



# The density ratio of Poisson binomial versus Poisson distributions

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## ABSTRACT

Let  $b(x)$  be the probability that a sum of independent Bernoulli random variables with parameters  $p_1, p_2, p_3, \dots \in [0, 1]$  equals  $x$ , where  $\lambda := p_1 + p_2 + p_3 + \dots$  is finite. We prove two inequalities for the maximum of the density ratio  $b(x)/\pi_\lambda(x)$ , where  $\pi_\lambda$  is the probability mass function of the Poisson distribution with parameter  $\lambda$ .

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## 1. Introduction and main results

We consider independent Bernoulli random variables  $Z_1, Z_2, Z_3, \dots \in \{0, 1\}$  with parameters  $\mathbb{P}(Z_i = 1) = \mathbb{E}(Z_i) = p_i \in [0, 1]$  and their sum  $X = \sum_{i \geq 1} Z_i$ . By the first and second Borel–Cantelli lemmas,  $X$  is almost surely finite if and only if the sequence  $\mathbf{p} = (p_i)_{i \geq 1}$  satisfies

$$\lambda := \sum_{k=1}^{\infty} p_k < \infty, \tag{1}$$

and we exclude the trivial case  $\lambda = 0$ . Under this assumption, the distribution  $Q = Q_{\mathbf{p}}$  of  $X$  is given by

$$b(x) = b_{\mathbf{p}}(x) := \mathbb{P}(X = x) = \sum_{J \in \mathcal{J}(x)} \prod_{i \in J} p_i \prod_{k \in J^c} (1 - p_k) \tag{2}$$

for integers  $x \geq 0$ , where  $\mathcal{J}(x) := \{J \subset \mathbb{N} : \#J = x\}$  and  $J^c := \mathbb{N} \setminus J$ .

It is well-known that the distribution  $Q$  may be approximated by the Poisson distribution  $\text{Pois}_\lambda$  with probability mass function  $\pi = \pi_\lambda$  given by  $\pi(x) = e^{-\lambda} \lambda^x / x!$ , provided that the quantity

$$\Delta := \lambda^{-1} \sum_{i \geq 1} p_i^2$$

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is small. Indeed, [Barbour and Hall \(1984\)](#) obtained the remarkable bound

$$d_{TV}(Q, \text{Poiss}_\lambda) \leq (1 - e^{-\lambda})\Delta$$

via a suitable version of Stein’s method developed by [Chen \(1975\)](#). Here  $d_{TV}(\cdot, \cdot)$  stands for total variation distance. Note also that  $\text{Var}(X) = \sum_{i \geq 1} p_i(1 - p_i) = \lambda(1 - \Delta)$ , and

$$\Delta \leq p_* := \max_{i \geq 1} p_i.$$

*Main results.* Motivated by [Dümbgen et al. \(2020\)](#), we are aiming at upper bounds for the maximal density ratio

$$\rho(Q, \text{Poiss}_\lambda) := \sup_{x \geq 0} r(x)$$

with  $r(x) = r_p(x) := b(x)/\pi(x)$ . Note that the probability mass functions  $b$  and  $\pi$  are densities (in the sense of the Radon–Nikodym theorem) of  $Q$  and  $\text{Poiss}_\lambda$  with respect to counting measure on the set  $\mathbb{N}_0$  of nonnegative integers. Thus  $r = b/\pi_\lambda$  is the “density ratio” in the title. For arbitrary sets  $A \subset \mathbb{N}_0$ , the probability  $Q(A) = \mathbb{P}(X \in A)$  is never larger than the corresponding Poisson probability times  $\rho(Q, \text{Poiss}_\lambda)$ , no matter how small the Poisson probability is. Hence,  $\rho(Q, \text{Poiss}_\lambda)$  is a strong measure of error when  $Q$  is approximated by  $\text{Poiss}_\lambda$ , see also [Remark 3](#). While [Dümbgen et al. \(2020\)](#) obtained explicit and essentially sharp bounds for  $\rho(Q, P)$  for various pairs of distributions  $P$  and  $Q$ , the present setting with the particular Poisson binomial distribution  $Q$  and  $P = \text{Poiss}_\lambda$  seems to be substantially more difficult. In this note we prove the following result:

**Theorem 1.** For any sequence  $\mathbf{p}$  of probabilities  $p_i \in [0, 1)$  with  $\lambda = \sum_{i \geq 1} p_i < \infty$ ,

$$\rho(Q, \text{Poiss}_\lambda) \leq (1 - p_*)^{-1}.$$

We conjecture that [Theorem 1](#) is true with  $\Delta$  in place of  $p_*$ . In the case of  $\lambda \leq 1$  we can prove the following result:

**Theorem 2.** For any sequence  $\mathbf{p}$  of probabilities  $p_i \in [0, 1)$  with  $\lambda = \sum_{i \geq 1} p_i \leq 1$ ,

$$\Delta \left( 1 - \frac{\Delta}{2} - \frac{\lambda}{2(1 - p_*)} \right) \leq \log \rho(Q, \text{Poiss}_\lambda) \leq \Delta.$$

In particular,  $\lambda \leq 1$  implies that  $\rho(Q, \text{Poiss}_\lambda) \leq e^\Delta < 1/(1 - \Delta)$ . And since  $\Delta \leq p_* \leq \lambda$ , [Theorem 2](#) implies that

$$\frac{\log \rho(Q, \text{Poiss}_\lambda)}{\Delta} \rightarrow 1 \text{ as } \lambda \rightarrow 0.$$

**Remark 3 (Total Variation Distance).** Proposition 1(a) of [Dümbgen et al. \(2020\)](#) implies that  $d_{TV}(Q, \text{Poiss}_\lambda) \leq Q(\{b > \pi\})(1 - \rho(Q, \text{Poiss}_\lambda)^{-1})$ . Since  $b(0) = \prod_{i \geq 1} (1 - p_i)$  satisfies the two inequalities  $1 - \lambda \leq b(0) < e^{-\lambda} = \pi(0)$ , we obtain the inequality  $Q(\{b > \pi\}) \leq 1 - b(0) \leq \min(1, \lambda)$  and the bounds

$$\begin{aligned} d_{TV}(Q, \text{Poiss}_\lambda) &\leq \min(1, \lambda)(1 - \rho(Q, \text{Poiss}_\lambda)^{-1}) \\ &\leq \begin{cases} \min(1, \lambda)p_* \\ \lambda(1 - e^{-\Delta}) \leq \lambda\Delta = \sum_{i \geq 1} p_i^2 \end{cases} \text{ if } \lambda \leq 1. \end{aligned}$$

The remainder of this note is structured as follows: In [Section 2](#) we provide some basic formulas for the probability masses  $b(x)$  and the ratios  $r(x)$ . Then we present the proofs of [Theorems 1](#) and [2](#) in [Section 3](#).

## 2. Auxiliary results

### 2.1. The probability mass function of $Q$

Since  $b(0) < 1$  (see [Remark 3](#)), we know that  $\rho(Q, \text{Poiss}_\lambda) = \sup_{x \geq 1} r(x)$ . Writing

$$\prod_{i \in J} p_i \prod_{k \in J^c} (1 - p_k) = \prod_{i \in J} \frac{p_i}{1 - p_i} \prod_{k \geq 1} (1 - p_k) = b(0) \prod_{i \in J} \frac{p_i}{1 - p_i},$$

Eq. (2) may be reformulated as

$$b(x) = b(0) \sum_{J \in \mathcal{J}(x)} W(J)$$

with

$$W(J) := \prod_{i \in J} q_i \text{ and } q_i := \frac{p_i}{1 - p_i} \in [0, \infty),$$

i.e.  $p_i = q_i/(1 + q_i)$ . Note also that the support of  $Q$  is equal to an integer interval containing 0. Precisely,

$$b(x) > 0 \text{ if and only if } x \leq \#\{i \geq 1 : p_i > 0\} \in \mathbb{N} \cup \{\infty\}.$$

2.2. Discrete scores

For any  $x \geq 0$ ,

$$\frac{\pi(x + 1)}{\pi(x)} = \frac{\lambda}{x + 1},$$

so the “scores”  $r(x + 1)/r(x)$  are given by

$$\frac{r(x + 1)}{r(x)} = \frac{(x + 1)b(x + 1)}{\lambda b(x)}$$

for  $x \geq 0$  with  $b(x) > 0$ . If  $x_0$  is a maximizer of  $r(\cdot)$ , then

$$\frac{(x_0 + 1)b(x_0 + 1)}{b(x_0)} \leq \lambda \leq \frac{x_0 b(x_0)}{b(x_0 - 1)} \tag{3}$$

with  $b(-1) := 0$ .

There are various ways to represent the ratios  $b(x + 1)/b(x)$ . The following notation will be useful for that task: For any set  $J \subset \mathbb{N}$ , we define

$$s(J) := \sum_{i \in J} p_i \text{ and } S(J) := \sum_{i \in J} q_i.$$

In case of  $x := \#J < \infty$  we set

$$\bar{s}(J) := s(J)/x, \quad \bar{S}(J) := S(J)/x \text{ and } \bar{W}(J) := W(J) / \sum_{L \in \mathcal{J}(x)} W(L)$$

with the convention  $0/0 := 0$ . The numbers  $\bar{W}(J)$  are probability weights in the sense that  $\sum_{J \in \mathcal{J}(x)} \bar{W}(J) = 1$  whenever  $b(x) > 0$ . In that case,

$$\begin{aligned} \frac{b(x + 1)}{b(0)} &= \sum_{L \in \mathcal{J}(x+1)} W(L) = \sum_{L \in \mathcal{J}(x+1)} \frac{1}{x + 1} \sum_{k \in L} W(L \setminus \{k\}) q_k \\ &= \frac{1}{x + 1} \sum_{J \in \mathcal{J}(x)} W(J) \sum_{k \in J^c} q_k \\ &= \frac{1}{x + 1} \sum_{J \in \mathcal{J}(x)} W(J) S(J^c). \end{aligned}$$

Consequently,

$$\frac{(x + 1)b(x + 1)}{b(x)} = \sum_{J \in \mathcal{J}(x)} \bar{W}(J) S(J^c). \tag{4}$$

Alternatively, if  $b(x + 1) > 0$ , then

$$\begin{aligned} \frac{b(x)}{b(0)} &= \sum_{J \in \mathcal{J}(x)} W(J) = \sum_{J \in \mathcal{J}(x)} W(J) \sum_{k \in J^c} \frac{q_k}{S(J^c)} \\ &= \sum_{J \in \mathcal{J}(x)} \sum_{k \in J^c} \frac{W(J \cup \{k\})}{q_k + S(J \cup \{k\})^c} \\ &= \sum_{L \in \mathcal{J}(x+1)} W(L) \sum_{k \in L} \frac{1}{q_k + S(L^c)}. \end{aligned}$$

Consequently,

$$\frac{b(x)}{(x + 1)b(x + 1)} = \sum_{L \in \mathcal{J}(x+1)} \bar{W}(L) \frac{1}{x + 1} \sum_{k \in L} \frac{1}{q_k + S(L^c)}. \tag{5}$$

One can repeat the previous arguments with the sums  $\sum_{k \in J^c} p_j / s(J^c) = 1$  in place of  $\sum_{k \in J^c} q_k / S(J^c) = 1$ . This leads to

$$\frac{b(x)}{b(0)} = \sum_{J \in \mathcal{J}(x)} \sum_{k \in J^c} \frac{W(J) p_k}{p_k + s(J \cup \{k\})^c} = \sum_{L \in \mathcal{J}(x+1)} W(L) \sum_{k \in L} \frac{1 - p_k}{p_k + s(L^c)},$$

because  $W(J)p_k = W(J \cup \{k\})(1 - p_k)$  for  $k \in J^c$ . Consequently,

$$\frac{b(x)}{(x+1)b(x+1)} = \sum_{L \in \mathcal{J}(x+1)} \bar{W}(L) \frac{1}{x+1} \sum_{k \in L} \frac{1-p_k}{p_k + s(L^c)}. \tag{6}$$

Analyzing Eq. (6) leads to a first result about the location of maximizers of  $r(\cdot)$ :

**Proposition 1.** Any maximizer  $x_0 \in \mathbb{N}_0$  of  $r(\cdot)$  satisfies the inequalities  $1 \leq x_0 \leq \lceil \lambda \rceil$ .

**Proof of Proposition 1.** The inequality  $x_0 \geq 1$  follows from  $r(0) < 1$ , see Remark 3. To verify the inequality  $x_0 \leq \lceil \lambda \rceil$ , it suffices to show that  $r(x+1)/r(x) < 1$  for any integer  $x \geq \lambda$  with  $b(x) > 0$ . This is equivalent to

$$\frac{b(x)}{(x+1)b(x+1)} > \lambda^{-1}. \tag{7}$$

If  $b(x+1) = 0$ , this inequality is trivial. Otherwise, the left hand side of (7) is given by (6). Since  $(1-y)/(y+s(L^c))$  is a strictly convex function of  $y \geq 0$ , Jensen's inequality implies that

$$\frac{1}{x+1} \sum_{k \in L} \frac{1-p_k}{p_k + s(L^c)} > \frac{1-\bar{s}(L)}{\bar{s}(L) + s(L^c)} = \frac{1-\bar{s}(L)}{\bar{s}(L) + \lambda - s(L)} = \frac{1-\bar{s}(L)}{\lambda - x\bar{s}(L)}.$$

But in case of  $x \geq \lambda$ ,

$$\frac{1-\bar{s}(L)}{\lambda - x\bar{s}(L)} \geq \frac{1-\bar{s}(L)}{\lambda - \lambda\bar{s}(L)} = \lambda^{-1},$$

whence (7) holds true.  $\square$

Finally, let us mention that the probability mass function  $b$  is ultra-log-concave in the sense that  $\log r = \log(b/\pi)$  is concave, i.e.  $r(x+1)/r(x)$  is monotone decreasing in  $x \in \{y \geq 0 : b(y) > 0\}$ , see Section 4 of Saumard and Wellner (2014) and the references therein. Equivalently,  $(x+1)b(x+1)/b(x)$  is monotone decreasing in  $x \in \{y \geq 0 : b(y) > 0\}$ . With a direct argument one can even show a stronger result.

**Proposition 2.** The ratio  $(x+1)b(x+1)/b(x)$  is strictly decreasing in  $x \in \{y \geq 0 : b(y) > 0\}$ .

**Proof of Proposition 2.** We have to show that for any integer  $x \geq 0$  with  $b(x+1) > 0$ ,

$$\frac{(x+2)b(x+2)}{b(x+1)} < \frac{(x+1)b(x+1)}{b(x)}.$$

It follows from (4) that the left hand side equals  $S(\mathbb{N}) - \sum_{L \in \mathcal{J}(x+1)} \bar{W}(L)S(L)$  while the right hand side equals  $S(\mathbb{N}) - \sum_{J \in \mathcal{J}(x)} \bar{W}(J)S(J)$ . Thus the assertion is equivalent to

$$\sum_{J \in \mathcal{J}(x), L \in \mathcal{J}(x+1)} W(J)W(L)(S(L) - S(J)) > 0. \tag{8}$$

But each pair  $(J, L) \in \mathcal{J}(x) \times \mathcal{J}(x+1)$  is uniquely determined by the three sets  $M := J \cap L$ ,  $K := (J \setminus M) \cup (L \setminus M)$  and  $L' := L \setminus M$ , and

$$W(J)W(L) = W(M)^2W(K) \quad \text{and} \quad S(L) - S(J) = 2S(L') - S(K).$$

Moreover,  $\#K = 2x + 1 - 2\#M$  and  $\#L' = x + 1 - \#M$ . Hence, the left hand side of (8) equals

$$\sum_{s=0}^x \sum_{M \in \mathcal{J}(s)} \sum_{K \in \mathcal{J}(2x+1-2s)} 1_{[M \cap K = \emptyset]} W(M)^2W(K)H(K) \tag{9}$$

with

$$\begin{aligned} H(K) &:= \sum_{L' \subset K : \#L' = x+1-s} (2S(L') - S(K)) \\ &= \sum_{i \in K} q_i \sum_{L' \subset K : \#L' = x+1-s} (2 \cdot 1_{L'}(i) - 1) \\ &= S(K) \binom{2x-2s}{x-s} / (x+1-s). \end{aligned}$$

Hence, all summands in (9) are non-negative, and  $W(M)^2W(K)S(K) > 0$  for suitable sets  $M \in \mathcal{J}(x)$  and  $K \in \mathcal{J}(1)$  with  $M \cap K = \emptyset$ .  $\square$

2.3. Log-density ratios along a ray

In what follows we consider the sequence  $t\mathbf{p}$  for arbitrary  $t \in (0, 1]$ , leading to the distributions  $Q_{t\mathbf{p}}$  with probability mass functions  $b_{t\mathbf{p}}$ , weights  $W_{t\mathbf{p}}(J)$  and sums  $S_{t\mathbf{p}}(J)$ . The corresponding Poisson probability mass functions are  $\pi_{t\lambda}$ , and this leads to the ratios  $r_{t\mathbf{p}}$ . According to Proposition 1,

$$f(t) := \log \rho(Q_{t\mathbf{p}}, \text{Poiss}_{t\lambda}) = \max_{1 \leq x \leq \lceil t\lambda \rceil} \log r_{t\mathbf{p}}(x) = \max_{1 \leq x \leq \lceil \lambda \rceil} \log r_{t\mathbf{p}}(x).$$

Now we analyze the functions  $L_x : (0, 1] \rightarrow \mathbb{R}$ ,

$$\begin{aligned} L_x(t) &:= \log r_{t\mathbf{p}}(x) \\ &= t\lambda + \log \left( (t\lambda)^{-x} x! \sum_{J \in \mathcal{J}(x)} \prod_{i \in J} \frac{tp_i}{1 - tp_i} \prod_{k \geq 1} (1 - tp_k) \right) \\ &= t\lambda + \sum_{k \geq 1} \log(1 - tp_k) + \log \left( \lambda^{-x} x! \sum_{J \in \mathcal{J}(x)} \prod_{i \in J} \frac{p_i}{1 - tp_i} \right), \end{aligned}$$

for integers  $x \geq 0$  with  $b(x) > 0$ . Note first that  $L_x(t)$  can be extended to a real-analytic function of  $t \in (-\infty, 1/p_*) \supset [0, 1]$ , and

$$\begin{aligned} L_x(0) &= \log \left( \lambda^{-x} x! \sum_{J \in \mathcal{J}(x)} \prod_{i \in J} p_i \right) \\ &\leq \log \left( \lambda^{-x} \sum_{i(1), \dots, i(x) \geq 1} \prod_{s=1}^x p_{i(s)} \right) = \log(\lambda^{-x} \lambda^x) = 0 \end{aligned}$$

with equality for  $x = 0, 1$  and strict inequality for  $x > 1$ . This shows already that  $f$  is a Lipschitz-continuous function on  $(0, 1]$  with limit  $f(0+) = 0$ .

Concerning the first derivative of  $L_x$ , for  $t \in (0, 1]$ ,

$$\frac{d}{dt} \prod_{i \in J} \frac{p_i}{1 - tp_i} = \sum_{k \in J} \frac{p_k^2}{(1 - tp_k)^2} \prod_{i \in J \setminus \{k\}} \frac{p_i}{1 - tp_i} = \prod_{i \in J} \frac{p_i}{1 - tp_i} \sum_{k \in J} \frac{p_k}{1 - tp_k},$$

whence

$$\begin{aligned} L'_x(t) &= \lambda - \sum_{k \geq 0} \frac{p_k}{1 - tp_k} + \sum_{J \in \mathcal{J}(x)} \prod_{i \in J} \frac{p_i}{1 - tp_i} \sum_{k \in J} \frac{p_k}{1 - tp_k} / \sum_{J \in \mathcal{J}(x)} \prod_{i \in J} \frac{p_i}{1 - tp_i} \\ &= \lambda - \frac{1}{t} \left( S_{t\mathbf{p}}(\mathbb{N}) - \sum_{J \in \mathcal{J}(x)} \bar{W}_{t\mathbf{p}}(J) S_{t\mathbf{p}}(J) \right) \\ &= \lambda - \frac{1}{t} \sum_{J \in \mathcal{J}(x)} \bar{W}_{t\mathbf{p}}(J) S_{t\mathbf{p}}(J^c). \end{aligned}$$

Combining this formula with (4) yields

$$\begin{aligned} L'_x(t) &= \lambda - \frac{1}{t} \frac{(x+1)b_{t\mathbf{p}}(x+1)}{b_{t\mathbf{p}}(x)} \\ &= \lambda - \lambda \frac{r_{t\mathbf{p}}(x+1)}{r_{t\mathbf{p}}(x)} \\ &= \lambda(1 - \exp(L_{x+1}(t) - L_x(t))). \end{aligned} \tag{10}$$

In particular,

$$L'_x(t) \begin{cases} > \\ = \\ < \end{cases} 0 \quad \text{if and only if} \quad L_x(t) \begin{cases} > \\ = \\ < \end{cases} L_{x+1}(t). \tag{11}$$

There is also an explicit expression for the second derivative of  $L_x$ : If  $b(x+1) = 0$ , then  $x = n = \#\{i \geq 1 : p_i > 0\}$  and  $L_x(t) = \lambda t + \log(\lambda^{-n} n! b(n))$ , whence  $L''_x \equiv 0$ . Otherwise, for  $0 < t \leq 1$ ,

$$L''_x(t) = \lambda \exp(L_{x+1}(t) - L_x(t)) (L'_x(t) - L'_{x+1}(t)),$$

and

$$L''_x(t) - L''_{x+1}(t) = \frac{1}{t} \left( \frac{(x+2)b_{t\mathbf{p}}(x+2)}{b_{t\mathbf{p}}(x+1)} - \frac{(x+1)b_{t\mathbf{p}}(x+1)}{b_{t\mathbf{p}}(x)} \right) < 0$$

by Proposition 2. Hence  $L_x$  defines a smooth concave function on  $[0, 1]$ .

### 3. Proofs of the main results

**Proof of Theorem 1.** We know that  $f(t) = \log \rho(Q_{t\mathbf{p}}, \text{Poiss}_{t\lambda})$  is equal to the maximum of  $L_x(t)$  over  $x \in \{1, \dots, \lceil \lambda \rceil\}$ , and that  $f(0+) = 0$ . Note also that

$$f'(t+) = \max_{x \in N(t)} L'_x(t)$$

where

$$N(t) := \arg \max_{x \in \{1, \dots, \lceil \lambda \rceil\}} r_{t\mathbf{p}}(x).$$

Since  $g(t) := -\log(1 - tp_*)$  satisfies  $g(0) = 0$  and  $g'(t) = p_*/(1 - tp_*)$ , it suffices to show that

$$L'_x(t) \leq \frac{p_*}{1 - tp_*} \quad \text{for any } x \in N(t).$$

According to (10), the latter requirement is equivalent to

$$\frac{(x+1)b_{t\mathbf{p}}(x+1)}{b_{t\mathbf{p}}(x)} \geq t\lambda - \frac{tp_*}{1 - tp_*} \quad \text{for any } x \in N(t).$$

Note that  $x \in N(t)$  implies that  $L_{x-1}(t) \leq L_x(t)$ . But the latter inequality is equivalent to  $L'_{x-1}(t) \leq 0$ , see (11), and by (10), this is equivalent to

$$\frac{xb_{t\mathbf{p}}(x)}{b_{t\mathbf{p}}(x-1)} \geq t\lambda.$$

Consequently, it suffices to show that

$$\frac{(x+1)b_{t\mathbf{p}}(x+1)}{b_{t\mathbf{p}}(x)} \geq t\lambda - \frac{tp_*}{1 - tp_*} \quad \text{whenever} \quad \frac{xb_{t\mathbf{p}}(x)}{b_{t\mathbf{p}}(x-1)} \geq t\lambda.$$

We may simplify notation by replacing  $t\mathbf{p}$  with  $\mathbf{p}$  and prove that

$$\frac{(x+1)b(x+1)}{b(x)} \geq \lambda - \frac{p_*}{1 - p_*} \quad \text{whenever} \quad \frac{xb(x)}{b(x-1)} \geq \lambda. \tag{12}$$

Note that for  $1 \leq x \leq \lceil \lambda \rceil$ , the representation (5) with  $x - 1$  in place of  $x$  reads

$$\frac{b(x-1)}{xb(x)} = \sum_{J \in \mathcal{J}(x)} \bar{W}(J) \frac{1}{x} \sum_{i \in J} \frac{1}{q_i + S(J^c)}.$$

By Jensen's inequality,

$$\frac{1}{x} \sum_{i \in J} \frac{1}{q_i + S(J^c)} \geq \left( \frac{1}{x} \sum_{i \in J} (q_i + S(J^c)) \right)^{-1} = (\bar{S}(J) + S(J^c))^{-1},$$

so

$$\frac{b(x-1)}{xb(x)} \geq \sum_{J \in \mathcal{J}(x)} \bar{W}(J) (\bar{S}(J) + S(J^c))^{-1}.$$

A second application of Jensen's inequality yields that

$$\frac{b(x-1)}{xb(x)} \geq \left( \sum_{J \in \mathcal{J}(x)} \bar{W}(J) (\bar{S}(J) + S(J^c)) \right)^{-1}.$$

Consequently, if  $xb(x)/b(x-1) \geq \lambda$ , then

$$\sum_{J \in \mathcal{J}(x)} \bar{W}(J) (\bar{S}(J) + S(J^c)) \geq \lambda.$$

On the other hand, (4) yields

$$\begin{aligned} \frac{(x+1)b(x+1)}{b(x)} &= \sum_{J \in \mathcal{J}(x)} \bar{W}(J) (\bar{S}(J) + S(J^c)) - \sum_{J \in \mathcal{J}(x)} \bar{W}(J) \bar{S}(J) \\ &\geq \lambda - \frac{p_*}{1 - p_*}, \end{aligned}$$

because  $\bar{S}(J) = x^{-1} \sum_{i \in J} p_i / (1 - p_i) \leq p_*/(1 - p_*)$  for any set  $J \in \mathcal{J}(x)$ . This proves (12).  $\square$

**Proof of Theorem 2.** We know from Proposition 1 that in case of  $\lambda \leq 1$ ,

$$\log \rho(Q, \text{Poiss}_\lambda) = \log r(1) = L_1(1)$$

with

$$L_1(t) = t\lambda + \sum_{i \geq 1} \log(1 - tp_i) + \log\left(\lambda^{-1} \sum_{i \geq 1} \frac{p_i}{1 - tp_i}\right).$$

First of all,  $L_1(0) = 0$ , and

$$\begin{aligned} L_1'(t) &= \lambda - \sum_{i \geq 1} \frac{p_i}{1 - tp_i} + \sum_{i \geq 1} \frac{p_i^2}{(1 - tp_i)^2} \bigg/ \sum_{i \geq 1} \frac{p_i}{1 - tp_i} \\ &= -t \sum_{i \geq 1} \frac{p_i^2}{1 - tp_i} + \sum_{i \geq 1} \frac{p_i^2}{(1 - tp_i)^2} \bigg/ \sum_{i \geq 1} \frac{p_i}{1 - tp_i}, \end{aligned}$$

whence  $L_1'(0) = \Delta$ . Moreover, we have seen before that  $L_1'' \leq 0$  by ultra-log-concavity of the probability mass functions  $b_{tp}$ . Consequently, for some  $\xi \in (0, 1)$ ,

$$L_1(1) = L_1(0) + L_1'(0) + 2^{-1}L_1''(\xi) = 0 + \Delta + 2^{-1}L_1''(\xi) \leq \Delta.$$

As to the lower bound, recall that

$$L_1(1) = \sum_{i \geq 1} (p_i + \log(1 - p_i)) + \log\left(\lambda^{-1} \sum_{i \geq 1} \frac{p_i}{1 - p_i}\right).$$

On the one hand,

$$p_i + \log(1 - p_i) = - \sum_{k \geq 2} \frac{p_i^k}{k} \geq - \frac{p_i^2}{2} \sum_{\ell \geq 0} p_*^\ell = - \frac{p_i^2}{2(1 - p_*)},$$

so

$$\sum_{i \geq 1} (p_i + \log(1 - p_i)) \geq - \frac{1}{2(1 - p_*)} \sum_{i \geq 1} p_i^2 = - \frac{\lambda}{2(1 - p_*)} \Delta.$$

Moreover,

$$\log\left(\lambda^{-1} \sum_{i \geq 1} \frac{p_i}{1 - p_i}\right) \geq \log\left(\lambda^{-1} \sum_{i \geq 1} (p_i + p_i^2)\right) = \log(1 + \Delta) \geq \Delta - \Delta^2/2,$$

and this implies the asserted lower bound for  $L_1(1)$ .  $\square$

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