

Loyalty Rewards and Monopoly Pricing

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DISCUSSION PAPERS

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Abstract

This article examines the impact of customer reward programs on the competitive outcome in duopolistic markets. We argue that loyalty discounts for repeat customers constitute a commitment device beneficial to suppliers rather than customers. Analyzing a two-period Bertrand model we show that the use of loyalty discounts makes it possible for duopolists to attain the fully collusive outcome in both periods. By offering generous loyalty discounts, the firms can credibly commit to refrain from second period poaching given that they attract enough customers in period one. Loyalty discounts invite firms to collude in the first period.

Keywords: switching costs, customer reward programs, loyalty discounts, repeat purchases, coupons, mixed equilibria.

JEL-Classification: C72, D43, L13, L14, L41

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1 Introduction

Customer loyalty is one of the most important qualities that every company strives to achieve. In recent years, the world's marketing departments have developed a huge variety of pricing schemes to promote customer loyalty. For example, most supermarket chains in Switzerland have launched loyalty cards that offer special advantages to consumers who have registered for a customer program. By presenting their loyalty card at the checkout lane, these customers are typically entitled to an allotment of shopping credits that they can use for future purchases. Similar programs are also run by credit card companies, hotel groups, and car rental agencies. The best-known customer programs are the frequent-flyer programs of the major airlines. To date, more than 350 million travelers registered in a frequent-flyer program worldwide.¹ However, loyalty-rewarding pricing schemes are not only used by big companies. For instance, coffee shops and pizza deliveries use simple, but effective punch cards to offer a free purchase to customers who have made a certain number of purchases. Other firms rely on repeat purchase coupons: they distribute a coupon along with their product which customers can use to get a discount for their next purchase of the same product.² Henceforth, we will refer to a loyalty-rewarding pricing scheme as reward program.

A *reward program* is a promotional tool that offers a loyalty reward on the basis of accumulated purchases from a firm. We observe many kinds of loyalty rewards: some firms allow for lump sum or proportional price discounts, other firms offer a free purchase of their products or services, and again other firms offer free products or services from other firms in different markets. Note, however, that reward programs are different from quantity discounts because firms do not commit to future prices.

One of the main reasons why firms launch reward programs is to create switching costs for consumers.³ The economic literature defines switching

¹Data from webflyer.com; December 2009.

²Another example for a loyalty-rewarding pricing scheme is tour operators' practice of rewarding travel agents who resell package holidays.

³More sophisticated reward programs have also emerged as effective tools to collect consumer data. For instance, loyalty cards from supermarkets feature a bar code or a magnetic stripe that can be easily scanned and thereby creates a detailed shopping profile

costs as the real or perceived costs that a consumer must incur when changing supplier but which he does not incur by remaining with the current supplier.⁴ Reward programs perfectly fit this definition because a customer who participates in such a program forfeits his loyalty reward if he changes the supplier.⁵ This means that he incurs a cost in purchasing an otherwise identical product from a new supplier even if that product is sold at the same unit price. Hence, reward programs generate switching costs that make otherwise identical products heterogeneous *ex post*, and should therefore be treated as a tool for “artificial” product differentiation.

As noted by Klemperer (1995), the most obvious effect of switching costs is to give firms some market power over their existing customers. Once the consumers have made an initial purchase, and thus built up switching costs, then—from the perspective of any particular firm—the market becomes segmented into two groups of customers with different price elasticities. The cost of attracting a customer who has previously bought from a rival firm increases because the price cut necessary to attract such a customer must cover his switching costs. For a firm that cannot price discriminate between first-time and repeat customers, this price cut might mean giving up more profits on repeat customers than gaining on any additional customer. Hence, if switching costs are large enough, each firm can improve unilaterally by exploiting its existing customers. It is evident that the proper conditions, when and to what extent a firm can benefit from switching costs, depend on the details of the respective model. In the context of reward programs, it turns out that a firm can only earn extra profits on existing customers if also all the other firms run reward programs and have already attracted customers.⁶ This gives the firms an incentive to compete less aggressively for

of the cardholder. This practice has been fiercely criticized by consumer organizations arguing that the data and shopping information collected by loyalty cards violates the privacy rights of consumers. For a critical discussion of supermarket reward programs from a privacy perspective see Albrecht (2002).

⁴Surveys of the switching costs literature are by Klemperer (1995), NERA (2003), or Farrell and Klemperer (2007).

⁵In the classification of Klemperer (1987), switching costs generated by reward programs are called artificial switching costs which arise entirely at firms’ discretion. Other authors refer to this switching cost also as endogenous switching cost, so for instance NERA (2003).

⁶On this point, switching costs generated by reward programs differ with other switch-

consumers who are not yet affiliated with a supplier since a higher market share makes the competitors less aggressive for the future.

This paper contributes to a better understanding of how firms can benefit from reward programs. We suggest that reward programs constitute a commitment device for firms with the potential to facilitate price coordination. The focus of the analysis is on reward programs offering lump sum loyalty discounts. However, qualitative results carry over to proportional discounts too.

A Brief Overview of the Model. We consider a simple two-period Bertrand model. Two symmetric firms supply a homogeneous good at zero marginal cost to a consumer population of mass one. Each consumer has unit demand per period and a reservation price of one. It is evident that in this competitive environment the firms earn zero profits in both periods because of Bertrand competition with undifferentiated products. However, we show that the introduction of reward programs allows the firms to attain the fully collusive outcome in both periods.

The idea is the following. Both firms provide a repeat purchase coupon along with the first-period product which consumer can use to get a minimum lump sum loyalty discount on their second-period purchase from the same supplier. The firms are free to increase their discount commitment after the first period.⁷ A consumer will redeem his coupon in the second period if the net price (regular price net of loyalty discount) charged by the current supplier does not exceed the other firm's regular price. This implies that each firm could price up to its loyalty discount above the competitor's price without losing a single customer to the rival. Such a price increase, aimed at exploiting existing customers, will henceforth be referred to as *rip-off strategy*. However, a firm could also go for the whole market and cut its

ing costs. For example, if existing customers incur a transaction cost s when changing supplier, then a firm can price up to the amount s above the competitor's marginal costs and earn a positive markup per existing customer, independent of the allocation of customers.

⁷This assumption accounts for the often observed fact that firms enlarge the variety of loyalty rewards in ongoing reward programs.

price to attract the customers who have previously bought from the other firm. Such a strategy will be named *poaching strategy*.

Poaching turns out to be less appealing for a firm with a big market share and a high loyalty discount. Moreover, it actually turns out that rip-off is the dominant strategy for firm i if its loyalty discount exceeds a certain threshold which is given by the ratio of the competitor's to firm i 's customers. By offering a loyalty discount of one, firm i can signal its intention to abstain from poaching, if the competitor grants it at least half the first-period market. We will show that the firms use this self-commitment to coordinate on the fully collusive outcome. The Pareto optimal Nash-equilibrium has both firms setting the monopoly price in the first period and coordinating on a loyalty discount higher or equal to one. This leaves the consumers indifferent and prompts them to choose their supplier at random. In consequence, the first-period market splits equally and both firms have rip-off as their dominant strategy for the second period. As a result, the firms will exploit their customers from the first period. The firms have no incentive to deviate since they earn zero second-period profits if (at least) one firm has no customers from the first period. This implies that any first-period price/discount pair, with the price not exceeding the monopoly price, could be sustained as a collusive equilibrium.

To sum up, our model suggests that repeat purchase coupons are a commitment device beneficial to duopolists rather than customers. The announcement of a generous loyalty discount is a credible commitment to weaken competition in the future and enables price coordination in the current period. Since the commitment value of such a loyalty discount is nil if the committing firm fails to attract consumers, the firms intend to share the market in the first period.

Related Literature. To our knowledge, Banerjee and Summers (1987) are the first who study reward programs as an endogenous decision to create switching costs. They consider a two period duopoly model with homogeneous products where firms in the first period simultaneously announce repeat purchase coupons and alternate in the price setting afterwards. Equilibrium prices coincide with monopoly prices in both periods. The present

setup is related to Banerjee and Summers (1987), but implements simultaneous instead of alternating price setting which changes the outcome significantly. The monopoly result is still an equilibrium, but it is not unique anymore.

Caminal and Matutes (1990) and Kim, Shi, and Srinivasan (2001) focus on differentiated duopoly models. They find that loyalty rewards are likely to relax price competition, but the specific form of the reward program is crucial. A more challenging result comes from Caminal and Claici (2007). For a monopolistic competition model with a large number of firms they show that reward programs are business stealing devices and, hence, have pro-competitive effects. For a large number of firms the commitment value with respect to rival firms is nil and firms will fail to collude. Caminal and Claici (2007) also argue that loyalty programs are anti-competitive only if the number of firms is sufficiently small and if firms are restricted to use specific reward designs that involve low commitment value for consumers.

Our result goes in the same direction as Caminal and Matutes (1990) and Kim et al. (2001). However, the effects become clearer and the statement stronger since we have Bertrand competition with homogeneous products and not competition with differentiated products à la Hotelling.

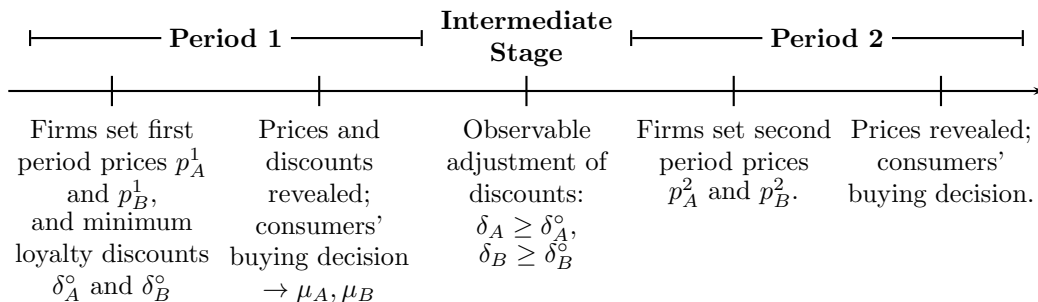
The remainder of the paper is organized as follows. Section 2 describes the basic setup of the model. Sections 3, 4, and 5 solve the model. Section 6 contains a discussion of the results and some informal extensions. Section 7 concludes. Long proofs are relegated to the Appendix 7.

2 The Model

We consider a two-period price competition model where firms can implement a reward program. Two otherwise identical firms, A and B , produce a homogeneous good at constant marginal cost. They distribute a coupon along with their first-period products which the customers can use to obtain a loyalty discount on their second period purchase. The good is non-storable. We normalize the mass of the consumer population to one and the firms' marginal cost to zero. Reservation prices are one and each consumer buys

at most one unit per period. Firms make simultaneous decisions to choose loyalty discounts and prices. The following figure illustrates the timing.

Figure 1: Timing



In period 1, the firms set first-period prices $p_A^1 \geq 0$ and $p_B^1 \geq 0$ and commit to minimum lump sum loyalty discounts $\delta_A^\circ \geq 0$ and $\delta_B^\circ \geq 0$.⁸ Then each consumer makes his first-period purchase. Whenever both firms charge the same price and offer identical minimum loyalty discounts, a fair coin is flipped to decide which firm to purchase from. This assumption rules out that an indifferent consumer would either always buy from firm A or always buy from firm B . However, this assumption does not rule out that a consumer could go for the firm with the lower first period price and/or the firm with the higher minimum loyalty discount. The first period ends with the proportion $\mu_A \geq 0$ of the consumers buying from firm A and the proportion $\mu_B \geq 0$ buying from firm B ; $\mu_A + \mu_B \leq 1$.

At the intermediate stage, both firms can upgrade their reward program and commit to a higher lump sum discount $\delta_A \geq \delta_A^\circ$ and $\delta_B \geq \delta_B^\circ$. These discount commitments are binding in the sense that each firm must accept its own coupons as means of payment. The upgrades are observable and verifiable to both customers and firms.

In period 2, the firms set second-period prices p_A^2 and p_B^2 . Consumers with the repeat purchase coupon compare their previous supplier's net price (regular price net of the lump sum discount) with the other firm's regular price. Indifferent consumers buy from their previous supplier. The firms cannot distinguish between first time and repeat buyers but the consumers

⁸Note that the firms cannot (or do not want to) commit to future prices at this stage of the game.

can prove their loyalty by presenting the repeat purchase coupon at the counter. Both firms and consumers are risk neutral and neither of them discounts the future. Thus, a firm's total expected profits at the beginning of the game are just the sum of the expected profits in each period, i.e., $\Pi_A = \pi_A^1 + \pi_A^2$ and $\Pi_B = \pi_B^1 + \pi_B^2$. A consumer's total expected utility is the sum of the expected utilities in each period.

3 Second-period Price Competition

We solve the model by backward induction. The solution of the second-period subgame depends on the loyalty discounts δ_A and δ_B (defined at the intermediate stage) and on the first-period market shares μ_A and μ_B . The fraction $0 \leq \mu_A \leq 1$ of consumers has previously purchased from firm A and has a repeat purchase coupon from firm A ; the fraction $0 \leq \mu_B \leq 1 - \mu_A$ has previously purchased from firm B and has a coupon from firm B .

It is not necessary to solve the second-period subgame for any arbitrary market allocation since there are only four ways the first period market can split: no firm has consumers, firm A has all consumers, firm B has all consumers, or both firms share the market equally. This simplification is a consequence of the fact that in period one each consumer either prefers not to buy, to buy from firm A , to buy from firm B , or flips a fair coin. Since consumers are identical, the sum of the individual buying decisions must lead to one of the following market allocations:⁹

$$(\mu_A, \mu_B) = (0, 0), (1, 0), (0, 1), \text{ or } (1/2, 1/2).$$

Henceforth we shall refer to the allocation $(0, 0)$ as *zero-zero allocation*, to the allocations $(1, 0)$ and $(0, 1)$ as *all-or-nothing allocations*, and to the allocation $(1/2, 1/2)$ as *fifty-fifty allocation*.

It should be clear that the *fifty-fifty allocation* arises if firm A and B both charge the same first-period price $p_A^1 = p_B^1 \leq 1$ and offer identical

⁹Note that if each consumer individually and independently flips a fair coin to decide whether to buy from firm A or from firm B , then the market share of customers buying from either firm converges to $1/2$ if the number of customers enlarge. That is, aggregation wipes out the uncertainty resulting from the coin flips on the individual level.

minimum loyalty discounts $\delta_A^\circ = \delta_B^\circ \geq 0$. In this case, the firms are indistinguishable from each other and a fair coin is flipped. However, if prices and minimum discounts are not the same, the analysis is not so straightforward. A consumer's expected utility at the outset of the game not only depends on first-period prices but also on price expectations for the second period. These price expectations, in turn, crucially depend on the consumer's belief about the outcome of the first-period market. As we will show in a next step below, the *zero-zero allocation* and the *all-or-nothing allocations* lead to fierce Bertrand competition in the second period. This implies that each consumer's expected second-period utility would be maximized if all consumers were to buy from the same supplier in the first period. Anticipating this result, it is rational for each consumer to rely on a public coordination device. If all consumers were to buy from the lower priced firm, everyone could rationally anticipate that an *all-or-nothing allocation* emerges and both firms charge a net price of zero in the second period. A similar argument applies to the case where the firms charge the same first-period price but offer different minimum loyalty discounts. For this case the consumers are assumed to buy from the firm offering the higher loyalty discount.

3.1 Zero-zero Allocation

We start the analysis of the second period with the *zero-zero allocation*. Because no consumer has a repeat purchase coupon from the first period, the firms compete with undifferentiated products. Therefore, a zero-profit Bertrand equilibrium will arise. This motivates the first proposition.

Proposition 1. *Any second-period subgame that starts after the **zero-zero allocation**, $(\mu_A, \mu_B) = (0, 0)$, has a unique Nash-equilibrium. The equilibrium prices are $p_A^{2*} = 0$ and $p_B^{2*} = 0$, leading to equilibrium profits $\pi_A^2 = \pi_B^2 = 0$.*

3.2 All-or-nothing Allocations

Next we focus on the *all-or-nothing allocations*. Without loss of generality suppose that each consumer has a repeat purchase coupon from firm A and

nobody a coupon from firm B . The firms simultaneously set p_A^2 and p_B^2 to maximize their second period profits

$$\pi_A^2(p_A^2, p_B^2) = \begin{cases} p_A^2 - \delta_A & \text{if } p_A^2 - \delta_A \leq \min\{p_B^2, 1\}, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\pi_B^2(p_A^2, p_B^2) = \begin{cases} p_B^2 & \text{if } p_B^2 < p_A^2 - \delta_A \text{ and } p_B^2 \leq 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

This non-cooperative price game has the unique zero-profit Nash-equilibrium $(p_A^{2*}, p_B^{2*}) = (\delta_A, 0)$. The result is based on the fact that firm B has no customers from the first period. At any combination of prices (p_A^2, p_B^2) where firm A earns positive profits, firm B has an incentive to cut its price to slightly below $p_B^{2'} = p_A^2 - \delta_A$ to attract the entire market. Proposition 2 generalizes the result.

Proposition 2. *Any second-period subgame that starts after the **all-or-nothing allocation**, $(\mu_i, \mu_j) = (1, 0)$, has a unique Nash-equilibrium. The equilibrium prices are $p_i^{2*} = \delta_i$ and $p_j^{2*} = 0$, with $i, j \in \{A, B\}$ and $i \neq j$, leading to equilibrium profits $\pi_A^2 = \pi_B^2 = 0$.*

3.3 Fifty-fifty Allocation

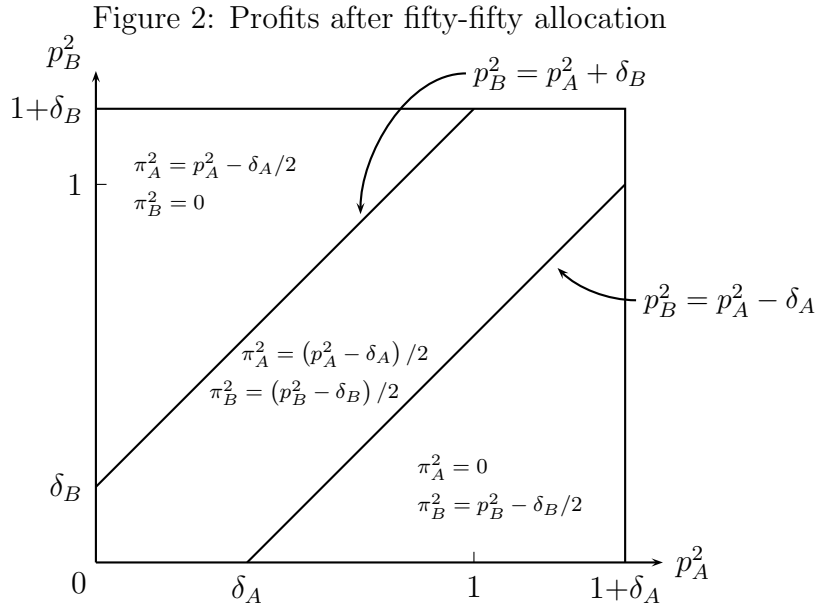
In the following we will analyze the subgames that start after the *fifty-fifty allocation*. Half the consumers have a repeat purchase coupon from firm A and qualify for the discount δ_A on the next purchase from A ; the remaining consumers have a coupon from firm B and qualify for the discount δ_B from B . Because the firms cannot distinguish between first-time and repeat customers, it is impossible to specially target the customers who have previously bought from the other firm. Consequently, each firm can only charge one price to all consumers.

By charging a price $p_A^2 > \min\{p_B^2, 1\} + \delta_A$, firm A prices itself out of the market: at such a high price the customers qualified for the discount δ_A either switch to firm B or leave the market. Conversely, by charging $p_A^2 < \min\{p_B^2 - \delta_B, 1\}$, firm A also attracts the customers qualified for the

discount δ_B and serves all consumers. The analogue is true firm B . For arbitrary $\delta_A \geq 0$ and $\delta_B \geq 0$ the profit functions of firm A and B are therefore given by

$$\pi_i^2(p_A^2, p_B^2) = \begin{cases} p_i^2 - \delta_i/2 & \text{if } p_i^2 < p_j^2 - \delta_j \text{ and } p_i^2 \leq 1, \\ (p_i^2 - \delta_i)/2 & \text{if } p_j^2 - \delta_j \leq p_i^2 \leq \min\{p_j^2, 1\} + \delta_i, \text{ and} \\ 0 & \text{otherwise;} \end{cases}$$

with $i, j \in \{A, B\}$ and $i \neq j$. Figure 2 shows the profit distribution graphically for different combinations of prices p_A^2 and p_B^2 .



It is well known that in such a differentiated product environment a pure-strategy Nash-equilibrium might not exist due to a price cycle à la Edgeworth. At every price combination (p_A^2, p_B^2) where firm B profitably serves all consumers, firm A (which makes zero profits because its price is too high) would like to lower its price and win back its customers from the first period. Then, at every price combination (p_A^2, p_B^2) where firm A serves only its customers from the first period, it can unilaterally improve by either charging $p_A^{2'} = \min\{p_B^2, 1\} + \delta_A$ in order to exploit its previous customers (*rip-off strategy*), or cutting its price to slightly below $\min\{p_B^2 - \delta_B, 1\}$ to also attract the consumers who have previously bought from firm B (*poaching strategy*).

By applying the latter strategy to the price $p_B^2 = 1 + \delta_B$ (the maximum price firm B is willing to charge), firm A expects almost $1/2$ in profits on new customers, against the loss of $\delta_A/2$ in the rip-off profits on repeat customers. Hence, for firm A never to have the incentive to poach, it must be the case that $\delta_A \geq 1$. The analog is true for firm B .

As a result, the non cooperative price setting game that starts after the *fifty-fifty allocation* has a pure strategy Nash-equilibrium only under the condition that $\delta_A \geq 1$ and $\delta_B \geq 1$. This condition implies that the forgone profits by cutting the price to A 's existing customers always exceed the additional profits gained by poaching from B , and vice versa. Hence, firm A and B can act as a monopolist on their respective customer base and set $p_A^{2*} = 1 + \delta_A$ and $p_B^{2*} = 1 + \delta_B$. They earn $\pi_A^2 = \pi_B^2 = 1/2$ in second period profits.

Conversely, if the condition $\delta_A \geq 1$ and $\delta_B \geq 1$ is not satisfied, firm A could guarantee itself only the profits $\tilde{\pi}_A^2 = \min\{\delta_B/4, 1/2\}$ by charging the price $\tilde{p}_A^2 = \delta_A + \min\{\delta_B/2, 1\}$. Firm B never undercuts \tilde{p}_A^2 since this leads to negative profits. In analog, firm B could charge $\tilde{p}_B^2 = \delta_B + \min\{\delta_A/2, 1\}$ which is good for the profits $\tilde{\pi}_B^2 = \min\{\delta_A/4, 1/2\}$. However, the price combination $(\tilde{p}_A^2, \tilde{p}_B^2)$ is not a Nash-equilibrium, as both firms have an incentive to increase their price. We will show in the Appendix that the firms can expect better profits by using fully mixed strategies. Proposition 3 summarizes the results.

Proposition 3. *Any second-period subgame that starts after the **fifty-fifty allocation**, $(\mu_A, \mu_B) = (1/2, 1/2)$, has a unique Nash-equilibrium either in pure or in mixed strategies:*

a) *Given $\delta_A \geq 1$ and $\delta_B \geq 1$, the pure strategy equilibrium prices are $p_A^{2*} = 1 + \delta_A$ and $p_B^{2*} = 1 + \delta_B$ leading to equilibrium profits $\pi_A^2 = \pi_B^2 = 1/2$.*

b) *Given $\delta_A < 1$ and $\delta_B < 1$, there exists unique mixed pricing (given in the Appendix) leading to expected nonnegative equilibrium profits*

$\pi_A^2(\delta_A, \delta_B) < 1/2$ and $\pi_B^2(\delta_A, \delta_B) < 1/2$ with $\partial\pi_A^2/\partial\delta_A > 0$ and $\partial\pi_B^2/\partial\delta_B > 0$.

c) *Given $\delta_i \geq 1$ and $\delta_j < 1$ with $i, j \in \{A, B\}$ and $i \neq j$, there exists unique mixed pricing (given in the Appendix) leading to expected equilibrium profits $\pi_i^2 = (1 + \delta_j)/4$ and $\pi_j^2 = 1/2$.*

4 Adjustment of Loyalty Discounts

Propositions 1 to 3 fully characterize the outcome of the second period. We now focus on the intermediate stage where the first-period market allocations have already emerged and the firms consider a discount increase to $\delta_A \geq \delta_A^\circ$ and $\delta_B \geq \delta_B^\circ$ respectively.

Note that Propositions 1 and 2 indicate that both firms earn zero second-period profits if (at least) one firm has failed to attract customers in the first period. Since this result applies to any combination of δ_A and δ_B , it is useless to increase the loyalty discount after the *zero-zero allocation* or an *all-or-nothing allocation*.

A converse result applies to the *fifty-fifty allocation*. Using weak dominance, it is always optimal for firm A to increase its discount commitment to $\delta_A^* = 1$ whenever $\delta_A^\circ < 1$ and $\delta_B^\circ < 1$, and to leave δ_A° unchanged otherwise. The analog is true for firm B . The reason for this result follows from Proposition 3 which implies that $\partial\pi_A^2/\partial\delta_A > 0$ and $\partial\pi_B^2/\partial\delta_B > 0$ if $\delta_A < 1$ and $\delta_B < 1$; and $\partial\pi_A^2/\partial\delta_A = 0$ and $\partial\pi_B^2/\partial\delta_B = 0$ otherwise. The next proposition summarizes the subgame-perfect equilibrium outcome of the intermediate stage.

Proposition 4. *Given that the first-period game has resulted in the fifty-fifty allocation and the firms are committed to low minimum loyalty discounts $\delta_A^\circ < 1$ and $\delta_B^\circ < 1$, the firms increase their discount commitments at the intermediate stage to $\delta_A^* = 1$ and $\delta_B^* = 1$. Otherwise, the discount commitment stay unchanged at $\delta_A^* = \delta_A^\circ$ and $\delta_B^* = \delta_B^\circ$.*

5 First-period Price Competition

In this section, we analyze the first-period game. At the outset of period one both firms simultaneously decide on their first-period price and the minimum loyalty discount, that is they set the nonnegative price/discount pairs (p_A^1, δ_A°) and (p_B^1, δ_B°) . Consumers observe these offers and come up with their first-period buying decision. In doing so, they also rely on price expectations for the second period. This makes the decision process quite complicated since the price expectations depend on beliefs about the outcome of the

first-period game. Nevertheless, buying from the firm with the lower first-period price turns out to be a rational consumption strategy. As mentioned earlier, a consumer's second-period utility will be maximized if all consumers were to buy from the same supplier in the first period. Therefore, if all consumers buy from the lower priced firm in the first period, everyone can rationally anticipate that both firms would charge a net price of zero in the second period. A similar argument applies to the case where the firms charge the same price, but offer different minimum loyalty discounts. This suggests the following coordination device. First, consumers shall coordinate on the firm charging the lower first-period price. Second, if prices coincide, consumers shall coordinate on the firm with the higher minimum loyalty discount.¹⁰ However, if both prices and discounts coincide, it is not possible to coordinate on a firm and each consumer individually flips a fair coin. Given this coordination device we are able to show:

Proposition 5. *Any nonnegative price/discount pair (p^1, δ°) with $p^1 \leq 1$ and $\delta^\circ \geq 0$ constitutes a symmetric Nash-equilibrium $(p_A^1, \delta_A^\circ) = (p_B^1, \delta_B^\circ) = (p^1, \delta^\circ)$ to the first-period game. Furthermore, there are no other Nash-equilibria.*

Proof. First, note that no firm ever charges a price greater than one since this would lead to zero first-period demand and (due to Propositions 1 and 2) to zero second-period profits. Then, with no loss of generality, suppose that firm A sets $0 \leq p_A^1 \leq 1$ and $\delta_A^\circ \geq 0$. If firm B exactly matches (p_A^1, δ_A°) , it comes to the *fifty-fifty allocation* and the firms share in a cooperative outcome in the second period. Due to Propositions 3 and 4, B 's profits will be:

$$\Pi_B = \pi_B^1 + \pi_B^2 = p_A^1/2 + 1/2.$$

If on the other hand firm B deviates from (p_A^1, δ_A°) , it gets $\pi_B^2 = p_A^1$ in first-period profits and no profits afterwards (since we have already shown that if at least one firm starts without customers from the first period, second-period profits must be zero). From this follows that it is the best response

¹⁰Actually it is not important whether the consumers coordinate on the firm with the higher or the lower loyalty discount. However, one might argue that common sense suggests to buy from the firm with the higher discount.

for firm B to set the same price and the same loyalty discount as firm A . Therefore, any price/discount pair (p^1, δ°) with $p^1 \leq 1$ and $\delta^\circ \geq 0$ can be implemented as a symmetric Nash-equilibrium. ■

As always if there are multiple Nash-equilibria, it is natural to ask which of them, if any, is the most reasonable solution. Although there is no conclusive answer to this question, we can partially dissolve the problem by assuming that the firms only coordinate on Nash-equilibria with $p_A^{1*} = p_B^{1*} = 1$. This assumption can be justified by a Pareto optimality argument: by coordinating on a first-period price/discount pair $(1, \delta^\circ)$, the firms attain the fully collusive outcome in both periods. In addition, it seems plausible that the firms coordinate on a minimum loyalty discount $\delta_A^\circ = \delta_B^\circ \geq 1$ as this makes it unnecessary to increase the discount commitment at the intermediate stage.

Proposition 6. *The two-period price competition model with loyalty discounts has the following most preferred Nash-equilibrium outcome: in the first period, the firms simultaneously set $(p_A^{1*}, \delta_A^{\circ*}) = (p_B^{1*}, \delta_B^{\circ*}) = (1, \delta^{\circ*})$ with $\delta^{\circ*} \geq 1$, at the intermediate stage, they choose $\delta_A^* = \delta_B^* = \delta^{\circ*}$, and, finally, in the second period they charge $p_A^{2*} = p_B^{2*} = 1 + \delta^{\circ*}$. This allows the firms to attain the fully collusive outcome in both periods. Total profits are $\Pi_A^* = \Pi_B^* = 1$.*

6 Discussion

The present model characterizes loyalty discounts as a powerful instrument for price coordination. In the absence of loyalty discounts, both firms would earn zero profits because of Bertrand competition with undifferentiated products. Conversely, the possibility to grant loyalty discounts allows the firms to attain the fully collusive outcome in both periods. The reason for this result is that loyalty discounts create switching costs which make it more difficult to attract customers who have previously bought from the rival firm. To be more specific, firm A must charge significantly below B 's regular price if it wants to poach customers from B . However, since price discrimination is not possible, firm A must also charge the same price to all its existing customers and, moreover, must grant them the promised loyalty discount.

Depending on the market allocation and the values of the discount offers, the forgone profits by cutting the price to A 's existing customers may exceed the additional profits gained by poaching from B . This phenomenon makes it possible to use loyalty discounts as a self-commitment device to less aggressive pricing. With a discount higher than one, each firm can credibly commit to refrain from second-period poaching provided that it succeeds in attracting at least half the first-period market. Together with the fact that no firm would earn positive second-period profits if at least one of them has no existing customers, this provides both firms with the right incentives to coordinate their first-period pricing. The best they can do is to charge the monopoly price and to coordinate on a minimum loyalty discount $\delta_A^{\circ*} = \delta_B^{\circ*} \geq 1$. In consequence, the market splits and each firm will act as a monopolist on its own respective market.

Up to this point we have looked at loyalty discounts within a two-period duopoly model. In the following we will discuss the robustness of our results along several dimensions. We consider in turn the effect of an increase in the number of firms, an increase in the number of periods, allowing for new consumers, and the possibility of entry.

More firms. Consider the case with N firms. It is straightforward to show that the second-period subgame will end in a zero-profit equilibrium if at least one firm has no customers from the first period. That is, in order to attain positive second-period profits, the firms will have to coordinate on a first-period price/discount pair that splits the market equally. Since in a symmetric equilibrium no firm shall expect more than $1/N$ in second-period profits, it is impossible for more than two firms to attain the fully collusive outcome in the first period. The intuition is simple. Suppose that all $N > 2$ firms charge the monopoly price $p^1 = 1$. Then, firm i has an incentive to cut its price since it can immediately gain $(N - 1)/N$ in first-period profits against the loss of $1/N$ in second-period profits. Hence, the collusive price must be lower than $1/(N - 1)$ to prevent undercutting. It can easily be checked that all the definitions for $N = 2$ have natural extensions for $N > 2$. The most preferred equilibrium outcome has each firm charging the

first-period price $p^{1*} = 1/(N - 1)$ and offering a minimum loyalty discount $\delta^{0*} \geq N - 1$. Each firm earns $1/(N - 1)$ in total profits.

More periods. There are two different ways of extending the model to $T > 2$ periods. One useful extension is to assume that the firms only give out T period coupons and that a consumer must buy in $T - 1$ periods from the same firm to get a loyalty discount in period T .¹¹ It is quite evident that if any firm fails to attract consumers in any period there will be no cooperation for the rest of the game. This property enforces cooperation in all T periods. The outcome of the extended model is similar to the standard two period model.

Another extension is to assume that the consumers stay in the market for T periods and get with each purchase a discount coupon for the next period. It should be clear that in the two firm case both firms earn zero profits in period T (the last period) if at least one firm has no customers from period $T - 1$. From this follows by a backward induction argument that it is a dominant strategy for both firms to cooperate in any previous period. The Pareto optimal equilibrium outcome has both firms coordinating on the t -th period price/discount pair (p^{t*}, δ^{t*}) , with $p^{t*} = 1 + \delta_{t-1}$ and $\delta^{t*} \geq 1$ for $t = 1, \dots, T$ and $\delta^{0*} = 0$. Each firm earns $T/2$ in total profits.

New consumers. The basic model assumes that consumers are identical and stay in the market for two periods. Given this assumptions, both firms have the right incentives to offer high loyalty discounts which makes it possible to exploit repeat customers in the second period. It turns out, however, that allowing for some new consumers who buy only in the second period has an important impact on the equilibrium outcome. New consumers reinforce the tradeoff between the loyalty inducing effect of larger discounts and the adverse effect of higher gross discount prices on sales to new customers. Offering a high loyalty discount needs not necessarily be the best way to proceed if new consumers enter in the second period.

¹¹Examples for such coupons include punch cards offered by coffee bars and pizza deliveries.

In the following we will extend the standard model to an overlapping generations model with three periods. The first generation of consumers enters in period one and leaves after period two; the second generation enters in period two and leaves after period three. First generation consumers obtain repeat purchase coupons in period one and use them in period two; second generation consumers obtain coupons in period two and use them in period three. Suppose that both generations are exactly the same as the consumer population in the standard model. Then, there will be one unit of demand in period one, two units of demand in period two and one unit of demand in period three.

The overlapping generations model turns out to have multiple equilibria. However, the fully collusive outcome is not attained in any equilibrium. The intuition is as follows. In the second period, there are two generations of consumers on the market: old consumers (first-generation consumers) with a repeat purchase coupon and young consumers (second-generation consumers) without a coupon. Given that a repeat purchase coupon qualifies for a positive loyalty discount it is easy to see that the second-period price must be higher than one if the firms wish to extract the full surplus from old consumers. This, however, means that the young generation will not buy at all. Hence, monopoly profits are not to be realized if the old consumers are entitled for positive loyalty discounts. On the other hand, however, monopoly profits are not to be realized without loyalty discounts either. In this case there are two generations of consumers on the second-period market and none of them qualifies for a loyalty discount. If firm A charges the monopoly price $p_A^2 = 1$, firm B can immediately gain profits of almost 1 by undercutting, against the expected loss of $1/2$ in third period profits.

There is a fundamental dilemma with overlapping generations. Offering a high loyalty discount to old consumers in the second period implies that the profits on repeat customers will be unattractively low if the price is set to keep young consumers in the market. On the other hand, however, it is also unfavorable to price young customers out of the market since this reduces third period profits to zero. Hence, in the overlapping generations model it is costly to increase loyalty discounts. The optimal discount for first

generation customers is determined by the conflicting interests of exploiting repeat customers and trying to capture new consumers in the second period. The symmetric Pareto optimal equilibrium has both firms offering the loyalty discounts $\delta_A^{1*} = \delta_B^{1*} = 1/4$ to first generation consumers, and $\delta_A^{2*} = \delta_B^{2*} = \delta^{2*} \geq 1$ to second generation consumers. The equilibrium prices increase over the three periods: $p_A^{1*} = p_B^{1*} = -1/4$, $p_A^{2*} = p_B^{2*} = 1$, and $p_A^{3*} = p_B^{3*} = 1 + \delta^{2*}$. It is most notable that the firms accept a negative markup in the first period. This reflects the value of locked in first generation customers in the second period. Each firm earns $5/4$ in total profits.

Entry. The consequence of potential entry on the equilibrium outcome depends on the timing assumption. To begin with, consider the case with two incumbents and an entrant considering to enter in the second period. If entry is free, Bertrand competition will drive all firms' second-period profits to zero since the situation of the entrant is the same as that of an established firm without customers. On the other hand, if costs of entry are $0 < s < 1$, incumbents can deter entry by charging the second-period net price $p_i^2 - \delta_i \leq s$. This, however, erodes the fully collusive outcome in the first period. The maximum first-period price that can be sustained as collusive price is $p^{1*} = s$. That is, each incumbent earns s in total profits if entry is possible in the second period.

Next consider the case where entry is also possible in the first period. Suppose that incumbents have already coordinated on the price/discount pair (p^1, δ°) and that the fully informed entrant may enter before consumers come up with their first-period buying decision. Then, the entrant's expected total profits are $p^1/3 + 1/3 - s$ if it matches (p^1, δ°) , and $p^1 - s$ if it undercuts. From this follows that any first-period collusive price $p^1 < \min\{3s - 1, s\}$ deters entry. However, entry deterrence might be too expensive for incumbents. In an entry deterring equilibrium, each incumbent earns $s/2$ in second-period profits and $\min\{3s - 1, s\}/2$ in first period profits. In the Pareto optimal equilibrium with $N = 3$, however, each firm earns $1/2$ in total profits. As a result, it is not optimal to deter entry if $s \leq 1/2$.

To summarize the discussion: While the second-period monopoly result is robust against an increase in the number of firms, the first-period monopoly result is not. If there are more than two firms in the market, the collusive first-period price must be lower than the monopoly price. This, however, does not change the result that loyalty discounts constitute a powerful instrument for price coordination among oligopolists. The same conclusion holds true for an increase in the number of periods. However, if we allow for some consumers who buy only in the second but not in the first period, the equilibrium outcome changes significantly. The discussion shows that the average loyalty discount is lower in an overlapping generations framework than in the standard model. Finally, it turns out that loyalty discounts have not the power to deter entry if the costs of entry are small.

7 Conclusion

The present model characterizes reward programs as a powerful instrument for price coordination in a highly competitive environment. Our analysis demonstrates that firms achieve otherwise unattainable cooperative outcomes in finite period games by creating switching costs through loyalty programs. The focus of the paper lies on reward programs offering lump sum discounts to repeat customers. We analyze a simple two-period Bertrand model with two duopolists providing a repeat purchase coupon along with their first period product. Consumers can use this coupon to obtain a minimum loyalty discount on the second-period purchase from the same supplier. The offer of a large minimum discount turns out to be an invitation to the other firm to collude on pricing. The reason behind the result is that the discount offer will become a credible self-commitment to refrain from second-period poaching, given that the competitor is willing to share the first-period market. In anticipation of a zero-profit result that will arise otherwise, the firms have the right incentives to coordinate on the same first period price: the market splits and each firm will act as monopolist on its own respective market in the second period.

At first sight one might expect that the second-period monopoly result leads to vigorous competition for market shares in the first period. The na-

ture of switching costs, however, leads to an opposite conclusion. In contrast to models where switching costs are paid for by consumers (e.g. Klemperer, 1995), in our model the switching costs are paid for by firms. Consequences are that a duopolist can only earn extra profits on repeat customers if the other duopolist has repeat customers as well. This provides the strong incentives to share the initial market. Hence, it seems quite safe to conclude that reward programs constitute a commitment device beneficial to competitors rather than consumers.

Appendix

Proposition 3 states that any subgame that starts after the *fifty-fifty allocation* has a unique Nash-equilibrium either in pure or in mixed strategies. The proof of this result breaks up into three Lemmas. For notational reasons we will write p_A and p_B instead of p_A^2 and p_B^2 for second-period prices; and π_A and π_B instead of π_A^2 and π_B^2 for expected second-period profits.

Lemma a). *Given $\delta_A \geq 1$ and $\delta_B \geq 1$, the pure strategy equilibrium prices are $p_A^* = 1 + \delta_A$ and $p_B^* = 1 + \delta_B$ leading to equilibrium profits $\pi_A = \pi_B = 1/2$.*

Proof. A pure-strategy Nash-equilibrium is the nonnegative price combination (p_A^*, p_B^*) such that, for a given p_B^* , firm A chooses p_A^* to maximize π_A and, for a given p_A^* , firm B chooses p_B^* to maximize π_B .

Remember that profits are given by

$$\pi_i(p_A, p_B) = \begin{cases} p_i - \delta_i/2 & \text{if } p_i < p_j - \delta_j \text{ and } p_i \leq 1, \\ (p_i - \delta_i)/2 & \text{if } p_j - \delta_j \leq p_i \leq \min\{p_j, 1\} + \delta_i, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

with $i, j \in \{A, B\}$ and $i \neq j$, and suppose that (p_A^*, p_B^*) is a Nash-equilibrium. Then, it must be satisfied that $p_A^* = \min\{p_B^*, 1\} + \delta_A$ since for any price $p_A^* \neq \min\{p_B^*, 1\} + \delta_A$ firm A could unilaterally improve by charging the rip-off price $p_A' = \min\{p_B, 1\} + \delta_A$. The analog is true for firm B . From this follows that the price combination $(1 + \delta_A, 1 + \delta_B)$ is the only candidate for a pure-strategy Nash-equilibrium. At this price combination, both firms serve only repeat customers and earn $1/2$ in second-period profits and

no firm can gain by raising its price because a higher net price for repeat customers would exceed the reservation value. However, for $(1 + \delta_A, 1 + \delta_B)$ to be a Nash-equilibrium, it must also be satisfied that no firm can improve by cutting its price to slightly below 1, in which case it earns almost 1/2 in profits on new customers against the loss of $\delta_A/2$ and $\delta_B/2$, respectively, in profits on repeat customers. This condition holds for $\delta_A \geq 1$ and $\delta_B \geq 1$ only, which completes the proof of Lemma a). \blacksquare

Lemma b). *Given $\delta_A < 1$ and $\delta_B < 1$, there exists unique mixed pricing leading to expected nonnegative equilibrium profits $\pi_A(\delta_A, \delta_B) < 1/2$ and $\pi_B(\delta_A, \delta_B) < 1/2$ with $\partial\pi_A/\partial\delta_A > 0$ and $\partial\pi_B/\partial\delta_B > 0$.*

Proof. The following cumulative distribution function $F_i(\cdot)$, when employed by firm A and B simultaneously, constitutes a unique mixed-strategy Nash-solution.

$$F_i(p_i) = \begin{cases} 0, & \text{for } p_i < \underline{p}_i, \\ 1 - \frac{2\pi_j}{p_i}, & \underline{p}_i \leq p_i \leq \bar{p}_j - \delta_j, \\ 1 - \frac{2\pi_j}{\bar{p}_j - \delta_j}, & \bar{p}_j - \delta_j \leq p_i < \underline{p}_j + \delta_i, \\ 1 - \frac{2\pi_j - p_i + \delta_i + \delta_j}{p_i - \delta_i}, & \underline{p}_j + \delta_i \leq p_i < \bar{p}_i, \\ 1, & \bar{p}_i \leq p_i, \end{cases}$$

with $i, j \in \{A, B\}, i \neq j$; where $\underline{p}_i = 2\pi_j$ is the lowest price charged by firm i and $\bar{p}_i = \min \{\underline{p}_i + \delta_i + \delta_j, 1 + \delta_i\}$ the highest; π_j , denoting the expected profits of firm j , are given as follows:

$$\pi_j = \begin{cases} \frac{2\delta_i + (\sqrt{5}-1)\delta_j}{4}, & \text{for } 0 \leq \delta_j \leq \min \left\{ \frac{(\sqrt{5}-1)(1-\delta_i)}{2}, \frac{2-(1+\sqrt{5})\delta_i}{2} \right\}, \\ \frac{1+\delta_i+(1-\delta_i)(\delta_j - \sqrt{(1+\delta_j)^2 + 4\delta_i})}{4\delta_i}, & \frac{(\sqrt{5}-1)(1-\delta_i)}{2} < \delta_j < \frac{(2-\delta_i)^2 - 2}{2-\delta_i}, \\ \frac{\sqrt{(1+\delta_i)^2 + 4\delta_j} - (1-\delta_i)}{4}, & \frac{2-(1+\sqrt{5})\delta_i}{2} < \delta_j < \frac{4-\delta_i - \sqrt{\delta_i^2 + 8}}{2}, \\ \frac{\delta_i - \delta_j + \sqrt{(\delta_i - \delta_j)^2 + 8(\delta_i + \delta_j)}}{8}, & \max \left\{ \frac{(2-\delta_i)^2 - 2}{2-\delta_i}, \frac{4-\delta_i - \sqrt{\delta_i^2 + 8}}{2} \right\} \leq \delta_j < 1. \end{cases}$$

We proof this result by construction and derive the stated solution by examining the sufficient conditions a mixed strategy equilibrium must satisfy.¹²

¹²The outline of the proof follows Shilony (1977) who offers a game-theoretic hypothesis to account for the phenomenon of price dispersion.

The assumptions made during the process of construction are proved to be necessary.

Suppose that firm i employs the cdf $F_i(\cdot)$ as its mixed strategy and firm j employs the cdf $F_j(\cdot)$, with $i, j \in \{A, B\}$ and $i \neq j$. Let \underline{p}_i and \underline{p}_j be the lowest, and \overline{p}_i and \overline{p}_j the highest price in the support of F_i and F_j , respectively. Given that F_i is continuous at $p_i = p_j - \delta_j$, firm j expects the following profits by charging the price p_j :

$$E[\pi_j(p_j, F_i)] = (1 - F_i(p_j - \delta_j)) \frac{p_j - \delta_j}{2} + (1 - F_i(p_j + \delta_i)) \frac{p_j}{2}$$

Since in a mixed equilibrium all firms must be indifferent between all the prices they are willing to charge, F_i must satisfy, for almost every p_j in the support of F_j , that $E[\pi_j(p_j, F_i)] = \pi_j$. This yields the difference equation

$$(1 - F_i(p_j - \delta_j))(p_j - \delta_j) + (1 - F_i(p_j + \delta_i))p_j = 2\pi_j \quad (2)$$

which F_i must satisfy at almost every point $p_i \in [\underline{p}_i, \overline{p}_i]$, such that F_i is continuous at $p_i = p_j - \delta_j$. We shall find such a cdf whose support is contained in an interval not larger than $\delta_i + \delta_j$. Later one we will prove that there are no solutions with a wider range.

Bounding the supports. Note that for $\delta_i < 1$ and $\delta_j < 1$ both firms are basically interested in poaching customers from the competitor. From this follows that $\underline{p}_i < \overline{p}_j - \delta_j$ and $\underline{p}_j < \overline{p}_i - \delta_i$. Together, this implies for the support of F_i and F_j

$$\begin{aligned} \delta_i/2 &\leq \underline{p}_i < \overline{p}_j - \delta_j \leq \underline{p}_j + \delta_i < \overline{p}_i \leq 1 + \delta_i, \quad \text{and} \\ \delta_j/2 &\leq \underline{p}_j < \overline{p}_i - \delta_i \leq \underline{p}_i + \delta_j < \overline{p}_j \leq 1 + \delta_j, \end{aligned} \quad (3)$$

where the lower and upper bounds follow from equation (1).

Defining F_i . For $\underline{p}_i + \delta_j \leq p_j < \overline{p}_j$, it applies that $p_j + \delta_i \geq \overline{p}_i$ and so $F_i(p_j + \delta_i) = 1$. Equation (2) becomes $(1 - F_i(p_j - \delta_j))(p_j - \delta_j) = 2\pi_j$ from which we get

$$\begin{aligned} F_i(p_j - \delta_j) &= 1 - \frac{2\pi_j}{p_j - \delta_j} \quad \text{for } \underline{p}_i + \delta_j \leq p_j < \overline{p}_j, \\ \text{or } F_i(p_i) &= 1 - \frac{2\pi_j}{p_i} \quad \text{for } \underline{p}_i \leq p_i < \overline{p}_j - \delta_j. \end{aligned} \quad (4)$$

For $\underline{p}_j \leq p_j < \bar{p}_i - \delta_i$, it applies that $p_j - \delta_j < \underline{p}_i$ and so $F_i(p_j - \delta_j) = 0$. Equation (2) simplifies to $p_j - \delta_j + (1 - F_i(p_j + \delta_i)) p_j = 2\pi_j$ from which we get

$$\begin{aligned} F_i(p_j + \delta_i) &= 1 - \frac{2\pi_j - p_j + \delta_j}{p_j} && \text{for } \underline{p}_j \leq p_j < \bar{p}_i - \delta_j, \\ \text{or } F_i(p_i) &= 1 - \frac{2\pi_j - p_i + \delta_i + \delta_j}{p_i - \delta_i} && \text{for } \underline{p}_j + \delta_i \leq p_i < \bar{p}_i. \end{aligned} \quad (5)$$

Finally, for $\bar{p}_j - \delta_j \leq p_i < \underline{p}_j + \delta_i$, the cdf has to be flat because at any price in this range firm i sells only to its customers from the first period and can unilaterally improve by increasing the price to $\tilde{p}_i = \underline{p}_j + \delta_i$. Together with (4) and (5) we get the following cdf:

$$F_i(p_i) = \begin{cases} 0, & \text{for } p_i < \underline{p}_i, \\ 1 - \frac{2\pi_j}{p_i}, & \underline{p}_i \leq p_i \leq \bar{p}_j - \delta_j, \\ 1 - \frac{2\pi_j}{\bar{p}_j - \delta_j}, & \bar{p}_j - \delta_j \leq p_i < \underline{p}_j + \delta_i, \\ 1 - \frac{2\pi_j - p_i + \delta_i + \delta_j}{p_i - \delta_i}, & \underline{p}_j + \delta_i \leq p_i < \bar{p}_i, \\ 1, & \bar{p}_i \leq p_i. \end{cases} \quad (6)$$

This equation defines F_i for any price p_i . There are three candidates for a discontinuity: \underline{p}_i , $\underline{p}_j + \delta_i$, and \bar{p}_i . To share the properties of a cdf, equation (6) must be continuous from the right (which is already satisfied) and non-decreasing in p_i . This demands

$$0 \leq x = 1 - \frac{2\pi_j}{p_i} \text{ to rule out a negative jump at } \underline{p}_i, \quad (7)$$

$$1 - \frac{2\pi_j}{\bar{p}_j - \delta_j} \leq 1 - \frac{2\pi_j - p_j + \delta_j}{p_j} \text{ to rule out a negative jump at } \underline{p}_j + \delta_i, \quad (8)$$

$$\text{and } 1 - \frac{2\pi_j - \bar{p}_i + \delta_i + \delta_j}{p_i - \delta_i} \leq 1 \text{ to rule out a negative jump at } \bar{p}_i. \quad (9)$$

In the following we show that F_i may only have a discontinuity at point \bar{p}_i .

No jump at \underline{p}_i . By charging the price $\tilde{p}_j = \underline{p}_i + \delta_j$, firm j only attracts repeat customers and earns $\tilde{\pi}_j = \frac{p_i}{2} \leq \pi_j$. Together with $2\pi_j = (1 - x)p_i$, from (7), this yields $\underline{p}_i \leq (1 - x)\underline{p}_i$ which has the unique solution $x = 0$. Hence, a jump at \underline{p}_i is ruled out and we get, using (6),

$$\underline{p}_i = 2\pi_j. \quad (10)$$

Since the analogue is true for F_j , it applies $\underline{p}_j = 2\pi_i$.

No prices outside the support of F_j . To be a solution, F_i must satisfy the condition that it does not pay for firm j to charge $\tilde{p}_j < \underline{p}_j$ or $\tilde{p}_j > \overline{p}_j$. For $\tilde{p}_j > \overline{p}_j$, this demands, using (2),

$$(1 - F_i(\tilde{p}_j - \delta_j))(\tilde{p}_j - \delta_j) \leq 2\pi_j, \quad \forall \tilde{p}_j > \overline{p}_j. \quad (11)$$

Assume that $\overline{p}_j - \delta_j < \underline{p}_j + \delta_i$. Then, for small deviations, the point $\tilde{p}_j - \delta_j$ lies in the flat segment of F_i . This means that inequality (11) can be simplified to $\frac{2\pi_j}{\overline{p}_j - \delta_j}(\tilde{p}_j - \delta_j) \leq 2\pi_j \forall \tilde{p}_j > \overline{p}_j$, which is obviously impossible because $\pi_j > 0$. Thus, for a deviation to $\tilde{p}_j > \overline{p}_j$ to be unprofitable it must be the case that $\overline{p}_j = 1 + \delta_j$ whenever $\overline{p}_j - \delta_j < \underline{p}_j + \delta_i$. Moreover, we have to distinguished two cases: a) $\overline{p}_j - \underline{p}_j = \delta_i + \delta_j$ and b) $\overline{p}_j - \underline{p}_j < \delta_i + \delta_j$.

Case a) $\overline{p}_j - \underline{p}_j = \delta_i + \delta_j$.

Note that this condition implies $\overline{p}_j - \delta_j = \underline{p}_j + \delta_i$ which means that F_i has no flat middle range. Moreover, since prices are bounded above, $\overline{p}_j \leq 1 + \delta_j$, this also implies that $\underline{p}_j = 2\pi_i \leq 1 - \delta_i$. To avoid deviations to $\tilde{p}_j > \overline{p}_j$ we then require, using (5) and (11):

$$\frac{2\pi_j - \tilde{p}_j + \delta_i + 2\delta_j}{\tilde{p}_j - \delta_i - \delta_j}(\tilde{p}_j - \delta_j) \leq 2\pi_j, \quad \forall \tilde{p}_j > \overline{p}_j.$$

The left-hand side is decreasing in \tilde{p}_j and so it is enough to check for $\tilde{p}_j = \overline{p}_j = \underline{p}_j + \delta_i + \delta_j = 2\pi_i + \delta_i + \delta_j$. This yields

$$\begin{aligned} \frac{2\pi_j - 2\pi_i + \delta_j}{2\pi_i}(2\pi_i + \delta_i) &\leq 2\pi_j, \quad \text{or} \\ (2\pi_j - 2\pi_i + \delta_j)(2\pi_i + \delta_i) &\leq 4\pi_i\pi_j. \end{aligned} \quad (12)$$

Next, to rule out $\tilde{p}_j < \underline{p}_j$ we want, using (2),

$$\tilde{p}_j - \delta_j + (1 - F_i(\tilde{p}_j + \delta_i))p_j \leq 2\pi_j, \quad \forall \tilde{p}_j < \underline{p}_j,$$

which, for $\overline{p}_j - \underline{p}_j = \delta_i + \delta_j$, is

$$\tilde{p}_j - \delta_j + \frac{2\pi_j}{\tilde{p}_j + \delta_i}\tilde{p}_j \leq 2\pi_j, \quad \forall \tilde{p}_j < \underline{p}_j$$

The left-hand side is increasing in \tilde{p}_j and so it is enough to check for $\tilde{p}_j = \underline{p}_j = 2\pi_i$. This yields

$$2\pi_i - \delta_j + \frac{2\pi_j}{2\pi_i + \delta_i} 2\pi_i \leq 2\pi_j, \quad \text{or}$$

$$4\pi_i\pi_j \leq (2\pi_j - 2\pi_i + \delta_j)(2\pi_i + \delta_i). \quad (13)$$

Equations (12) and (13) give a necessary condition for π_i and π_j , such that $2\pi_i \leq 1 - \delta_i$, which implies that $\overline{p}_j - \underline{p}_j = \delta_i + \delta_j$:

$$4\pi_i\pi_j = (2\pi_j - 2\pi_i + \delta_j)(2\pi_i + \delta_i). \quad (14)$$

Equation (14) also implies that inequality (8) holds with equality. This means that F_i has no jump at point $\underline{p}_j + \delta_i$.

Case b) $\overline{p}_j - \underline{p}_j < \delta_i + \delta_j$.

Due to the fact that $\overline{p}_j - \underline{p}_j = \delta_i + \delta_j$ may only apply if $2\pi_i \leq 1 - \delta_i$, it must now be the case that $2\pi_i > 1 - \delta_i$. Moreover, we need not to worry about deviations to $\tilde{p}_j > \overline{p}_j$ since, as mentioned earlier, it must now be the case that $\overline{p}_j = 1 + \delta_j$. To rule out deviations to $\tilde{p}_j < \underline{p}_j$ we then require

$$\tilde{p}_j - \delta_j + (1 - F_i(\tilde{p}_j + \delta_i)) p_j \leq 2\pi_j, \quad \forall \tilde{p}_j < \underline{p}_j.$$

Note that for small deviations the point $\tilde{p}_j + \delta_i$ lies in the flat segment of F_i and so we want, using (6),

$$\tilde{p}_j - \delta_j + \frac{2\pi_j}{\underline{p}_j - \delta_j} \tilde{p}_j \leq 2\pi_j, \quad \forall \tilde{p}_j < \underline{p}_j.$$

The left-hand side is increasing in \tilde{p}_j and so it is enough to check for $\tilde{p}_j = \underline{p}_j = 2\pi_i$. Together with $\overline{p}_j = 1 + \delta_j$ this then yields the inequality

$$2\pi_i - \delta_j + 2\pi_j 2\pi_i \leq 2\pi_j,$$

$$4\pi_i\pi_j \leq 2\pi_j - 2\pi_i + \delta_j. \quad (15)$$

We still need to rule out a negative jump at $\underline{p}_j + \delta_i = 2\pi_i + \delta_i$. Simplifying condition (8), which rules out such a jump, we therefore require

$$2\pi_j - 2\pi_i + \delta_j \leq 4\pi_i\pi_j. \quad (16)$$

Equations (15) and (16) together give a necessary condition for π_i and π_j , such that $2\pi_i > 1 - \delta_i$, which implies that $\overline{p_j} - \underline{p_j} < \delta_i + \delta_j$:

$$4\pi_i\pi_j = 2\pi_j - 2\pi_i + \delta_j, \quad (17)$$

Similar to case a), equation (17) implies that inequality (8) holds with equality and so a jump at $\underline{p_j} + \delta_i$ is ruled out.

Solving for profits. Condition (14) and (17) imply that we require

$$4\pi_i\pi_j = (2\pi_i - 2\pi_j + \delta_i)(2\pi_j + \delta_j) \quad \text{for } 2\pi_j \leq 1 - \delta_j, \quad (18)$$

$$4\pi_i\pi_j = (2\pi_i - 2\pi_j + \delta_i) \quad \text{for } 2\pi_j > 1 - \delta_j, \quad (19)$$

$$4\pi_i\pi_j = (2\pi_j - 2\pi_i + \delta_j)(2\pi_i + \delta_i) \quad \text{for } 2\pi_i \leq 1 - \delta_i \text{ and } \quad (20)$$

$$4\pi_i\pi_j = (2\pi_j - 2\pi_i + \delta_j) \quad \text{for } 2\pi_i > 1 - \delta_i. \quad (21)$$

First, solving (18) and (20) simultaneously for $2\pi_j \leq 1 - \delta_j$ and $2\pi_i \leq 1 - \delta_i$ yields

$$\begin{aligned} \pi_i &= \frac{2\delta_j + (\sqrt{5}-1)\delta_i}{2\delta_i + (\sqrt{5}-1)\delta_j} \\ \pi_j &= \frac{4}{4} \end{aligned} \quad \text{for } 0 \leq \delta_j \leq \min \left\{ \frac{(\sqrt{5}-1)(1-\delta_i)}{2}, \frac{2-(1+\sqrt{5})\delta_i}{2} \right\}$$

which implies that the cdfs F_i and F_j are continuous everywhere and have no flat segments.

Second, solving (19) and (20) for $2\pi_j > 1 - \delta_j$ and $2\pi_i \leq 1 - \delta_i$ yields

$$\begin{aligned} \pi_i &= \frac{\sqrt{(1+\delta_B)^2 + 4\delta_i} - (1-\delta_j)}{4} \\ \pi_j &= \frac{1+\delta_i + (1-\delta_i)(\delta_j - \sqrt{(1+\delta_j)^2 + 4\delta_i})}{4\delta_i} \end{aligned} \quad \text{for } \frac{(\sqrt{5}-1)(1-\delta_i)}{2} < \delta_j \leq \frac{(2-\delta_i)^2 - 2}{2-\delta_i}$$

which implies that the cdf F_i has no flat segment, but is discontinuous at $1 + \delta_i$ where it exhibits a jump of size $2\pi_j - 1 + \delta_j > 0$; the cdf F_j has a flat segment, but no discontinuity point.

Third, solving (18) and (21) simultaneously for $2\pi_j \leq 1 - \delta_j$ and $2\pi_i > 1 - \delta_i$ yields

$$\begin{aligned} \pi_i &= \frac{1+\delta_j + (1-\delta_j)(\delta_i - \sqrt{(1+\delta_i)^2 + 4\delta_j})}{4\delta_j} \\ \pi_j &= \frac{\sqrt{(1+\delta_A)^2 + 4\delta_j} - (1-\delta_i)}{4} \end{aligned} \quad \text{for } \frac{2-(1+\sqrt{5})\delta_i}{2} < \delta_j \leq \frac{4-\delta_i - \sqrt{\delta_i^2 + 8}}{2}$$

which implies that the cdf F_i has a flat segment and no discontinuity point, while the cdf F_j has no flat segment but is discontinuous at point $1 + \delta_j$ where it exhibits a jump of size $2\pi_i - 1 + \delta_i > 0$.

Finally, solving (19) and (21) simultaneously for $2\pi_j > 1 - \delta_j$ and $2\pi_i > 1 - \delta_i$ yields

$$\begin{aligned} \pi_i &= \frac{\delta_j - \delta_i + \sqrt{(\delta_i - \delta_j)^2 + 8(\delta_i + \delta_j)}}{8} \\ \pi_j &= \frac{\delta_i - \delta_j + \sqrt{(\delta_i - \delta_j)^2 + 8(\delta_i + \delta_j)}}{8} \end{aligned} \quad \text{for } \max \left\{ \frac{(2 - \delta_i)^2 - 2}{2 - \delta_i}, \frac{4 - \delta_i - \sqrt{\delta_i^2 + 8}}{2} \right\} < \delta_j < 1$$

which implies that the cdfs F_i and F_j are discontinuous at the points $1 + \delta_i$ and $1 + \delta_j$, where they exhibit a jump of size $2\pi_j - 1 + \delta_j > 0$ and $2\pi_i - 1 + \delta_i > 0$, respectively; moreover, both cdfs have a flat segment.

No wider range than $\delta_i + \delta_j$. To complete the proof of Lemma b) we have to show that there are no solutions with $\overline{p_i} - \underline{p_i} > \delta_i + \delta_j$ and/or $\overline{p_j} - \underline{p_j} > \delta_i + \delta_j$.

To establish a contradiction, suppose that $\overline{p_i} - \underline{p_i} \leq \delta_i + \delta_j$ and $\overline{p_j} - \underline{p_j} > \delta_i + \delta_j$. Then, both equations (4) and (5) define the cdf F_i for the range $\underline{p_j} + \delta_i \leq p_i < \overline{p_j} - \delta_j$. Since for F_i to be a cdf these definitions must coincide, we want

$$1 - \frac{2\pi_j}{p_i} = 1 - \frac{2\pi_j - p_i + \delta_i + \delta_j}{p_i - \delta_i} \quad \text{for } \underline{p_j} + \delta_i \leq p_i < \overline{p_j} - \delta_j$$

which is impossible.

Now suppose that $\overline{p_i} - \underline{p_i} > \delta_i + \delta_j$, and $\overline{p_j} - \underline{p_j} > \delta_i + \delta_j$. Then, for $\overline{p_i} - \delta_i \leq p_j < \overline{p_j}$ it applies that that $F_i(p_j + \delta_i) = 1$ and the difference equation (2) becomes $(1 - F_i(p_j - \delta_j))(p_j - \delta_j) = 2\pi_j$. From this we get

$$F_i(p_i) = 1 - \frac{2\pi_j}{p_i} \quad \text{for } \overline{p_i} - \delta_i - \delta_j \leq p_i < \overline{p_j} - \delta_j. \quad (22)$$

Similarly, for $\underline{p_j} \leq p_j < \underline{p_i} + \delta_j$ it applies that $F_i(p_j - \delta_j) = 0$ from which we get, using (2),

$$F_i(p_i) = 1 - \frac{2\pi_j - p_i + \delta_i + \delta_j}{p_i - \delta_i} \quad \text{for } \underline{p_j} + \delta_i \leq p_i < \underline{p_i} + \delta_i + \delta_j. \quad (23)$$

Both (22) and (23) define the cdf F_i in the range $\underline{p_j} + \delta_i \leq p_i < \overline{p_j} - \delta_j$. This is a contradiction that completes the proof of Lemma b). ■

Lemma c). Given $\delta_i \geq 1$ and $\delta_j < 1$, with $i, j \in \{A, B\}$ and $i \neq j$, there exists unique mixed pricing leading to expected equilibrium profits $\pi_i = (1 + \delta_j)/4$ and $\pi_j = 1/2$.

Proof. With no loss of generality, assume that $\delta_A \geq 1$ and $\delta_B < 1$. Then, the following cumulative distribution functions $F_A(\cdot)$ and $F_B(\cdot)$, when employed by firm A and B simultaneously, constitute a unique mixed-strategy Nash-solution:

$$F_A(p_A) = \begin{cases} 0 & \text{for } p_A < \frac{1+2\delta_A+\delta_B}{2} \\ 1 - \frac{2\pi_B - p_A + \delta_A + \delta_B}{p_A - \delta_A} & \frac{1+2\delta_A+\delta_B}{2} \leq p_A < 1 + \delta_i \\ 1 & 1 + \delta_A \leq p_A \end{cases}$$

$$F_B(p_B) = \begin{cases} 0 & \text{for } p_B < \frac{1+\delta_B}{2} \\ 1 - \frac{2\pi_A}{p_B} & \frac{1+\delta_B}{2} \leq p_B \leq 1 \\ 1 - 2\pi_A & 1 \leq p_B < 1 + \delta_B \\ 1 & 1 + \delta_B \leq p_B \end{cases}$$

$\pi_A = \frac{1+\delta_B}{4}$ denotes the expected profits for firm A ; and $\pi_B = \frac{1}{2}$ denotes the expected profits for firm B .

In analogy to the proof of Lemma b), suppose that firm A employs the cdf $F_A(\cdot)$ as its mixed strategy and firm B employs the cdf $F_B(\cdot)$. Let \underline{p}_A and \underline{p}_B be the lowest, and \overline{p}_A and \overline{p}_B the highest price in the support of F_A and F_B , respectively. To be a solution, F_A must satisfy the difference equation

$$(1 - F_A(p_B - \delta_B))(p_B - \delta_B) + (1 - F_A(p_B + \delta_A))p_B = 2\pi_B \quad (24)$$

at almost every point $p_A \in [\underline{p}_A, \overline{p}_A]$, such that F_A is continuous at $p_A = p_B - \delta_B$; similarly, F_B must satisfy the difference equation

$$(1 - F_B(p_A - \delta_A))(p_A - \delta_A) + (1 - F_B(p_A + \delta_B))p_A = 2\pi_A \quad (25)$$

at almost every point $p_B \in [\underline{p}_B, \overline{p}_B]$, such that F_B is continuous at $p_B = p_A - \delta_A$.

Bounding the supports. Given $\delta_A \geq 1$ and $\delta_B < 1$, firm A has no incentive to poach, but firm B is potentially interested in poaching. From

this follows that $\underline{p}_A = \underline{p}_B + \delta_A \geq \overline{p}_B - \delta_B$ and $\underline{p}_B < \overline{p}_A - \delta_A$. Together, this implies for the support of F_A and F_B

$$\begin{aligned} \overline{p}_B - \delta_B &\leq \underline{p}_A = \underline{p}_B + \delta_A < \overline{p}_A && \text{and} \\ \underline{p}_B &< \overline{p}_A - \delta_A \leq \overline{p}_B &\leq \underline{p}_A + \delta_B. \end{aligned}$$

Defining F_A . For $\underline{p}_B \leq p_B < \overline{p}_A - \delta_A$ it applies that $F_A(p_B - \delta_B) = 0$. Equation (24) simplifies to $p_B - \delta_B + (1 - F_A(p_B + \delta_A))p_B = 2\pi_B$, from which we get

$$\begin{aligned} F_A(p_B + \delta_A) &= 1 - \frac{2\pi_B - p_B + \delta_B}{p_B} && \text{for } \underline{p}_B \leq p_B < \overline{p}_A - \delta_A, \\ \text{or } F_A(p_A) &= 1 - \frac{2\pi_B - p_A + \delta_A + \delta_B}{p_A - \delta_A} && \text{for } \underline{p}_B + \delta_A \leq p_A < \overline{p}_A. \end{aligned}$$

This yields the cdf for firm A :

$$F_A(p_A) = \begin{cases} 0, & \text{for } p_A < \underline{p}_A = \underline{p}_B + \delta_A, \\ 1 - \frac{2\pi_B - p_A + \delta_A + \delta_B}{p_A - \delta_A}, & \underline{p}_B + \delta_A \leq p_A < \overline{p}_A, \\ 1, & \overline{p}_A \leq p_A. \end{cases} \quad (26)$$

There are two candidates for a discontinuity: $\underline{p}_A = \underline{p}_B + \delta_A$, and \overline{p}_A . To share the properties of a cdf, F_A must be continuous from the right (which is already satisfied) and non-decreasing in p_A . This demands

$$0 \leq 1 - \frac{2\pi_B - \underline{p}_B + \delta_B}{\underline{p}_B} \text{ to rule out a negative jump at } \underline{p}_B + \delta_A, \quad (27)$$

$$\text{and } 1 - \frac{2\pi_B - \overline{p}_A + \delta_A + \delta_B}{\overline{p}_A - \delta_A} \leq 1 \text{ to rule out a negative jump at } \overline{p}_A. \quad (28)$$

Defining F_B . For $\underline{p}_B + \delta_A \leq p_A \leq \overline{p}_A$ it applies that $F_B(p_A + \delta_B) = 1$. Equation (25) simplifies to $(1 - F_B(p_A - \delta_A))(p_A - \delta_A) = 2\pi_A$ from which we get

$$\begin{aligned} F_B(p_A - \delta_A) &= 1 - \frac{2\pi_A}{p_A - \delta_A} && \text{for } \underline{p}_B + \delta_A \leq p_A \leq \overline{p}_A, \\ \text{or } F_B(p_B) &= 1 - \frac{2\pi_A}{p_B} && \text{for } \underline{p}_B \leq p_B \leq \overline{p}_A - \delta_A. \end{aligned}$$

Moreover, note for $\overline{p}_A - \delta_A \leq p_B < 1 + \delta_B$, F_B has to be flat because in this range firm B sells only to repeat customers and so can unilaterally improve by increasing its price to $\tilde{p}_B = 1 + \delta_B$. From this follows that $\overline{p}_B = 1 + \delta_B$.

Firm B 's cdf is given as follows.

$$F_B(p_B) = \begin{cases} 0, & \text{for } p_B < \underline{p}_B, \\ 1 - \frac{2\pi_A}{p_B}, & \underline{p}_B \leq p_B \leq \overline{p}_A - \delta_A, \\ 1 - \frac{2\pi_A}{\overline{p}_A - \delta_A}, & \overline{p}_A - \delta_A \leq p_B < 1 + \delta_B, \\ 1, & 1 + \delta_B \leq p_B. \end{cases} \quad (29)$$

There are two candidates for a discontinuity: \underline{p}_B , and $\overline{p}_B = 1 + \delta_B$. To share the properties of a cdf, F_B must be continuous from the right (which is already satisfied) and non-decreasing in p_B . This demands

$$0 \leq x = 1 - \frac{2\pi_A}{\underline{p}_B} \text{ to rule out a negative jump at } \underline{p}_B, \quad (30)$$

$$\text{and } 1 - \frac{2\pi_A}{\overline{p}_A - \delta_A} \leq 1 \text{ to rule out a negative jump at } 1 + \delta_B. \quad (31)$$

No jump of F_B at the point \underline{p}_B . By charging the price $\tilde{p}_A = \underline{p}_B + \delta_A$, firm A only attracts repeat customers and earns $\tilde{\pi}_A = \frac{p_B}{2} \leq \pi_A$. Together with $2\pi_A = (1-x)\underline{p}_B$, from (30), this yields $\underline{p}_B \leq (1-x)\underline{p}_B$ which has the unique solution $x = 0$. Hence, a jump at \underline{p}_B is ruled out and we get

$$\underline{p}_B = 2\pi_A.$$

No prices outside the supports. Obviously, the firms must not have an incentive to charge a price outside of their support. We therefore require

$$(1 - F_A(\tilde{p}_B - \delta_B))(\tilde{p}_B - \delta_B) + (1 - F_A(\tilde{p}_B + \delta_A))\tilde{p}_B \leq 2\pi_B,$$

for all $\tilde{p}_B < \underline{p}_B$ and $\tilde{p}_B > \overline{p}_B$, and

$$(1 - F_B(\tilde{p}_A - \delta_A))(\tilde{p}_A - \delta_A) + (1 - F_B(\tilde{p}_A + \delta_B))\tilde{p}_A \leq 2\pi_A,$$

for all $\tilde{p}_A < \underline{p}_A$ and $\tilde{p}_A > \overline{p}_A$.

First note that firm B will not charge a price $\tilde{p}_B > \overline{p}_B$ because it is already satisfied that $\overline{p}_B = 1 + \delta_B$. Using (29) we know that firm A will not charge a price $\tilde{p}_A > \overline{p}_A$ if $\frac{2\pi_A}{\overline{p}_A - \delta_A}(\tilde{p}_A - \delta_A) \leq 2\pi_A$. Since this inequality has no solution for $\tilde{p}_A > \overline{p}_A$ it must be the case that $\overline{p}_A = 1 + \delta_A$.

To preclude firm B from charging a price $\tilde{p}_B < \underline{p}_B$ we then demand, using (26), that $2\tilde{p}_B - \delta_B \leq 2\pi_B$. This condition is satisfied for all $\tilde{p}_B < \underline{p}_B$ if it holds for $\tilde{p}_B = \underline{p}_B$. Hence we require

$$2\underline{p}_B - \delta_B \leq 2\pi_B. \quad (32)$$

Similarly, to preclude firm A from charging a price $\tilde{p}_A < \underline{p}_A$, we demand, using (29), that $\tilde{p}_A - \delta_A \leq 2\pi_A$. This condition is satisfied for all $\tilde{p}_A < \underline{p}_A$ if it holds for $\tilde{p}_A = \underline{p}_A$. Hence we require $\underline{p}_A \leq 2\pi_A + \delta_A$ which is already satisfied since $\underline{p}_A = \underline{p}_B + \delta_A$ and $\underline{p}_B = 2\pi_A$. Moreover, F_A must not exhibit a negative jump at $\underline{p}_A = \underline{p}_B + \delta_A$. Using (27) we therefore require that $F_A(\underline{p}_B + \delta_A) \geq 0$, which yields

$$2\underline{p}_B - \delta_B \geq 2\pi_B. \quad (33)$$

Together with equation (32), this implies $\underline{p}_B = 2\pi_B + \frac{\delta_B}{2}$ which rules out both a jump at $\underline{p}_B + \delta_A$ and deviations to $\tilde{p}_B < \underline{p}_B$.

Finally, we can evaluate equation (25) at $p_B = 1 + \delta_B$ which yields $\pi_B = 1/2$. From this follows that $\underline{p}_B = \frac{1+\delta_B}{2}$, $\underline{p}_A = \frac{1+2\delta_A+\delta_B}{2}$, and $\pi_A = \frac{1+\delta_B}{4}$. This completes the proof of Lemma c). ■

Equilibrium outcome after fifty-fifty allocation. Lemmas a) to c) characterize the equilibrium outcome of any subgame that starts after the *fifty-fifty-allocation*. The expected profits π_A and π_B are summarized in Table 1. It is straightforward to show that π_A is nondecreasing in δ_A and π_B nondecreasing δ_B . Figure 3 depicts the different regions for the different classes of Nash-equilibria.

Figure 3: Regions for different classes of Nash-equilibria

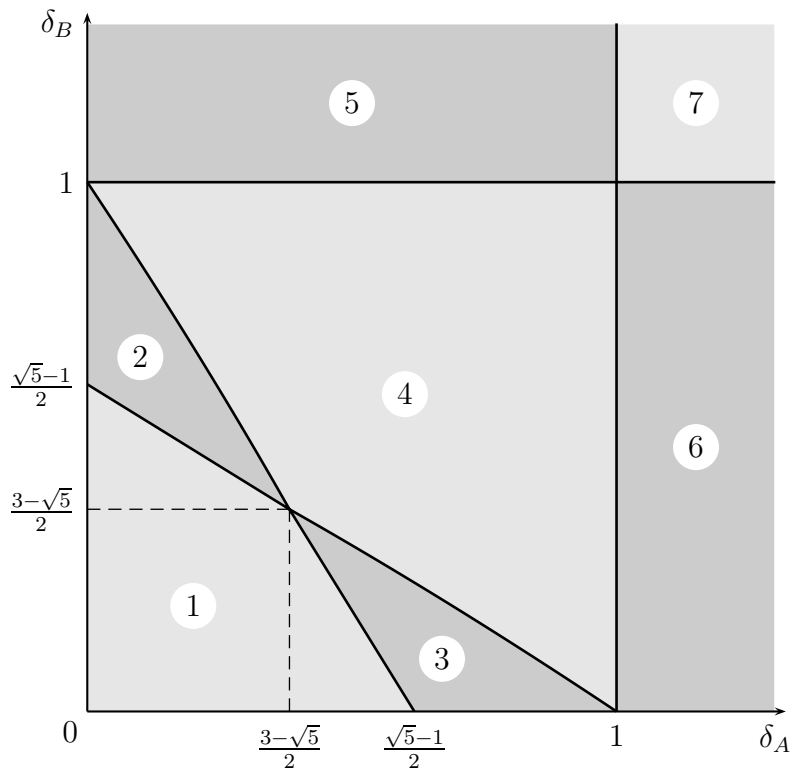


Table 1: Expected profits after fifty-fifty allocation, with different loyalty discounts

Region	π_A	π_B	
1	$\frac{2\delta_B + (\sqrt{5}-1)\delta_A}{4}$	$\frac{2\delta_A + (\sqrt{5}-1)\delta_B}{4}$	for $0 \leq \delta_B \leq \min \left\{ \frac{(\sqrt{5}-1)(1-\delta_A)}{2}, \frac{2-(1+\sqrt{5})\delta_A}{2} \right\}$
2	$\frac{\sqrt{(1+\delta_B)^2 + 4\delta_A} - (1-\delta_B)}{4}$	$\frac{1+\delta_A + (1-\delta_A)(\delta_B - \sqrt{(1+\delta_B)^2 + 4\delta_A})}{4\delta_A}$	$\frac{(\sqrt{5}-1)(1-\delta_A)}{2} < \delta_B < \frac{(2-\delta_A)^2 - 2}{2-\delta_A}$
3	$\frac{1+\delta_B + (1-\delta_B)(\delta_A - \sqrt{(1+\delta_A)^2 + 4\delta_B})}{4\delta_B}$	$\frac{\sqrt{(1+\delta_A)^2 + 4\delta_B} - (1-\delta_A)}{4}$	$\frac{2-(1+\sqrt{5})\delta_A}{2} < \delta_B < \frac{4-\delta_A - \sqrt{\delta_A^2 + 8}}{2}$
4	$\frac{\delta_B - \delta_A + \sqrt{(\delta_A - \delta_B)^2 + 8(\delta_A + \delta_B)}}{8}$	$\frac{\delta_A - \delta_B + \sqrt{(\delta_A - \delta_B)^2 + 8(\delta_A + \delta_B)}}{8}$	$\max \left\{ \frac{(2-\delta_A)^2 - 2}{2-\delta_A}, \frac{4-\delta_A - \sqrt{\delta_A^2 + 8}}{2} \right\} \leq \delta_B < 1$ and $\delta_A < 1$
5	$\frac{1}{2}$	$\frac{1+\delta_A}{4}$	$\delta_A < 1$ and $\delta_B \geq 1$
6	$\frac{1+\delta_B}{4}$	$\frac{1}{2}$	$\delta_A \geq 1$ and $\delta_B < 1$
7	$\frac{1}{2}$	$\frac{1}{2}$	$\delta_A \geq 1$ and $\delta_B \geq 1$

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