EXPONENTIAL FACTORIZATIONS OF HOLOMORPHIC MAPS

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Abstract. We show that any element of the special linear group SL₂(R) is a product of two exponentials if the ring R is either the ring of holomorphic functions on an open Riemann surface or the disc algebra. This is sharp: one exponential factor is not enough since the exponential map corresponding to SL₂(C) is not surjective. Our result extends to the linear group GL₂(R).

1. Introduction

For a Stein space X, a complex Lie group G and its exponential map exp : g → G we say that a holomorphic map f : X → G is a product of k exponentials if there are holomorphic maps f₁, . . . , fₖ : X → g such that

f = exp(f₁) · · · exp(fₖ).

It is easy to see that any map f which is a product of exponentials (for some sufficiently large k) is null-homotopic. In the case where G is the special linear group SLₙ(C) the converse follows from [6] as explained in [1]. However, it turns out to be a difficult problem to determine the minimal number k of needed factors in dependence of the dimensions of X and SLₙ(C). We solve this problem for dim X = 1 and n = 2.

Theorem 1. Any holomorphic map from an open Riemann surface to the special linear group SL₂(C) is a product of two exponentials.

Theorem 1 improves a result of Doubtsov and Kutzschebauch, who showed the same result with three instead of two factors in the conclusion, see Proposition 3 in [1]. Stated differently, Theorem 1 says that every element of SL₂(O(X)) can be written as a product of two exponentials, where O(X) denotes the ring of holomorphic functions on a given open Riemann surface X. Our second result is of similar flavor, but the ring O(X) is replaced by the disc algebra A. By definition, the disc algebra A is the C-Banach algebra of continuous functions on the closed disc \{z ∈ C : |z| ≤ 1\} which are holomorphic on the interior, equipped with the supremum norm.

Theorem 2. For the disc algebra A, any element of SL₂(A) is a product of two exponentials.

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Recall that the exponential map \( \exp : \mathfrak{sl}_2(\mathbb{C}) \to \text{SL}_2(\mathbb{C}) \) is not surjective. In this sense Theorem 1 and 2 are sharp. It is worth mentioning that \( \text{SL}_2(\mathbb{C}) \) is simply connected implying that holomorphic maps from open Riemann surfaces to \( \text{SL}_2(\mathbb{C}) \) and elements of \( \text{SL}_2(\mathcal{A}) \) are null-homotopic. This is the reason that the map in question being null-homotopic is a redundant assumption in Theorem 1 and 2. As corollaries of Theorem 1 and 2 we get the analogous results if the special linear group is replaced by the linear group with the corresponding entries.

**Corollary 1.** Any null-homotopic holomorphic map from an open Riemann surface to the linear group \( \text{GL}_2(\mathbb{C}) \) is a product of two exponentials.

**Proof.** Let \( X \) be an open Riemann surface and \( \text{M}_2(\mathbb{C}) \) the complex \( 2 \times 2 \)-matrices. If a given holomorphic map \( A : X \to \text{GL}_2(\mathbb{C}) \) is null-homotopic, then \( \det A : X \to \mathbb{C}^* \) is null-homotopic as well. Therefore \( \det A \) has a holomorphic logarithm \( \log : X \to \mathbb{C} \), satisfying \( e^{\log} = \det A \). In particular, if \( D : X \to \text{M}_2(\mathbb{C}) \) is the diagonal matrix with diagonal entries \( \log \) divided by 2, \( \exp(-D)A \) has values in \( \text{SL}_2(\mathbb{C}) \). By Theorem 1 there are holomorphic \( B, C : X \to \text{M}_2(\mathbb{C}) \) such that

\[
A = e^D e^{-D} A = e^{D} e^{B} e^{C} = e^{D+B} e^{C},
\]

where we used in the last equality that \( D \) commutes with all other matrices. This finishes the proof. \( \square \)

Unlike in Theorem 1 in Corollary 1 the assumption that \( f \) is null-homotopic is not redundant. For instance,

\[
A(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \ z \in \mathbb{C}^*
\]

is not null-homotopic since otherwise \( \det A : \mathbb{C}^* \to \mathbb{C}^* \), \( z \mapsto z^2 \) would be null-homotopic as well.

**Corollary 2.** For the disc algebra \( \mathcal{A} \), any element of \( \text{GL}_2(\mathcal{A}) \) is a product of two exponentials.

**Proof.** This follows from Theorem 2 in the same way as Corollary 1 follows from Theorem 1. Here, we need in addition that any unit in \( \mathcal{A} \) has a logarithm, which follows from the fact that the disc (and thereby the domain of the elements of \( \mathcal{A} \)) is contractible. In particular, the map in question being null-homotopic is again a redundant assumption. \( \square \)

Corollary 2 improves a result of Mortini and Rupp, who showed the same with four instead of two factors in the conclusion, see Theorem 7.1 in [8].

Also Corollary 1 and 2 are sharp in the sense that one exponential factor is not enough. An example is the matrix

\[
A(z) = \begin{pmatrix} 1 & 1 \\ 0 & e^{4\pi i z} \end{pmatrix}, \ z \in \Delta.
\]

One can show that the second entry of any lift of \( z \mapsto A(z) \), \( |z| < 1/2 \) via the exponential map tends to infinity if \( z \to 1/2 \). For details see [8], Example 6.4.

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2. Proof of Theorem 1

An important ingredient in the proof is an Oka principle due to Forstnerič, which follows essentially from Theorem 2.1 in [2]. The version, which we use in this text is the below stated Theorem 4. It is used to show Proposition 1 which is the main ingredient in the proof of Theorem 1. Throughout this section \( X \) denotes an open Riemann surface.

**Proposition 1.** Let \( A : X \to \text{SL}_2(\mathbb{C}) \) be holomorphic and assume that \( A(x) \) has distinct eigenvalues for some \( x \in X \). Then \( A = BC \) for suitable holomorphic \( B, C : X \to \text{SL}_2(\mathbb{C}) \), both of which have vanishing trace.

Note that the conclusion of Proposition 1 is equivalent to finding a holomorphic \( B : X \to \text{SL}_2(\mathbb{C}) \) such that \( B \) and \( AB \) have vanishing trace, simply since taking the inverse of a \( 2 \times 2 \)-matrix with trace zero has again trace zero. Expressed differently, Proposition 1 is proved if we can show the existence of a global section of the bundle

\[
Z := \{(x, B) \in X \times \text{SL}_2(\mathbb{C}) : \text{tr}(B) = \text{tr}(A(x)B) = 0\}
\]

over \( X \). If \( a, b, c, d \) denote the coefficients of \( A \), and \( u, v, w, -u \) denote the coefficients of \( B \), we can express \( Z \) more explicitly as

\[
\{(x, u, v, w) \in X \times \mathbb{C}^3 : (a(x) - d(x))u + b(x)v + c(x)w = 0, \ u^2 + vw = -1\}.
\]

More concretely, Proposition 1 is proved if we manage the proof the following reformulation.

**Proposition 2.** Let \( A : X \to \text{SL}_2(\mathbb{C}) \) be holomorphic and assume that \( A(x) \) has distinct eigenvalues for some \( x \in X \). Then the restriction \( h \) of the projection \( X \times \mathbb{C}^3 \to X \) to \( Z \) has a holomorphic section.

For an open subset \( U \subset X \), \( Z|U \) denotes the restriction of the bundle \( h : Z \to X \) to \( h^{-1}(U) \). We start the proof of Proposition 2 with the following simple

**Lemma 1.** For every \( x \in X \) there is a neighborhood \( U \) of \( x \) and a holomorphic section \( F : U \to Z|U \) of \( Z|U \).

**Proof.** After passing to a local chart we may assume that \( X \) is the unit disc \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \) and \( x = 0 \). Finding a local holomorphic section in a neighborhood of 0 is equivalent to finding a neighborhood \( 0 \in U \subset \Delta \) and holomorphic maps \( u, v, w : U \to \mathbb{C} \), which satisfy

\[
(a - d)u + bv + cw = 0, \ u^2 + vw = -1.
\]

Local holomorphic solutions to (1) exist if and only if there are local holomorphic solutions to the less restrictive problem

\[
(a - d)u + bv + cw = 0, \ u^2 + vw \in \mathcal{O}_0^*.
\]

The reason is that if \( u, v, w \) are local solutions in a neighborhood of the origin to (2), we can rescale these solutions with a local holomorphic square root of \( u^2 + vw \), or more precisely, by defining new solutions by \( \frac{iu}{r^n}, \frac{iv}{r^n}, \frac{iw}{r^n} \) for some \( r : U \to \mathbb{C}^* \) satisfying \( r^2 = u^2 + vw \) defined on a sufficiently small neighborhood \( U \) of the origin. To find solutions to (2) we distinguish three cases. Let \( n(f) \in \mathbb{Z}_{\geq 0} \) denote the vanishing order of a holomorphic function...
This shows that the fiber is given by \( c \). Since \( u, v, w \) to determine all \( u \) is holomorphic in a neighborhood of 0 and \( u = 1, v = \frac{a - d}{b} \) and \( w = 0 \) is a solution to \( 4. \) The second case \( n(a - d) \geq n(c) \) we find similarly a solution \( u = 1, v = 0 \) and \( w = -\frac{a - d}{c} \) to \( 4. \) The remaining case is \( n(a - d) < \min(n(b), n(c)) \), which implies \( n(a - d) < n(b + c) \) and hence \( -\frac{b + c}{a - d} \) is holomorphic in a neighborhood of the origin and vanishes at the origin. Then \( u = -\frac{b + c}{a - d}, v = 1, w = 1 \) solves \( 4. \) This finishes the proof. \( \square \)

Let \( D \) denote the discriminant of \( A \), that is \( D := (a + d)^2 - 4 \). By isomorphic fiber bundles we mean isomorphic as complex analytic fiber bundles.

**Lemma 2.** Let \( U \subset X \setminus \{(D = 0) \cup \{c = 0\}\} \) be an open neighborhood where \( D : U \to \mathbb{C} \) has a holomorphic square root \( \sqrt{D} \), and set \( f := \frac{d - a + \sqrt{D}}{2c} \). Then \( Z \mid U \) is isomorphic to \( U \times \mathbb{C}^* \), and an isomorphism is given by

\[
\phi : Z \mid U \to U \times \mathbb{C}^*, \quad \phi(x, u, v, w) = (x, u + f(x)v).
\]

**Proof.** First we do the necessary computations at the level of a single fiber. For this, we think of the coefficients \( a, b, c, d \) of \( A \) as elements of \( \mathbb{C} \). We want to determine all \( u, v, w \in \mathbb{C} \) such that

\[ (a - d)u + bv + cw = 0, \quad -u^2 - vw = 1. \]

Since \( c \neq 0 \), we can solve for \( w \) and get equivalently

\[
-1 = u^2 + vw = u^2 + v \frac{(d - a)u - bv}{c} = u^2 + \frac{d - a}{c}uv - \frac{b}{c}v^2 = \left( u + \frac{d - a}{2c}v \right)^2 - \left( \frac{(d - a)^2}{4c^2} + \frac{b}{c} \right)v^2.
\]

Furthermore we have

\[
\frac{(d - a)^2}{4c^2} + \frac{b}{c} = \frac{(d + a)^2 - 4ad}{4c^2} + \frac{4bc}{4c^2} = \frac{(d + a)^2 - 4(ad - bc)}{4c^2} = \frac{D}{4c^2}.
\]

Fix a square root \( \sqrt{D} \) of \( D \) and note that

\[
\tilde{u} = u + \frac{d - a + \sqrt{D}}{2c}v, \quad \tilde{v} = u + \frac{d - a - \sqrt{D}}{2c}v
\]

defines a linear coordinate change of \( \mathbb{C}^2 \), which translates the above equation to

\[
-1 = \left( u + \frac{d - a}{2c}v \right)^2 - \frac{D}{4c^2}v^2 = \left( \frac{d - a}{2c}v \right)^2 - \left( \frac{\sqrt{D}}{2c}v \right)^2 = \left( u + \frac{d - a + \sqrt{D}}{2c}v \right) \left( u + \frac{d - a - \sqrt{D}}{2c}v \right) = \tilde{u} \tilde{v}.
\]

This shows that the fiber is given by \( \{(\tilde{u}, \tilde{v}) \in \mathbb{C}^2 : \tilde{u} \tilde{v} = -1\} = \mathbb{C}^* \) and that \( (u, v, w) \to u + \frac{d - a + \sqrt{D}}{2c}v \) is an isomorphism of the fiber onto \( \mathbb{C}^* \). Moreover, our computations yield a trivialization of \( Z \mid U \), which is defined similarly, or more precisely, as in the assumption of the Lemma. This is the case since
Lemma 3. Over $X \setminus \{D = 0\}$, $h : Z \to X$ is a fiber bundle with fiber $\mathbb{C}^*$.

Proof. At points $x \in X \setminus \{D = 0\}$ with $c(x) \neq 0$, choose a neighborhood $U \subset X$ of $x$ such that $c|U$ does not vanish, and such that $D$ has a square root on $U$. Then a trivialization of $Z|U$ is given by Lemma 2. In the case $c(x) = 0$, let us reduce the problem to the case $c(x) \neq 0$ with the following observation. Our bundle is given by

$$Z = \{(x, B) \in X \times \text{SL}_2(\mathbb{C}) : \text{tr}(B) = \text{tr}(A(x)B) = 0\}.$$

Define for $P \in \text{SL}_2(\mathbb{C})$ a bundle

$$Z_P = \{(x, PBP^{-1}) \in X \times \text{SL}_2(\mathbb{C}) : \text{tr}(B) = \text{tr}(A(x)B) = 0\}.$$

Clearly $Z$ and $Z_P$ are isomorphic over $X$. Since conjugation with a matrix does not change the trace, we obtain with the substitution $C = PBP^{-1}$

$$Z_P = \{(x, C) \in X \times \text{SL}_2(\mathbb{C}) : \text{tr}(P^{-1}CP) = \text{tr}(A(x)P^{-1}CP) = 0\}
= \{(x, C) \in X \times \text{SL}_2(\mathbb{C}) : \text{tr}(C) = \text{tr}(PA(x)P^{-1}C) = 0\}.$$

Note that if the third entry $c$ of $A$ equals 0 at $x$, then, since $D(x) \neq 0$ and hence $A(x) \neq \pm id$, there is $P \in \text{SL}_2(\mathbb{C})$ such that the third entry of $PA(x)P^{-1}$ does not vanish. Using that $Z$ and $Z_P$ are isomorphic and that we can solve the problem for $Z_P$ close to $x$, the statement follows. □

To finish the proof of Proposition 2, we need the following special case of Theorem 6.14.6, p. 310 in [3].

Theorem 3. Let $h : Z \to X$ be a holomorphic map of a reduced complex space $Z$ onto a reduced Stein space $X$. Let $X' \subset X$ be a complex analytic subvariety and let $Z' := h^{-1}(X')$ and assume that the restriction $h : Z \setminus Z' \to X \setminus X'$ is an elliptic submersion. Moreover, let $f : X \to Z$ be a continuous section of $h$ which is holomorphic in a neighborhood of $X'$. Then $f$ is homotopic through continuous sections of $h$ which are holomorphic in a fixed small neighborhood of $X'$ to a holomorphic section of $h$.

A consequence of this is the following

Proposition 3. Let $h : Z \to X$ be a holomorphic map from a reduced complex space onto an open Riemann surface. Moreover, assume that there is a discrete set $X' \subset X$ such that for $Z' = h^{-1}(X')$, the restriction $h : Z \setminus Z' \to X \setminus X'$ is a fiber bundle with fiber $\mathbb{C}^*$ and assume that there is a local holomorphic section in a neighborhood of every point of $X'$. Then $h$ has a global holomorphic section $f : X \to Z$.

Proof. First we show the existence of a continuous section which is holomorphic in a neighborhood $U$ of $X'$. By assumption there is a local holomorphic section $f : U \to Z$ of $h$ defined on a neighborhood $U$ of $X'$. By possibly shrinking $U$ we may assume that every connected component of $U$ contains exactly one point of $X'$ and is homeomorphic to a disc, and that $f$ extends continuously to $\overline{U}$. $X \setminus X'$ is an open Riemann surface and thus deformation retracts onto a 1-dimensional CW-complex $K$, see e.g. [4]. After possibly modifying a fixed deformation retract $r$ of $X \setminus X'$ onto $K$ by a
conjugation with a suitable homeomorphism of $X \setminus X'$ we can assume that $\partial U \subset K$. Since the fiber $\mathbb{C}^*$ of $Z$ is connected we can extend $f|\partial U$ to a section $\tilde{f} : K \to Z[K]$. Since $K$ is a deformation retract of $X \setminus X'$ and $h : Z \setminus Z' \to X \setminus X'$ is a fiber bundle, the section $\tilde{f}$ extends to a continuous section $F : X \setminus X' \to Z \setminus Z'$, see e.g. Theorem 7.1, p. 21 in [3]. Since $f$ and $F|X \setminus U$ agree on $\partial U$, these two sections define a continuous section $X \to Z$ which agrees with the holomorphic section $f$ on the neighborhood $U$ of $X'$.

The existence of a global holomorphic section follows now from the above Oka principle due to Forstnerič, see Theorem 3. This finishes the proof. □

Proof of Proposition 3. Let $h : Z \to X$ be the bundle over $X$ from Proposition 2. With Lemma 1 we proved that there are local sections of $h$ at every point $x \in X$, in particular also at points of the discrete set $X' = \{D = 0\}$. Moreover, with Lemma 5 we showed that $h$ is a locally trivial $\mathbb{C}^*$-bundle over $X \setminus \{D = 0\}$. It follows now from Proposition 4 that there is a holomorphic section of $h$. This finishes the proof. □

Lemma 4. Let $X$ be an open Riemann surface and let $A : X \to \text{SL}_2(\mathbb{C})$ be holomorphic with vanishing trace. Then $A = e^B$ for some holomorphic $B : X \to \text{M}_2(\mathbb{C})$ with vanishing trace.

Proof. The characteristic polynomial of $A$ equals $T^2 + 1$. In particular $\pm i$ are the eigenvalues (at every point $x \in X$). There are line bundles $E(i)$ and $E(-i)$ over $X$, whose non-vanishing sections correspond to holomorphic eigenvectors of $i$ and $-i$ respectively. Explicitly, we have

$$E(i) := \{(x, z) \in X \times \mathbb{C}^2 : A(x)z = iz\},$$

$$E(-i) := \{(x, z) \in X \times \mathbb{C}^2 : A(x)z = -iz\}.$$ 

Since every line bundle over an open Riemann surface is trivial, we have $E(i) \cong X \times \mathbb{C} \cong E(-i)$ as complex analytic line bundles. This implies that there are two holomorphic eigenvectors $v : X \to E(i)$, $w : X \to E(-i)$ with $v(x) \neq 0 \neq w(x)$ for all $x \in X$. In particular

$$P : X \to \text{M}_2(\mathbb{C}), P(x) := (v(x) \ w(x))$$

takes values in $\text{GL}_2(\mathbb{C})$ since $v(x)$ and $w(x)$ are eigenvectors of $A(x)$ to the distinct eigenvalues $\pm i$. This implies that $A$ is holomorphically diagonalizable with

$$A = PDP^{-1}, \ D := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$ 

For the diagonal matrix $\tilde{D}$ with entries $\pm \frac{i\pi}{2}$ we have $e^{\tilde{D}} = D$. We get for $B := P\tilde{D}P^{-1}$ the equality

$$A = PDP^{-1} = Pe^{\tilde{D}}P^{-1} = e^{P\tilde{D}P^{-1}} = e^B,$$

as desired. Note that $B$ has vanishing trace since $\tilde{D}$ has vanishing trace. This finishes the proof. □

Proof of Theorem 7. Let $X$ be an open Riemann surface and let $A : X \to \text{SL}_2(\mathbb{C})$ be a holomorphic map. If the characteristic polynomial of $A$ equals $(T - 1)^2$, then, since $(A - id)^2 = \chi_A(A) = 0$ by Cayley-Hamilton, we have

$$\exp(A - id) = id + (A - id) = A.$$
Moreover, the trace of $A$ is equal to minus the second coefficient of the characteristic polynomial, which implies in our case that $\text{tr}(A-id) = 0$, as desired. This shows that $A$ can be written as a single exponential factor. If the characteristic polynomial is $(T+1)^2$, then the characteristic polynomial of $-A$ is $(T-1)^2$ and since $-id$ is equal to the exponential of the diagonal matrix with diagonal entries $\pi i$ and $-\pi i$, $A$ is a product of at most two exponentials with vanishing trace. Otherwise there is $x \in X$ such that $A(x)$ has distinct eigenvalues. In that case it follows from Proposition [1] that $A = BC$ for holomorphic $B, C : X \to \text{SL}_2(\mathbb{C})$ with vanishing trace. In particular, the characteristic polynomials of $B$ and $C$ are both $(T-i)(T+i)$. Since $B$ and $C$ have a logarithm by Lemma [1], we are done. \[\square\]

3. Proof of Theorem [2]

The proof depends essentially on three ingredients. The first ingredient is that the Bass stable rank of the disc algebra $A$ equals 1. This is needed to reduce the problem to matrices with an invertible first entry. The second and third ingredient are the simple facts that the elements of $A$ are bounded, and that $\exp : A \to A$ is onto to units of $A$. In the following $\overline{A} \subset \mathbb{C}$ denotes the closed unit disc centered at the origin. We use the following notation. If $f : \overline{A} \to \mathbb{C}$ is a function, then $|f| : \overline{A} \to \mathbb{R}$ denotes the absolute value $z \mapsto |f(z)|$. In particular, the symbol $|f|$ should not be confused with the sup-norm on $A$, which is not used explicitly in the proof. Moreover, for $f, g : \overline{A} \to \mathbb{R}$ we write $f > g$ if $f(z) > g(z)$ for all $z \in \overline{A}$. The proof depends on the following elementary lemma.

**Lemma 5.** Let $f \in A$ be such that $|f| > 2$. Then the polynomial $T^2 - fT + 1$ has roots $\lambda, \lambda^{-1} \in A$ such that $|\lambda| > 1$.

**Proof.** First note that our assumption implies that the discriminant $f^2 - 4$ does not vanish. Therefore $f^2 - 4$ has a square root in $A$, which implies that there are roots $\lambda, \lambda^{-1} \in A$ of $T^2 - fT + 1$. We have to show that one of $|\lambda|$ and $|\lambda^{-1}|$ is strictly larger than 1. Note that if $T^2 - zT + 1$, $z \in \mathbb{C}$ has a root $r \in \mathbb{C}$ with $|r| = 1$, then we get $|z| = |r^2 + 1|/|r| = |r^2 + 1| \leq 2$. Expressed differently, if $|z| > 2$, then $T^2 - zT + 1$ has no root on the unit circle. This implies that $\lambda$ and $\lambda^{-1}$ avoid the unit circle, and moreover – by continuity of $\lambda$ and $\lambda^{-1}$ – that exactly one of the two is strictly bigger than 1 in absolute value. \[\square\]

**Proof of Theorem [2]** Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(A).$$

It is well-known that the Bass stable rank of $A$ equals 1, see [7]. By definition of the Bass stable rank this means that for any pair $f, g \in A$ with $fA + gA = A$, there is $h \in A$ such that $f + hg$ is a unit in $A$. In particular, since $ad - bc = 1$, there is $h \in A$ such that $a + hc = 1$. Consequently the first entry of

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + hc & * \\ * & * \end{pmatrix}$$
is a unit. Since conjugation with matrices in $GL_2(\mathcal{A})$ does not change the number of needed exponential factors to represent a given matrix, this shows that it suffices to consider the case where the first entry $a$ of $A$ is a unit. For such $A$, the strategy is as follows: for $\delta > 0$ set

$$B := \begin{pmatrix} \delta & 0 \\ 0 & 1/\delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta a & \delta b \\ c/\delta & d/\delta \end{pmatrix} \in SL_2(\mathcal{A}).$$

If we find $\delta$ such that $B = B(\delta)$ has a logarithm, then – since $A$ is the product of the diagonal matrix with entries $1/\delta, 0, 0, \delta$ and $B$ – we know that $A$ is a product of two exponentials. Our claim is that $B$ has a logarithm for any sufficiently large $\delta > 0$. To see this, let $\delta \geq 1$ be an upper bound of the (bounded) function

$$\beta = \frac{3 + |d|}{|a|}.$$  

From the fact that $\delta \geq 1$ is an upper bound of $\beta$ it follows that

$$|\text{tr}(B)| = |\delta a + d/\delta| \geq \delta|a| - \frac{|d|}{\delta} \geq (3 + |d|) - |d| > 2.$$  

By Lemma 5 we know that the characteristic polynomial $\chi_B = T^2 - \text{tr}(B)T + 1$ has roots $\lambda, \lambda^{-1} \in \mathcal{A}$ with $|\lambda| > 1$. Since $\lambda$ is a unit in $\mathcal{A}$, the matrix $D$ with diagonal entries $\lambda$ and $\lambda^{-1}$ has a logarithm given by the diagonal matrix with diagonal entries $\log(\lambda) \in \mathcal{A}$ and $-\log(\lambda) \in \mathcal{A}$ for some fixed logarithm of $\lambda$. Moreover, since conjugation with an element in $GL_2(\mathcal{A})$ does not change the number of needed exponential factors, it suffices to find $P \in GL_2(\mathcal{A})$ with

$$B = PDP^{-1}.$$  

Our claim is that

$$P = \begin{pmatrix} d/\delta - \lambda & -\delta b \\ -c/\delta & \delta a - \lambda^{-1} \end{pmatrix} \in M_2(\mathcal{A})$$

does the job. To show this it suffices to show that the columns $v$ resp. $w$ of $P = (v \ w)$ satisfy $(B - \lambda id)v = (B - \lambda^{-1} id)w = 0$ and that $|\det B| \geq 1$. For the first part we get

$$(B - \lambda id)v = \begin{pmatrix} \delta a - \lambda & \delta b \\ c/\delta & d/\delta - \lambda \end{pmatrix} \begin{pmatrix} d/\delta - \lambda \\ -c/\delta \end{pmatrix} = \begin{pmatrix} \chi_B(\lambda) \\ 0 \end{pmatrix} = 0,$$

and similarly

$$(B - \lambda^{-1} id)w = \begin{pmatrix} \delta a - \lambda^{-1} & \delta b \\ c/\delta & d/\delta - \lambda^{-1} \end{pmatrix} \begin{pmatrix} -\delta b \\ \delta a - \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_B(\lambda^{-1}) \end{pmatrix} = 0.$$  

For the second part, we get with $ad - bc = 1$

$$\det P = -\delta \lambda a - \delta^{-1} \lambda^{-1} d + 2.$$  

It follows from $|\lambda| > 1$ that

$$|\det P| \geq \delta|\lambda||a| - \delta^{-1} |\lambda^{-1}||d| - 2 \geq \delta|a| - \delta^{-1} |d| - 2.$$  

Furthermore, the fact that $\delta \geq 1$ bounds $\beta = (3 + |d|)/|a|$ from above yields

$$\delta|a| - \delta^{-1} |d| - 2 \geq (3 + |d|) - |d| - 2 = 1,$$

which shows that $|\det P| \geq 1$. This finishes the proof. \qed
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