The dimension of an amoeba

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Abstract

Answering a question by Nisse and Sottile, we derive a formula for the dimension of the amoeba of an irreducible algebraic variety.

1. Introduction and main result

Let $X \subseteq (\mathbb{C}^*)^n$ be an irreducible, closed algebraic subvariety. We define

$$\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$$

and $\mathcal{A}(X) := \mathrm{Log}(X)$, the *amoeba* of X. The amoeba is the image of the semi-algebraic set (*algebraic amoeba*)

$$|X| := \{(|z_1|, \ldots, |z_n|) \mid (z_1, \ldots, z_n) \in X\} \subseteq \mathbb{R}^n_{>0},$$

under a diffeomorphism and thus has an obvious notion of dimension, denoted $\dim_{\mathbb{R}} \mathcal{A}(X)$. Clearly, $\dim_{\mathbb{R}} \mathcal{A}(X) \leqslant 2 \dim_{\mathbb{C}} X$. In [3], Nisse and Sottile raise the question when this inequality is strict, as happens in the following two examples.

EXAMPLE 1 (hypersurfaces). Suppose that $n > 2$ and that X is a hypersurface. Then $\dim_{\mathbb{R}} \mathcal{A}(X) \leqslant n < 2(n-1) = 2 \dim_{\mathbb{C}} X$. ♣

EXAMPLE 2 (torus-invariant varieties). Suppose that X is stable under a subtorus $\mathrm{S} \subseteq (\mathbb{C}^*)^n$ of dimension $k > 0$. Denote by Y the image of X in the algebraic torus $(\mathbb{C}^*)^n/\mathrm{S} \cong (\mathbb{C}^*)^{n-k}$. The map $X \to Y$ has fibers of complex dimension k, and the corresponding map $\mathcal{A}(X) \to \mathcal{A}(Y)$ has fibers of real dimension k — namely, translates of $\mathcal{A}(\mathrm{S})$, which is a linear subspace of $\mathbb{R}^n$ spanned by its intersection with $\mathbb{Q}^n$. Thus, we have

$$\dim_{\mathbb{R}} \mathcal{A}(X) = k + \dim_{\mathbb{R}} \mathcal{A}(Y) \leqslant k + 2 \dim_{\mathbb{C}} Y = -k + 2 \dim_{\mathbb{C}} X < 2 \dim_{\mathbb{C}} X. \quad ♣$$

Our theorem says that these are two instances of the same phenomenon, and that this phenomenon is responsible for all drops in dimension.

THEOREM 3. *Let $X \subseteq (\mathbb{C}^*)^n$ be an irreducible, closed algebraic subvariety. Then*

$$\dim_{\mathbb{R}} \mathcal{A}(X) = \min\{ \, 2 \dim_{\mathbb{C}} X + 2 \dim_{\mathbb{C}} \mathrm{T} - \dim_{\mathbb{C}} \mathrm{S} \mid$$

$$\mathrm{T} \subseteq \mathrm{S} \subseteq (\mathbb{C}^*)^n \text{ subtori and } \mathrm{S} \cdot (\overline{\mathrm{T} \cdot X}) = \overline{\mathrm{T} \cdot X} \, \}.$$

An equivalent but more concise formula can then be given as

$$\dim_{\mathbb{R}} \mathcal{A}(X) = \min\{2 \dim_{\mathbb{C}} \overline{S \cdot X} - \dim_{\mathbb{C}} S \mid S \subseteq (\mathbb{C}^*)^n \ subtorus\}.$$

In this theorem, $\overline{T \cdot X}$ (respectively, $\overline{S \cdot X}$) is the Zariski closure of the set of all tz with $t \in T$ (respectively, all rz with $r \in S$) and $z \in X$; note that whenever $S \cdot (\overline{T \cdot X}) = \overline{T \cdot X}$ as in the formula, the set is also equal to $\overline{S \cdot X}$. Naturally, S and T may be taken zero dimensional, in which case we recover the upper bound $2 \dim_{\mathbb{C}} X$.

EXAMPLE 4 (hypersurfaces revisited). If X is a hypersurface, then most one-dimensional tori $T \subseteq (\mathbb{C}^*)^n$ will satisfy $\overline{T \cdot X} = (\mathbb{C}^*)^n$ (see Lemma 10), so we may take $S = (\mathbb{C}^*)^n$. The bound in the theorem is $2(n-1) + 2 - n = n$. ♣

To motivate the structure of this paper, we now prove the easy inequality $\leqslant$ in our main theorem.

Proof of $\leqslant$ in Theorem 3. Let $T \subseteq S \subseteq (\mathbb{C}^*)^n$ be subtori such that $Y := \overline{T \cdot X}$ is S-stable. Then

$$\dim_{\mathbb{R}} \mathcal{A}(X) \leqslant \dim_{\mathbb{R}} \mathcal{A}(Y) \leqslant 2 \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} S \leqslant 2(\dim_{\mathbb{C}} X + \dim_{\mathbb{C}} T) - \dim_{\mathbb{C}} S,$$

where the second equality follows from Example 2. □

If we want equality to hold in the proof above, then we need that first, $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X + \dim_{\mathbb{C}} T$; second, the bound in Example 2 for the pair (Y, S) is tight; and third, $\dim_{\mathbb{R}} \mathcal{A}(X) = \dim_{\mathbb{R}} \mathcal{A}(Y) = \dim_{\mathbb{R}}(\mathcal{A}(X) + \mathcal{A}(T))$. Our proof of Theorem 3 consists of first finding a torus T with the latter property (see Section 2):

PROPOSITION 5. *Let $X \subseteq (\mathbb{C}^*)^n$ be a closed, irreducible variety. Then the Zariski-closure $\overline{|X|}$ in $(\mathbb{R}^*)^n$ of the algebraic amoeba is stable under a subtorus of the real algebraic torus $(\mathbb{R}^*)^n$ of dimension at least $2 \dim_{\mathbb{C}} X - \dim_{\mathbb{R}} \mathcal{A}(X)$.*

In particular, if the amoeba has dimension strictly less than $2 \dim_{\mathbb{C}} X$, then a positive-dimensional real torus acts on $\overline{|X|}$. Using this positive-dimensional torus, we prove Theorem 3 by induction in Section 3.

Theorem 3 implies [**3**, Conjecture 4.4], which proposes *near torus actions* (Definition 12 below) as the only cause of dimension drops for the amoeba.

COROLLARY 6. *For an irreducible, closed subvariety $X \subseteq (\mathbb{C}^*)$, we have*

$$\dim_{\mathbb{R}} \mathcal{A}(X) < \min\{n, 2 \dim_{\mathbb{C}} X\}$$

if and only if some nontrivial subtorus $S \subseteq (\mathbb{C}^*)^n$ has a near action on X.

We conclude this introduction with a relation to the tropical variety of X, also to be proved in Section 3.

COROLLARY 7. *For any irreducible, closed subvariety $X \subseteq (\mathbb{C}^*)$ the dimension $\dim_{\mathbb{R}} \mathcal{A}(X)$ is determined by the tropical variety $\mathrm{Trop}(X) \subseteq \mathbb{R}^n$ of X via*

$$\dim_{\mathbb{R}} \mathcal{A}(X) = \min\{2 \dim_{\mathbb{R}} \mathrm{Trop}(X) + 2 \dim_{\mathbb{R}} T - \dim_{\mathbb{R}} S \mid$$

$$T \subseteq S \subseteq \mathbb{R}^n \ rational \ linear \ subspaces \ with \ S + (T + \mathrm{Trop}(X)) = T + \mathrm{Trop}(X)\},$$

where a subspace of $\mathbb{R}^n$ is called *rational* if it is spanned by vectors in $\mathbb{Q}^n$. Similar to Theorem 3, we have the equivalent formula

$$\dim_{\mathbb{R}} \mathcal{A}(X) = \min\{2\dim_{\mathbb{R}}(S + \mathrm{Trop}(X)) - \dim_{\mathbb{R}} S \mid S \subseteq \mathbb{R}^n \text{ rational linear subspace}\}.$$

2. In search of a positive-dimensional torus

Throughout this section, we fix an irreducible, closed subvariety $X \subseteq (\mathbb{C}^*)^n$. If $\dim_{\mathbb{R}} \mathcal{A}(X) < 2\dim_{\mathbb{C}} X$, then we will find a one-dimensional torus $\mathrm{T} \subseteq (\mathbb{C}^*)^n$ such that $\mathrm{T} \cap (\mathbb{R}^*)^n$ preserves the Zariski-closure $\overline{X}$ and $\dim_{\mathbb{R}}(\mathcal{A}(X) + \mathcal{A}(\mathrm{T})) = \dim_{\mathbb{R}} \mathcal{A}(X)$.

Preliminaries

We write $S^1 \subseteq \mathbb{C}^*$ for the unit circle. Recall that this is a real form of the algebraic group $\mathbb{C}^*$: indeed, tensoring the coordinate ring $\mathbb{R}[c, s]/(c^2 + s^2 - 1)$ of S^1 with $\mathbb{C}$ yields the coordinate ring $\mathbb{C}[c, s]/((c + is)(c - is) - 1)$, which we recognize as the coordinate ring of an algebraic torus with standard coordinate $c + is$; moreover, the inverse morphism $S^1 \to S^1, (c, s) \mapsto (c, -s)$ complexifies to the inverse morphism $\mathbb{C}^* \to \mathbb{C}^*, (c + is) \mapsto 1/(c + is) = (c - is)$; and similarly for the multiplication morphism $S^1 \times S^1 \to S^1$. Both S^1 and the other real form of $\mathbb{C}^*$, the real algebraic group $\mathbb{R}^*$, will play fundamental roles in our proof.

We write $(S^1)^n \subseteq (\mathbb{C}^*)^n$, where the former is a real form of the latter algebraic group. For $p \in (\mathbb{C}^*)^n$ and Q any subset of $\mathbb{C}^n$, we write pQ for the for the result of coordinate-wise multiplication of p with each element of Q. Writing 1 for the unit element in $(\mathbb{C}^*)^n$ and $T_\bullet \bullet$ for (real or complex) tangent spaces, we have

$$T_1(\mathbb{C}^*)^n = \mathbb{C}^n = \mathbb{R}^n \oplus_{\mathbb{R}} i\mathbb{R}^n = T_1\mathbb{R}^n \oplus_{\mathbb{R}} T_1(S^1)^n.$$

Component-wise multiplication by $p \in (\mathbb{C}^*)^n$ yields

$$T_p(\mathbb{C}^*)^n = p\mathbb{R}^n \oplus_{\mathbb{R}} ip\mathbb{R}^n = T_p p\mathbb{R}^n \oplus_{\mathbb{R}} T_p p(S^1)^n.$$

Note that $p^{-1}T_p p(S^1)^n$ is naturally identified with (the same) $i\mathbb{R}^n$ for all $p \in (\mathbb{C}^*)^n$, and $p^{-1}T_p p\mathbb{R}^n$ is identified with (the same) $\mathbb{R}^n$ for all p.

Rather than directly working with the amoeba of X, we will work with the algebraic amoeba $|X|$, the image of X under the semi-algebraic map

$$\mathrm{abs} : (\mathbb{C}^*)^n \to \mathbb{R}^n_{>0}, \ (z_1, \ldots, z_n) \mapsto (|z_1|, \ldots, |z_n|).$$

The reason for this is that $|X|$ is, by real quantifier elimination, a semi-algebraic set, hence analyzable with methods from real algebraic geometry. The following is immediate.

LEMMA 8. *At $p \in (\mathbb{C}^*)^n$, the derivative $d_p \mathrm{Log}$ (respectively, $d_p \mathrm{abs}$) sends the real vector space $T_p p(S^1)^n$ to zero and an element pv with $v \in \mathbb{R}^n$ to v (respectively, to $|p|v$).*

Subvarieties of real tori

We prove an auxiliary result on subvarieties of real tori. We will use the term *real-Zariski* to refer to the real Zariski topology on a real algebraic variety or, more generally, on a semi-algebraic set. We write Z_{ns} for the nonsingular locus of a real algebraic variety.

LEMMA 9. *Let Z be a real-Zariski-closed subset of $(S^1)^n \subseteq (\mathbb{C}^*)^n$. Then the real subspace $\sum_{p \in Z_{\mathrm{ns}}} p^{-1}T_p Z \subseteq i\mathbb{R}^n$ is spanned by its intersection with $i\mathbb{Q}^n$.*

Proof. That subspace is additive under union of irreducible components, so we may assume that Z is irreducible. Let $Z_{\mathbb{C}} \subseteq (\mathbb{C}^*)^n$ be the complexification of Z, an irreducible algebraic

variety. After multiplying with p^{-1} for any fixed $p \in Z_{\mathbb{C}}$, we may assume that $1 \in Z_{\mathbb{C}}$. By [**2**, Proposition 2.2], there exist a natural number m and exponents $e_1, \ldots, e_m \in \{\pm 1\}$ such that the image T of the multiplication map

$$\mu_Z : Z_{\mathbb{C}}^m \to (\mathbb{C}^*)^n, \quad \mathbf{z} = (z_1, \ldots, z_m) \mapsto z_1^{e_1} \cdots z_m^{e_m}$$

is a closed, connected algebraic subgroup T of $(\mathbb{C}^*)^n$, that is, a subtorus. Since Z_{ns} is Zariski-dense in $Z_{\mathbb{C}}$, there exists a point $\mathbf{z} = (z_1, \ldots, z_m) \in Z_{\text{ns}}^m$ (no complexification!) such that the complex-linear map $d_{\mathbf{z}} \mu_Z : T_{\mathbf{z}} Z_{\mathbb{C}}^m \to T_{\mu(\mathbf{z})} \text{T}$ is surjective. Now μ_Z is the restriction to $Z_{\mathbb{C}}^m$ of the multiplication map $\mu : ((\mathbb{C}^*)^n)^m \to (\mathbb{C}^*)^n$ with the same definition. We have

$$\mu = L_{\mu(\mathbf{z})} \circ \mu \circ (L_{z_1^{-1}} \times \cdots \times L_{z_m^{-1}}),$$

where L_x is left multiplication with $x \in (\mathbb{C}^*)^n$, and accordingly,

$$d_{\mathbf{z}} \mu = d_1 L_{\mu(\mathbf{z})} \circ d_{(1,\ldots,1)} \mu \circ (d_{z_1} L_{z_1^{-1}} \times \cdots \times d_{z_m} L_{z_m^{-1}}).$$

Using that the derivative of multiplication is addition and the derivative of inverse is negation, we find

$$d_{\mu(\mathbf{z})} L_{\mu(\mathbf{z})^{-1}} \circ d_{\mathbf{z}} \mu_Z : T_{\mathbf{z}} Z_{\mathbb{C}}^m \to T_1 \text{T}, \ (v_1, \ldots, v_m) \mapsto e_1 z_1^{-1} v_1 + \cdots + e_m z_m^{-1} v_m;$$

and by the choice of $\mathbf{z}$ this map is surjective. For each j, we have $T_{z_j} Z_{\mathbb{C}} = T_{z_j} Z \oplus_{\mathbb{R}} i T_{z_j} Z$ and the complex-linear map $d_{\mu(\mathbf{z})} L_{\mu(\mathbf{z})^{-1}} \circ d_{\mathbf{z}} \mu_Z$ sends the real direct sum $\bigoplus_j T_{z_j} Z$ surjectively onto $T_1(\text{T} \cap (S^1)^n) = (T_1 \text{T}) \cap (i\mathbb{R})^n =: Q$. Since T is an algebraic torus, Q is spanned by its intersection with $i\mathbb{Q}^n$, and the space in the lemma contains Q. Moreover, for all $z \in Z$, we have $z^{-1} T_z Z \subseteq Q$, so that the space in the lemma is in fact equal to Q. $\qquad \square$

A real torus action

We return to our irreducible variety $X \subseteq (\mathbb{C}^*)^n$. By standard results in real algebraic geometry, X is also irreducible when regarded as a real algebraic variety of dimension $2 \dim_{\mathbb{C}} X$. Then the semialgebraic set $|X|$ is irreducible in the sense that its (real) Zariski closure in $\mathbb{R}^n$ is irreducible. To see that, first note that the square

$$|X|^2 := \{(|z_1|^2, \ldots, |z_n|^2) \mid (z_1, \ldots, z_n) \in X\} \subseteq \mathbb{R}_{>0}^n$$

is irreducible, since it is the image of X under an algebraic morphism. Now, since the map $(x_1, \ldots, x_n) \mapsto (x_1^2, \ldots, x_n^2)$ on $(\mathbb{R}^*)^n$ is a finite flat morphism, there exists exactly one irreducible component of the preimage of the Zariski closure $\overline{|X|^2}$ which intersects the positive orthant. Hence, $|X|$ is irreducible.

Proof of Proposition 5. For $q \in |X|$, write $Z_q := q^{-1} X \cap (S^1)^n$, which is a real Zariski-closed subset of $(S^1)^n$ such that $q Z_q = \text{abs}^{-1}(q) \cap X$ is the fiber of $\text{abs}|_X$ over q. By Sard's theorem, there is an open subset U of $|X|$, dense in $|X|$ in the real Zariski-topology, such that Z_q has dimension equal to the expected dimension $c := 2 \dim_{\mathbb{C}} X - \dim_{\mathbb{R}} |X| = 2 \dim_{\mathbb{C}} X - \dim_{\mathbb{R}} \mathcal{A}(X)$. For each $q \in U$, define

$$Q_q := \sum_{p \in (Z_q)_{\text{ns}}} p^{-1} T_p Z_q \subseteq i\mathbb{R}^n,$$

which is a real vector space of dimension at least c, spanned by $Q_p \cap i\mathbb{Q}$ by Lemma 9.

Fix $q \in U$. For each $p \in Z_q$, we have $qp \in X$ and $qp(p^{-1} T_p Z_q) \subseteq T_{qp} X$ and hence, since X is a complex algebraic variety, also $(qp)(ip^{-1} T_p Z_q) \subseteq T_{qp} X$. The space on the left is contained

in $qp\mathbb{R}^n$, and hence, by Lemma 8, d_{qp} abs maps it onto $|qp|(ip^{-1}T_pZ_q) = q(ip^{-1}T_pZ_q)$. We conclude that the latter space is contained in T_qU for each $p \in Z_q$. Therefore,

$$T_qU \supseteq q \sum_{p \in (Z_q)_{\mathrm{ns}}} ip^{-1}T_pZ_q = q(iQ_q);$$

here $iQ_q \subseteq \mathbb{R}^n$ is spanned by its intersection with $\mathbb{Q}^n$. Now for each vector space $R \subseteq \mathbb{R}^n$ of dimension at least c and spanned by its intersection with $\mathbb{Q}^n$, the set

$$V_R := \{q \in U \mid T_qU \supseteq qR\}$$

is a real-Zariski-closed subset of U. There are only countably many such R, and the above discussion shows that the closed sets V_R cover the semialgebraic set U.

But then one of them must have dimension equal to that of U, and in fact, since the Zariski closure of U is irreducible, be equal to U. We conclude that there exists a real vector space $R \subseteq \mathbb{R}^n$, of dimension at least c and spanned by $R \cap \mathbb{Q}^n$, such that $qR \subseteq T_qU$ for all $q \in U$. But then $qR \subseteq T_q\overline{|X|}$ for all q in the real algebraic variety $\overline{|X|}$. Since R is spanned by its intersection with $\mathbb{Q}^n$, there exists a real algebraic torus $\mathrm{R}_\mathbb{R} \subseteq (\mathbb{R}^*)^n$ with Lie algebra R. The subbundle of the tangent bundle of $(\mathbb{R}^*)^n$ that arises by differentiating the action of $\mathrm{R}_\mathbb{R}$ on $(\mathbb{R}^*)^n$ is tangent to $\overline{|X|}$. This implies that $\overline{|X|}$ is $\mathrm{R}_\mathbb{R}$-stable. $\qquad\square$

3. Proofs of the main results

We begin with a lemma that was already used in the introduction (Example 4).

LEMMA 10.   *Let $X \subseteq (\mathbb{C}^*)^n$ be a closed, irreducible subvariety and $\mathrm{S} \subseteq (\mathbb{C}^*)^n$ a subtorus. Then there exists a subtorus $\mathrm{T} \subseteq \mathrm{S}$ with $\dim_\mathbb{C} \mathrm{T} = \dim_\mathbb{C} \overline{\mathrm{S} \cdot X} - \dim_\mathbb{C} X$ such that $\overline{\mathrm{T} \cdot X} = \overline{\mathrm{S} \cdot X}$.*

Proof.   We prove the statement by induction on $k = \dim_\mathbb{C} \overline{\mathrm{S} \cdot X} - \dim_\mathbb{C} X$. If $k = 0$, then $\overline{\mathrm{S} \cdot X} = X$ and $\mathrm{T} = \{1\}$ will do. If $k > 0$, choose a one-dimensional subtorus $\mathrm{R} \subset \mathrm{S}$ such that $\dim_\mathbb{C} \overline{\mathrm{R} \cdot X} > \dim_\mathbb{C} X$. Such R exists since otherwise X would be invariant under all such R and hence under S. Then the statement follows from the induction assumption applied to $X' = \overline{\mathrm{R} \cdot X}$ and a torus S' such that $\mathrm{S} = \mathrm{R} \times \mathrm{S}'$. $\qquad\square$

We now use Proposition 5 to establish our dimension formula for the (ordinary or algebraic) amoeba.

Proof of Theorem 3.   Let $X \subseteq (\mathbb{C}^*)^n$ be Zariski-closed and irreducible. Since we have already proved the inequality $\leqslant$ of the theorem, it suffices to establish the existence of subtori $\mathrm{T} \subseteq \mathrm{S}$ of $(\mathbb{C}^*)^n$ such that $\overline{\mathrm{T} \cdot X}$ is S-stable and $\dim_\mathbb{R} \mathcal{A}(X) = 2\dim_\mathbb{C} X + 2\dim_\mathbb{C} \mathrm{T} - \dim_\mathbb{C} \mathrm{S}$. We proceed by induction on n. For $n = 0$, we have $X = (\mathbb{C}^*)^0$ and we can take $\mathrm{S} = \mathrm{T} = \{1\}$. So we assume that $n > 0$ and that the statement holds for subvarieties of tori of dimension $n - 1$.

If $\dim_\mathbb{R} \mathcal{A}(X) = 2\dim_\mathbb{C} X$, then we may take $\mathrm{T} = \mathrm{S} = \{1\}$ and we are done. So we may assume that $\dim_\mathbb{R} \mathcal{A}(X) < 2\dim_\mathbb{C} X$. Then, by Proposition 5, there exists a one-dimensional, real algebraic torus $\mathrm{R}_\mathbb{R} \subseteq (\mathbb{R}^*)^n$ which stabilizes the Zariski-closure $\overline{|X|}$ of the algebraic amoeba. Let $\mathrm{R} \subseteq (\mathbb{C}^*)^n$ be the complexification of $\mathrm{R}_\mathbb{R}$. Then we find an open subset $U \subseteq \mathcal{A}(X)$ whose complement has positive codimension such that U is a smooth manifold with $\mathcal{A}(\mathrm{R}) \subseteq T_uU$ for each $u \in U$ (use Lemma 8 for the tangent vectors coming from the action of $\mathrm{R}_\mathbb{R}$). We find that the fibers of the map $U \to \mathbb{R}^n/\mathcal{A}(\mathrm{R})$ have dimension 1, and this implies

$$\dim_\mathbb{R} \overline{\mathcal{A}(R) + \mathcal{A}(X)}/\mathcal{A}(R) = -1 + \dim_\mathbb{R} \mathcal{A}(X).$$

(We note that we are working with the closure with respect to the Euclidean topology of $\mathbb{R}^n$ in the above formula.)

Define
$$\tilde{X} := \overline{\mathrm{R} \cdot X}/\mathrm{R} \subseteq (\mathbb{C}^*)^n/\mathrm{R} \cong (\mathbb{C}^*)^{n-1}.$$
Then the previous equation implies that the amoeba
$$\mathcal{A}(\tilde{X}) = \overline{\mathcal{A}(R) + \mathcal{A}(X)}/\mathcal{A}(R)$$
has real dimension equal to $-1 + \dim_{\mathbb{R}} \mathcal{A}(X)$.

By the induction hypothesis, there exist subtori $\tilde{\mathrm{T}} \subseteq \tilde{\mathrm{S}}$ of $(\mathbb{C}^*)^n/\mathrm{R}$ such that $\overline{\tilde{\mathrm{T}} \cdot \tilde{X}}$ is $\tilde{\mathrm{S}}$-stable and
$$\dim_{\mathbb{R}} \mathcal{A}(\tilde{X}) = 2 \dim_{\mathbb{C}} \tilde{X} + 2 \dim_{\mathbb{C}} \tilde{\mathrm{T}} - \dim_{\mathbb{C}} \tilde{\mathrm{S}}.$$
We distinguish two cases. First, assume that X is not stable under R, so that $\dim_{\mathbb{C}} \tilde{X} = \dim_{\mathbb{C}} X$. Then let T, S be the pre-images in $(\mathbb{C}^*)^n$ of $\tilde{\mathrm{T}}, \tilde{\mathrm{S}} \subseteq (\mathbb{C}^*)^n/\mathrm{R}$, respectively. Then $\mathrm{T} \subseteq \mathrm{S}$ are subtori such that $\overline{\mathrm{T} \cdot X}$ is S-stable, and we find
$$\begin{aligned}
\dim_{\mathbb{R}} \mathcal{A}(X) &= 1 + \dim_{\mathbb{R}} \mathcal{A}(\tilde{X}) \\
&= 1 + 2 \dim_{\mathbb{C}} \tilde{X} + 2 \dim_{\mathbb{C}} \tilde{\mathrm{T}} - \dim_{\mathbb{C}} \tilde{\mathrm{S}} \\
&= 1 + 2 \dim_{\mathbb{C}} X + 2(-1 + \dim_{\mathbb{C}} \mathrm{T}) - (-1 + \dim_{\mathbb{C}} \mathrm{S}) \\
&= 2 \dim_{\mathbb{C}} X + 2 \dim_{\mathbb{C}} \mathrm{T} - \dim_{\mathbb{C}} \mathrm{S}.
\end{aligned}$$

Second, assume that X is stable under R. As before, let S be the pre-image of $\tilde{\mathrm{S}}$ in $(\mathbb{C}^*)^n$, but now let T be any torus in $(\mathbb{C}^*)^n$ of complex dimension *equal* to $\dim_{\mathbb{C}} \tilde{\mathrm{T}}$ that projects surjectively onto T. Using that X is R-stable and $\overline{\tilde{\mathrm{T}} \cdot \tilde{X}}$ is $\tilde{\mathrm{S}}$-stable, we find that $\overline{\mathrm{T} \cdot X}$ is S-stable. Furthermore,
$$\begin{aligned}
\dim_{\mathbb{R}} \mathcal{A}(X) &= 1 + \dim_{\mathbb{R}} \mathcal{A}(\tilde{X}) \\
&= 1 + 2 \dim_{\mathbb{C}} \tilde{X} + 2 \dim_{\mathbb{C}} \tilde{\mathrm{T}} - \dim_{\mathbb{C}} \tilde{\mathrm{S}} \\
&= 1 + 2(-1 + \dim_{\mathbb{C}} X) + 2 \dim_{\mathbb{C}} \mathrm{T} - (-1 + \dim_{\mathbb{C}} \mathrm{S}) \\
&= 2 \dim_{\mathbb{C}} X + 2 \dim_{\mathbb{C}} \mathrm{T} - \dim_{\mathbb{C}} \mathrm{S},
\end{aligned}$$
as desired.

For the second formula, let T, S be subtori as in the first formula. We then have
$$2 \dim_{\mathbb{C}} \overline{\mathrm{S} \cdot X} - \dim_{\mathbb{C}} \mathrm{S} = 2 \dim_{\mathbb{C}} \overline{\mathrm{T} \cdot X} - \dim_{\mathbb{C}} \mathrm{S} \leqslant 2 \dim_{\mathbb{C}} X + 2 \dim_{\mathbb{C}} \mathrm{T} - \dim_{\mathbb{C}} \mathrm{S},$$
so the second formula is a lower bound to the first formula. Conversely, if S is any subtorus, then by Lemma 10 there exists a subtorus $\mathrm{T} \subseteq \mathrm{S}$ such that $\overline{\mathrm{T} \cdot X} = \overline{\mathrm{S} \cdot X}$ and $\dim_{\mathbb{C}} \mathrm{T} = \dim_{\mathbb{C}} \overline{\mathrm{S} \cdot X} - \dim_{\mathbb{C}} X$, and we find
$$2 \dim_{\mathbb{C}} X + 2 \dim_{\mathbb{C}} \mathrm{T} - \dim_{\mathbb{C}} \mathrm{S} = 2 \dim_{\mathbb{C}} \overline{\mathrm{S} \cdot X} - \dim_{\mathbb{C}} \mathrm{S},$$
hence the first formula is a lower bound to the second formula. $\qquad\square$

EXAMPLE 11. We give an alternative proof of [**3**, Theorem 4.5], which says that if $\dim_{\mathbb{R}} \mathcal{A}(X) = \dim_{\mathbb{C}} X$, then X is a single orbit under a subtorus of $(\mathbb{C}^*)^n$. Take a subtorus $\mathrm{S} \subseteq (\mathbb{C}^*)^n$ that achieves the minimum in the second formula of Theorem 3. Since we always have $\dim_{\mathbb{C}} \overline{\mathrm{S} \cdot X} \geqslant \dim_{\mathbb{C}} \mathrm{S}, \dim_{\mathbb{C}} X$, from our choice of S, we have
$$\dim_{\mathbb{C}} X = 2 \dim_{\mathbb{C}} \overline{\mathrm{S} \cdot X} - \dim_{\mathbb{C}} \mathrm{S} \geqslant \dim_{\mathbb{C}} X.$$
Hence, $\dim_{\mathbb{C}} \mathrm{S} = \dim_{\mathbb{C}} X = \dim_{\mathbb{C}} \overline{\mathrm{S} \cdot X}$. But $\overline{\mathrm{S} \cdot X}$ is irreducible and contains both X and an orbit of S, so X must be equal to such an orbit. ♣

Near Torus Actions

We start by reviewing Nisse–Sottile's notion of near torus actions [**3**, Definition 4.1].

DEFINITION 12. Let $X \subseteq (\mathbb{C}^*)$ be an irreducible closed subvariety and $\mathrm{S} \subseteq (\mathbb{C}^*)^n$. We set $Y := (\mathrm{S} \cdot X)/\mathrm{S}$. Then S *has a near action on* X if

$$2 \dim_{\mathbb{C}} X > \dim_{\mathbb{C}} \mathrm{S} + 2 \dim_{\mathbb{C}} Y \quad \text{and} \quad n > \dim_{\mathbb{C}} \mathrm{S} + 2 \dim_{\mathbb{C}} Y.$$

We now show that, as conjectured in [**3**], unexpected amoeba dimension is equivalent to a near torus action. The implication $\Leftarrow$ is [**3**, Theorem 4.3].

Proof of Corollary 6. For any subtorus $\mathrm{S} \subseteq (\mathbb{C}^*)^n$, setting $Y := \overline{\mathrm{S} \cdot X}/\mathrm{S}$ we have $\dim_{\mathbb{C}}(Y) = \dim_{\mathbb{C}}(\mathrm{S} \cdot X) - \dim_{\mathbb{C}}(\mathrm{S})$. Note that $2 \dim_{\mathbb{C}}(\mathrm{S} \cdot X) - \dim_{\mathbb{C}}(\mathrm{S}) < 2 \dim_{\mathbb{C}}(X)$ and $2 \dim_{\mathbb{C}}(\mathrm{S} \cdot X) - \dim_{\mathbb{C}}(\mathrm{S}) < n$ imply $\mathrm{S} \neq \{1\}$ and $\mathrm{S} \neq (\mathbb{C}^*)^n$, respectively. Hence, the statement follows directly from the second formula of Theorem 3. In particular, in case of a dimension drop, a torus S providing the minimum in this formula has a near action on X. $\square$

Proof of Corollary 7. We start by presenting a well-known fact in tropical geometry.

LEMMA 13. *Let* $X \subseteq (\mathbb{C}^*)$ *be an irreducible* (*in particular, reduced*) *closed subvariety and denote by* $\mathrm{Trop}(X) \subseteq \mathbb{R}^n$ *its tropicalization. Let* $\mathrm{S} \subset (\mathbb{C}^*)^n$ *be a subtorus and* $S = \mathrm{Trop}(\mathrm{S}) = \mathcal{A}(\mathrm{S}) \subset \mathbb{R}^n$ *the associated* (*rational*) *linear subspace. Then* $\mathrm{S} \cdot X = X$ *if and only if* $S + \mathrm{Trop}(X) = \mathrm{Trop}(X)$.

Proof. By basic tropical geometry, $\mathrm{Trop}(\overline{\mathrm{S} \cdot X}) = S + \mathrm{Trop}(X)$. Hence, $\mathrm{S} \cdot X = X$ implies $S + \mathrm{Trop}(X) = \mathrm{Trop}(X)$. Let us assume $S + \mathrm{Trop}(X) = \mathrm{Trop}(X)$ now. Note that for irreducible varieties $Y \subseteq (\mathbb{C}^*)^n$, we have $\dim_{\mathbb{C}} Y = \dim_{\mathbb{R}} \mathrm{Trop}(Y)$, see [**1**, Theorem A]. Since both X and $\overline{\mathrm{S} \cdot X}$ are irreducible, it follows that $\dim_{\mathbb{C}} \overline{\mathrm{S} \cdot X} = \dim_{\mathbb{C}} X$. Since $X \subseteq \overline{\mathrm{S} \cdot X}$, this implies $X = \overline{\mathrm{S} \cdot X} = \mathrm{S} \cdot X$. $\square$

Proof of Corollary 7. By Lemma 13, the pairs of subtori $\mathrm{T} \subseteq \mathrm{S} \subseteq (\mathbb{C}^*)^n$ such that $\mathrm{S} \cdot (\overline{\mathrm{T} \cdot X}) = \overline{\mathrm{T} \cdot X}$ are in bijection to the pairs of rational linear subspaces $T \subseteq S \subseteq \mathbb{R}^n$ such that $S + (T + \mathrm{Trop}(X)) = T + \mathrm{Trop}(X)$, via $T = \mathrm{Trop}(\mathrm{T})$, $S = \mathrm{Trop}(\mathrm{S})$. Using the relation $\dim_{\mathbb{C}} Y = \dim_{\mathbb{R}} \mathrm{Trop}(Y)$ again, we have

$$2 \dim_{\mathbb{C}} X + 2 \dim_{\mathbb{C}} \mathrm{T} - \dim_{\mathbb{C}} \mathrm{S} = 2 \dim_{\mathbb{R}} \mathrm{Trop}(X) + 2 \dim_{\mathbb{R}} T - \dim_{\mathbb{R}} S.$$

Hence, the two minima agree. The second formula follows similarly as in the proof of Theorem 3. $\square$

We conclude this paper with a question on computability.

QUESTION 14. Does there exist an algorithm that, on input a balanced, pure-dimensional, rational polyhedral complex $\Sigma \subseteq \mathbb{R}^n$ which is connected in codimension 1, computes the expression

$$\min\{2 \dim_{\mathbb{R}}(S + \Sigma) - \dim_{\mathbb{R}} S \mid S \subseteq \mathbb{R}^n \text{ rational subspace}\}$$

from Corollary 7?

The first term is the maximum, over all maximal cones C of Σ, of $\dim_{\mathbb{R}} \Sigma + \dim_{\mathbb{R}} S - \dim_{\mathbb{R}}(\langle C \rangle_{\mathbb{R}} \cap S)$, and hence it is minimized by an S have certain incidences with given linear subspaces of $\mathbb{R}^n$. If the rationality assumption is dropped, then real quantifier elimination

answers the question in the affirmative. However, similar incidence problems often have real but no rational solutions. For instance, a classical result in enumerative geometry says that the number of two-dimensional subspaces in $\mathbb{R}^4$ (lines in projective three-space) that nontrivially intersect four given two-dimensional subspaces in general position is either zero (in which case there are two complex conjugate solutions) or two. In the latter case, even if the four given spaces are rational, the two solutions will typically not be. We do not know whether the existence of rational solutions for such incidence problems is decidable in general, nor whether the additional conditions on Σ force that real solutions imply rational solutions. On the other hand, if X is a variety given by equations with coefficients in, say, some number field, then of course, by real quantifier elimination, there does exist an algorithm for computing $\dim_{\mathbb{R}} \mathcal{A}(X) = \dim_{\mathbb{R}} |X|$.

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