# FACTORIZATION BY ELEMENTARY MATRICES, NULL-HOMOTOPY AND PRODUCTS OF EXPONENTIALS FOR INVERTIBLE MATRICES OVER RINGS 

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#### Abstract

Let $R$ be a commutative unital ring. A well-known factorization problem is whether any matrix in $\mathrm{SL}_{n}(R)$ is a product of elementary matrices with entries in $R$. To solve the problem, we use two approaches based on the notion of the Bass stable rank and on construction of a null-homotopy. Special attention is given to the case, where $R$ is a ring or Banach algebra of holomorphic functions. Also, we consider a related problem on representation of a matrix in $\mathrm{GL}_{n}(R)$ as a product of exponentials.


## 1. Introduction

Let $R$ be an associative, commutative, unital ring. A well-known factorization problem is whether any matrix in $\mathrm{SL}_{n}(R)$ is a product of elementary (equivalently, unipotent) matrices with entries in $R$. Here the elementary matrices are those which have units on the diagonal and zeros outside the diagonal, except one nonzero entry. In particular, for $n=3,4, \ldots$, Suslin 20 proved that the problem is solvable for the polynomials rings $\mathbb{C}\left[\mathbb{C}^{m}\right], m \geq 1$. For $n=2$, the required factorization for $R=\mathbb{C}\left[\mathbb{C}^{m}\right]$ does not always exist; the first counterexample was constructed by Cohn (4).

In the present paper, we primarily consider the case, where $R$ is a functional Banach algebra. So, let $\mathcal{O}(\mathbb{D})$ denote the space of holomorphic functions on the unit disk $\mathbb{D}$ of $\mathbb{C}$. Recall that the disk-algebra $A(\mathbb{D})$ consists of $f \in \mathcal{O}(\mathbb{D})$ extendable up to continuous functions on the closed disk $\overline{\mathbb{D}}$. The disk-algebra $A(\mathbb{D})$ and the space $H^{\infty}(\mathbb{D})$ of bounded holomorphic functions on $\mathbb{D}$ may serve as good working examples for the algebras under consideration.

In fact, we propose two approaches to the factorization problem. The first one is based on construction of a null-homotopy; see Section 2. This method applies to the disk-algebra and similar algebras. The second approach is applicable to rings whose Bass stable rank is equal to one; see Section 3. This methods applies, in particular, to $H^{\infty}(\mathbb{D})$.

Also, the factorization problem is closely related to the following natural question: whether a matrix $F \in \mathrm{GL}_{n}(R)$ is representable as a product of exponentials, that is, $F=\exp G_{1} \ldots \exp G_{k}$ with $G_{j} \in M_{n}(R)$. For $n=2$ and matrices with entries in a Banach algebra, this question was recently considered in [15. In Section 4. we obtain results related to this question with emphasis on the case, where $R=\mathcal{O}(\Omega)$ and $\Omega$ is an open Riemann surface.

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## 2. FACTORIZATION AND NULL-HOMOTOPY

Given $n \geq 2$ and an associative, commutative, unital ring $R$, let $E_{n}(R)$ denote the set of those $n \times n$ matrices which are representable as products of elementary matrices with entries in $R$.

For a unital commutative Banach algebra $R$, an element $X \in \mathrm{SL}_{n}(R)$ is said to be null-homotopic if $X$ is homotopic to the unity matrix, that is, there exists a homotopy $X_{t}:[0,1] \rightarrow \mathrm{SL}_{n}(R)$ such that $X_{1}=X$ and $X_{0}$ is the unity matrix.

We will use the following theorem:
Theorem 1 ([13, §7]). Let $A$ be a unital commutative Banach algebra and let $X \in \mathrm{SL}_{n}(A)$. The following properties are equivalent:
(i) $X \in E_{n}(A)$;
(ii) $X$ is null-homotopic.

To give an illustration of Theorem 1, consider the disk-algebra $A(\mathbb{D})$.
Corollary 1. For $n=2,3, \ldots, E_{n}(A(\mathbb{D}))=\operatorname{SL}_{n}(A(\mathbb{D}))$.
Proof. We have to show that $E_{n}(A(\mathbb{D})) \supset \operatorname{SL}_{n}(A(\mathbb{D}))$. So, assume that

$$
F=F(z)=\left(\begin{array}{ccc}
f_{11}(z) & & f_{1 n}(z) \\
& \ddots & \\
f_{n 1}(z) & & f_{n n}(z)
\end{array}\right) \in \operatorname{SL}_{n}(A(\mathbb{D}))
$$

Define

$$
\begin{equation*}
F_{t}(z)=F(t z) \in \mathrm{SL}_{n}(A(\mathbb{D})), \quad 0 \leq t \leq 1, z \in \mathbb{D} \tag{2.1}
\end{equation*}
$$

Given an $f \in A(\mathbb{D})$, let $f_{t}(z)=f(t z), 0 \leq t \leq 1, z \in \mathbb{D}$. Observe that $\left\|f_{t}-f\right\|_{A(\mathbb{D})} \rightarrow$ 0 as $t \rightarrow 1-$. Applying this observation to the entries of $F_{t}$, we conclude that $F$ is homotopic to the constant matrix $F(0)$. Since $\mathrm{SL}_{n}(\mathbb{C})$ is path-connected, the constant matrix $F(0)$ is homotopic to the unity matrix. So, it remains to apply Theorem 1

## 3. Factorization and Bass stable Rank

3.1. Definitions. Let $R$ be a commutative unital ring. An element $\left(x_{1}, \ldots, x_{k}\right) \in$ $R^{k}$ is called unimodular if

$$
\sum_{j=1}^{k} x_{j} R=R
$$

Let $U_{k}(R)$ the set of all unimodular elements in $R^{k}$.
An element $x=\left(x_{1}, \ldots, x_{k+1}\right) \in U_{k+1}(R)$ is called reducible if there exists $\left(y_{1}, \ldots, y_{k}\right) \in R^{k}$ such that

$$
\left(x_{1}+y_{1} x_{k+1}, \ldots, x_{k}+y_{k} x_{k+1}\right) \in U_{k}(R)
$$

The Bass stable rank of $R$, denoted by $\operatorname{bsr}(R)$ and introduced in 1], is the least $k \in \mathbb{N}$ such that every $x \in U_{k+1}(R)$ is reducible. If there is no such $k \in \mathbb{N}$, then we set $\operatorname{bsr}(R)=\infty$.

Remark 1. The identity $\operatorname{bsr}(R)=1$ is equivalent to the following property: For any $x_{1}, x_{2} \in R$ such that $x_{1} R+x_{2} R=R$, there exists $y \in R$ such that $x_{1}+y x_{2} \in R^{*}$.

### 3.2. A sufficient condition for factorization.

Theorem 2. Let $R$ be a unital commutative ring and $n \geq 2$. If $\operatorname{bsr}(R)=1$, then $E_{n}(R)=\operatorname{SL}_{n}(R)$.

Proof. First, assume that $n=2$. Let

$$
X=\left(\begin{array}{ll}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{array}\right) \in \mathrm{SL}_{2}(R)
$$

Since $\operatorname{det} X=1$, we have

$$
x_{21} R+x_{11} R=R .
$$

Hence, using the assumption $\operatorname{bsr}(X)=1$ and Remark [1 we conclude that there exists $y \in R$ such that

$$
\begin{equation*}
\alpha=x_{21}+y x_{11} \in R^{*} \tag{3.1}
\end{equation*}
$$

Now, we have

$$
\left(\begin{array}{ll}
1 & 0 \\
y & 1
\end{array}\right) X=\left(\begin{array}{cc}
x_{11} & x_{12} \\
\alpha & *
\end{array}\right)
$$

Next, using (3.1) we obtain

$$
\left(\begin{array}{cc}
1 & \left(1-x_{11}\right) \alpha^{-1} \\
0 & 1
\end{array}\right)\left(\begin{array}{cc}
x_{11} & x_{12} \\
\alpha & *
\end{array}\right)=\left(\begin{array}{cc}
1 & * \\
\alpha & *
\end{array}\right)
$$

Finally, we have

$$
\left(\begin{array}{cc}
1 & 0 \\
-\alpha & 1
\end{array}\right)\left(\begin{array}{cc}
1 & * \\
\alpha & *
\end{array}\right)=\left(\begin{array}{cc}
1 & * \\
0 & x_{0}
\end{array}\right)
$$

Since the determinant of the last matrix is equal to one, we conclude that $x_{0}=1$. Therefore, the $X$ is representable as a product of four multipliers.

For $n \geq 3$, let

$$
X=\left(\begin{array}{cc}
x_{11} & \\
\vdots & * \\
x_{n 1} &
\end{array}\right) \in \mathrm{SL}_{n}(R)
$$

Since $\operatorname{det} X=1$, there exist $\alpha_{1}, \ldots, \alpha_{n} \in R$ such that $\alpha_{1} x_{11}+\cdots+\alpha_{n-1} x_{n-11}+$ $\alpha_{n} x_{n 1}=1$. Therefore,

$$
x_{n 1} R+\left(\sum_{i=1}^{n-1} \alpha_{i} x_{i 1}\right) R=R
$$

Applying the property $\operatorname{bsr} R=1$, we obtain $y \in R$ such that

$$
x_{n 1}+y\left(\sum_{i=1}^{n-1} \alpha_{i} x_{i 1}\right):=\alpha \in R^{*}
$$

Put

$$
L=\left(\begin{array}{cccc}
1 & & & \\
& 1 & \mathbf{0} & \\
& \mathbf{0} & \ddots & \\
\alpha_{1} y & \ldots & \alpha_{n-1} y & 1
\end{array}\right)
$$

Then

$$
L X=\left(\begin{array}{cc}
x_{11} & \\
\vdots & * \\
x_{n-11} & \\
\alpha &
\end{array}\right)
$$

Multiplying by the upper triangular matrix

$$
U_{1}=\left(\begin{array}{ccccc}
1 & & & & \left(1-x_{11}\right) \alpha^{-1} \\
& 1 & & \mathbf{0} & -x_{21} \alpha^{-1} \\
& \mathbf{0} & \ddots & & \ldots \\
& & & 1 & -x_{n-11} \alpha^{-1} \\
& & & & 1
\end{array}\right)
$$

we obtain

$$
U_{1} L X=\left(\begin{array}{cc}
1 & \\
0 & \\
\vdots & * \\
0 & \\
\alpha &
\end{array}\right)
$$

Now, put

$$
\widetilde{L}=\left(\begin{array}{cccc}
1 & & & \\
& 1 & \mathbf{0} & \\
0 & \mathbf{0} & \ddots & \\
-\alpha & 0 & & 1
\end{array}\right)
$$

We have

$$
\widetilde{L} U_{1} L X=\left(\begin{array}{cccc}
1 & * & * & * \\
0 & & & \\
\vdots & & Y_{1} & \\
0 & & &
\end{array}\right)
$$

Observe that $Y_{1} \in \mathrm{SL}_{n-1}(R)$. So, arguing by induction, we obtain

$$
\left(\prod_{i=1}^{n-1} \widetilde{L}_{i} U_{i} L_{i}\right) X=\left(\begin{array}{lll}
1 & & * \\
& \ddots & \\
\mathbf{0} & & 1
\end{array}\right):=U
$$

or, equivalently,

$$
\left(\prod_{i=1}^{n-1} \mathcal{L}_{i} U_{i}\right) L_{n-1} X=U
$$

where $\mathcal{L}_{i}$ are lower triangular matrices. So, we conclude that every $X \in \mathrm{SL}_{n}(R)$ is a product of $2 n$ unipotent upper and lower triangular matrices.

Corollary 2. Let $A$ be a unital commutative Banach algebra such that $\operatorname{bsr}(A)=1$. If $X \in \mathrm{SL}_{n}(A)$, then $X$ is null-homotopic.
Proof. It suffices to combine Theorems 1 and 2,
3.3. Examples of algebras $A$ with $\operatorname{bsr}(A)=1$.
3.3.1. Disk-algebra $A(\mathbb{D})$. By Corollary $1 E_{n}(A(\mathbb{D}))=\operatorname{SL}_{n}(A(\mathbb{D}))$. Theorem 2 provides a different proof of this property. Indeed, Jones, Marshall and Wolff 12 and Corach and Suárez [5] proved that $\operatorname{bsr}(A(\mathbb{D}))=1$, so Theorem 2 applies.
3.3.2. Algebra $H^{\infty}(\mathbb{D})$. Let $f \in H^{\infty}(\mathbb{D})$. If $\left\|f_{r}-f\right\|_{\infty} \rightarrow 0$ as $r \rightarrow 1-$, then clearly $f \in A(\mathbb{D})$. So the homotopy argument used for $A(\mathbb{D})$ is not applicable to $H^{\infty}(\mathbb{D})$. However, Treil 22] proved that $\operatorname{bsr}\left(H^{\infty}(\mathbb{D})\right)=1$, hence, Theorem 2 holds for $R=H^{\infty}(\mathbb{D})$. Also, Corollary 2 guarantees that any $F \in \operatorname{SL}_{n}\left(H^{\infty}(\mathbb{D})\right)$ is null-homotopic.
3.3.3. Generalizations of $H^{\infty}(\mathbb{D})$. Tolokonnikov 21] proved that $\operatorname{bsr}\left(H^{\infty}(G)\right)=1$ for any finitely connected open Riemann surface $G$ and for certain infinitely connected planar domains $G$ (Behrens domains). In particular, any $F \in \mathrm{SL}_{n}\left(H^{\infty}(G)\right)$ is null-homotopic. However, even in the case $G=\mathbb{D}$ the homotopy in question is not explicit. So, probably it would be interesting to give a more explicit construction of the required homotopy.

Let $\mathbb{T}=\partial \mathbb{D}$ denote the unit circle. Given a function $f \in H^{\infty}(\mathbb{D})$, it is wellknown that the radial $\operatorname{limit} \lim _{r \rightarrow 1-} f(r \zeta)$ exists for almost all $\zeta \in \mathbb{T}$ with respect to Lebesgue measure on $\mathbb{T}$. So, let $H^{\infty}(\mathbb{T})$ denote the space of the corresponding radial values. It is known that $H^{\infty}(\mathbb{T})+C(\mathbb{T})$ is an algebra, moreover, $\operatorname{bsr}\left(H^{\infty}(\mathbb{T})+\right.$ $C(\mathbb{T}))=1$; see 18 .

Now, let $B$ denote a Blaschke product in $\mathbb{D}$. Then $\mathbb{C}+B H^{\infty}(\mathbb{D})$ is an algebra. It is proved in [16] that $\operatorname{bsr}\left(\mathbb{C}+B H^{\infty}(\mathbb{D})\right)=1$.

### 3.4. Examples of algebras $A$ with $\operatorname{bsr}(A)>1$.

3.4.1. Algebra $A_{\mathbb{R}}(\mathbb{D})$. Each element $f$ of the disk-algebra $A(\mathbb{D})$ has a unique representation

$$
\begin{equation*}
f(z)=\sum_{j=0}^{\infty} a_{j} z^{j}, \quad z \in \mathbb{D} \tag{3.2}
\end{equation*}
$$

The space $A_{\mathbb{R}}(\mathbb{D})$ consists of those $f \in A(\mathbb{D})$ for which $a_{j} \in \mathbb{R}$ for all $j=0,1 \ldots$ in (3.2). As shown in [17], $\operatorname{bsr}\left(A_{\mathbb{R}}(\mathbb{D})\right)=2$. Nevertheless, the following result holds.

Proposition 1. For $n=2,3, \ldots, E_{n}\left(A_{\mathbb{R}}(\mathbb{D})\right)=\operatorname{SL}_{n}\left(A_{\mathbb{R}}(\mathbb{D})\right)$.
Proof. For a function $f \in A_{\mathbb{R}}(\mathbb{D})$, we have $f_{t} \in A_{\mathbb{R}}(\mathbb{D})$ or all $0 \leq t<1$. Hence, given a matrix $F \in \mathrm{SL}_{n}\left(A_{\mathbb{R}}(\mathbb{D})\right)$, we have $F_{t} \in \mathrm{SL}_{n}\left(A_{\mathbb{R}}(\mathbb{D})\right)$, where $F_{t}$ is defined by (2.1). Since $\left\|f_{t}-f\right\|_{A_{\mathbb{R}}(\mathbb{D})} \rightarrow 0$ as $t \rightarrow 1-, F$ is homotopic to the constant matrix $F_{0} \in \mathrm{SL}_{n}(\mathbb{C})$. Hence, $F$ is homotopic to the unity matrix. Therefore, $F \in E_{n}\left(A_{\mathbb{R}}(\mathbb{D})\right)$ by Theorem 1
3.4.2. Ball algebra $A\left(B^{m}\right)$, polydisk algebra $A\left(\mathbb{D}^{m}\right)$, $m \geq 2$, and infinite polydisk algebra $A\left(\mathbb{D}^{\infty}\right)$. Let $B^{m}$ denote the unit ball of $\mathbb{C}^{m}, m \geq 2$. The ball algebra $A\left(B^{m}\right)$ and the polydisk algebra $A\left(\mathbb{D}^{m}\right)$ are defined analogously to the disk-algebra $A(\mathbb{D})$. By [6, Corollary 3.13],

$$
\operatorname{bsr}\left(A\left(B^{m}\right)\right)=\operatorname{bsr}\left(A\left(\mathbb{D}^{m}\right)\right)=\left[\frac{m}{2}\right]+1, \quad m \geq 2
$$

The infinite polydisk algebra $A\left(\mathbb{D}^{\infty}\right)$ is the uniform closure of the algebra generated by the coordinate functions $z_{1}, z_{2}, \ldots$ on the countably infinite closed polydisk
$\overline{\mathbb{D}}^{\infty}=\overline{\mathbb{D}} \times \overline{\mathbb{D}} \ldots$ Proposition 1 from [14] guarantees that $\operatorname{bsr}\left(A\left(\mathbb{D}^{\infty}\right)\right)=\infty$. Large or infinite Bass stable rank of the algebras under consideration is compatible with the following result.
Proposition 2. Let $n=2,3, \ldots$ Then

$$
\begin{array}{ll}
E_{n}\left(A\left(B^{m}\right)\right)=\operatorname{SL}_{n}\left(A\left(B^{m}\right)\right), & m=2,3, \ldots, \infty \\
E_{n}\left(A\left(\mathbb{D}^{m}\right)\right)=\operatorname{SL}_{n}\left(A\left(\mathbb{D}^{m}\right)\right), & m=2,3, \ldots, \infty
\end{array}
$$

Proof. It suffices to repeat the argument used in the proof of Corollary 1 or Proposition 1
3.4.3. Algebra $H_{\mathbb{R}}^{\infty}(\mathbb{D})$. It is proved in 17 that $\operatorname{bsr}\left(H_{\mathbb{R}}^{\infty}(\mathbb{D})\right)=2$. We have not been able to determine the connected component of the identity in $\mathrm{SL}_{n}\left(H_{\mathbb{R}}^{\infty}(\mathbb{D})\right)$.
Problem 1. Is any element in $\mathrm{SL}_{n}\left(H_{\mathbb{R}}^{\infty}(\mathbb{D})\right)$ null-homotopic?

## 4. Invertible matrices as products of exponentials

Let $R$ be a commutative unital ring. In the present section, we address the following problem: whether a matrix $F \in \mathrm{GL}_{n}(R)$ is representable as a product of exponentials, that is, $F=\exp G_{1} \ldots \exp G_{k}$ with $G_{j} \in M_{n}(R)$. For $n=2$ and matrices with entries in a Banach algebra, this problem was recently studied in [15].
4.1. Basic results. There is a direct relation between the problem under consideration and factorization of matrices in $\mathrm{GL}_{n}(R)$.

Lemma 1. Let $X \in \operatorname{SL}_{n}(R)$ be a unipotent upper or lower triangular matrix. Then $X$ is an exponential.

Proof. For $n=2$, we have

$$
\exp \left(\begin{array}{ll}
0 & a \\
0 & 0
\end{array}\right)=\left(\begin{array}{ll}
1 & a \\
0 & 1
\end{array}\right)
$$

Let $n \geq 3$. Given $\alpha_{1}, \alpha_{2}, \ldots ; \beta_{1}, \beta_{2}, \ldots ; \gamma_{1}, \gamma_{2}, \ldots$, we will find $a_{1}, a_{2}, \ldots$; $b_{1}, b_{2}, \ldots ; c_{1}, c_{2}, \ldots$ such that

$$
\left(\begin{array}{ccccc}
1 & \alpha_{1} & \alpha_{2} & \alpha_{3} & \cdots \\
& 1 & \beta_{1} & \beta_{2} & \ddots \\
& & 1 & \gamma_{1} & \ddots \\
& \mathbf{0} & & 1 & \ddots \\
& & & & \ddots
\end{array}\right)=\exp \left(\begin{array}{ccccc}
0 & a_{1} & a_{2} & a_{3} & \cdots \\
& 0 & b_{1} & b_{2} & \ddots \\
& & 0 & c_{1} & \ddots \\
& \mathbf{0} & & 0 & \ddots \\
& & & & \ddots
\end{array}\right) .
$$

Put $a_{1}=\alpha_{1}, b_{1}=\beta_{1}, \ldots$ Next, we have $a_{2}=\alpha_{2}-f\left(a_{1}, b_{1}\right)=\alpha_{2}-f\left(\alpha_{1}, \beta_{1}\right)$. Analogously, we find $b_{2}, c_{2}, \ldots$ To find $a_{3}$, observe that $a_{3}=\alpha_{3}-f\left(a_{1}, a_{2}, b_{1}, c_{2}\right)$. Since $f$ depends on $a_{i}, b_{i}, c_{i}$ with $i<3$, we obtain $a_{3}=\alpha_{3}-\widetilde{f}\left(\alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2}, \gamma_{1}, \gamma_{2}\right)$, and the procedure continues. So, the equation under consideration is solvable for any $\alpha_{1}, \alpha_{2}, \ldots ; \beta_{1}, \beta_{2}, \ldots$

Corollary 3. Assume that $\mathrm{SL}_{n}(R)=E_{n}(R)$ and every element in $E_{n}(R)$ is a product of $N(R)$ unipotent upper or lower triangular matrices. Then every element in $\mathrm{SL}_{n}(R)$ is a product of $N(R)$ exponentials.

Corollary 4. Let the assumptions of Corollary 3 hold. Suppose in addition that every invertible element in $R$ admits a logarithm. Then every $X \in \operatorname{GL}_{n}(R)$ is a product of $N(R)$ exponentials.

Proof. Let $X \in \mathrm{GL}_{n}(R)$. So, $\operatorname{det} X \in R^{*}$ and $\ln \operatorname{det} X$ is defined. Therefore, $\operatorname{det} X=f^{n}$ for appropriate $f \in R^{*}$ and

$$
\left(\begin{array}{ccc}
f^{-1} & & \mathbf{0} \\
& \ddots & \\
\mathbf{0} & & f^{-1}
\end{array}\right) X \in \mathrm{SL}_{n}(R)
$$

Applying Corollary 3, we obtain

$$
\begin{aligned}
X & =\left(\begin{array}{lll}
f & & \mathbf{0} \\
& \ddots & \\
\mathbf{0} & & f
\end{array}\right) \exp Y_{1} \ldots \exp Y_{N} \\
& =\exp \left[\left(\begin{array}{ccc}
\ln f & & \mathbf{0} \\
& \ddots & \\
\mathbf{0} & & \ln f
\end{array}\right)+Y_{1}\right] \exp Y_{2} \ldots \exp Y_{N}
\end{aligned}
$$

as required.

### 4.2. Rings of holomorphic functions on Stein spaces.

Corollary 5. Let $\Omega$ be a Stein space of dimension $k$ and let $X \in \operatorname{GL}_{n}(\mathcal{O}(\Omega))$. Then there exists a number $E(k, n)$ such that the following properties are equivalent:
(i) $X$ is null-homotopic;
(ii) $X$ is a product of $E(k, n)$ exponentials.

Proof. By [10, Theorem 2.3], any null-homotopic $F \in \mathrm{SL}_{n}(\mathcal{O}(\Omega))$ is a product of $N(k, n)$ unipotent upper or lower triangular matrices. So, arguing as in the proof of Corollary 4, we conclude that (i) implies (ii) with $E(k, n) \leq N(k, n)$ The reverse implication is straightforward.

The numbers $N(k, n)$ are not known in general. If the dimension $k$ of the Stein space is fixed, then the dependence of $N(k, n)$ on the size $n$ of the matrix is easier to handle. Certain $K$-theory arguments guarantee that the number of unipotent matrices needed for factorizing an element in $\operatorname{SL}_{n}(\mathcal{O}(\Omega))$ is a non-increasing function of $n$ (see [7]). So, as done in [3], combining the above property and results from [11], we obtain the following estimates:

$$
\begin{aligned}
& E(1, n) \leq N(1, n)=4 \text { for all } n, \\
& E(2, n) \leq N(2, n) \leq 5 \text { for all } n, \text { and }
\end{aligned}
$$

for each $k$, there exists $n(k)$ such that $E(k, n) \leq N(k, n) \leq 6$ for all $n \geq n(k)$.
In Section 4.4 we in fact improve on that: we show $E(1,2) \leq 3$. In general, it seems that the number of exponentials $E(k, n)$ to factorize an element in $\mathrm{GL}_{n}(\mathcal{O}(\Omega))$ is less than the number $N(k, n)$ needed to write an element in $\operatorname{SL}_{n}(\mathcal{O}(\Omega))$ as a product of unipotent upper or lower triangular matrices.

Also, remark that (ii) implies (i) in Corollary 5 for any algebra $R$ in the place of the ring of holomorphic functions. Assume that the algebra $R$ has a topology. Then a topology on $\mathrm{GL}_{n}(R)$ is naturally induced and the implication (i) $\Rightarrow$ (ii) means that
any product of exponentials is contained in the connected component of the identity (also known as the principal component) of $\mathrm{GL}_{n}(R)$. The reverse implication is a difficult question, even without a uniform bound on the number of exponentials.
4.3. Rings $R$ with $\operatorname{bsr}(R)=1$. Combining Theorem 2 and Corollary 4 , we recover a more general version of Theorem 7.1(3) from [15], where $R$ is assumed to be a Banach algebra. Moreover, we obtain similar results for larger size matrices.

Corollary 6. Let $R$ be a commutative unital ring, $\operatorname{bsr} R=1$, and let every $x \in R^{*}$ admit a logarithm. Then every element in $\mathrm{GL}_{2}(R)$ is a product of 4 exponentials.

Corollary 7. Let $R$ be a commutative unital ring, $\operatorname{bsr} R=1$, and let every $x \in R^{*}$ admit a logarithm. Then every element in $\mathrm{GL}_{n}(R), n \geq 3$, is a product of 6 exponentials.
Proof. For $n=3$, it suffices to combine Theorem 2 and Corollary 4 ,
Now, assume that $n \geq 4$. Let $\mathrm{ut}_{m}$ denote the number of unipotent matrices needed to factorize any element in $\mathrm{SL}_{m}(R)$ starting with an upper triangular matrix. Theorem 20(b) in [7] says that any element in $\mathrm{SL}_{n}(R)$ is a product of 6 exponentials for

$$
n \geq \min \left(m\left[\frac{\operatorname{ut}_{m}(R)+1}{2}\right]\right)
$$

where the minimum is taken over all $m \geq \operatorname{bsr} R+1$. In our case the minimum is taken over $m \geq 2$ and the number $\mathrm{ut}_{2}(R)=4$ by the proof of Theorem 2. Since $n \geq 4$, the proof is finished.

Corollary 6 applies to the disk algebra and also to the rings $\mathcal{O}(\mathbb{C})$ and $\mathcal{O}(\mathbb{D})$ of holomorphic functions. Indeed, the identity $\operatorname{bsr}(\mathcal{O}(\Omega))=1$ for an open Riemann surface follows from the strengthening of the classical Wedderburn lemma (see [19, Chapter 6, Section 3]; see also [10] or [2]). However, for $R=\mathcal{O}(\mathbb{C})$ and $R=\mathcal{O}(\mathbb{D})$, the number 4 is not optimal; see Section 4.4 below. Also, it is known that the optimal number is at least 2 (see [15]). So, we arrive at the following natural question:

Problem 2. Is any element of $\mathrm{GL}_{2}(\mathcal{O}(\mathbb{D}))$ or $\mathrm{GL}_{2}(\mathcal{O}(\mathbb{C}))$ a product of two exponentials?
4.4. Products of 3 exponentials. In this section, we prove the following result.

Proposition 3. Let $\Omega$ be an open Riemann surface. Then every element in $\mathrm{SL}_{2}(\mathcal{O}(\Omega))$ is a product of 3 exponentials.

We will need several auxiliary results. The first theorem is a classical one 8 .
Theorem 3 (Mittag-Leffler Interpolation Theorem). Let $\Omega$ be an open Riemann surface and let $\left\{z_{i}\right\}_{i=1}^{\infty}$ be a discrete closed subset of $\Omega$. Assume that a finite jet

$$
\begin{equation*}
J_{i}(z)=\sum_{j=1}^{N_{i}} b_{j}^{(i)}\left(z-z_{i}\right)^{j} \tag{4.1}
\end{equation*}
$$

is defined in some local coordinates for every point $z_{i}$. Then there exists $f \in \mathcal{O}(\Omega)$ such that

$$
\begin{equation*}
f(z)-J_{i}(z)=o\left(\left|z-z_{i}\right|^{N_{i}}\right) \quad \text { as } z \rightarrow z_{i}, i=1,2 \ldots \tag{4.2}
\end{equation*}
$$

Corollary 8. Under assumptions of Theorem (3, suppose that $b_{0}^{(i)} \neq 0$ in (4.1) for $i=1,2, \ldots$. Then there exist $f, g \in \mathcal{O}(\Omega)$ such that (4.2) holds and $f=e^{g}$.

Proof. Let $b_{0}=b_{0}^{(i)}$ for some $i$. Since $b_{0} \neq 0$, there exists a logarithm $\ln$ in a neighborhood of $b_{0}$. So, $\ln$ is a local biholomorphism which induces a bijection between jets of $f$ and $g:=\ln f$.

In "modern" language, the proof of Corollary 8 uses the fact that $\mathbb{C}^{*}$ is an Oka manifold (we refer the interested reader to 9 ). Thus for any Stein manifold $X$ and an analytic subset $Y \subset X$, a (jet of) holomorphic map $f: Y \rightarrow \mathbb{C}^{*}$ (along $Y$ ) extends to a holomorphic map $f: X \rightarrow \mathbb{C}^{*}$ if and only if it extends continuously. The obstruction for a continuous extension is an element of the relative homology group $H_{2}(X, Y, \mathbb{Z})$. Observe that, for any discrete subset $Y$ of a 1-dimensional Stein manifold $X$, we have $H_{2}(X, Y, \mathbb{Z})=0$ because $H_{2}(X, \mathbb{Z})=H_{1}(Y, \mathbb{Z})=0$. This is the point where the proof of Proposition 3 below breaks down when we replace the Riemann surface $\Omega$ by a Stein manifold of higher dimension. Even a nowhere vanishing continuous function $\alpha$, as in the proof, does not exist in general.
Lemma 2. Let $\Omega$ be an open Riemann surface and $X \in \mathrm{GL}_{2}(\mathcal{O}(\Omega))$. Assume that $\lambda \in \mathcal{O}^{*}(\Omega)$ is the double eigenvalue of $X$ and $\operatorname{det} X$ has a logarithm in $\mathcal{O}(\Omega)$. Then $X$ is an exponential.

Proof. We consider two cases.
Case 1: $X(z)$ is a diagonal matrix for all $z \in \Omega$.
We have

$$
X(z)=\left(\begin{array}{cc}
\lambda(z) & 0 \\
0 & \lambda(z)
\end{array}\right)=\exp \left(\begin{array}{cc}
\alpha(z) & 0 \\
0 & \alpha(z)
\end{array}\right)
$$

Case 2: $X(z)$ is not identically diagonal.
Either the first or the second line in $X(z)-\lambda(z) I$, say $(h(z), g(z))$, is not identical zero. So,

$$
v_{1}(z)=\binom{-g(z)}{h(z)}
$$

is a holomorphic eigenvector for $X(z)$ except those points $z \in \Omega$ for which $v_{1}(z)=$ 0. Construct a function $f(z) \in \mathcal{O}(\Omega)$ such that its vanishing divisor is exactly $\min ($ ord $g$, ord $h)$. Then

$$
v(z)=\frac{1}{f(z)} v_{1}(z)
$$

is a holomorphic eigenvector for $X(z), z \in \Omega$.
Now, choose a matrix $P(z) \in \mathrm{GL}_{2}(\mathcal{O}(\Omega))$ with first column $v(z)$. Then the matrix $P^{-1}(z) X(z) P(z)$ has the following form:

$$
\left(\begin{array}{cc}
\lambda(z) & \beta(z) \\
0 & \lambda(z)
\end{array}\right)=\exp \left(\begin{array}{cc}
\frac{1}{2} \gamma(z) & \frac{\beta(z)}{\lambda(z)} \\
0 & \frac{1}{2} \gamma(z)
\end{array}\right)
$$

Thus,

$$
X(z)=\exp P(z)\left(\begin{array}{cc}
\frac{1}{2} \gamma(z) & \frac{\beta(z)}{\lambda(z)} \\
0 & \frac{1}{2} \gamma(z)
\end{array}\right) P^{-1}(z)
$$

as required.

Proof of Proposition 3. Let

$$
X=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in \mathrm{SL}_{2}(R)
$$

that is, $a d-b c=1$. We are looking for $\alpha \in R^{*}$ and $\beta \in R$ such that the matrix

$$
X\left(\begin{array}{cc}
\alpha^{2} & \beta \\
0 & 1
\end{array}\right)=\left(\begin{array}{cc}
\alpha^{2} a & \beta a+b \\
\alpha^{2} c & \beta c+d
\end{array}\right):=Y
$$

has a double eigenvalue.
Case 1: $c=0$. We have

$$
X=\left(\begin{array}{cc}
a & b \\
0 & a^{-1}
\end{array}\right) .
$$

It suffice to observe that

$$
\left(\begin{array}{cc}
a & b \\
0 & a^{-1}
\end{array}\right)\left(\begin{array}{cc}
a^{-2} & 0 \\
0 & 1
\end{array}\right)=\left(\begin{array}{cc}
a^{-1} & b \\
0 & a^{-1}
\end{array}\right)
$$

has the double eigenvalue $a^{-1}$.
Case 2: $c \neq 0$. The matrix $Y$ has a double eigenvalue if $4 \operatorname{det} Y=(\operatorname{tr} Y)^{2}$, that is,

$$
\begin{equation*}
\left(\alpha^{2} a+\beta c+d\right)^{2}=4 \alpha^{2} \tag{4.3}
\end{equation*}
$$

Put

$$
\beta=\frac{2 \alpha-a \alpha^{2}-d}{c} .
$$

Clearly, $\beta$ is a formal solution of (4.3). Below we show how to construct $\alpha(z)=$ $\exp (\widetilde{\alpha}(z)) \in \mathcal{O}^{*}(\Omega)$ such that $\beta$ is holomorphic.

Let $\left\{z_{i}\right\} \subset \Omega$ be the zero set of $c(z)$. Fix $i$ and $z_{i} \in \Omega$. Let $c\left(z_{i}\right)=\cdots=$ $c^{(k)}\left(z_{i}\right)=0$, and $c^{(k+1)}\left(z_{i}\right) \neq 0$. Observe that $a\left(z_{i}\right) \neq 0$. So, define $\alpha(z)$, in a neighborhood of $z_{i}$, as $1 / a(z)$ up to a sufficiently high order, namely,

$$
\begin{equation*}
a(z) \alpha(z)=1+\left(z-z_{i}\right)^{k} h(z) \tag{4.4}
\end{equation*}
$$

where $h(z)$ is holomorphic in a neighborhood of $z_{i}$. Since $a d-b c=1$, we have $1-a d=\left(z-z_{i}\right)^{k} g(z)$. Therefore,

$$
\begin{aligned}
2 a \alpha-a^{2} \alpha^{2}-a d & =-\left(1-a \alpha^{2}\right)^{2}+1-a d \\
& =-\left(z-z_{0}\right)^{2 k} h^{2}(z)+\left(z-z_{0}\right)^{k} g(z)
\end{aligned}
$$

vanishes of order $k$ at $z_{i}$. Hence, $2 \alpha-a \alpha^{2}-d$ also vanishes of order $k$ at $z_{i}$.
So, we have constructed $\alpha(z)$ locally as finite jets $J_{i}(z)$ defined by (4.1) with $b_{0}^{(i)} \neq 0$ in some local coordinates for every point $z_{i}, i=1,2, \ldots$ Now, Corollary 8 provides $\widetilde{\alpha} \in \mathcal{O}(\Omega)$ such that $\alpha(z)=\exp (\widetilde{\alpha}(z)) \in \mathcal{O}^{*}(\Omega)$ and (4.4) holds. Hence, $\beta$ is holomorphic.

So, the matrix

$$
X\left(\begin{array}{cc}
\alpha^{2} & \beta \\
0 & 1
\end{array}\right):=Y
$$

has a double eigenvalue and $\operatorname{det} Y$ admits a logarithm. Thus, applying Lemma 2 we conclude that $Y$ is an exponential. To finish the proof of the proposition, it remains observe that

$$
\left(\begin{array}{cc}
\alpha^{2} & \beta \\
0 & 1
\end{array}\right)=\left(\begin{array}{cc}
\alpha & 0 \\
0 & \alpha^{-1}
\end{array}\right)\left(\begin{array}{cc}
\alpha & \beta \alpha^{-1} \\
0 & \alpha
\end{array}\right)
$$

where both multipliers on the right hand side are exponentials.

Corollary 9. Let $X \in \mathrm{GL}_{2}(\mathcal{O}(\Omega))$. The following properties are equivalent:
(i) $X$ is a product of 3 exponentials;
(ii) $\operatorname{det} X$ is an exponential;
(iii) $X$ is null-homotopic.

Proof. Clearly, (i) $\Rightarrow$ (iii). Now, assume that $X$ is null-homotopic. Then $\operatorname{det} X$ is homotopic to the function $f \equiv 1$. Since $\exp : \mathbb{C} \rightarrow \mathbb{C}^{*}$ is a covering, we conclude that $\operatorname{det} X(z)=\exp (h(z))$ with $h \in \mathcal{O}(\Omega)$. So, (iii) implies (ii). The implication $($ ii $) \Rightarrow(\mathrm{i})$ is standard; see, for example, the proof of Corollary 4 .

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