. Phys.* **A 9** (1994) 3631 [hep-th/9308080] [INSPIRE].

[42] C.G. Callan Jr., E.J. Martinec, M.J. Perry and D. Friedan, *Strings in Background Fields*, *Nucl. Phys.* **B 262** (1985) 593 [INSPIRE].

[43] A.A. Tseytlin, *Sigma model approach to string theory*, *Int. J. Mod. Phys.* **A 4** (1989) 1257 [INSPIRE].

[44] A.A. Tseytlin, *On the form of the black hole solution in $D = 2$ theory*, *Phys. Lett.* **B 268** (1991) 175 [INSPIRE].

[45] I. Jack, D.R.T. Jones and J. Panvel, *Exact bosonic and supersymmetric string black hole solutions*, *Nucl. Phys.* **B 393** (1993) 95 [hep-th/9201039] [INSPIRE].

[46] A.A. Tseytlin, *On field redefinitions and exact solutions in string theory*, *Phys. Lett.* **B 317** (1993) 559 [hep-th/9308042] [INSPIRE].

[47] R. Dijkgraaf, H.L. Verlinde and E.P. Verlinde, *String propagation in a black hole geometry*, *Nucl. Phys.* **B 371** (1992) 269 [INSPIRE].

[48] A.A. Tseytlin, *Effective action of gauged WZW model and exact string solutions*, *Nucl. Phys.* **B 399** (1993) 601 [hep-th/9301015] [INSPIRE].

[49] I. Bars and K. Sfetsos, *Exact effective action and space-time geometry n gauged WZW models*, *Phys. Rev.* **D 48** (1993) 844 [hep-th/9301047] [INSPIRE].

[50] A.A. Tseytlin, *Conformal sigma models corresponding to gauged Wess-Zumino-Witten theories*, *Nucl. Phys.* **B 411** (1994) 509 [hep-th/9302083] [INSPIRE].

[51] B. Hoare and A.A. Tseytlin, *On the perturbative S-matrix of generalized sine-Gordon models*, *JHEP* **11** (2010) 111 [arXiv:1008.4914] [INSPIRE].

[52] L. O'Raifeartaigh and A. Wipf, *Conformally reduced WZNW theories and two-dimensional gravity*, *Phys. Lett.* **B 251** (1990) 361 [INSPIRE].

[53] J. Balog, L. Feher, L. O'Raifeartaigh, P. Forgacs and A. Wipf, *Toda Theory and W Algebra From a Gauged WZNW Point of View*, *Annals Phys.* **203** (1990) 76 [INSPIRE].

[54] L. Feher, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, *On Hamiltonian reductions of the Wess-Zumino-Novikov-Witten theories*, *Phys. Rept.* **222** (1992) 1 [INSPIRE].

JHEP09(2019)036

- [55] C. Klimčík and A.A. Tseytlin, *Exact four-dimensional string solutions and Toda like sigma models from 'null gauged' WZNW theories*, Nucl. Phys. **B 424** (1994) 71 [hep-th/9402120] [INSPIRE].
- [56] A. Gerasimov, A. Morozov, M. Olshanetsky, A. Marshakov and S.L. Shatashvili, *Wess-Zumino-Witten model as a theory of free fields*, Int. J. Mod. Phys. **A 5** (1990) 2495 [INSPIRE].
- [57] C. Ford, *Quantum parafermions in the* $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ *WZNW black hole model*, Rept. Math. Phys. **48** (2001) 67 [hep-th/0010123] [INSPIRE].
- [58] C. Kruger, *Exact operator quantization of the Euclidean black hole CFT*, hep-th/0411275 [INSPIRE].
- [59] I. Bakas and E. Kiritsis, *Beyond the large N limit: Nonlinear* W_∞ *as symmetry of the* $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ *coset model*, Int. J. Mod. Phys. **A 7S1A** (1992) 55 [hep-th/9109029] [INSPIRE].
- [60] A. Sevrin and W. Troost, *Extensions of the Virasoro algebra and gauged WZW models*, Phys. Lett. **B 315** (1993) 304 [hep-th/9306033] [INSPIRE].
- [61] F.A. Dolan and H. Osborn, *Conformal Partial Waves: Further Mathematical Results*, arXiv:1108.6194 [INSPIRE].
- [62] E. D'Hoker, D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Graviton exchange and complete four point functions in the AdS/CFT correspondence*, Nucl. Phys. **B 562** (1999) 353 [hep-th/9903196] [INSPIRE].
- [63] F.A. Dolan and H. Osborn, *Conformal four point functions and the operator product expansion*, Nucl. Phys. **B 599** (2001) 459 [hep-th/0011040] [INSPIRE].
- [64] F.A. Dolan and H. Osborn, *Conformal partial waves and the operator product expansion*, Nucl. Phys. **B 678** (2004) 491 [hep-th/0309180] [INSPIRE].
- [65] V.A. Fateev and S.L. Lukyanov, *The Models of Two-Dimensional Conformal Quantum Field Theory with* Z_n *Symmetry*, Int. J. Mod. Phys. **A 3** (1988) 507 [INSPIRE].
- [66] E. Braaten, T. Curtright and C.B. Thorn, *An Exact Operator Solution of the Quantum Liouville Field Theory*, Annals Phys. **147** (1983) 365 [INSPIRE].
- [67] V. Schomerus, *Lectures on branes in curved backgrounds*, Class. Quant. Grav. **19** (2002) 5781 [hep-th/0209241] [INSPIRE].
- [68] A. Recknagel and V. Schomerus, *Boundary Conformal Field Theory and the Worldsheet Approach to D-Branes*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, (2013), [https://doi.org/10.1017/CBO9780511806476].
- [69] P. Menotti and E. Tonni, *The tetrahedron graph in Liouville theory on the pseudosphere*, Phys. Lett. **B 586** (2004) 425 [hep-th/0311234] [INSPIRE].

JHEP09(2019)036