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Abstract

We analyze evolutionary games with replicator dynamics that have frequency dependent stage games. In such an evolutionary game, the payoffs of a strategy at any point in time are functions of the strategy shares given by the players' strategy choices at that time. This framework is suited to model feedback effects between population variables and individual incentives, indirect network effects, and behavior under social norms. We show that the replicator dynamics with frequency dependent stage games is well behaved, i.e. has unique solutions and is simplex invariant for all initial strategy states. Moreover, we present an extension of Liapunov's Theorem that facilitates the analysis of evolutionary equilibria for frequency dependent evolutionary games.

Keywords: Replicator Dynamics, Frequency Dependent, State Dependent, Evolutionary Games, Liapunov.

JEL-Classification: C73.

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1 Introduction

We analyze evolutionary games with replicator dynamics that have frequency dependent stage games. A stage game that is frequency dependent features strategy payoffs which are functions of the shares of strategies chosen by the players at a given point in time. We show that the replicator dynamics with frequency dependent stage games is well behaved, i.e., has unique solutions and is simplex invariant for all initial strategy states. Moreover, we present an extension of Liapunov's Theorem that facilitates the analysis of evolutionary equilibria for evolutionary games with frequency dependent stage games (called frequency dependent evolutionary games in the following).

The class of frequency dependent evolutionary games, which includes the standard evolutionary games as a special case, applies to a broader range of economic contexts than the standard framework. In the following we present some interesting examples which can be analyzed with a model based on a frequency dependent evolutionary game.

Whenever feedback effects between population variables and individual incentives exist, a frequency dependent evolutionary game framework is better suited for an analysis than the standard framework. The reason is that payoffs for certain strategies can be modelled as functions of the population behavior, i.e., as functions of the strategy shares prevalent in the population. For example, when analyzing the evolution of corruption in a society, such a feedback effect could play an important role. The individual incentives to corrupt depend on the prevalence of corruption: When corruption is widely spread, corrupt behavior may have greater benefits (gains from corruption need not to be hidden) and lower costs (corrupt activities are less likely to be punished).

In the same way corrupt activities become less costly the more corruption there is, social norms are felt the more intensely by an individual the larger the population share adhering to it is. A frequency dependent evolutionary game is well suited to include the changing costs of offending against a social norm and the resulting dynamics in the behavior under social norms (see Lindbeck et al., 1999, for the example of the social norm to live off one's own work).

Similar in structure, but with a different interpretation, indirect network effects as studied in Caillaud and Jullien (2003) can be grasped as frequency dependent evolutionary games. The utility of customers choosing between two market intermediaries offering matching services on the web does not only depend on how many others subscribe to a certain intermediary, but also on how the two intermediaries adjust their pricing schemes in reaction to present market shares in order to attract a larger customer community. Frequency dependent evolutionary game representing markets of two-sided interaction and different "locations" will reproduce the results of Ellison and Fudenberg (2003) in which markets do not tip for one location.

Frequency dependent evolutionary games may also help to explain certain puzzles in

biology as the following example shows. Left-handers happen to be lighter and smaller than right-handers, which should put left-handers in an evolutionary disadvantage. However, the share of left-handers has not been driven to zero in any human population. Biologists now argue that this is because benefits from left-handedness balance its costs out: Left-handers have a strategic advantage in fights, because most right-handers have very little experience of fighting left-handers, but not vice versa.[1] To model this hypothesis, the advantage from the experience to fight a certain type must be captured as a strategy stage dependent payoff.

We have chosen to demonstrate the analysis of frequency dependent stage games in evolutionary games with the replicator dynamics. The reason is that the replicator dynamics, together with fictitious play and partial best-response dynamics, is the most widely used and studied strategy adjustment rule in evolutionary games. Although it originated as a concept in evolutionary biology (Maynard Smith and Price, 1973), economists have studied many individual behavioral rules that result in the replicator dynamics on the population level: Imitation driven by dissatisfaction and imitation of successful agents (Weibull, 1995, pp. 188), imitation rules motivated by regret theory (Alos-Ferrer and Nermuth, 2003; Cubitt and Sugden, 1998), expedient and monotone learning rules (Borgers et al., 2004), and stochastic forms of fictitious play and reinforcement learning (Hopkins, 2002; Borgers and Sarin, 1997; Gaunersdorfer and Hofbauer, 1995). It is this work which makes frequency dependent evolutionary games with the replicator dynamics an attractive framework whenever learning or imitation behavior seems adequate.

The plan of the paper is as follows. In Section 2 we present the standard evolutionary games with replicator dynamics. In Section 3 we introduce the extension to frequency dependent evolutionary games, prove the conditions under which the replicator dynamics with frequency dependent stage games are well behaved and discuss the 2×2-games with linear payoff functions. In the last section we present a Liapunov Theorem which facilitates the analysis of frequency dependent evolutionary games. Note that all proofs of Propositions and Theorems are collected in Appendix A.

2 Evolutionary Games with the Replicator Dynamics

An evolutionary game describes strategic interaction over time. It is defined by the populations of players, a state space of strategies, a stage game, and an adaptation rule which determines the dynamic adjustment process of strategy choices. Together with fictitious play and best-response dynamics, the replicator dynamics is the most common adaptation rule employed in evolutionary games. In this section, we briefly present the standard variant of an evolutionary game with continuous-time replicator

[1] See the article "Left-handedness: A sinister advantage" in The Economist, Dec 9th, 2004. It cites literature in biology which finds that the share of left-handers is higher in more violent societies.

dynamics.

The Population. We focus on one-population games. The population consists of a continuum of infinitely-lived players which implies a continuous state space of strategies.[2] Together with the assumption of continuous time, this allows us to specify the dynamics of a game as a system of ordinary differential equations.

The Strategies. Each player chooses to play one of n pure strategies of a given strategy set S.[3] If an individual plays strategy $i \in S$ we denote his strategy choice by $\sigma_i = (0, ..., 1, 0, ..., 0)^T \in \mathbb{R}^n$, where the 1 is the i-th component of σ_i.

The fraction of the population playing strategy i at time t is denoted by $x_i(t) \in [0, 1]$. The *strategy state* of the game, $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$, specifies the frequency of each of the n strategies in t. The set of feasible strategy states, the *strategy state space*, is the simplex Σ_{n-1} of dimension $n - 1$.[4] Note that we will omit the time index t if there is no risk of misunderstanding.

The Stage Game. The stage game characterizes the strategic interaction of two players at any point in time. It is defined by an expected payoff function $f(\sigma_i, x)$, $f : S \times \Sigma_{n-1} \to \mathbb{R}$, which gives the payoff of strategy σ_i given the strategy state x.

Most of the existing literature adopts a linear expected payoff function originally employed in Maynard Smith (1982) and depicts the stage game as a payoff matrix A. In every period players are drawn randomly and pairwise to play the stage game and receive the expected payoff $f(\sigma_i, x) = \sigma_i^T A x$.[5]

The Replicator Dynamics. Now let us describe how the strategy frequencies change over time. The replicator dynamics (see Taylor and Jonker, 1978; Schuster and Sigmund, 1983) are defined by the differential equation system

$$\dot{x}_i \quad = \quad x_i \left(f(\sigma_i, x) - f(x, x) \right) \quad \forall i \in S \,. \tag{1}$$

If we divide the equations in (1) by the respective x_i, the interpretation of the replicator dynamics is clear: The higher a strategy's payoff in comparison with the average payoff in the population, the higher its relative growth rate.

For a simplified notation, we define the system's right hand side as the function

[2] For other implications see Friedman (1998).

[3] Note that by changing the interpretation of strategy states, the replicator dynamics with mixed strategies takes the same functional form; see Hofbauer and Sigmund (1998) for the structure of such a setup.

[4] The simplex Σ_{n-1} is defined as $\{x(t) \in \mathbb{R}^n \mid x_i(t) \geq 0 \text{ and } \sum_{i=1}^n x_i(t) = 1 \text{ for } i = 1, ..., n\}$.

[5] It is common in the literature not to differentiate between the expected payoff against the population and the realized payoff of a specific stage game played. There are several reasons for that (Friedman, 1998): First, in large populations such as ours the expected payoff is a sufficient statistic. Second, payoffs are often not generated by random pairwise encounters, but by general interactions such as markets, and are therefore not stochastic.

$F : \Sigma \to \Sigma$, and can then write equation (1) as $\dot{x} = F(x)$.[6]

Equilibrium Concept. The most common equilibrium concept in biological literature concerned with evolutionary games is the Evolutionary Stable Strategy (Maynard Smith and Price, 1973) which is a static equilibrium concept that rests upon the the payoff function of the stage game. Since we will abandon the assumption of a constant stage game, an equilibrium concept which assures asymptotic stability of F in an equilibrium serves our purposes better: The *Evolutionary Equilibrium*[7] (EE) specifies an equilibrium in terms of the mathematical definitions of function F's asymptotic stability in a critical point.

Definition 1 *A strategy state $x^* \in \Sigma_{n-1}$ is an evolutionary equilibrium of an evolutionary game if x^* is an attractor*[8] *of the dynamical system $\dot{x} = F(x)$ defining the game's adjustment dynamics.*

In other words, an evolutionary equilibrium is a subset of state space Σ_{n-1} which a solution trajectory of $\dot{x} = F(x)$ does not leave once reached. Additionally, if a solution trajectory of the dynamics starts sufficiently close to the evolutionary equilibrium, it remains close and converges asymptotically to the evolutionary equilibrium over time. The open set of points in Σ_{n-1} converging to a given EE are called its basin of attraction.

3 The Replicator Dynamics with Frequency Dependent Stage Games

3.1 Why Extending the Standard Framework?

In many situations of economic interaction, the frequency of certain strategies not only matters because players are more or less likely to play against these strategies, but also because their frequencies may change the payoffs of the stage game. If payoffs of the stage game change with the frequency of certain strategies, it is unsatisfactory to apply a model based on a standard evolutionary game. In order to model these situations, we suggest the class of frequency dependent evolutionary games: The payoffs for all

[6] The replicator dynamics is simplex invariant:

$$\sum_{i \in S} \dot{x}_i = \sum_{i \in S} x_i \left(f(\sigma_i, x) - f(x, x) \right) = \sum_{i \in S} x_i f(\sigma_i, x) - f(x, x) \sum_{i \in S} x_i = 0 \,.$$

[7] The term evolutionary equilibrium was introduced by Hirshleifer (1982).

[8] An attractor is defined as an asymptotically stable non-wandering set (in our case the only possible non-wandering sets are critical points and points on limit cycles or graphics). For definitions of critical points (also called equilibrium points or fixed points), limit cycles, graphics (also called separatrix cycles), asymptotic stability, and non-wandering sets see Perko (2000) or any textbook on dynamic systems.

strategies are nonlinear functions of the strategy frequencies, i.e., the elements of the payoff matrix of the frequency dependent evolutionary games are functions of the distribution of strategies at any point in time.

Such an extension broadens the applicability of evolutionary games to a variety of interesting topics. For instance, economic situations in which feedback effects play an important role may rather be analyzed as frequency dependent evolutionary games than as standard evolutionary games. Feedback effects between variables defined by population behavior and individual decisions are present in many economic situations: The payoff of a certain strategy may well depend on how many others choose the strategy as well.

We give an example to clarify our point. In an evolutionary game modelling corruption, government employees choose between the strategy to be corrupt and the strategy to be fair. The payoff for corrupt behavior contains the benefits and the costs for a corrupt act. So far we could comprehend the situation as a standard evolutionary game. However, the costs of corruption may very likely depend on the frequency of corrupt agents, the reason being that the more corruption there is, the more likely it is that a corrupt government employee gets around his punishment. Since the costs of corruption are part of the payoff of the corrupt strategy, we have to employ a frequency dependent evolutionary game framework.[9]

Other applications for which the frequency dependent evolutionary game may be a suitable framework are the decisions for a level or type of education and the issue of technology adaptation. Whenever agents decide between different educational levels or different professions, their future salary will most certainly depend on how the distribution of the educational levels is, respectively, how big the number of people educated for a certain profession is. A large number of people with the same kind of education may decrease their salary, a very small number could prevent the existence of a certain industry. A similar reasoning can be made for the issue of technology adaptation: The returns to investment of a certain technology depend on the frequency of all technologies adopted in a population of firms, for instance because price setting depends on the degree of competition and cost structures present.

Very little research has been done on frequency dependent evolutionary games. The notion *game with frequency dependent payoffs* is due to Brenner and Witt (2003) who look at two-strategy, two-player stage games with a strategy adaptation rule motivated by reinforcement learning. The concept itself dates back to Joosten et al. (1994), who first described games with changing payoffs, in a different context though. A few other examples of frequency dependent games (not evolutionary ones) are provided by Joosten et al. (2000) who introduce frequency dependent payoffs in the setup of stochastic games.

[9] See Bruegger (2005) for such a model of corruption.

3.2 Definition

We first define the class of frequency dependent evolutionary games.

Definition 2 *An evolutionary game consisting of a population, a strategy state space, a strategy state dependent stage game, and a dynamic adjustment process, belongs to the class of frequency dependent evolutionary games.*

In this paper, we are only concerned with frequency dependent evolutionary games that have replicator dynamics as an adjustment process. According to Definition 2, the replicator dynamics of a frequency dependent evolutionary game can be written as

$$\dot{x}_i \quad = \quad x_i \left(\sigma^T A(x)\, x - x\, A(x)\, x \right) \quad \forall\, i \in S\,. \tag{2}$$

In the next section we find the conditions under which the general features of the replicator dynamics still hold for frequency dependent evolutionary games.

3.3 Some General Results

The replicator dynamics of a frequency dependent evolutionary game is a system of differential equations as specified in (2). Frequency dependent evolutionary games can only be used as economic models if the solutions of (2) have a meaningful economic interpretation. This is the case if (2) is a *well-defined* dynamics on the state space Σ_{n-1}, i.e., if the following two conditions are satisfied:

(I) there are unique solutions $\phi_t(x^0)$ for all initial conditions $x^0 \in \Sigma_{n-1}$,

(II) these solutions must remain in the strategy state space for all initial conditions, i.e. $\phi_t(x^0) \in \Sigma_{n-1}\ \forall\ t$ and $\forall\ x^0 \in \Sigma_{n-1}$.

The following two propositions comprise the conditions for the system of differential equations (2) to be well-defined. Proposition 1 is concerned with (I), it states the sufficient conditions for the existence and the uniqueness of the solutions of system (2).

Proposition 1 *If all elements of $A(x)$ are Lipschitz-continuous functions, the replicator dynamics of a frequency dependent evolutionary game has a unique solution for every initial condition in the state space.*

The next proposition is concerned with (II), it states the conditions under which the unique solutions of the replicator dynamics of a frequency dependent evolutionary games lie in the interior of the game's state space.

Proposition 2 *If all elements of $A(x)$ are continuous functions, then the interior of simplex Σ and the boundary of the simplex Σ are both invariant under the replicator dynamics of a frequency dependent evolutionary game.*

Since Lipschitz-continuity implies continuity (see e.g. Walter, 1991), the differential equation system (2) induces a well-defined dynamics if we assume the elements of $A(x)$ to be Lipschitz-continuous functions of x.

Proposition 2 implies in particular

$$\sum_{i=1}^{N} x_i = 1 \quad \Rightarrow \quad \sum_{i=1}^{N} \dot{x}_i = 0 \quad \Rightarrow \quad \dot{x}_k = -\sum_{\substack{j=1 \\ j \neq k}}^{N} \dot{x}_j \,.$$

The change in the frequency of any strategy can be expressed through the changes in frequencies of the remaining strategies. This allows us to reduce the differential equation system of the replicator dynamics for frequency dependent evolutionary games by one equation.

In Proposition 3, we state one more property of the replicator dynamics, which often simplifies the calculations of the solution.

Proposition 3 *The replicator dynamics of a frequency dependent evolutionary game is invariant under positive continuous transformations of payoffs.*

Invariance under a positive continuous payoff transformation means that the functions in $A(x)$ can be multiplied by a positive real number without changing the solutions of the system. Similarly, adding or subtracting a continuous function from the columns of $A(x)$ does not change the replicator dynamics of a frequency dependent evolutionary game.

In the next section we focus on the class of evolutionary games that have stage games with two strategies. We compare the evolutionary equilibria of the standard evolutionary games with those of the frequency dependent evolutionary games.

3.4 The Two-Strategy Case

From Proposition 3 follows that we can write the payoff matrix of a two-strategy (2×2) frequency dependent evolutionary game as

$$B(x) \;=\; \begin{pmatrix} a(x) & 0 \\ 0 & b(x) \end{pmatrix},$$

where $a(x)$ and $b(x)$ are Lipschitz-continuous by assumption. The replicator dynamics (2) can be written as

$$\begin{aligned} \dot{x}_1 &= a(x)x_1 - \left(a(x)x_1^2 + b(x)x_2^2\right) \\ \dot{x}_2 &= b(x)x_2 - \left(a(x)x_1^2 + b(x)x_2^2\right) \,. \end{aligned}$$

By Proposition 2, $\dot{x}_2 = -\dot{x}_1$, which leaves us with

$$\dot{x}_1 \;=\; x_1(1-x_1)\left(x_1 a(x_1) - (1-x_1)\, b(x_1)\right) \;=\; F(x_1)\,. \tag{3}$$

For arbitrary functions $a(x_1)$ and $b(x_1)$ the replicator dynamics can yield very complicated behavior because $F(x_1)$ may have many critical points which qualify as evolutionary equilibria.

Most economic interpretations might not need highly nonlinear and non-monotonic payoff functions. For many applications it may be sufficient to model whether a payoff increases or decreases with the frequency of a strategy, and if these changes become stronger or weaker the higher the frequency of the strategy. However, even with quite simple functions for the payoffs, the number of critical points of $F(x_1)$ cannot be determined generally. The only class of payoff functions that allows for a more precise description of equilibrium behavior is the class of linear functions, i.e., $a(x_1) = a_1 + a_2 x_1$ and $b(x_1) = b_1 + b_2 x_1$.

Proposition 4 *If $a(x_1)$ and $b(x_1)$ are linear functions, a 2×2 frequency dependent evolutionary game with replicator dynamics has at most one EE and at most one unstable critical point in the interior of the simplex. Apart from the interior EE, it can additionally have an EE at $x_1 = 0$ (if $F(0) \leq 0$) and an EE at $x_1 = 1$ (if $F(1) \geq 0$).*

We now compare the standard games with their frequency dependent counterparts. Of the former, we distinguish between three categories: Prisoners' Dilemma (Type I and II), Coordination Games, and Hawk-Dove Games (see Weibull, 1995, p. 75). Analogous to these categories we assume for the frequency dependent Prisoners' Dilemma I $a(x_1) > 0$ and $b(x_1) < 0$, for the frequency dependent Prisoners' Dilemma II $a(x_1) < 0$ and $b(x_1) > 0$, for the frequency dependent Coordination Game $a(x_1) > 0$ and $b(x_1) > 0$, and for the frequency dependent Hawk-Dove Game $a(x_1) < 0$ and $b(x_1) < 0$, $x_1 \in [0, 1]$. Table 1 summarizes our findings, which are proved in Appendix A. We have $p_{cg}, p_{hd,1}, p_{hd,2} \in]0, 1[$, the explicit expressions can be found in

Table 1: Evolutionary Equilibria of standard and FD-games.

Game Category	EE of Standard Game (values for x_1)	EE of FD-Game (values for x_1)
PD I	$\{0\}$	$\{0\}$
PD II	$\{1\}$	$\{1\}$
CG	$\{0, 1\}$	$\{0\}$ or $\{0, 1\}$ or $\{0, p_{cg}\}$
HD	$\{\frac{b}{a+b}\}$	$\{1\}$ or $\{p_{hd,1}, 1\}$ or $\{p_{hd,2}\}$

Appendix A.

We see that no matter how the payoffs in a Prisoners' Dilemma change with the frequency of a strategy, the EE will be the same as in a game with constant payoffs. However, for Coordination Games the situation is different: depending on the payoff function, it is now possible that $x_1 = 1$ is no longer an EE, or that it is replaced

by an EE in the interior of the strategy space. In the case of the Hawk-Dove Game, frequency dependent payoffs can change the dynamics of the game too. While the game with constant payoffs featured a unique interior EE, the frequency dependent game can either have a (different) unique interior EE, too or have an EE at $x_1 = 1$, or both.

For Table 1 we have assumed that the functions $a(x_1)$ and $b(x_1)$ either take negative or positive values. However, frequency dependent evolutionary games also allow to conjoin the different categories of games. We demonstrate this with an example.

Example 1

Let us consider a general situation of competition: Two players compete for a prize, by either choosing competition technology 1 or 2. Technology 1 is more expensive, but wins over technology 2. If two players with the same technology compete, they share the prize. We assume that the costs for the technologies depends on how many agents use a certain technology. This setup can be summarized with the following payoff matrix:[10]

$$A(x) = \begin{pmatrix} \frac{p}{2} - c_1(x) & p - c_1(x) \\ -c_2(x) & \frac{p}{2} - c_2(x) \end{pmatrix}. \tag{4}$$

Let us normalize the prize p to 1, and assume that the costs for technology 1, $c_1(x)$, increase, the more players use it. We assume $c_1(x_1) = 0.8x_1$. Technology 2, being the less sophisticated technology, becomes cheaper the more players adopt it, $c_2(x_2) = 0.2 - 0.2x_2$. On the interval $x_1 \in [0, \frac{5}{6}]$ this game is a Prisoners' Dilemma I, on the interval $x_1 \in [\frac{5}{6}, 1]$ it is a Prisoners' Dilemma II. The only EE in this game is $x_1^* = \frac{5}{6}$. Independent of how many players use each of the technologies initially, they converge to the equilibrium in which five out of six players use technology 1. Note that an equilibrium in the interior of the state space, $]0, 1[$, can neither appear in a Prisoners' Dilemma I nor II.

The EE is found by solving the differential equation

$$\dot{x}_1 = 0.1x_1(1 - x_1)(5 - 6x)\,,$$

which we derive by first using Proposition 3 to transform $A(x)$ in (4) into a diagonal matrix, and then plugging the respective elements of this new matrix into (3). The critical point x_1^*, $F(x_1^*) = 0$, is asymptotically stable because of $DF(x_1^*) < 0$.

In the next section, we attend to an important complicacy which often appears in the analysis of frequency dependent evolutionary games: Nonhyperbolic critical points on the boundary of the simplex. We present an extension of Liapunov's Theorem which can be very helpful for analyzing nonlinear dynamical systems with a simplex as a state space.

[10] Top left entry of matrix is payoff for technology 1 when competing with technology 2, top right is payoff for technology 1 when competing with technology 2. Bottom row similar for technology 2.

4 An Extention of Liapunov's Theorem for Evolutionary Games

To find the evolutionary equilibria of a frequency dependent evolutionary game, we have to find the asymptotically stable equilibrium points of the corresponding differential equation system. In many cases this is not an easy task, since the differential equation systems are highly nonlinear.

Local Theory of Nonlinear Differential Equation Systems distinguishes between hyperbolic and nonhyperbolic critical points of a system.[11] In order to evaluate the stability of hyperbolic critical points, the Hartman-Grobman Theorem can be used, [12] i.e., the stability of a critical point x^* is determined by the signs of the real parts of the eigenvalues of $DF(x^*)$.[13] This is the method we have applied to find the EE in Example 1.

However, differential equation systems of frequency dependent evolutionary games may have nonhyperbolic critical points. One method to show stability for nonhyperbolic critical points is due to Liapunov. The Liapunov Theorem (see e.g. Perko, 2000, p. 131, Theorem 3) which states under which conditions the existence of a Liapunov function (defined below) implies (asymptotic) stability of a critical point, only applies to critical points that are interior points of the definition space of $F(x)$. Hence, the conditions of Liapunov's Theorem rule out all equilibrium points on the simplex boundary. To clarify the problem, we now give an example of a critical point which cannot be analyzed by Liapunov's method.

Example 2

Let us again look at the 2-strategy-2-player case with linear functions in the payoff matrix. We have calculated

$$\begin{aligned} DF(x_1) &= (1-2x_1)((a_2+b_2)x_1^2+(a_1+b_1-b_2)x_1-b_1) \\ &\quad + x_1(1-x_1)(2(a_2+b_2)+a_1+b_1-b_2)\,. \end{aligned}$$

From (3) we know, that $x_1^* = 0$ is a critical point for this category of games. We see that $DF(0) = 0$ if $b_1 = 0$. Consequently, if $b_1 = 0$, the game cannot be analyzed by using the Hartman-Grobman Theorem and we would suggest Liapunov's method to analyze the stability of x_1^*. However, Liapunov's method does not apply either because $x_1^* = 0$ is a point on the boundary of the simplex.

[11] Critical point x^* is hyperbolic, if none of the eigenvalues of $DF(x^*)$ has a zero real part.

[12] The Hartman-Grobman Theorem states that if F is differentiable then there exists a homeomorphism that maps the trajectories in an open set around a hyperbolic critical point x^* onto trajectories near x^* of the linear system $\dot{x} = Ax$ with $A = DF(x^*)$. That is to say that near a hyperbolic critical point x^* the nonlinear system $\dot{x} = F(x)$ has the same qualitative structure as the linear system $\dot{x} = Ax$ with $A = DF(x^*)$.

[13] Sources (at least one eigenvalue of $DF(x^*)$ has a positive and at least one has a negative real part) and saddles (all eigenvalues of $DF(x^*)$ have positive real parts) are unstable, sinks (all eigenvalues of $DF(x^*)$ have negative real parts) are asymptotically stable.

In some cases, Liapunov's method can still be applied if we just consider the dynamical system on a set containing the simplex, making a point on the simplex boundary an interior point. However, since the behavior of the solutions to the dynamical system of an evolutionary game tend to be complicated on simplex boundaries (remember that the simplex as well as its boundaries are invariant under the dynamics of frequency dependent evolutionary games), it might become impossible to find a Liapunov function for this extended definition set. In order to facilitate the analysis of frequency dependent evolutionary games in these cases, we present a theorem which extends Liapunov's method to critical points on the simplex boundary.

Theorem 1 *Let E be an open subset of Σ and $x^* \in \overline{E}$.*[14] *Suppose that $F(x) \in C^1(\overline{E})$ and $F(x^*) = 0$, where the simplex is invariant under $\dot{x} = F(x)$. Suppose further that there exists a real valued function $V \in C^1(\overline{E})$ satisfying $V(x^*) = 0$ and $V(x) > 0$, $\forall\, x \in \overline{E} \setminus x^*$. If $\dot{V}(x) < 0\ \forall\, x \in E$, x^* is asymptotically stable.*

Theorem 1 states that we do not have to find a Liapunov function for an environment of x^*, but only a Liapunov function for the environment of x^* *within* the simplex. With Example 3 we show how this can ease the proof that a critical point is an EE.

Example 3

To make the example as simple as possible, assume that $a_1 = -2$, $a_2 = 2$, and $b_2 = -1$ in a 2×2 frequency dependent evolutionary game with linear payoff functions. The differential equation for the replicator dynamics, see (3), can then be written as

$$\dot{x}_1 \quad = \quad -x_1^2(1-x_1)^2\,.$$

We want to find out if $x_1^* = 0$ is an EE. The obvious Liapunov function for this nonlinear system is $V(x_1) = x_1^2$ because $V(0) = 0$ and $V(x_1) > 0$ for x_1 in the ρ-neighborhood of 0, i.e. $]-\rho, \rho\,[$. We now have to show that $\dot{V}(x_1) < 0$ for the ρ-neighborhood of 0 (Liapunov Theorem) or that $\dot{V}(x_1) < 0$ for the ρ-neighborhood *within* the simplex, i.e. $]\,0, \rho\,[$, according to Theorem 1.

$$\dot{V}(x_1) \quad = \quad 2x_1\dot{x}_1 \quad = \quad 2x_1 \cdot (-x_1^2(1-x_1)^2) \quad = \quad -2x_1^3(1-x_1)^2\,.$$

We see that $\dot{V}(x_1) < 0$ for $x_1 \in]\,0, \rho\,[$, but not $x_1 \in]-\rho, \rho\,[$. This shows, how Theorem 1 can facilitate a proof for asymptotic stability.

The Liapunov method does not provide us with a procedure or a hint on how to find a Liapunov function, neither does our extension in Theorem 1. In many cases a Liapunov function may be guessed with a little bit of experience and consideration of the function $F(x)$. However, Theorem 1 certainly makes the search for a Liapunov function easier since it has to comply with the conditions of Liapunov's Theorem on a smaller set.

[14] Notation: $\overline{E}$ is the set of osculation points of E. Point x is an osculation point of E if $E \cap U_\rho(x) \neq \emptyset, \forall \rho \in \mathbb{R}_+$. The set $U_\rho(x)$ is the ρ-neighborhood of x (or the open sphere around x). It is defined as $U_\rho(x) = \{y \in \mathbb{R}^n; |x - y| < \rho\}$.

5 Conclusions

If the share of players using a certain strategy in an evolutionary game has an impact on the payoff of any strategy in the game, we deal with a frequency dependent evolutionary game. In this paper, we have studied frequency dependent evolutionary games which have the replicator dynamics as a strategy adjustment rule.

We have shown that under very general assumptions, the replicator dynamics with strategy state dependent payoff functions is well-behaved, that is, has unique solutions for all initial startegy states and is simplex invariant. We suggest the evolutionary equilibrium as an equilibrium concept for frequency dependent evolutionary games.

Very few statements can be made in general about the evolutionary equilibria of frequency dependent evolutionary games. The functional forms of the strategy state dependent payoff functions are decisive for the set of equilibria. Nevertheless, we think that the framework is suited for interesting applications in economics.

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A Appendix

Proof of Proposition 1

Proposition 1 follows directly from the Fundamental Existence-Uniqueness Theorem (Picard-Lindelöf, see e.g. Perko (2000)).

Proof of Proposition 2

For simplex invariancy of the replicator dynamics of a frequency dependent evolutionary game the following three conditions must be satisfied:

$$\sum_{i=1}^{N} \dot{x}_i = 0 \tag{5}$$

$$\lim_{x_i \to 0^+} \dot{x}_i = 0 \tag{6}$$

$$\lim_{x_i \to 1^-} \dot{x}_i = 0 \tag{7}$$

Condition (5) guarantees that the solution of the system satisfies $\sum_{i=1}^{N} x_i = 1$ if the initial condition is an element of simplex Σ. Conditions (6) and (7) impose the upper bound 1 and the lower bound 0 on the solution $x(t)$. The three together limit the solution $x_i(t)$ to the simplex Σ.

We introduce the following notation for row i of matrix A:

$$A_i = (a_{i1}, a_{i2}, ..., a_{i\,n-1}, a_{i\,n}) .$$

Condition (6) can be written as

$$\lim_{x_i \to 0^+} \dot{x}_i = \lim_{x_i \to 0^+} x_i \left((A_i x) - \left(\sum_{j=1}^{N} x_j (A_j x) \right) \right) = \lim_{x_i \to 0^+} x_i g(x) ,$$

where we denote the function in brackets by $g(x)$. If all elements of A are continuous functions on simplex Σ, then $g(x)$ is a continuous function on Σ because sums and products of continuous functions are continuous functions. The simplex Σ is compact, from this it follows (Theorem of Weierstrass) that $g(x)$ is compact (and therefore bounded). So we have

$$\lim_{x_i \to 0^+} \dot{x}_i = 0 .$$

We next consider condition (7),

$$\lim_{x_i \to 1^-} \dot{x}_i = \lim_{x_i \to 1^-} x_i \left((A_i x) - \left(\sum_{j=1}^{N} x_j (A_j x) \right) \right) ,$$

under the assumption that $a_{ij}(x)$ are continuous functions. By the reasoning above we know that sums and products of the functions $a_{ij}(x)$ are bounded and that limits on Σ are finite therefore. Thus we can write

$$\begin{aligned}\lim_{x_i\to 1^-} \dot{x}_i &= \lim_{x_i\to 1^-} x_i(A_i x) - \lim_{x_i\to 1^-} \sum_{j=1}^{N} x_j(A_j x) \\ &= \lim_{x_i\to 1^-} (A_i x) - \sum_{j=1}^{N} \lim_{x_i\to 1^-} x_j(A_j x) \\ &= -\sum_{j\neq i} \lim_{x_i\to 1^-} x_j(A_j x)\end{aligned}$$

For $x_i \to 1^-$, we have that $x_j \to 0^+$ for $j \neq i$. Again continuity of the functions $a_{ij}(x)$ on Σ implies that

$$\lim_{x_j\to 0^+} x_j(A_j x) = 0$$

and therefore we have

$$\lim_{x_i\to 1^-} \dot{x}_i = 0\,.$$

Finally condition (5) can be shown by summation of all equations in (2):

$$\begin{aligned}\sum_{i=1}^{N} \dot{x}_i &= \sum_{i=1}^{N} x_i \left(e_i' A(x)x - x' A(x)x\right) \\ &= \sum_{i=1}^{N} x_i(e_i' A(x)x) - \sum_{i=1}^{N} x_i(x' A(x)x) \\ &= \sum_{i=1}^{N} (x_i e_i') A(x)x - (x' A(x)x) \sum_{i=1}^{N} x_i \\ &= \left(\sum_{i=1}^{N} x_i e_i'\right) A(x)x - x' A(x)x \\ &= x' A(x)x - x' A(x)x \;=\; 0\,.\end{aligned}$$

From the above it is clear that if $x_i = 0$, we have $\dot{x}_i = 0$. Thus the boundary of Σ is invariant. When rewriting equation (2) as

$$\frac{\dot{x}_i}{x_i} = \left(\sigma^T A(x)x - xA(x)x\right) \quad \forall\, i \in S$$

we see that the differential equation system intuitively describes the relative change of the solutions $x_i(t)$. From this it is obvious that if $x_i(0) > 0 \Leftrightarrow x_i(t) > 0$. So the interior of Σ is invariant, too.

Proof of Proposition 3

If we multiply all payoffs with $\lambda > 0$, we can write the replicator dynamics as

$$\dot{x}_i \;=\; x_i \left(\sigma^T \lambda A(x)x - x\lambda A(x)x\right) \;=\; \lambda x_i \left(\sigma^T A(x)x - xA(x)x\right) ,$$

which is a system with the same solutions $x_i(t)$ as (2).

Now let $B(x)$ be a matrix with n identical rows, which we denote by

$$b(x) = (b_1(x), b_2(x), ..., b_{n-1}(x), b_n(x)) .$$

The elements of $B(x)$ are continuous functions on Σ.

$$\begin{aligned} \dot{x}_i &= x_i \left(\sigma^T \left[A(x) + B(x)\right] x - x \left[A(x) + B(x)\right] x\right) \\ &= x_i \left(\sigma^T A(x)x + b(x)x - xA(x)x - xb(x)x\right) \\ &= x_i \left(\sigma^T A(x)x - xA(x)x\right) . \end{aligned}$$

Proof of Proposition 4

By plugging the linear functions into (3), we get

$$\begin{aligned} \dot{x}_1 &= x_1(1-x_1)\left(x_1 a(x_1) - (1-x_1)b(x_1)\right) \\ &= x_1(1-x_1)\left((a_2+b_2)x_1^2 + (a_1+b_1-b_2)x_1 - b_1\right) . \end{aligned}$$

The polynomial $(a_2 + b_2)x_1^2 + (a_1 + b_1 - b_2)x_1 - b_1$ has 2 roots. So it has at most 2 roots on the interval $]0, 1[$. Because $F(x_1)$ is continuously differentiable, not both of these potential roots can be EE: We need $DF(x_1^*) < 0$ for a root x_1^* to be an EE. So we can at most have one EE and one unstable critical point in the interior of the simplex. Furthermore, 0 and 1 are critical points too, each of which can be an EE or unstable.

Calculations for Table 1

Note that

$$\begin{aligned} DF(x_1) &= (1-2x_1)\left(x_1 a(x_1) - (1-x_1)b(x_1)\right) \\ &\quad + x_1(1-x_1)\left(a(x_1) + x_1 a'(x_1) + b(x_1) - (1-x_1)b(x_1)\right) \\ &= (1-2x_1)\left((a_2+b_2)x_1^2 + (a_1+b_1-b_2)x_1 - b_1\right) \\ &\quad + x_1(1-x_1)\left(2(a_2+b_2)x_1 + a_1 + b_1 - b_2\right) . \end{aligned}$$

Prisoners' Dilemma I: If $a(x_1) < 0$ and $b(x_1) > 0$, the only critical points are $x_1 = 0$ and $x_1 = 1$, because this implies $xa(x_1) - (1 - x_1)b(x_1) < 0$. $DF(0) < 0$ and

$DF(1) > 0$, so $x_1 = 0$ is the only EE.

Prisoners' Dilemma II: The critical point $x_1 = 1$ is the only EE, reasoning similar to Prisoners' Dilemma I.

Coordination Game: The Coordination Game requires $a(x_1) = a_1 + a_2 x > 0$ and $b(x_1) = b_1 + b_2 x > 0$. Hence, $a_1 > 0$ and $b_1 > 0$. The roots of $x_1 a(x_1) - (1 - x_1) b(x_1)$ are

$$r_{1,2} \quad = \quad \frac{-(a_1 + b_1 - b_2) \pm \sqrt{(a_1 + b_1 - b_2)^2 + 4(a_2 + b_2) b_1}}{2(a_2 + b_2)} \,.$$

Note that $DF(0) = -b_1 < 0$. So if $r_1 \in]0,1[$ and $r_2 \in]0,1[$, then $x_1 = 0$ and $x_1 = r_2$ are EE. We have named r_2 as p_{cd} in Table 1. If either $r_1 \in]0,1[$ or $r_2 \in]0,1[$ but not both, then this root is an unstable critical point and $\{x_1 = 0, x_1 = 1\}$ are the EE. If none of the roots lies in $]0,1[$, then $F(x_1) < 0$ on $]0,1[$ and $x_1 = 0$ is the only EE.

Hawk-Dove Game: We proceed analogously to the calculation for the coordination game. In a Hawk-Dove Game, the FD-payoffs satisfy $a(x_1) = a_1 + a_2 x < 0$ and $b(x_1) = b_1 + b_2 x < 0$. It follows that $a_1 < 0$ and $b_1 < 0$. Note that $DF(0) = -b_1 > 0$. The roots are the same as in the coordination game. If $r_1 \in]0,1[$ and $r_2 \in]0,1[$, then $x_1 = r_1$ and $x_1 = 1$ are EE. We have named r_1 as $p_{hd,1}$ in Table 1. If either $r_1 \in]0,1[$ or $r_2 \in]0,1[$ but not both, then this root is the sole EE. We have named that root $p_{hd,2}$ in Table 1. If none of the roots lies in $(0,1)$, then $F(x_1) > 0$ on $(0,1)$ and $x_1 = 1$ is the only EE.

Proof of Theorem 1

Note that we follow the proof of Theorem 3 in Perko (2000, p. 131) and make adjustments to our case where necessary.

Function $V(x)$ is called a Liapunov function. We define $\phi_t(x)$ as the flow of system $\dot{x} = F(x)$. We can write

$$\dot{V}(x) = \frac{d}{dt} V(\phi_t(x))|_{t=0} = DV(x)F(x) \,. \tag{8}$$

The first equation is due to the definition of the flow of a differential equation system, the second equation is due to the chain rule.

Choose $\varepsilon > 0$ sufficiently small that $\overline{N_\varepsilon(x^*)} = \overline{U_\varepsilon(x^*) \cap \Sigma_2} \subset \overline{E}$. We define the compact set S_ε,

$$S_\varepsilon \quad = \quad \{x \in \mathbb{R}^2 |\, |x - x^*| = \varepsilon\} \cap \overline{N_\varepsilon(x^*)} \,.$$

Since $V(x)$ is continuous there exists a minimum m_ε of $V(x)$ on S_ε and $V(x) > 0$ for $x \in \overline{E} \setminus x^*$ implies $m_\varepsilon > 0$. We also have $V(x^*) = 0$ and since $V(x)$ is continuous

there exists a δ such that $|x - x^*| < \delta$ implies $V(x) < m_\varepsilon$. Equations (8) imply that if $\dot{V}(x) < 0$ for $x \in E$, $V(x)$ is strictly decreasing along the trajectories of $\dot{x} = F(x)$. It follows that for all $\tilde{x} \in \overline{N_\delta(x^*)} = \overline{U_\delta(x^*) \cap \Sigma} \subset \overline{E}$ and $t > 0$ we have

$$V(\phi_t(\tilde{x})) < V(\tilde{x}) < m_\varepsilon \,. \tag{9}$$

Now suppose that for $\tilde{x}$ with $|\tilde{x} - x^*| < \delta$ there is a $t_1 > 0$ such that $|\phi_{t_1}(\tilde{x})| = \varepsilon$. Then since m_ε is the minimum of $V(x)$ on S_ε, this would imply that $V(\phi_{t_1}(\tilde{x})) \geq m_\varepsilon$ which contradicts (9). Thus for $\tilde{x}$ with $|\tilde{x} - x^*| < \delta$ and $t \geq 0$ it follows that $|\phi_t(\tilde{x})| < \varepsilon$.[15] Note that this is only true if the simplex is invariant under the dynamics of the differential equation system $\dot{x} = F(x)$. The reason is that simplex invariancy implies that the trajectories through $\tilde{x}$ can only leave $\overline{N_\varepsilon(x^*)}$ by crossing S_ε.

So for $\tilde{x}$ with $|\tilde{x} - x^*| < \delta$ and $t \geq 0$, $\phi_t(\tilde{x}) \subset \overline{N_\varepsilon(x^*)}$. Let $\{t_k\}$ be any sequence with $t \to \infty$. Then since $\overline{N_\varepsilon(x^*)}$ is compact, there is a subsequence $\{\phi_{t_n}(\tilde{x})\}$ of $\{\phi_{t_k}(\tilde{x})\}$ that converges to a point $y^* \in \overline{N_\varepsilon(x^*)}$.[16] Because $V(x)$ is a continuous function, $V(\phi_{t_n}(\tilde{x})) \to V(y^*)$. Since $V(x)$ is strictly decreasing along the trajectories of $\dot{x} = F(x)$ we have that

$$V(\phi_t(\tilde{x})) > V(y^*)$$

for $t \geq 0$. Now we have to determine y^*. Assume that $y^* \neq x^*$. Then for $s > 0$ we have $V(\phi_s(y^*)) < V(y^*)$. Continuity of $V(x)$ implies that for all y sufficiently close to y^* we have $V(\phi_s(y)) < V(y^*)$ for $s > 0$. But then for $y = \phi_{t_n}(\tilde{x})$ and n sufficiently large, we have $V(\phi_{s+t_n}(\tilde{x})) < V(y^*)$ which contradicts the above inequality. So by contradiction we have

$$y^* = x^* \,.$$

Since $V(x)$ is strictly decreasing along trajectories and since the subsequence $\phi_{t_n}(\tilde{x})$ converges to x^*, it follows for every sequence $t_k \to \infty$ that $\phi_{t_k}(\tilde{x}) \to x^*$. Therefore $\phi_t(\tilde{x}) \to x^*$ as $t \to \infty$, which means that x^* is asymptotically stable.

[15] By that, we have shown stability of x^*, which is weaker than asymptotic stability.

[16] By Bolzano-Weierstrass: Every sequence in a compact set of $\mathbb{R}^n$ has at least one convergent subsequence (e.g. Koenigsberger, 2001, p. 51).