FACTORIZATION BY ELEMENTARY MATRICES, NULL-HOMOTOPY AND PRODUCTS OF EXPONENTIALS FOR INVERTIBLE MATRICES OVER RINGS

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Abstract. Let R be a commutative unital ring. A well-known factorization problem is whether any matrix in $\mathrm{SL}_n(R)$ is a product of elementary matrices with entries in R. To solve the problem, we use two approaches based on the notion of the Bass stable rank and on construction of a null-homotopy. Special attention is given to the case, where R is a ring or Banach algebra of holomorphic functions. Also, we consider a related problem on representation of a matrix in $\mathrm{GL}_n(R)$ as a product of exponentials.

1. Introduction

Let R be an associative, commutative, unital ring. A well-known factorization problem is whether any matrix in $\mathrm{SL}_n(R)$ is a product of elementary (equivalently, unipotent) matrices with entries in R. Here the elementary matrices are those which have units on the diagonal and zeros outside the diagonal, except one non-zero entry. In particular, for $n = 3, 4, \dots$, Suslin [20] proved that the problem is solvable for the polynomials rings $\mathbb{C}[\mathbb{C}^m]$, $m \geq 1$. For $n = 2$, the required factorization for $R = \mathbb{C}[\mathbb{C}^m]$ does not always exist; the first counterexample was constructed by Cohn [4].

In the present paper, we primarily consider the case, where R is a functional Banach algebra. So, let $\mathcal{O}(\mathbb{D})$ denote the space of holomorphic functions on the unit disk $\mathbb{D}$ of $\mathbb{C}$. Recall that the disk-algebra $A(\mathbb{D})$ consists of $f \in \mathcal{O}(\mathbb{D})$ extendable up to continuous functions on the closed disk $\overline{\mathbb{D}}$. The disk-algebra $A(\mathbb{D})$ and the space $H^\infty(\mathbb{D})$ of bounded holomorphic functions on $\mathbb{D}$ may serve as good working examples for the algebras under consideration.

In fact, we propose two approaches to the factorization problem. The first one is based on construction of a null-homotopy; see Section 2. This method applies to the disk-algebra and similar algebras. The second approach is applicable to rings whose Bass stable rank is equal to one; see Section 3. This methods applies, in particular, to $H^\infty(\mathbb{D})$.

Also, the factorization problem is closely related to the following natural question: whether a matrix $F \in \mathrm{GL}_n(R)$ is representable as a product of exponentials, that is, $F = \exp G_1 \dots \exp G_k$ with $G_j \in M_n(R)$. For $n = 2$ and matrices with entries in a Banach algebra, this question was recently considered in [15]. In Section 4, we obtain results related to this question with emphasis on the case, where $R = \mathcal{O}(\Omega)$ and Ω is an open Riemann surface.

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2. Factorization and null-homotopy

Given $n \geq 2$ and an associative, commutative, unital ring R, let $E_n(R)$ denote the set of those $n \times n$ matrices which are representable as products of elementary matrices with entries in R.

For a unital commutative Banach algebra R, an element $X \in \mathrm{SL}_n(R)$ is said to be null-homotopic if X is homotopic to the unity matrix, that is, there exists a homotopy $X_t : [0,1] \to \mathrm{SL}_n(R)$ such that $X_1 = X$ and X_0 is the unity matrix.

We will use the following theorem:

Theorem 1 ([13, §7])**.** *Let A be a unital commutative Banach algebra and let $X \in \mathrm{SL}_n(A)$. The following properties are equivalent:*

(i) *$X \in E_n(A)$;*
(ii) *X is null-homotopic.*

To give an illustration of Theorem 1, consider the disk-algebra $A(\mathbb{D})$.

Corollary 1. *For $n = 2, 3, \dots,$ $E_n(A(\mathbb{D})) = \mathrm{SL}_n(A(\mathbb{D}))$.*

Proof. We have to show that $E_n(A(\mathbb{D})) \supset \mathrm{SL}_n(A(\mathbb{D}))$. So, assume that

$$F = F(z) = \begin{pmatrix} f_{11}(z) & & f_{1n}(z) \\ & \ddots & \\ f_{n1}(z) & & f_{nn}(z) \end{pmatrix} \in \mathrm{SL}_n(A(\mathbb{D})).$$

Define

$$F_t(z) = F(tz) \in \mathrm{SL}_n(A(\mathbb{D})), \quad 0 \leq t \leq 1, \ z \in \mathbb{D}. \tag{2.1}$$

Given an $f \in A(\mathbb{D})$, let $f_t(z) = f(tz)$, $0 \leq t \leq 1$, $z \in \mathbb{D}$. Observe that $\|f_t - f\|_{A(\mathbb{D})} \to 0$ as $t \to 1-$. Applying this observation to the entries of F_t, we conclude that F is homotopic to the constant matrix $F(0)$. Since $\mathrm{SL}_n(\mathbb{C})$ is path-connected, the constant matrix $F(0)$ is homotopic to the unity matrix. So, it remains to apply Theorem 1. □

3. Factorization and Bass stable rank

3.1. **Definitions.** Let R be a commutative unital ring. An element $(x_1, \dots, x_k) \in R^k$ is called *unimodular* if

$$\sum_{j=1}^{k} x_j R = R.$$

Let $U_k(R)$ the set of all unimodular elements in R^k.

An element $x = (x_1, \dots, x_{k+1}) \in U_{k+1}(R)$ is called *reducible* if there exists $(y_1, \dots, y_k) \in R^k$ such that

$$(x_1 + y_1 x_{k+1}, \dots, x_k + y_k x_{k+1}) \in U_k(R).$$

The *Bass stable rank* of R, denoted by $\mathrm{bsr}(R)$ and introduced in [1], is the least $k \in \mathbb{N}$ such that every $x \in U_{k+1}(R)$ is reducible. If there is no such $k \in \mathbb{N}$, then we set $\mathrm{bsr}(R) = \infty$.

Remark 1. *The identity $\mathrm{bsr}(R) = 1$ is equivalent to the following property: For any $x_1, x_2 \in R$ such that $x_1 R + x_2 R = R$, there exists $y \in R$ such that $x_1 + y x_2 \in R^*$.*

3.2. **A sufficient condition for factorization.**

Theorem 2. *Let R be a unital commutative ring and $n \geq 2$. If* $\operatorname{bsr}(R) = 1$*, then* $E_n(R) = \mathrm{SL}_n(R)$.

Proof. First, assume that $n = 2$. Let

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathrm{SL}_2(R).$$

Since $\det X = 1$, we have

$$x_{21}R + x_{11}R = R.$$

Hence, using the assumption $\operatorname{bsr}(X) = 1$ and Remark 1, we conclude that there exists $y \in R$ such that

$$\alpha = x_{21} + yx_{11} \in R^*. \tag{3.1}$$

Now, we have

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} X = \begin{pmatrix} x_{11} & x_{12} \\ \alpha & * \end{pmatrix}.$$

Next, using (3.1) we obtain

$$\begin{pmatrix} 1 & (1 - x_{11})\alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ \alpha & * \end{pmatrix} = \begin{pmatrix} 1 & * \\ \alpha & * \end{pmatrix}.$$

Finally, we have

$$\begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ \alpha & * \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & x_0 \end{pmatrix}.$$

Since the determinant of the last matrix is equal to one, we conclude that $x_0 = 1$. Therefore, the X is representable as a product of four multipliers.

For $n \geq 3$, let

$$X = \begin{pmatrix} x_{11} & \\ \vdots & * \\ x_{n1} & \end{pmatrix} \in \mathrm{SL}_n(R).$$

Since $\det X = 1$, there exist $\alpha_1, \ldots, \alpha_n \in R$ such that $\alpha_1 x_{11} + \cdots + \alpha_{n-1} x_{n-11} + \alpha_n x_{n1} = 1$. Therefore,

$$x_{n1}R + \left(\sum_{i=1}^{n-1} \alpha_i x_{i1} \right) R = R.$$

Applying the property $\operatorname{bsr} R = 1$, we obtain $y \in R$ such that

$$x_{n1} + y \left(\sum_{i=1}^{n-1} \alpha_i x_{i1} \right) := \alpha \in R^*.$$

Put

$$L = \begin{pmatrix} 1 & & & \\ & 1 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ \alpha_1 y & \ldots & \alpha_{n-1} y & 1 \end{pmatrix}.$$

Then

$$LX = \begin{pmatrix} x_{11} & \\ \vdots & * \\ x_{n-11} & \\ \alpha & \end{pmatrix}.$$

Multiplying by the upper triangular matrix

$$U_1 = \begin{pmatrix} 1 & & & (1-x_{11})\alpha^{-1} \\ & 1 & \mathbf{0} & -x_{21}\alpha^{-1} \\ & \mathbf{0} \ddots & & \dots \\ & & 1 & -x_{n-11}\alpha^{-1} \\ & & & 1 \end{pmatrix},$$

we obtain

$$U_1LX = \begin{pmatrix} 1 & \\ 0 & \\ \vdots & * \\ 0 & \\ \alpha & \end{pmatrix}.$$

Now, put

$$\widetilde{L} = \begin{pmatrix} 1 & & & \\ & 1 & \mathbf{0} & \\ 0 & \mathbf{0} & \ddots & \\ -\alpha & 0 & & 1 \end{pmatrix}.$$

We have

$$\widetilde{L}U_1LX = \begin{pmatrix} 1 & * & * & * \\ 0 & & & \\ \vdots & & Y_1 & \\ 0 & & & \end{pmatrix}.$$

Observe that $Y_1 \in \mathrm{SL}_{n-1}(R)$. So, arguing by induction, we obtain

$$\left(\prod_{i=1}^{n-1} \widetilde{L}_i U_i L_i\right) X = \begin{pmatrix} 1 & & * \\ & \ddots & \\ \mathbf{0} & & 1 \end{pmatrix} := U$$

or, equivalently,

$$\left(\prod_{i=1}^{n-1} \mathcal{L}_i U_i\right) L_{n-1} X = U,$$

where $\mathcal{L}_i$ are lower triangular matrices. So, we conclude that every $X \in \mathrm{SL}_n(R)$ is a product of $2n$ unipotent upper and lower triangular matrices. □

Corollary 2. *Let A be a unital commutative Banach algebra such that* $\mathrm{bsr}(A) = 1$. *If $X \in \mathrm{SL}_n(A)$, then X is null-homotopic.*

Proof. It suffices to combine Theorems 1 and 2. □

3.3. **Examples of algebras A with $\mathrm{bsr}(A) = 1$.**

3.3.1. *Disk-algebra $A(\mathbb{D})$.* By Corollary 1, $E_n(A(\mathbb{D})) = \mathrm{SL}_n(A(\mathbb{D}))$. Theorem 2 provides a different proof of this property. Indeed, Jones, Marshall and Wolff [12] and Corach and Suárez [5] proved that $\mathrm{bsr}(A(\mathbb{D})) = 1$, so Theorem 2 applies.

3.3.2. *Algebra $H^\infty(\mathbb{D})$.* Let $f \in H^\infty(\mathbb{D})$. If $\|f_r - f\|_\infty \to 0$ as $r \to 1-$, then clearly $f \in A(\mathbb{D})$. So the homotopy argument used for $A(\mathbb{D})$ is not applicable to $H^\infty(\mathbb{D})$. However, Treil [22] proved that $\mathrm{bsr}(H^\infty(\mathbb{D})) = 1$, hence, Theorem 2 holds for $R = H^\infty(\mathbb{D})$. Also, Corollary 2 guarantees that any $F \in \mathrm{SL}_n(H^\infty(\mathbb{D}))$ is null-homotopic.

3.3.3. *Generalizations of $H^\infty(\mathbb{D})$.* Tolokonnikov [21] proved that $\mathrm{bsr}(H^\infty(G)) = 1$ for any finitely connected open Riemann surface G and for certain infinitely connected planar domains G (Behrens domains). In particular, any $F \in \mathrm{SL}_n(H^\infty(G))$ is null-homotopic. However, even in the case $G = \mathbb{D}$ the homotopy in question is not explicit. So, probably it would be interesting to give a more explicit construction of the required homotopy.

Let $\mathbb{T} = \partial\mathbb{D}$ denote the unit circle. Given a function $f \in H^\infty(\mathbb{D})$, it is well-known that the radial limit $\lim_{r\to 1-} f(r\zeta)$ exists for almost all $\zeta \in \mathbb{T}$ with respect to Lebesgue measure on $\mathbb{T}$. So, let $H^\infty(\mathbb{T})$ denote the space of the corresponding radial values. It is known that $H^\infty(\mathbb{T}) + C(\mathbb{T})$ is an algebra, moreover, $\mathrm{bsr}(H^\infty(\mathbb{T}) + C(\mathbb{T})) = 1$; see [18].

Now, let B denote a Blaschke product in $\mathbb{D}$. Then $\mathbb{C} + BH^\infty(\mathbb{D})$ is an algebra. It is proved in [16] that $\mathrm{bsr}(\mathbb{C} + BH^\infty(\mathbb{D})) = 1$.

3.4. **Examples of algebras A with $\mathrm{bsr}(A) > 1$.**

3.4.1. *Algebra $A_\mathbb{R}(\mathbb{D})$.* Each element f of the disk-algebra $A(\mathbb{D})$ has a unique representation

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad z \in \mathbb{D}. \tag{3.2}$$

The space $A_\mathbb{R}(\mathbb{D})$ consists of those $f \in A(\mathbb{D})$ for which $a_j \in \mathbb{R}$ for all $j = 0, 1 \dots$ in (3.2). As shown in [17], $\mathrm{bsr}(A_\mathbb{R}(\mathbb{D})) = 2$. Nevertheless, the following result holds.

Proposition 1. *For $n = 2, 3, \dots$, $E_n(A_\mathbb{R}(\mathbb{D})) = \mathrm{SL}_n(A_\mathbb{R}(\mathbb{D}))$.*

Proof. For a function $f \in A_\mathbb{R}(\mathbb{D})$, we have $f_t \in A_\mathbb{R}(\mathbb{D})$ or all $0 \le t < 1$. Hence, given a matrix $F \in \mathrm{SL}_n(A_\mathbb{R}(\mathbb{D}))$, we have $F_t \in \mathrm{SL}_n(A_\mathbb{R}(\mathbb{D}))$, where F_t is defined by (2.1). Since $\|f_t - f\|_{A_\mathbb{R}(\mathbb{D})} \to 0$ as $t \to 1-$, F is homotopic to the constant matrix $F_0 \in \mathrm{SL}_n(\mathbb{C})$. Hence, F is homotopic to the unity matrix. Therefore, $F \in E_n(A_\mathbb{R}(\mathbb{D}))$ by Theorem 1. $\square$

3.4.2. *Ball algebra $A(B^m)$, polydisk algebra $A(\mathbb{D}^m)$, $m \ge 2$, and infinite polydisk algebra $A(\mathbb{D}^\infty)$.* Let B^m denote the unit ball of $\mathbb{C}^m$, $m \ge 2$. The ball algebra $A(B^m)$ and the polydisk algebra $A(\mathbb{D}^m)$ are defined analogously to the disk-algebra $A(\mathbb{D})$. By [6, Corollary 3.13],

$$\mathrm{bsr}(A(B^m)) = \mathrm{bsr}(A(\mathbb{D}^m)) = \left[\frac{m}{2}\right] + 1, \quad m \ge 2.$$

The infinite polydisk algebra $A(\mathbb{D}^\infty)$ is the uniform closure of the algebra generated by the coordinate functions $z_1, z_2, \dots$ on the countably infinite closed polydisk

$\overline{\mathbb{D}}^\infty = \overline{\mathbb{D}} \times \overline{\mathbb{D}} \dots$. Proposition 1 from [14] guarantees that $\mathrm{bsr}(A(\mathbb{D}^\infty)) = \infty$. Large or infinite Bass stable rank of the algebras under consideration is compatible with the following result.

Proposition 2. *Let $n = 2, 3, \dots$. Then*

$$E_n(A(B^m)) = \mathrm{SL}_n(A(B^m)), \quad m = 2, 3, \dots, \infty,$$
$$E_n(A(\mathbb{D}^m)) = \mathrm{SL}_n(A(\mathbb{D}^m)), \quad m = 2, 3, \dots, \infty.$$

Proof. It suffices to repeat the argument used in the proof of Corollary 1 or Proposition 1. □

3.4.3. *Algebra $H^\infty_{\mathbb{R}}(\mathbb{D})$.* It is proved in [17] that $\mathrm{bsr}(H^\infty_{\mathbb{R}}(\mathbb{D})) = 2$. We have not been able to determine the connected component of the identity in $\mathrm{SL}_n(H^\infty_{\mathbb{R}}(\mathbb{D}))$.

Problem 1. *Is any element in $\mathrm{SL}_n(H^\infty_{\mathbb{R}}(\mathbb{D}))$ null-homotopic?*

4. Invertible matrices as products of exponentials

Let R be a commutative unital ring. In the present section, we address the following problem: whether a matrix $F \in \mathrm{GL}_n(R)$ is representable as a product of exponentials, that is, $F = \exp G_1 \dots \exp G_k$ with $G_j \in M_n(R)$. For $n = 2$ and matrices with entries in a Banach algebra, this problem was recently studied in [15].

4.1. Basic results.

There is a direct relation between the problem under consideration and factorization of matrices in $\mathrm{GL}_n(R)$.

Lemma 1. *Let $X \in \mathrm{SL}_n(R)$ be a unipotent upper or lower triangular matrix. Then X is an exponential.*

Proof. For $n = 2$, we have

$$\exp \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Let $n \ge 3$. Given $\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots; \gamma_1, \gamma_2, \dots,$ we will find $a_1, a_2, \dots;$ $b_1, b_2, \dots; c_1, c_2, \dots$ such that

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ & 1 & \beta_1 & \beta_2 & \ddots \\ & & 1 & \gamma_1 & \ddots \\ & \mathbf{0} & & 1 & \ddots \\ & & & & \ddots \end{pmatrix} = \exp \begin{pmatrix} 0 & a_1 & a_2 & a_3 & \dots \\ & 0 & b_1 & b_2 & \ddots \\ & & 0 & c_1 & \ddots \\ & \mathbf{0} & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

Put $a_1 = \alpha_1$, $b_1 = \beta_1$, $\dots$. Next, we have $a_2 = \alpha_2 - f(a_1, b_1) = \alpha_2 - f(\alpha_1, \beta_1)$. Analogously, we find b_2, c_2, $\dots$. To find a_3, observe that $a_3 = \alpha_3 - f(a_1, a_2, b_1, c_2)$. Since f depends on a_i, b_i, c_i with $i < 3$, we obtain $a_3 = \alpha_3 - \widetilde{f}(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$, and the procedure continues. So, the equation under consideration is solvable for any $\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots$. □

Corollary 3. *Assume that $\mathrm{SL}_n(R) = E_n(R)$ and every element in $E_n(R)$ is a product of $N(R)$ unipotent upper or lower triangular matrices. Then every element in $\mathrm{SL}_n(R)$ is a product of $N(R)$ exponentials.*

Corollary 4. *Let the assumptions of Corollary 3 hold. Suppose in addition that every invertible element in R admits a logarithm. Then every $X \in \mathrm{GL}_n(R)$ is a product of $N(R)$ exponentials.*

Proof. Let $X \in \mathrm{GL}_n(R)$. So, $\det X \in R^*$ and $\ln\det X$ is defined. Therefore, $\det X = f^n$ for appropriate $f \in R^*$ and

$$\begin{pmatrix} f^{-1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & f^{-1} \end{pmatrix} X \in \mathrm{SL}_n(R).$$

Applying Corollary 3, we obtain

$$\begin{aligned} X &= \begin{pmatrix} f & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & f \end{pmatrix} \exp Y_1 \ldots \exp Y_N \\ &= \exp\left[\begin{pmatrix} \ln f & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \ln f \end{pmatrix} + Y_1\right] \exp Y_2 \ldots \exp Y_N, \end{aligned}$$

as required. □

4.2. **Rings of holomorphic functions on Stein spaces.**

Corollary 5. *Let Ω be a Stein space of dimension k and let $X \in \mathrm{GL}_n(\mathcal{O}(\Omega))$. Then there exists a number $E(k,n)$ such that the following properties are equivalent:*

(i) *X is null-homotopic;*
(ii) *X is a product of $E(k,n)$ exponentials.*

Proof. By [10, Theorem 2.3], any null-homotopic $F \in \mathrm{SL}_n(\mathcal{O}(\Omega))$ is a product of $N(k,n)$ unipotent upper or lower triangular matrices. So, arguing as in the proof of Corollary 4, we conclude that (i) implies (ii) with $E(k,n) \le N(k,n)$ The reverse implication is straightforward. □

The numbers $N(k,n)$ are not known in general. If the dimension k of the Stein space is fixed, then the dependence of $N(k,n)$ on the size n of the matrix is easier to handle. Certain K-theory arguments guarantee that the number of unipotent matrices needed for factorizing an element in $\mathrm{SL}_n(\mathcal{O}(\Omega))$ is a non-increasing function of n (see [7]). So, as done in [3], combining the above property and results from [11], we obtain the following estimates:

$$\begin{aligned} E(1,n) &\le N(1,n) = 4 \text{ for all } n, \\ E(2,n) &\le N(2,n) \le 5 \text{ for all } n, \text{ and} \\ \text{for each } k, \text{ there exists } n(k) \text{ such that } E(k,n) &\le N(k,n) \le 6 \text{ for all } n \ge n(k). \end{aligned}$$

In Section 4.4, we in fact improve on that: we show $E(1,2) \le 3$. In general, it seems that the number of exponentials $E(k,n)$ to factorize an element in $\mathrm{GL}_n(\mathcal{O}(\Omega))$ is less than the number $N(k,n)$ needed to write an element in $\mathrm{SL}_n(\mathcal{O}(\Omega))$ as a product of unipotent upper or lower triangular matrices.

Also, remark that (ii) implies (i) in Corollary 5 for any algebra R in the place of the ring of holomorphic functions. Assume that the algebra R has a topology. Then a topology on $\mathrm{GL}_n(R)$ is naturally induced and the implication (i)⇒(ii) means that any product of exponentials is contained in the connected component of the identity (also known as the principal component) of $\mathrm{GL}_n(R)$. The reverse implication is a difficult question, even without a uniform bound on the number of exponentials.

4.3. **Rings R with $\mathrm{bsr}(R) = 1$.** Combining Theorem 2 and Corollary 4, we recover a more general version of Theorem 7.1(3) from [15], where R is assumed to be a Banach algebra. Moreover, we obtain similar results for larger size matrices.

Corollary 6. *Let R be a commutative unital ring, $\mathrm{bsr} R = 1$, and let every $x \in R^*$ admit a logarithm. Then every element in $\mathrm{GL}_2(R)$ is a product of 4 exponentials.*

Corollary 7. *Let R be a commutative unital ring, $\mathrm{bsr} R = 1$, and let every $x \in R^*$ admit a logarithm. Then every element in $\mathrm{GL}_n(R)$, $n \geq 3$, is a product of 6 exponentials.*

Proof. For $n = 3$, it suffices to combine Theorem 2 and Corollary 4.

Now, assume that $n \geq 4$. Let ut_m denote the number of unipotent matrices needed to factorize any element in $\mathrm{SL}_m(R)$ starting with an upper triangular matrix. Theorem 20(b) in [7] says that any element in $\mathrm{SL}_n(R)$ is a product of 6 exponentials for

$$n \geq \min \left(m \left[\frac{\mathrm{ut}_m(R) + 1}{2} \right] \right),$$

where the minimum is taken over all $m \geq \mathrm{bsr} R + 1$. In our case the minimum is taken over $m \geq 2$ and the number $\mathrm{ut}_2(R) = 4$ by the proof of Theorem 2. Since $n \geq 4$, the proof is finished. □

Corollary 6 applies to the disk algebra and also to the rings $\mathcal{O}(\mathbb{C})$ and $\mathcal{O}(\mathbb{D})$ of holomorphic functions. Indeed, the identity $\mathrm{bsr}(\mathcal{O}(\Omega)) = 1$ for an open Riemann surface follows from the strengthening of the classical Wedderburn lemma (see [19, Chapter 6, Section 3]; see also [10] or [2]). However, for $R = \mathcal{O}(\mathbb{C})$ and $R = \mathcal{O}(\mathbb{D})$, the number 4 is not optimal; see Section 4.4 below. Also, it is known that the optimal number is at least 2 (see [15]). So, we arrive at the following natural question:

Problem 2. *Is any element of $\mathrm{GL}_2(\mathcal{O}(\mathbb{D}))$ or $\mathrm{GL}_2(\mathcal{O}(\mathbb{C}))$ a product of two exponentials?*

4.4. **Products of 3 exponentials.** In this section, we prove the following result.

Proposition 3. *Let Ω be an open Riemann surface. Then every element in $\mathrm{SL}_2(\mathcal{O}(\Omega))$ is a product of 3 exponentials.*

We will need several auxiliary results. The first theorem is a classical one [8].

Theorem 3 (Mittag-Leffler Interpolation Theorem). *Let Ω be an open Riemann surface and let $\{z_i\}_{i=1}^\infty$ be a discrete closed subset of Ω. Assume that a finite jet*

$$J_i(z) = \sum_{j=1}^{N_i} b_j^{(i)} (z - z_i)^j \tag{4.1}$$

is defined in some local coordinates for every point z_i. Then there exists $f \in \mathcal{O}(\Omega)$ such that

$$f(z) - J_i(z) = o(|z - z_i|^{N_i}) \quad \text{as } z \to z_i,\ i = 1, 2 \ldots. \tag{4.2}$$

Corollary 8. *Under assumptions of Theorem 3, suppose that* $b_0^{(i)} \neq 0$ *in* (4.1) *for* $i = 1, 2, \ldots$. *Then there exist* $f, g \in \mathcal{O}(\Omega)$ *such that* (4.2) *holds and* $f = e^g$.

Proof. Let $b_0 = b_0^{(i)}$ for some i. Since $b_0 \neq 0$, there exists a logarithm ln in a neighborhood of b_0. So, ln is a local biholomorphism which induces a bijection between jets of f and $g := \ln f$. □

In "modern" language, the proof of Corollary 8 uses the fact that $\mathbb{C}^*$ is an Oka manifold (we refer the interested reader to [9]). Thus for any Stein manifold X and an analytic subset $Y \subset X$, a (jet of) holomorphic map $f : Y \to \mathbb{C}^*$ (along Y) extends to a holomorphic map $f : X \to \mathbb{C}^*$ if and only if it extends continuously. The obstruction for a continuous extension is an element of the relative homology group $H_2(X, Y, \mathbb{Z})$. Observe that, for any discrete subset Y of a 1-dimensional Stein manifold X, we have $H_2(X, Y, \mathbb{Z}) = 0$ because $H_2(X, \mathbb{Z}) = H_1(Y, \mathbb{Z}) = 0$. This is the point where the proof of Proposition 3 below breaks down when we replace the Riemann surface Ω by a Stein manifold of higher dimension. Even a nowhere vanishing continuous function α, as in the proof, does not exist in general.

Lemma 2. *Let* Ω *be an open Riemann surface and* $X \in \mathrm{GL}_2(\mathcal{O}(\Omega))$. *Assume that* $\lambda \in \mathcal{O}^*(\Omega)$ *is the double eigenvalue of* X *and* $\det X$ *has a logarithm in* $\mathcal{O}(\Omega)$. *Then* X *is an exponential.*

Proof. We consider two cases.
Case 1: $X(z)$ is a diagonal matrix for all $z \in \Omega$.
We have

$$X(z) = \begin{pmatrix} \lambda(z) & 0 \\ 0 & \lambda(z) \end{pmatrix} = \exp \begin{pmatrix} \alpha(z) & 0 \\ 0 & \alpha(z) \end{pmatrix}.$$

Case 2: $X(z)$ is not identically diagonal.
Either the first or the second line in $X(z) - \lambda(z)I$, say $(h(z), g(z))$, is not identical zero. So,

$$v_1(z) = \begin{pmatrix} -g(z) \\ h(z) \end{pmatrix}$$

is a holomorphic eigenvector for $X(z)$ except those points $z \in \Omega$ for which $v_1(z) = \mathbf{0}$. Construct a function $f(z) \in \mathcal{O}(\Omega)$ such that its vanishing divisor is exactly $\min(\operatorname{ord} g, \operatorname{ord} h)$. Then

$$v(z) = \frac{1}{f(z)} v_1(z)$$

is a holomorphic eigenvector for $X(z)$, $z \in \Omega$.
Now, choose a matrix $P(z) \in \mathrm{GL}_2(\mathcal{O}(\Omega))$ with first column $v(z)$. Then the matrix $P^{-1}(z)X(z)P(z)$ has the following form:

$$\begin{pmatrix} \lambda(z) & \beta(z) \\ 0 & \lambda(z) \end{pmatrix} = \exp \begin{pmatrix} \frac{1}{2}\gamma(z) & \frac{\beta(z)}{\lambda(z)} \\ 0 & \frac{1}{2}\gamma(z) \end{pmatrix}$$

Thus,

$$X(z) = \exp P(z) \begin{pmatrix} \frac{1}{2}\gamma(z) & \frac{\beta(z)}{\lambda(z)} \\ 0 & \frac{1}{2}\gamma(z) \end{pmatrix} P^{-1}(z),$$

as required. □

Proof of Proposition 3. Let

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R),$$

that is, $ad - bc = 1$. We are looking for $\alpha \in R^*$ and $\beta \in R$ such that the matrix

$$X \begin{pmatrix} \alpha^2 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha^2 a & \beta a + b \\ \alpha^2 c & \beta c + d \end{pmatrix} := Y$$

has a double eigenvalue.
Case 1: $c = 0$. We have

$$X = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

It suffice to observe that

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} a^{-2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{-1} & b \\ 0 & a^{-1} \end{pmatrix}$$

has the double eigenvalue a^{-1}.
Case 2: $c \neq 0$. The matrix Y has a double eigenvalue if $4 \det Y = (\operatorname{tr} Y)^2$, that is,

$$(\alpha^2 a + \beta c + d)^2 = 4\alpha^2. \tag{4.3}$$

Put

$$\beta = \frac{2\alpha - a\alpha^2 - d}{c}.$$

Clearly, β is a formal solution of (4.3). Below we show how to construct $\alpha(z) = \exp(\widetilde{\alpha}(z)) \in \mathcal{O}^*(\Omega)$ such that β is holomorphic.

Let $\{z_i\} \subset \Omega$ be the zero set of $c(z)$. Fix i and $z_i \in \Omega$. Let $c(z_i) = \cdots = c^{(k)}(z_i) = 0$, and $c^{(k+1)}(z_i) \neq 0$. Observe that $a(z_i) \neq 0$. So, define $\alpha(z)$, in a neighborhood of z_i, as $1/a(z)$ up to a sufficiently high order, namely,

$$a(z)\alpha(z) = 1 + (z - z_i)^k h(z), \tag{4.4}$$

where $h(z)$ is holomorphic in a neighborhood of z_i. Since $ad - bc = 1$, we have $1 - ad = (z - z_i)^k g(z)$. Therefore,

$$\begin{aligned} 2a\alpha - a^2\alpha^2 - ad &= -(1 - a\alpha^2)^2 + 1 - ad \\ &= -(z - z_0)^{2k} h^2(z) + (z - z_0)^k g(z) \end{aligned}$$

vanishes of order k at z_i. Hence, $2\alpha - a\alpha^2 - d$ also vanishes of order k at z_i.

So, we have constructed $\alpha(z)$ locally as finite jets $J_i(z)$ defined by (4.1) with $b_0^{(i)} \neq 0$ in some local coordinates for every point z_i, $i = 1, 2, \dots$. Now, Corollary 8 provides $\widetilde{\alpha} \in \mathcal{O}(\Omega)$ such that $\alpha(z) = \exp(\widetilde{\alpha}(z)) \in \mathcal{O}^*(\Omega)$ and (4.4) holds. Hence, β is holomorphic.

So, the matrix

$$X \begin{pmatrix} \alpha^2 & \beta \\ 0 & 1 \end{pmatrix} := Y$$

has a double eigenvalue and $\det Y$ admits a logarithm. Thus, applying Lemma 2, we conclude that Y is an exponential. To finish the proof of the proposition, it remains observe that

$$\begin{pmatrix} \alpha^2 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta\alpha^{-1} \\ 0 & \alpha \end{pmatrix},$$

where both multipliers on the right hand side are exponentials. $\square$

Corollary 9. *Let $X \in \mathrm{GL}_2(\mathcal{O}(\Omega))$. The following properties are equivalent:*

(i) *X is a product of 3 exponentials;*
(ii) *$\det X$ is an exponential;*
(iii) *X is null-homotopic.*

Proof. Clearly, (i)⇒(iii). Now, assume that X is null-homotopic. Then $\det X$ is homotopic to the function $f \equiv 1$. Since $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering, we conclude that $\det X(z) = \exp(h(z))$ with $h \in \mathcal{O}(\Omega)$. So, (iii) implies (ii). The implication (ii)⇒(i) is standard; see, for example, the proof of Corollary 4. □

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