Math. Proc. Camb. Phil. Soc. (2020), **169**, 141–148 © *Cambridge Philosophical Society* 2019
doi:10.1017/S0305004119000136
First published online 10 April 2019

Symmetric quotients of knot groups and a filtration of the Gordian graph

BY SEBASTIAN BAADER
Mathematisches Institut, Universität Bern,
Sidlerstrasse 5, CH-3012 Bern, Switzerland.
e-mail: sebastian.baader@math.unibe.ch

ALEXANDRA KJUCHUKOVA
Mathematics Department, University of Wisconsin–Madison,
480 Lincoln Dr, Madison, WI 53703, U.S.A.
e-mail: kjuchukova@math.wise.edu

(*Received* 04 *December* 2017; *revised* 25 *February* 2019)

Abstract

We define a metric filtration of the Gordian graph by an infinite family of 1-dense subgraphs. The nth subgraph of this family is generated by all knots whose fundamental groups surject to a symmetric group with parameter at least n, where all meridians are mapped to transpositions. Incidentally, we verify the Meridional Rank Conjecture for a family of knots with unknotting number one yet arbitrarily high bridge number.

2010 Mathematics Subject Classification: 57M25

1. *Introduction*

The Gordian graph G is a countable graph whose vertices correspond to smooth knot types and whose edges correspond to pairs of knots related by a crossing change in a suitable diagram. The set of vertices of the Gordian graph carries a natural metric induced by the minimal path length between vertices. This metric, called the Gordian metric, generalises the classical unknotting number, which is defined as the minimal number of crossing changes needed to transform a knot K into the trivial knot $\bigcirc$, that is, $u(K) = d_G(K, \bigcirc)$ [9]. The Gordian graph G is locally infinite: for any given vertex of G, i.e. for any knot type, we obtain an infinite number of neighbouring vertices by taking the connected sum of that knot with all knots whose unknotting number is one. Furthermore, every vertex of G is contained in an arbitrarily large complete subgraph of G [6]. In contrast, little is known about the global structure of the Gordian graph, except for the fact that it contains lattices of arbitrarily high rank [5]. The purpose of this note is to construct an infinite descending sequence of subgraphs $G_n \subset G$ that captures the global geometry of the Gordian graph, by looking at finite symmetric quotients of the fundamental group of knot exteriors.

Fix a natural number $n \geq 2$. We say that a knot K is $\binom{n}{2}$-colorable, if there exists a surjective homomorphism from the knot group $\pi_1(\mathbb{R}^3 \setminus K)$ onto the symmetric group S_n, mapping meridians to transpositions. Let $G_n \subset G$ to be the induced subgraph whose vertices

Downloaded from https://www.cambridge.org/core. Universitätsbibliothek Bern, on 13 Jan 2021 at 09:49:51, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0305004119000136

correspond to knot types K which are $\binom{m}{2}$-colorable, for some $m \geq n$. By definition, the sequence of subgraphs G_n forms a descending chain, in fact strictly descending for $n \geq 2$, as we will see. A chain of graphs $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \cdots$ forms a 1-dense metric filtration of the graph Γ, if for all $n \in \mathbb{N}$ the following three statements hold:

(i) every vertex of Γ is connected to a vertex of Γ_n by an edge;
(ii) the maps $\Gamma_n \hookrightarrow \Gamma$ are isometric inclusions;
(iii) $\bigcap_{m \in \mathbb{N}} \Gamma_m = \emptyset$.

The existence of a 1-dense metric filtration for a graph Γ implies that every pair of non-neighbouring vertices is connected by infinitely many different shortest paths. In particular, Γ cannot be disconnected by removing finitely many vertices. The Gordian graph G was already known to have these features [1].

THEOREM 1. *The chain of subgraphs* $G = G_1 \supset G_2 \supset G_3 \cdots$ *defined above forms a* 1-*dense metric filtration of the Gordian graph* G.

Thanks to properties (i) and (ii) of a 1-dense metric filtration, determining the Gordian distance on G is equivalent to determining its restriction to any of the subgraphs G_n, up to an error of two. Distance in G_n can be studied via irregular branched covers of the 3-sphere along knots. In the first interesting case, $n = 3$, these covers have been used to define a very effective knot invariant, as follows. To a surjective homomorphism $\varphi : \pi_1(\mathbb{R}^3 \setminus K) \twoheadrightarrow S_3$ corresponds a three-fold irregular covering space M of S^3 branched along K. When the two connected lifts of K to M represent torsion elements in homology, their linking number is a well-defined rational number, determined by φ. The set $lk(K) \subset \mathbb{Q}$ of linking numbers associated to all surjective homomorphisms of $\pi_1(\mathbb{R}^3 \setminus K)$ onto the symmetric group S_3 is called the linking number invariant of K [11, 13]. We hope to obtain lower bounds on the Gordian distance of knots by estimating the effect of crossing changes on $lk(K)$. An outline of this method is contained in the final section of this paper.

2. *Symmetric quotients of knot groups*

The fundamental group of a knot complement admits a finite presentation via generators and relations, namely the Wirtinger presentation obtained from a knot diagram [3]. This allows for a simple algorithmic decision of whether a given finite group H is a quotient of $\pi_1(\mathbb{R}^3 \setminus K)$ or not. Consider the case $H = S_3$. In order to define a surjection onto S_3, the meridians of a knot need to be mapped to the three transpositions (12), (13), (23), commonly referred to as colours. The group relations thus translate into the famous Fox 3-colouring conditions at the crossings of a diagram [4]. More generally, surjections onto dihedral groups D_p correspond to Fox p-colourings of diagrams respecting similar rules.

The natural isomorphism $S_3 \simeq D_3$ gives rise to an alternative generalisation of Fox 3-colorings, namely $\binom{n}{2}$-colorings, introduced earlier. Such a coloring encodes an $\binom{n}{2}$-representation of the knot group, that is, a surjective homomorphism

$$\pi_1(\mathbb{R}^3 \setminus K) \longrightarrow\!\!\!\!\!\rightarrow S_n,$$

Downloaded from https://www.cambridge.org/core. Universitätsbibliothek Bern, on 13 Jan 2021 at 09:49:51, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0305004119000136

with the additional assumption that meridians be mapped to transpositions. We define the permutation number of a knot K as follows:

$$p(K) = \max\{n \in \mathbb{N} \mid \pi_1(\mathbb{R}^3 \setminus K) \text{ admits an } \binom{n}{2}\text{-representation}\}.$$

We observe that the trefoil knot 3_1 admits a Fox 3-colouring, in other words a $\binom{3}{2}$-representation, but no higher order $\binom{n}{2}$-representation, since the group $\pi_1(\mathbb{R}^3 \setminus 3_1)$ is generated by two meridians, whereas S_n is not generated by two transpositions for $n \geq 4$. Therefore, $p(3_1) = 3$. Similarly, $p(4_1) = 2$ for the figure-eight knot 4_1, since it does not admit a non-trivial 3-colouring, and its fundamental group is generated by two meridians. These simple observations extend to obtain a general upper bound on the permutation number of knots, from the minimal number of meridians needed to generate the knot group. Let $b(K)$ be the minimal bridge number of a knot K, defined as the minimal number of local maxima of the height function among all representatives of $K \subset \mathbb{R}^3$. The fundamental group of a knot K is generated by $b(K)$ meridians. Now the symmetric group S_n cannot be generated by fewer than $n-1$ transpositions. Indeed, in order to act transitively on n numbers, one needs at least $n-1$ transpositions. Therefore, we obtain the following upper bound on the permutation number $p(K)$:

$$p(K) \leq b(K) + 1.$$

Going back to the definition of the chain of subgraphs G_n of the Gordian graph, we conclude that $K \notin G_n$ for $n > b(K) + 1$, which implies property (3) of a metric filtration:

$$\bigcap_{n \in \mathbb{N}} G_n = \emptyset.$$

Our next goal is to construct knots with arbitrarily high permutation number yet unknotting number one. As a consequence, we will prove property (i) of a metric filtration, i.e. the 1-density of all subgraphs $G_n \subset G$. Fix a natural number n and let K_n be a knot with $p(K_n) \geq n$ and $u(K_n) = 1$. Then the connected sum of knots $K \# K_n$ of any knot K with K_n is contained in G_n, since we can extend the existing homomorphism $\pi_1(\mathbb{R}^3 \setminus K_n) \twoheadrightarrow S_n$ to $\pi_1(\mathbb{R}^3 \setminus K \# K_n)$ by mapping all the meridians of K to the transposition associated with the meridian of K_n to which K is attached. Moreover, the knot $K \# K_n$ is related to K by a single crossing change, since the knot K_n has unknotting number one. In other words, every knot (vertex) in G is connected by an edge to a knot (vertex) in G_n. We are left to construct, for each $n \geq 3$, a knot K_n with $p(K_n) \geq n$ and $u(K_n) = 1$.

We first construct a family of knots with increasing permutation number. In fact, the n-times iterated connected sum of trefoil knots $3_1^n = 3_1 \# \cdots \# 3_1$ will do, as $p(3_1^n) = n + 2$. This can be seen by representing the knot 3_1^n as the closure of the braid $\sigma_1^3 \sigma_2^3 \cdots \sigma_{n-1}^3$ in the braid group B_n. Mapping the meridians around the n bottom strands of that braid to the transpositions $(12), (13), \ldots, (1n)$, in this order, extends to a surjective homomorphism $\pi_1(\mathbb{R}^3 \setminus 3_1^n) \twoheadrightarrow S_{n+2}$. The case $n = 2$ is shown in Figure 1. This shows that $p(3_1^n) \geq n + 2$. On the other hand, Schubert's additivity result for the bridge number [15] implies $b(3_1^n) = n + 1$, so by the previous discussion we have $p(3_1^n) \leq n + 2$. In particular, $3_1^n \in G_{n+2} \setminus G_{n+3}$, establishing $G_n \supsetneq G_{n+1}$ for $n \geq 3$.

Downloaded from https://www.cambridge.org/core. Universitätsbibliothek Bern, on 13 Jan 2021 at 09:49:51, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0305004119000136

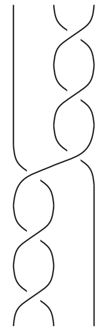

Fig. 1. A $\frac{4}{2}$-colouring of the knot $3_1\#3_1$.

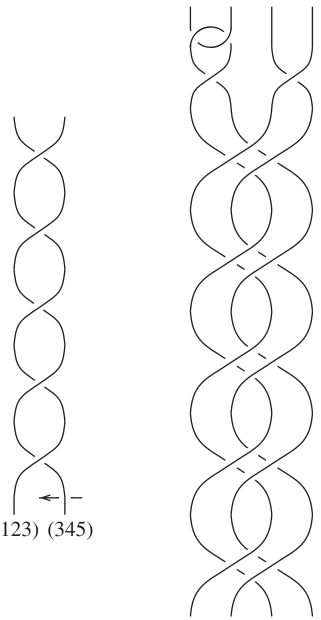

Fig. 2. Colourings of $T(2, 5)$ and its Whitehead double.

In order to construct knots with unknotting number one and $\binom{n}{2}$-representations for n large, we take suitable Whitehead doubles of iterated connected sums of the torus knot with 5 crossings. Let $T(2, 5)$ be the torus knot represented as the closure of the braid $\sigma_1^5 \in B_2$. Mapping the meridians of the two bottom strands of this braid to the two 3-cycles $(123), (345) \in A_5$ extends to a unique homomorphism $\pi_1(\mathbb{R}^3 \setminus T(2, 5)) \to A_5$. Here the orientation of the meridians matters; we choose the convention that meridians cross under braid strands from the right to the left, as indicated by the arrow in Figure 2. Now let K_1 be the twisted Whitehead double of $T(2, 5)$, defined as the closure of the braid

$$\beta_1 = (\sigma_2\sigma_1\sigma_3\sigma_2)^5\sigma_3\sigma_1 \in B_4,$$

with an additional clasp, as depicted on the right of Figure 2. The same figure exhibits a homomorphism $\pi_1(\mathbb{R}^3 \setminus K_1) \twoheadrightarrow S_5$, which maps the meridians of the bottom four strands

Downloaded from https://www.cambridge.org/core. Universitätsbibliothek Bern, on 13 Jan 2021 at 09:49:51, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0305004119000136

of β_1 to the transpositions (12),(23),(34),(45). This doubling construction works since each 3-cycle can be written as a product of two transpositions.

The above argument generalises to provide a homomorphism

$$\pi_1(\mathbb{R}^3 \setminus T(2,5)^m) \longrightarrow A_{2m+3}$$

mapping the meridian about the kth bottom strand of the braid $\sigma_1^5 \ldots \sigma_m^5 \in B_{m+1}$ to the cycle $(2k-1\ 2k\ 2k+1)$. We define K_m to be the Whitehead double of the knot $T(2,5)^m$, represented as the closure of the following braid in B_{2m+2}:

$$\beta_m = (\sigma_2\sigma_1\sigma_3\sigma_2)^5\sigma_3(\sigma_4\sigma_3\sigma_5\sigma_4)^5\sigma_5 \ldots (\sigma_{2m}\sigma_{2m-1}\sigma_{2m+1}\sigma_{2m})^5\sigma_{2m+1}\sigma_1,$$

again with an additional clasp on top of the first two strands. By construction, the sequence of knots K_m have unknotting number one and are contained in $G_n \subset G$, for all $n \leq 2m+3$, since they admit an $\binom{2m+3}{2}$-colouring. This concludes the proof of the 1-density for all subgraphs $G_n \subset G$.

Remark. The inequality $p(K) \leq b(K) + 1$ between the permutation number and the bridge number of a knot implies that the bridge number of the knot K_m defined above is at least $2m+2$. This bound is in fact sharp, since these knots are represented as closures of braids with $2m+2$ strands, i.e. by diagrams with precisely $2m+2$ local maxima. Moreover, we note that the permutation number provides the same lower bound for the meridional rank $\mu(K)$ of a knot, defined as the minimal number of meridians needed to generate the knot group:

$$p(K) \leq \mu(K) + 1.$$

Since $\mu(K) \leq b(K)$, this implies equality between the bridge number and the meridional rank of the knots K_m, and settles the Meridional Rank Conjecture for this family of knots (see Kirby's list [**8**, Problem 1.11]). The latter conclusion could not have been drawn by using an analogous argument with Fox p-colourings. The mere existence of a non-trivial p-colouring does not provide an effective lower bound for the bridge number, due to the fact that every dihedral group is generated by two reflections. Moreover, counting p-colourings does not help either, since Whitehead doubles do not admit multiple independent non-trivial p-colourings. Indeed, any p-colouring of a Whitehead double is completely determined by choosing two colours near the clasp.

3. *Constructing coloured shortest paths*

In this section we prove that the inclusion maps $G_n \hookrightarrow G$ are isometric inclusions, for all $n \in \mathbb{N}$. In other words, the intrinsic path metric on G_n coincides with the path metric on the ambient space G. The strategy of proof is as follows: let $A, B \in G_n$ be two knots which admit $\binom{a}{2}$- and $\binom{b}{2}$-colourings $(a, b \geq n)$, respectively, and let $d_G(A, B) = m \in \mathbb{N}$ be the Gordian distance between them. Then there exists a sequence of knots $K_0, K_1, \ldots, K_m$ successively related by crossing changes, i.e. $d_G(K_i, K_{i+1}) = 1$ for all $i \leq m-1$, with $K_0 = A$, $K_m = B$. We will construct a knot $\widetilde{K}_1 \in G_n$ with $d_G(K_0, \widetilde{K}_1) = d_G(\widetilde{K}_1, K_2) = 1$. By repeating this argument inductively, we will end up with a path from A to B in G_n of the same length as the original one.

In order to construct the knot $\widetilde{K}_1$, we will make use of the fact that the two crossing changes relating K_1 to each of K_0 and K_2 can be realised in the same diagram of K_1. This is based on a sliding disc argument (see [**14**, proposition 1·5]). Here is an outline of the

Downloaded from https://www.cambridge.org/core. Universitätsbibliothek Bern, on 13 Jan 2021 at 09:49:51, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0305004119000136

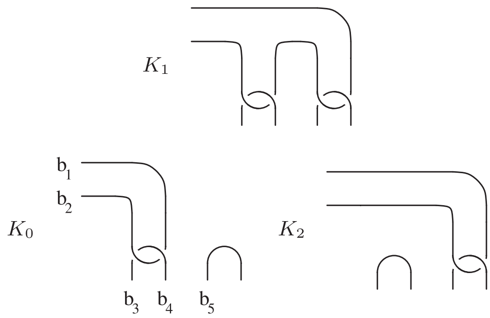

Fig. 3. Diagrams of the knots K_1, K_0, K_2.

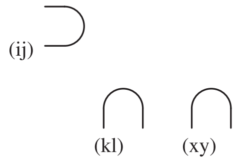

Fig. 4. $\widetilde{K}_1$ in case 1.

argument: a crossing change between two strands can be represented by a framed chord with endpoints on the knot. The endpoints of these chords can be moved along the knot, and moreover they can be contracted to small segments. We may therefore assume that K_1 has a diagram with a section as depicted in Figure 3, containing two neighbouring clasps, a crossing change at which transforms K_1 into K_0 and K_2, respectively.

Our assumption on $A = K_0$ tells us that the diagram of K_0 admits a colouring by transpositions that generate the symmetric group S_a. The section of the diagram on the bottom left of Figure 3 has five connected arcs b_1, b_2, b_3, b_4, b_5, whose meridians are sent to five transpositions, some of which may coincide. Depending on these transpositions, we will define a new knot $\widetilde{K}_1$, which admits an $\binom{a}{2}$-representation, as well. We distinguish two cases:

(1) The bridges b_1, b_2 are sent to the same transposition (ij). Then the Wirtinger relations at the clasp imply that b_3, b_4 are also sent to one transposition (kl), which commutes with (ij). In this case, we define the knot $\widetilde{K}_1$ as in Figure 4, where it is assumed that $\widetilde{K}_1$ coincides with K_1 outside the depicted region. Observe that the $\binom{a}{2}$-colouring of K_0 carries over to an $\binom{a}{2}$-colouring of $\widetilde{K}_1$. Therefore, $\widetilde{K}_1$ is in G_a, in turn in G_n, since $a \geq n$. Moreover, this new knot is still related to each of K_0 and K_2 by a single crossing change.

(2) The arcs b_1, b_2 are sent to different transpositions, whose support overlaps in one number, (jk) and (ij), since b_1 and b_2 meet at a crossing. Then the Wirtinger relations at the clasp imply that b_3, b_4 are sent to (jk) and (ik). We distinguish three subcases, depending on the transposition (xy) associated with the arc b_5.

(a) The transposition (xy) commutes with (jk), that is, $(xy) = (jk)$ or $\{x, y\} \cap \{j, k\} = \emptyset$. In this case, the $\binom{a}{2}$-colouring of K_0 carries over to an $\binom{a}{2}$-colouring of K_1, so we keep $\widetilde{K}_1 = K_1$.

(b) The transposition (xy) commutes with (ij), that is, $(xy) = (ij)$ or $\{x, y\} \cap \{i, j\} = \emptyset$. We define the knot $\widetilde{K}_1$ as in Figure 5. By construction, the $\binom{a}{2}$-colouring of K_0 carries over to an $\binom{a}{2}$-colouring of $\widetilde{K}_1$. Moreover, $\widetilde{K}_1$ is related to each of K_0 and K_2 by a single crossing change.

Downloaded from https://www.cambridge.org/core. Universitätsbibliothek Bern, on 13 Jan 2021 at 09:49:51, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0305004119000136

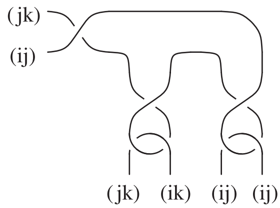

Fig. 5. $\widetilde{K}_1$ in case 2b, with $(xy) = (ij)$.

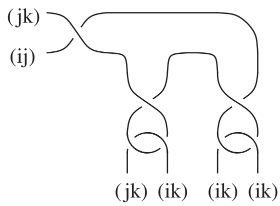

Fig. 6. $\widetilde{K}_1$ in case 2c, with $(xy) = (ik)$.

(c) The transposition (xy) commutes with (ik), that is, $(xy) = (ik)$ or $\{x, y\} \cap \{i, k\} = \emptyset$. We define the knot $\widetilde{K}_1$ as in Figure 6. As before, the $\binom{a}{2}$-colouring of K_0 carries over to an $\binom{a}{2}$-colouring of $\widetilde{K}_1$ and $\widetilde{K}_1$ is related to each of K_0 and K_2 by a single crossing change.

Note that the above cases exhaust all possibilities, with some redundancy. For example, (xy) with $\{x, y\} \cap \{i, j, k\} = \emptyset$ is covered by 2(a), (b), (c). Where there is overlap, one can choose which construction to use. This concludes the proof of Theorem 1.

4. *Speculation on the Gordian distance*

In this final section we restrict our attention to 3-colourable knots. We say a knot is twice three-colourable if it admits two independent non-trivial 3-colourings, i.e. 3-colourings that are not related by a permutation of the three transpositions (12), (13), (23). Examples of twice 3-colourable knots are the square knot $3_1\#\overline{3}_1$, the granny knots $3_1\#3_1$ and $\overline{3}_1\#\overline{3}_1$, and the pretzel knot $P(3, 3, 3)$. A well-known feature of the 3-colourings of a knot is that they from an $\mathbb{F}_3$-vector space, after any bijective identification of the three transpositions (12), (13), (23) with the numbers $0, 1, 2 \in \mathbb{F}_3$. This means that the colouring conditions at crossings are preserved under addition, a fact that is easily checked. Now suppose that a knot K has two independent non-trivial 3-colourings C_1, C_2. Then the sum $C_1 + C_2$ and the difference $C_1 - C_2$ are also non-trivial 3-colourings of K. Moreover, given any crossing X in a diagram of K, one of the four 3-colourings $C_1, C_2, C_1 + C_2, C_1 - C_2$ will have two coinciding colours at X, hence it will be monochromatic at X. As a consequence, every knot K_2 that is related to a twice 3-colourable knot K_1 by a single crossing change will be related to K_1 by a monochromatic crossing change with respect to suitable non-trivial 3-colourings of K_1, K_2. This fact could possibly obstruct the existence of a crossing change between (twice) 3-colourable knots. A concrete example we have in mind is the pair of knots 8_{21} and $3_1\#\overline{3}_1$, which we suspect to have Gordian distance two. This is a hard case, since the knot 8_{21} is related to the trivial knot by a single crossing change, and the square knot is indistinguishable from the trivial knot by the signature or other well-known concordance knot invariants like s and τ, because it is a ribbon knot.

Downloaded from https://www.cambridge.org/core. Universitätsbibliothek Bern, on 13 Jan 2021 at 09:49:51, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0305004119000136

We propose to study Gordian distance by investigating the effect of a monochromatic crossing change between K_1 and K_2 on the 3-fold irregular branched coverings associated with the representations $\pi_1(\mathbb{R}^3 \setminus K_{1,2}) \twoheadrightarrow S^3$. We refer the reader to [10] for a precise definition of these branched covering spaces. Whenever the branch curve lifts to a curve of finite order in homology, the linking number of the pair of lifts of the branch curve is a well-defined rational number. The set of linking numbers obtained by considering all non-trivial 3-colourings of a knot K forms the knot invariant $\mathrm{lk}(K) \subset \mathbb{Q}$ mentioned in the introduction. Using the algorithm implemented in [2], we computed a list of linking numbers for the knots 3_1, 8_{20}, 8_{21} and their mirror images, as well as for the twice 3-colourable granny and square knots:

$$\mathrm{lk}(3_1) = \{2\},\ \mathrm{lk}(\overline{3}_1) = \{-2\},\ \mathrm{lk}(8_{20}) = \mathrm{lk}(\overline{8}_{20}) = \{0\},$$

$$\mathrm{lk}(8_{21}) = \{4\},\ \mathrm{lk}(\overline{8}_{21}) = \{-4\},$$

$$\mathrm{lk}(3_1\#3_1) = \{2, 4\},\ \mathrm{lk}(\overline{3}_1\#\overline{3}_1) = \{-4, -2\},\ \mathrm{lk}(3_1\#\overline{3}_1) = \{-2, 0, 2\}.$$

We found crossing changes relating knots from the first two lines to knots from the third line in all the cases that have a linking number in common. This raises the question whether monochromatic crossing changes do not affect linking numbers in branched coverings. Perko informed us that this is true for special types of monochromatic crossing changes, but not in general [12]. We believe a refined analysis of the effect of monochromatic crossing changes on the corresponding 3-fold irregular branched coverings can be used to rule out single crossing changes between twice 3-colourable knots.

REFERENCES

[1] S. BAADER. Note on crossing changes. *Quart. J. Math.* **57** (2006), 139–142.
[2] P. CAHN and A. KJUCHUKOVA. Linking numbers in three-manifolds. arXiv: 1611.10330.
[3] R. H. CROWELL and R. H. FOX. *Introduction to Knot Theory*, reprint of the 1963 original (Springer-Verlag, 1977).
[4] R. H. FOX. Metacyclic invariants of knots and links. *Canad. J. of Math.* **22** (1970), 193–201.
[5] J. M. GAMBAUDO and E. GHYS. Braids and signatures. *Bull. Soc. Math. France* **133** (2005), no. 4, 541-579.
[6] M. HIRASAWA and Y. UCHIDA. The Gordian complex of knots. *J. Knot Theory Ramifications* **11** (2002), no. 3, 36–368.
[7] L. H. KAUFFMAN. Knots and physics, 3rd edition Series on Knots and Everything 1 (World Scientific Publishing Co., River Edge, NJ 2001).
[8] R. KIRBY (editor). *Problems in low-dimensional topology* in Proceedings of Georgia Topology Conference, Part 2 (Citeseer, 1995).
[9] H. MURAKAMI. Some metrics on classical knots. *Math. Ann.* **270** (1985), 35–45.
[10] M. PERKO. An invariant of certain knots, Undergraduate Thesis (1964).
[11] M. PERKO. On the classification of knots, *Proc. Amer. Math. Soc* **45** (1974), 262–266.
[12] M. PERKO. Visualising linking numbers in 3-coloured knot covers. Private correspondence (2018).
[13] K. REIDEMEISTER. Knoten und Verkettungen. *Mathematische Zeitschrift* **29** (1929), no. 1, 713–729.
[14] M. SCHARLEMANN. Crossing changes, Knot theory and its applications. *Chaos Solitons Fractals* **9** (1998), no. 4–5, 693–704.
[15] H. SCHUBERT. Über eine numerische Knoteninvariante. *Math. Z.* **61** (1954), 245–288.

Downloaded from https://www.cambridge.org/core. Universitätsbibliothek Bern, on 13 Jan 2021 at 09:49:51, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0305004119000136