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Equilibrium Mobility ⋆

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Abstract

This paper proposes to measure the mobility of a stochastic process as the expected value of a "mobility functional" with respect to its stationary distribution. The mobility functional thereby measures the valuation of movements between states. We not only highlight the conceptual advantages of our approach, but also show how this so-called "equilibrium mobility index" relates to the axiomatic approach by Fields and Ok (7) and the approach based on monotone transition matrices by Conlisk (4) and Dardanoni (5)).

Key words: Mobility indices, mobility functional, equilibrium mobility

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1 Introduction

Following the seminal contribution of Shorrocks (16), mobility indices have been defined directly on the set of stochastic processes represented by their transition function, respectively transition matrix in the context of discrete time finite state space Markov processes. This approach led, at least in our view, to some confusion as the weighting of movements between states is mixed up with the properties of the stochastic process. In this paper we propose a new approach based on "mobility functionals" which just assign values to movements between states. The mobility indices, we suggest, are then defined as the expected value of this mobility functional with respect to the stationary distribution of the underlying stochastic process. We therefore term these indices as "equilibrium mobility" indices.

This approach has several conceptual advantages. First, several popular mobility indices can actually be represented in this way. Second, our approach ties in with several recently developed theoretical approaches to mobility measurement. In particular, we relate our approach to the axiomatic approach to income movements as developed by Fields and Ok (7) and Mitra and Ok (11) and, by introducing so-called "2-decreasing" mobility functionals, to Dardanoni's ordering (Dardanoni (5)) and the weak D-criterion of Conlisk (4). Third, although our approach just measures what is called exchange mobility, we show in a companion paper (see Aebi, Neusser, and Steiner (1)) how the equilibrium mobility index gives naturally rise to a mobility index measuring the degree to which future states do not depend on the initial state. Finally, although we restrict our presentation to finite state space discrete time Markov processes to make our point as clear as possible, it will become clear, as we go along, that the approach can be easily extended with little additional effort to more general stochastic processes (e.g. ARMA processes).

2 Mobility functionals and equilibrium mobility

Let $\{X_t\}_{t=0,1,\ldots}$ denote a discrete-time stochastic process with values in a finite state space $\mathcal{E}$. The state space consists of $K \geq 2$ elements denoted by $i, j, k, \ldots$. Given some arbitrary initial probability distribution μ on $\mathcal{E}$, we assume that $\{X_t\}$ is a homogenous Markov chain with $K \times K$ transition matrix $\mathrm{P} = \mathrm{P}(i,j)$. The Markov chain is denoted by (μ, P). Following the literature on mobility indices, we consider only irreducible transition matrices. Thus there exists a unique invariant or stationary probability distribution $\pi > 0$.[1] In addition we assume that the transition matrix is primitive so that the Markov chain becomes regular (irreducible and aperiodic). Under these assumptions, $\lim_{T\to\infty} \mu'\mathrm{P}^T = \pi'$ for any probability distribution μ or equivalently $\lim_{T\to\infty} \mathrm{P}^T = \mathrm{P}^\infty$ where P^∞ is a transition matrix whose rows are all equal to π'.[2]

In contrast to Shorrocks (16) or Geweke , Marshall and Zarkin (9), we do not define our mobility index directly on the set of transition matrices. Instead, more in the spirit of Bartholomew (2, 24–30), we base our concept on the valuation of movements between states where the valuation is represented by a *mobility functional*.

Definition 1. *A* mobility functional f *is a nonnegative functional on* $\mathcal{E} \times \mathcal{E}$ *such that*

(1) $f(i,i) = 0$ *for all* $i \in \mathcal{E}$
(2) $f(i,j) > 0$ *for all* i *and* j *in* $\mathcal{E}$ *with* $i \neq j$.

[1] For matrices, and analogously for vectors, we employ the notation: $\mathrm{A} \geq \mathrm{B}$ if $\mathrm{A}(i,j) \geq \mathrm{B}(i,j)$ for all i and j; $\mathrm{A} > \mathrm{B}$ if $\mathrm{A} \geq \mathrm{B}$ and $\mathrm{A} \neq \mathrm{B}$; $\mathrm{A} \gg \mathrm{B}$ if $\mathrm{A}(i,j) > \mathrm{B}(i,j)$ for all i and j.

[2] The proofs of these implications can be found in any standard textbook on Markov chains (for example Berman and Plemmons(3), Norris (14), or Seneta (15)).

Hence, the mobility functional attaches positive weights to movements from one state to another and zero when no movement occurs. The structure imposed on f is very loose. Depending on the application in mind, more restrictive specifications may be considered (see section 3.2). An interesting subset of mobility functionals is given by the set of distance functions on $\mathcal{E}$. As we will see below some prominent mobility indices arise from mobility functionals which are actually distance functions.

Given a mobility functional, we then define the equilibrium mobility index as the expected value of this functional where the expectation is taken with respect to the invariant probability distribution:

Definition 2. *For any given mobility functional f on $\mathcal{E}\times\mathcal{E}$ and any irreducible transition matrix* P *with its unique invariant distribution π,*

$$\mathbf{M}^e_f(\mathrm{P}) = \sum_{i\in\mathcal{E}} \pi(i) \sum_{j\in\mathcal{E}} \mathrm{P}(i,j) f(i,j) \tag{2.1}$$

is called the equilibrium f-mobility index *of P.*

For any given mobility functional f, this definition introduces a total quasi-ordering $\succeq_f$ on the union of the set of irreducible transition matrices and the identity matrix $\{I_K\}$. For any two irreducible transition matrices P and Q, we say that P is *more mobile than* Q, denoted by P $\succeq_f$ Q, if and only if $\mathbf{M}^e_f(\mathrm{P}) \geq \mathbf{M}^e_f(\mathrm{Q})$. The definition of equilibrium mobility leads to the following considerations.

Remark 1. *The properties of f guarantee that $\mathbf{M}^e_f(\mathrm{P}) \geq 0$.*

Remark 2. *As the values of f are not restricted to lie in the interval $[0,1]$, the equilibrium f-mobility is also not restricted to lie in the unit interval. A normalization to the unit interval can be achieved if $\mathbf{M}^e_f(\mathrm{P})$ is divided by $\max_{i,j\in\mathcal{E}} f(i,j)$, a number which obviously depends on f only.*

Remark 3. *It is easy to see that the equilibrium f-mobility of the identity matrix I_K equals zero, i.e. $\mathbf{M}^e_f(\mathrm{I}_K) = 0$, so that the index fulfills Shorrocks' (16, 1015) Immobility axiom. As we restrict ourselves to ergodic transition matrices* *(which do not include I_K), the equilibrium mobility index is always strictly greater than zero. Hence the Strong Immobility axiom is fulfilled on the union of the set of ergodic transition matrices and $\{\mathrm{I}_K\}$.*

Remark 4. *Because the equilibrium index measures mobility in a situation where the probability distribution remains unchanged over time (i.e. remains equal to π), it measures what is called pure exchange mobility in the sociologically oriented literature (see Dardanoni (5), Fields and Ok (8), and Maasoumi (10)).*

The definition of the equilibrium mobility index encompasses several specifications encountered in the literature. Consider first the class of *power mobility functionals* where $\mathcal{E} = \{1, 2, \ldots, K\}$ and f is given by

$$f(i,j) = |i-j|^{\alpha}, \qquad \alpha \geq 1.$$

For $\alpha = 1$, the equilibrium mobility index specializes to Bartholomew's index:[3]

$$\mathbf{M}^e_B(\mathrm{P}) = \sum_{i\in\mathcal{E}} \pi(i) \sum_{j\in\mathcal{E}} \mathrm{P}(i,j)|i-j| \tag{2.2}$$

Another interesting choice for the mobility functional is $f(i,j) = 1 - \delta(i,j)$ where $\delta(i,j)$ denotes Kronecker's delta. An advantage of this functional is that the state space $\mathcal{E}$ can be an arbitrary set (e.g. social classes). The implied mobility index is just the unconditional probability of leaving the current class which is nothing but the expected number of class changes:[4]

$$\mathbf{M}^e_{CC}(\mathrm{P}) = \sum_{i\in\mathcal{E}} \pi(i)\,(1-\mathrm{P}(i,i)) = \sum_{i\in\mathcal{E}} \pi(i) \sum_{j\in\mathcal{E}} \mathrm{P}(i,j)\,(1-\delta(i,j)) \tag{2.3}$$

The above mobility functionals actually define distance functions on the state space $\mathcal{E}$. While in the case of Bartholomew's index the functional expresses the

[3] Bartholomew (2) scaled the index by $1/(K-1)$ to confine it to the interval $[0,1]$.
[4] In the literature, this index is scaled by $K/(K-1)$.

ordinary distance between states i and j, the functional corresponding to the index of leaving the current class is known in topology as the trivial metric.

3 Relations to the Literature

3.1 Importance of power mobility functionals

The measurement of equilibrium mobility as the expected value of a mobility functional lies in the spirit of Fields and Ok (7) and Mitra and Ok (11). To see this, suppose that the population consists of N individuals and that the state, each individual is in, evolves according to the stationary Markov chain (π, P). Replacing the expectation by the corresponding ensemble average (i.e. the average over all individuals) then leads to the following measure of mobility between two periods:

$$\frac{1}{N}\sum_{i=1}^{N} f(x_i, y_i)$$

where x_i and y_i denote the realizations of $\{X_t\}$ in two consecutive periods for the i-th individual. But this is nothing but the per capita version of "total absolute income mobility" where the distance function between $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$, in their terminology, is just given by $d_N(x, y) = \sum_{i=1}^{N} f(x_i, y_i)$. The interest in this interpretation of the equilibrium mobility index is that the axioms proposed by Fields and Ok (7) and Mitra and Ok (11) for $d_N(x, y)$ restrict the set of possible mobility functionals. Their axioms imply that the class of power mobility functionals is the only relevant class. Out of this class only the Bartholomew functional ($\alpha = 1$) fulfills all axioms.

3.2 Monotone transition matrices and 2-decreasing mobility functionals

The definition of the mobility functional f is quite general and consequently does not impose enough structure on equilibrium mobility indices to lead to interesting properties. Besides the axioms of Fields and Ok (7), there exists a different strand in the literature that turns out to be beneficial in restricting the class of mobility functionals f appropriately. To link our approach to these concepts, we examine more closely the special class of, so-called, *2-decreasing mobility functionals* (see Nelsen (12)). As it turns out, this class provides interesting relations to existing criteria and partial orderings of transition matrices.

Definition 3. *A mobility functional f on $\mathcal{E} \times \mathcal{E}$ with $\mathcal{E} = \{1, 2, \ldots, K\}$ is* 2-decreasing *if*

$$\mathrm{V}(i,j) = f(i+1, j+1) - f(i+1, j) - f(i, j+1) + f(i,j) \leq 0 \tag{3.1}$$

for all $i, j \in \{1, 2, \ldots, K-1\}$.

Note that the inequality is strict if $i = j$. 2-decreasing functions are the two-dimensional analogues of non-increasing functions in one variable. $-V(i,j)$ can be interpreted as the area assigned by f to the rectangle with vertices $(i,j), (i+1,j), (i,j+1), (i+1,j+1)$ (see Nelsen (12)). The above definition immediately implies that $f(i+1,j) - f(i,j)$ and $f(i,j+1) - f(i,j)$ are nonincreasing functions of j and i, respectively. The power functional is 2-decreasing for $\alpha \geq 1$ whereas the functional $f(i,j) = 1 - \delta(i,j)$ is not.

Recently, the class of monotone transition matrices attracted special attention (see Conlisk (4), Dardanoni (5), Dardanoni (6), and Fields and Ok (8)). It is argued that these matrices have theoretically plausible properties and are supported empirically. Monotone transition matrices are transition matrices where row $i+1$ stochastically dominates row i for all $i = 1, \ldots, K-1$. This condition can be written compactly as $T^{-1}\mathrm{P}T \geq 0$ where T denotes the summation matrix.[5]

Theorem 1. *For any two irreducible transition matrices P and Q with the same invariant distribution π and any 2-decreasing mobility functional f,*

$$T'\text{diag}(\pi)(\mathrm{P}-\mathrm{Q})T \leq 0 \quad \textit{implies} \quad \mathrm{P} \succeq_f \mathrm{Q}.$$

Proof. First note that $\mathbf{M}_f^e(\mathrm{P}) = \text{tr}(\mathrm{P}' \text{ diag}(\pi)f)$ where f denotes the matrix with elements $f(i,j)$ and where the diag operator transforms any K-vector x into a $K \times K$ diagonal matrix with the elements of x on the diagonal. Using the properties of the trace operator, we get:

$$\begin{aligned}
\mathbf{M}_f^e(\mathrm{P}) - \mathbf{M}_f^e(\mathrm{Q}) &= \text{tr}\left((\mathrm{P}-\mathrm{Q})'\text{diag}(\pi)f\right) \\
&= \text{tr}\left(\left(T'\text{diag}(\pi)(\mathrm{P}-\mathrm{Q})T\right)\left(T^{-1}f'T'^{-1}\right)\right)
\end{aligned}$$

The fact that $\sum_i \pi(i)\mathrm{P}((i,j) - \mathrm{Q}(i,j) = 0$ for all j and that $\sum_j \left(P(i,j) - Q(i,j)\right) = 0$ for all i implies that $T'\text{diag}(\pi)(\mathrm{P}-\mathrm{Q})T$ can be expressed as

$$T'\text{diag}(\pi)(\mathrm{P}-\mathrm{Q})T = \begin{pmatrix} N & 0_{(K-1)\times 1} \\ 0_{1\times(K-1)} & 0 \end{pmatrix}.$$

Because $T'\text{diag}(\pi)(\mathrm{P}-\mathrm{Q})T \leq 0$ by assumption, $N \leq 0$. The matrix $T^{-1}f'T'^{-1}$ on the other hand is of the form

$$T^{-1}f'T'^{-1} = \begin{pmatrix} \mathrm{V}' & c \\ b' & 0 \end{pmatrix}$$

where b and c are nonnegative $K-1$ vectors. The $(K-1)\times(K-1)$ matrix V has typical element:

$$\mathrm{V}(i,j) = f(i,j) - f(i,j+1) + f(i+1,j+1) - f(i+1,j) \leq 0$$

[5] The summation matrix T is an upper triangular matrix with all elements on the diagonal and above equal to one. Its inverse T^{-1} is the matrix with ones on the diagonal, minus ones on the first superdiagonal and zeros elsewhere.

where the inequality follows from f being 2-decreasing. This finally leads to:

$$\mathbf{M}^e_f(\mathrm{P}) - \mathbf{M}^e_f(\mathrm{Q}) = \mathrm{tr}\left(\begin{pmatrix} N & 0 \\ & \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{V}' & c \\ & \\ b' & 0 \end{pmatrix}\right) = \mathrm{tr}(\mathrm{NV}') \geq 0$$

which is by definition $\mathrm{P} \succeq_f \mathrm{Q}$. □

Note that the implication goes only in one direction as we can give examples such that $\mathrm{P} \succeq_f \mathrm{Q}$ with $T'\mathrm{diag}(\pi)(\mathrm{P}-\mathrm{Q})T$ not being nonpositive. Theorem 1 implies that the equilibrium mobility index induced by a 2-decreasing mobility functional is coherent with Dardanoni's partial ordering of monotone transition matrices sharing the same invariant distribution π (see Dardanoni (5)). Thus the welfare implications considered by Dardanoni are also applicable. Furthermore, in this class the perfect mobility matrix $\iota\pi'$, $\iota = (1,\ldots,1)'$, is a maximal element with respect to equilibrium mobility because $\mathrm{T}'\mathrm{diag}(\pi)(\iota\pi' - \mathrm{P})T \leq 0$ for all monotone transition matrices P with stationary probability distribution π (see Dardanoni (5, theorem 2)) and therefore $(\iota\pi') \succeq_f \mathrm{P}$ whenever f is 2-decreasing.

Corollary 1. *If* P *and* Q *are two monotone transition matrices with the same invariant distribution* π *such that* $\mathrm{P}(i,j) \geq \mathrm{Q}(i,j)$ *for all* $i \neq j$ *and* $\mathrm{P}(i,j) > \mathrm{Q}(i,j)$ *for some* $i \neq j$, *then* $\mathrm{P} \succeq_f \mathrm{Q}$ *if the mobility functional* f *is 2-decreasing.*

Proof. The assumptions of the corollary imply that $T'\mathrm{diag}(\pi)(\mathrm{P}-\mathrm{Q})T \leq 0$ (Dardanoni (5, Appendix 2)). For 2-decreasing functionals, $\mathrm{P} \succeq_f \mathrm{Q}$ follows from Theorem 1. □

Corollary 2. *Let* P *and* Q *be two monotone transition matrices with the same invariant distribution* π. *If the upper left* $(K-1)\times(K-1)$ *matrices of* $T^{-1}\mathrm{P}T$ *and* $T^{-1}\mathrm{P}T$ *are denoted by* $\Delta(\mathrm{P})$ *and* $\Delta(\mathrm{Q})$, *respectively, then* $\Delta(\mathrm{Q}) \geq \Delta(\mathrm{P})$ *implies* $\mathrm{P} \succeq_f \mathrm{Q}$ *if the mobility functional* f *is 2-decreasing.*

Proof. $\Delta(\mathrm{Q}) \geq \Delta(\mathrm{P})$ implies $T'\mathrm{diag}(\pi)(\mathrm{P}-\mathrm{Q})T \leq 0$ (Dardanoni (5, Appendix 2)). For 2-decreasing functionals, $\mathrm{P} \succeq_f \mathrm{Q}$ follows from Theorem 1. $\square$

These corollaries show that in the case of monotone transition matrices, our equilibrium mobility index with a 2-decreasing mobility functional is coherent with the weak D-criterion of Conlisk (4) as well as with the monotonicity axiom of Shorrocks (16) and thus satisfies all persistence criteria listed by Geweke, Marshall and Zarkin (9).

4 Conclusion

This paper showed that, by taking expectations of a mobility functional, one can construct meaningful mobility indices. Indeed, some of the most prominent mobility indices can actually be stated in this form. The mobility functional approach provides a link to two different strands in the literature of income mobility, namely the literature based on monotone transition matrices (Conlisk (4), Dardanoni (5)) as well as the axiomatic approach represented by Fields and Ok (7). The class of power mobility functionals turns out to be compatible with both strands and thus provides a direct connection between these strands.

Our way of introducing equilibrium mobility presents several virtues. First, the weighting of movements between states by a mobility functional seems to us a natural starting point which facilitates the interpretation and evaluation of mobility. Second, although we presented our approach in the context of discrete-time finite state space Markov chains, the definition of equilibrium mobility can be readily extended to more general stochastic processes (see e.g. Neusser (13)). Finally, as shown by Aebi, Neusser, and Steiner (1), the choice of the mobility functional pins down automatically a mobility index measuring the degree to which future states do not depend on the initial state. This suggests that with the choice of a mobility functional, one is led to measure the two most important aspects of mobility simultaneously: equilibrium or exchange mobility and convergence or origin independence mobility.

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