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THE HESSIAN DISCRIMINANT

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We express the Hessian discriminant of a cubic surface in terms of fundamental invariants. This answers Question 15 from the *27 questions on the cubic surface*.

Introduction

A cubic surface is $\mathbb{P}^3$ is the vanishing locus of a degree 3 polynomial

$$f(x_0,x_1,x_2,x_3) := \sum_{0\le i\le j\le k\le 3} c_{ijk}x_ix_jx_k \in \mathbb{C}[x_0,x_1,x_2,x_3]$$

in 4 variables. The study of cubic surfaces is an important research topic in classical algebraic geometry. Recently, Anna Seigal [14] introduced a new invariant of cubic surfaces called the *Hessian discriminant HD*. It is a homogeneous degree 120 polynomial in the 20 variables c_{ijk}, which is defined as a specialization of the so-called Hurwitz form of the variety of rank 2 symmetric 4×4 matrices.

The main motivation for introducing the Hessian discriminant comes from its relation with the rank of cubic surfaces. We recall that the (Waring) rank of a homogeneous degree d polynomial f is the smallest number r for which f can be written in the form

$$f = L_1^d + \cdots + L_r^d,$$

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where the L_i are linear forms. The rank of a cubic surface $V(f)$ is simply the Waring rank of f; for a general cubic surface the rank is equal to 5. It can be shown that (the vanishing locus of) the Hessian discriminant is precisely the Zariski closure of the set of rank 6 cubic surfaces. There is also a connection between the Hessian discriminant and the more classical study of cubic surfaces: it is related to the singular points of the Hessian surface.

By construction, the Hessian discriminant defines a hypersurface in $\mathbb{P}^{19}$ which is invariant under the action of $PGL(3)$. In other words, HD is an invariant of cubic surfaces. The generators of the invariant ring of cubic surfaces are known [10], so it is natural to ask how to express HD in terms of these fundamental invariants. This was Question 15 in the *27 questions on the cubic surface* [8]. The main result of this article provides an answer to this question:

Theorem 4.1. $HD = I_{40}^3$, where I_{40} is the degree 40 Salmon invariant.

In fact, it is not very hard to deduce this result from known facts about cubic surfaces. However, the required results appear to be quite scattered in the literature. In this article, we present a proof that only relies on two very classical results: the classification of cubic surfaces by Schläfli [11], and the computation of the invariant ring by Salmon [10]. We also spend some time explaining connections with Hessian surfaces and with apolar schemes, and we give an independent argument why we should expect HD to be a cube.

The organization of the article is as follows:

In Section 1, we review the definition of the Hurwitz form and the Hessian discriminant. We also explain how to use software to verify whether a given cubic lies on the Hessian discriminant, and explain connections with Hessian surfaces and with apolar schemes.

In Section 2, we use the classical theory of normal forms for cubic surfaces to decide for every cubic surface outside of a certain codimension 2 locus whether or not it lies on the Hessian discriminant.

In Section 3, we recall the invariant theory of cubic surfaces, and give a computational proof that the vanishing locus of the invariant I_{40} is the Zariski closure of the set of smooth rank 6 cubic surfaces.

Finally, in Section 4 we put together the results of the preceding two sections to prove Theorem 4.1.

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1. The Hessian discriminant

1.1. The Hurwitz form

Let X be an irreducible variety in projective space $\mathbb{P}^n$ of codimension $d \geq 1$ and degree $p \geq 2$. Let $\mathbb{G}(d, \mathbb{P}^n)$ denote the Grassmannian of dimension d subspaces of $\mathbb{P}^n$. Following [16], we define $\mathcal{H}_X \subset \mathbb{G}(d, \mathbb{P}^n)$ to be the set of all subspaces L for which $L \cap X$ does not consist of p reduced points.

If L is the row space of a matrix $B = (b_{i,j})_{0 \leq i \leq d, 0 \leq j \leq n}$, then the entries $b_{i,j}$ are the *Stiefel coordinates* of L, and the maximal minors of B are the *Plücker coordinates*.

One can obtain the *sectional genus* of X by intersecting the variety with a general subspace of codimension $d-1$ and then taking the arithmetic genus of the obtained curve.

Theorem 1.1. [16, Theorem 1.1] $\mathcal{H}_X$ is an irreducible hypersurface in $\mathbb{G}(d, \mathbb{P}^n)$, defined by an irreducible element Hu_X in the homogeneous coordinate ring of $\mathbb{G}(d, \mathbb{P}^n)$. If X is regular in codimension 1, then the degree of Hu_X in Plücker coordinates equals $2p + 2g - 2$, where g is the sectional genus of X.

The polynomial Hu_X defined above is called the *Hurwitz form* of X. Interesting examples of Hurwitz forms in computational algebraic geometry can be consulted in [16].

To define the Hessian discriminant, we will need to consider the Hurwitz form of the variety X_2 of symmetric 4×4 matrices of rank at most 2. If we write $\mathbb{P}^9$ for the space of all symmetric 4×4 matrices, then $X_2 \subset \mathbb{P}^9$ is an irreducible subvariety defined by the vanishing of the 3×3 minors. It has dimension 6, degree 10, and sectional genus 6. By Theorem 1.1, the Hurwitz form Hu_{X_2} is an irreducible hypersurface of degree 30 in the Plücker coordinates of $\mathbb{G}(3, \mathbb{P}^9)$. In [16], there is an algorithm to compute the polynomial Hu_X, but it does not finish in a reasonable amount of time in this case.

1.2. The Hessian discriminant

For the rest of the paper, we fix a 4-dimensional $\mathbb{C}$-vector space V. Let $\mathcal{C} = V(f)$ be a cubic surface in $\mathbb{P}^3 = \mathbb{P}(V)$, defined by a quaternary cubic

$$f = \sum_{0 \leq i \leq j \leq k \leq 3} c_{ijk} x_i x_j x_k \in \mathbb{C}[x_0, x_1, x_2, x_3]_3 = S^3(V^*).$$

The 20 coefficients c_{ijk} determine a point in $\mathbb{P}(S^3(\mathbb{C}^4)) \cong \mathbb{P}^{19}$. We will use the notions of "cubic surfaces", "quaternary cubics (up to scaling)", and "points in $\mathbb{P}^{19}$" interchangeably. If $\mathcal{C}$ is not a cone over a plane cubic, we can associate to f a 3-plane $H(f)$ in the space $\mathbb{P}^9 = \mathbb{P}(S^2(V^*))$ of symmetric 4×4 matrices. The points of $H(f)$ are called *polar quadrics* of f. There are several equivalent ways to define $H(f)$. We leave it to the reader to check that they are indeed equivalent.

- The Hessian of f is the 4×4 matrix of linear forms whose (i,j)-th entry is $\frac{\partial^2 f}{\partial x_i \partial x_j}$. It defines an injective linear map $i_f : \mathbb{P}^3 \to \mathbb{P}^9$, sending a point $p = [x_0 : x_1 : x_2 : x_3]$ to the Hessian matrix evaluated in that point. We define $H(f)$ to be the image of i_f.

- We can also define $H(f)$ as the linear span of the four partial derivatives $\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}$, seen as points in $\mathbb{P}(S^2(V^*))$. Note that these 4 points are well-defined and not coplanar, unless after change of coordinates f is a polynomial in 3 variables. This explains our assumption that $\mathcal{C}$ is not a cone over a plane cubic.

- We can view f as a symmetric three-way tensor $T = (T_{ijk})_{i,j,k}$. (I.e. $c_{ijk} = \lambda T_{ijk}$, where λ is the number of distinct permutations of i,j,k. Then $f = \sum_{i,j,k} T_{ijk} x_i x_j x_k$.) For $m \in \{0,1,2,3\}$, the m-th *slice* of T is defined to be the symmetric matrix obtained by fixing the first index to be m. Then $H(f)$ is the linear span of the four slices of T. From this description we see immediately the Stiefel coordinates of $H(f)$: they are the entries of a 4×10 matrix with colums indexed by pairs (j,k) with $j<k$, whose $i,(j,k)$-th entry is T_{ijk}.

Now we can take the Hurwitz polynomial Hu_{X_2} from the previous subsection, and evaluate it in the Plücker coordinates of $H(f)$, where f is a general cubic surface. The result is a degree 120 polynomial in the 20 variables c_{ijk}, called the *Hessian discriminant HD*. By construction, the Hessian discriminant vanishes at $f \in \mathbb{P}^{19}$ if and only if $H(f)$ does not intersect the variety of rank 2 matrices in 10 reduced points. Clearly, $V(HD)$ is invariant under linear changes of coordinates. It follows that HD is invariant under the natural action of $SL(4)$ on $\mathbb{C}[c_{000},\dots,c_{333}]$.

The following observation connects the Waring rank of cubic forms with the Hessian discriminant:

Observation 1.2. (See [14, Section 2.4].) If f has Waring rank at least 6 and defines a smooth cubic surface, then f lies on the Hessian discriminant.

It will be easy to verify this, once we have recalled the normal forms of smooth cubic surfaces in Section 2.

In [15, Corollary 4.4], it is proven that the vanishing locus of the Hessian discriminant is the Zariski closure of the set of all rank 6 cubic surfaces; in particular, all rank ≥ 6 cubics (not just the smooth ones) lie on the Hessian discriminant.

1.3. The Hessian surface

In this section we investigate how the Hessian discriminant is related to the singular locus the Hessian surface of a cubic surface. Nothing further in the paper logically depends on this section.

The determinant locus of the Hessian of f defines a quartic surface $\mathrm{Hess}(f)$ in $\mathbb{P}^3$, called the *Hessian surface* of f. It can be identified with the intersection of $H(f)$ and the variety X_3 of singular 4×4 matrices. Since the singular locus of X_3 is equal to X_2, the locus $H(f)\cap X_2$ of rank 2 matrices in $\mathrm{Hess}(f)$ is contained in the singular locus of $\mathrm{Hess}(f)$. For smooth cubic surfaces, this is an equality:

Proposition 1.3. For a smooth cubic surface, the singular locus $\mathrm{Sing}(\mathrm{Hess}(f))$ of its Hessian surface is equal to $H(f)\cap X_2$.

It follows that a smooth cubic surface lies on the Hessian discriminant if and only if its Hessian surface has strictly less than 10 singular points. This result appears to be well-known, but we were not able to find a complete proof in the literature. In Section 2.4, we will give a proof of Proposition 1.3 relying on the classification of smooth cubic surfaces.

For singular cubic surfaces, the situation is somewhat more subtle: first of all, the following result (which also appears to be folklore) shows that the Hessian surface might have additional singular points.

Proposition 1.4. If a cubic surface f is singular at a point p, then p is a singular point of $\mathrm{Hess}(f)$.

Proof. Without loss of generality, we may assume that $p=[1:0:0:0]$. Then $f=x_0g+h$, where g and h are homogenous polynomials of respective degrees 2 and 3 containing only the variables x_1,x_2,x_3. In particular it holds that $\frac{\partial^2 f}{\partial {x_0}^2}=0$ and $\frac{\partial^2 f}{\partial x_0 \partial x_i}=\frac{\partial^2 g}{\partial x_i}$ for $i=1,2,3$. Hence the determinant of the Hessian matrix is a homogeneous degree 4 polynomial that does not contain any monomials divisible by x_0^3, which implies that $[1:0:0:0]$ is a singular point of $\mathrm{Hess}(f)$. □

An explicit example of a cubic surface whose Hessian surface has more than 10 (but finitely many) singularities is the Cayley cubic; see Section 2.3.

In [9, Proposition 4.5], it is shown that for a certain class of (possibly singular) cubic surfaces, the singular locus of the Hessian surface is precisely equal to $(H(f)\cap X_2)\cup \mathrm{Sing}(f)$. However, the following example shows that this is not true for all cubic surfaces.

Example 1.5. Consider the cubic surface defined by the equation

$$f := \frac{1}{6}x_0^3 + x_1x_2x_3 = 0.$$

The Hessian matrix of f is equal to

$$\begin{pmatrix} x_0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & x_2 \\ 0 & x_3 & 0 & x_1 \\ 0 & x_2 & x_1 & 0 \end{pmatrix},$$

hence its Hessian surface a union of 4 planes defined by $x_0x_1x_2x_3 = 0$, whose singular points are the 6 lines $x_i = x_j = 0$, $0 \leq i < j \leq 3$. However, $H(f) \cap X_2$ only consists of the 3 lines $x_0 = x_i = 0$, $i = 1,2,3$, and $\mathrm{Sing}(f)$ consists of 3 points $[0:1:0:0]$, $[0:0:1:0]$, $[0:0:0:1]$.

1.4. Computational methods

While computing an expression for the Hessian discriminant is a computationally difficult task, it is easy to verify for a given cubic whether or not the Hessian discriminant vanishes at that cubic: one simply needs to compute the ideal defining the intersection of $H(f) \cap X_2$, and check whether or not it is zero-dimensional and radical. Some code in `Macaulay2` [5] for computing this can be found below:

```
R=QQ[x_0..x_3,z_0..z_9]
X={x_0,x_1,x_2,x_3};
A=genericSymmetricMatrix(R,z_0,4)
I2=minors(3,A)
hessRank2 = f ->(
    hess = diff(transpose matrix{X},diff(matrix{X},f));
    I=eliminate(X,ideal(flatten entries (A-hess)));
    return (I+I2);
)
isOnHessianDiscriminant = f ->(
    J=hessRank2(f);
    return not ((codim J==9) and (J==radical J));
)
--Examples:
f=x_0*x_1*x_2+x_0*x_1*x_3+x_0*x_2*x_3+x_1*x_2*x_3
isOnHessianDiscriminant(f)
--false
```

```
f=x_0^3+x_1^3+x_2^3+x_3^2*(3*x_0+3*x_1+3*x_2+x_3)
isOnHessianDiscriminant(f)
--true
```

Remark 1.6. The above algorithm can also be used to simultaneously compute $H(f) \cap X$ for all f in a family of cubic surfaces. For more details, see Remark 2.5, as well as the supplementary code available at [1].

1.5. Apolarity

There is a beautiful connection between the 3-plane $H(f)$ associated to f and apolar schemes of f. Although not logically necessary for the proof of our main theorem, it can provide some insight in the nature of the singularities of the Hessian surface of a smooth cubic.

We will identify the symmetric algebra $S(V)$ of V with the polynomial ring $\mathbb{C}[y_0, y_1, y_2, y_3]$. For every $d, m \in \mathbb{N}$, there is a natural pairing

$$\circ : \mathbb{C}[y_0, y_1, y_2, y_3]_d \times \mathbb{C}[x_0, x_1, x_2, x_3]_m \to \mathbb{C}[x_0, x_1, x_2, x_3]_{m-d}$$

defined by $g \circ f = g(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}) f(x_0, x_1, x_2, x_3)$.

Note that $H(f)$ can be identified with the image of the map $V \to S^2(V^*)$: $g \mapsto g \circ f$.

Definition 1.7. For f in $\mathbb{C}[x_0, x_1, x_2, x_3]$, we define the *annihilator* of f to be the ideal

$$\operatorname{Ann}(f) = \{g | g \circ f = 0\} \subseteq \mathbb{C}[y_0, y_1, y_2, y_3].$$

If $I \subseteq \operatorname{Ann}(f)$ is a saturated ideal, we say that I is an *apolar ideal* to f, and $V(I)$ is an *apolar scheme* to f. In other words, $Y \subseteq \mathbb{P}(V^*)$ is an apolar scheme to f if every polynomial that vanishes on Y also annihilates f.

Observation 1.8. Denote the coordinates on $\mathbb{P}^9 = \mathbb{P}(S^2(V^*))$ by z_{ij}. Then defining equations $\sum_{i \le j} a_{ij} z_{ij} = 0$ of $H(f)$ are in one-to-one correspondence with degree 2 elements $\sum_{i \le j} a_{ij} y_i y_j$ of $\operatorname{Ann}(f)$: $\sum_{i \le j} a_{ij} y_i y_j$ is in $\operatorname{Ann}(f)$ if and only if $\sum_{i \le j} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} = 0$, if and only if $\sum_{i \le j} a_{ij} z_{ij}$ vanishes on $H(f)$. As a corollary of this, *if Y is an apolar scheme to f, then $H(f)$ is contained in the linear span of $v_2(Y)$*, the image of Y under the second Veronese embedding $v_2 : \mathbb{P}(V^*) \to \mathbb{P}(S^2(V^*))$. Indeed: every linear equation $\sum_{i \le j} a_{ij} z_{ij}$ on $v_2(Y)$ comes from a quadratic equation $\sum_{i \le j} a_{ij} y_i y_j$ on Y, which by the above also vanishes on $H(f)$.

2. Normal forms for cubics

It is possible to classify the cubic surfaces up to linear transformation, and use this to provide a list of normal forms so that every quaternary cubic can be brought in one of the normal forms by a linear change of coordinates. This was first done by Schläfli [11], we refer the interested reader to [12] for an overview.

We first recall the classification of smooth cubic surfaces:

Theorem 2.1 (See [13, §§84 - 91]). Every smooth cubic surface can after a linear change of coordinates be written in one of the following 4 normal forms:

1. Sylvester's pentahedral form:
$$c_0x_0^3 + c_1x_1^3 + c_2x_2^3 + c_3x_3^3 + c_4(-x_0 - x_1 - x_2 - x_3)^3 = 0, \tag{2.1}$$
with $c_i \in \mathbb{C}^*$, and $\sum_i \pm\frac{1}{\sqrt{c_i}} \neq 0$.

2. General rank 6 cubic surfaces:
$$x_1^3 + x_2^3 + x_3^3 - x_0^2(\mu x_0 + 3\lambda_1 x_1 + 3\lambda_2 x_2 + 3\lambda_3 x_3) = 0 \tag{2.2}$$
with $\lambda_i \in \mathbb{C}^*$, $\mu \in \mathbb{C}$, and $\mu + 2(\lambda_1^{\frac{3}{2}} + \lambda_2^{\frac{3}{2}} + \lambda_3^{\frac{3}{2}}) \neq 0$.

3. Special rank 6 cubic surfaces:
$$2\mu_0x_0^3 + x_1^3 + x_2^3 - 3x_0(\mu_1 x_0 x_1 + x_0 x_2 + x_3^2) = 0, \tag{2.3}$$
with $\mu_1(\mu_0 \pm \mu_1^{\frac{3}{2}} \pm 1) \neq 0$.

4. Cyclic cubic surfaces:
$$x_0^3 + x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1 x_2 x_3 = 0, \tag{2.4}$$
with $(\lambda^3 + 8)(\lambda^3 - 1) \neq 0$.

The families of cubics with these normal forms have respective codimensions 0,1,2,3 in $\mathbb{P}^{19}$. Cubics of the form (2.1) have Waring rank 5, cubics of the form (2.2) or (2.3) have rank 6, and cubics of the form (2.4) have rank 5 if $\lambda \neq 0$, and rank 4 if $\lambda = 0$.

A detailed discussion on normal forms for singular cubic surfaces can be found in [2]. For our purposes, it suffices to know the following result:

Theorem 2.2 (See [2, Lemma 2]). A general singular cubic surface can be written in the form

$$x_3(x_1^2 - x_0x_2) + x_1(x_0 - (1+\rho_0)x_1 + \rho_0 x_2)(x_0 - (\rho_1 + \rho_2)x_1 + \rho_1\rho_2 x_2) = 0, \tag{2.5}$$

where $\rho_i \in \mathbb{C} \setminus \{0, 1\}$ are pairwise different.

In fact, all we need to know to prove our main theorem is the following:

Corollary 2.3. Every cubic, outside of a certain codimension > 1 set in $\mathbb{P}^{19}$, can after a linear change of coordinates be written in one of the forms (2.1), (2.2) or (2.5).

2.1. Sylvester's pentahedral form

Proposition 2.4 (See also [4, Chapter 9.4.2])**.** No cubic of the form (2.1) lies on the Hessian discriminant, as long as all c_i are nonzero.

Proof. Write $x_4 = -x_0 - x_1 - x_2 - x_3$. The set of rank 2 quadratic forms in $H(f)$ is given by

$$\{c_i x_i^2 - c_j x_j^2 | i \leq j\}.$$

This can easily be verified by hand, or computationally by using the algorithm described below. Since we assumed that all c_i are nonzero, we find that $H(f) \cap X_2$ consists of 10 distinct points, proving the result. □

Remark 2.5. We can use our `Macaulay2` code to simultaneously analyze $H(f) \cap X_2$ for all cubics f of the form (2.1), including the ones where one of the c_i is zero (if two or more of them are zero then $H(f)$ is not defined). The code (available at [1]) computes a primary decomposition of the ideal defining $H(f) \cap X_2$ (where the c_i are variables). The primary decomposition of our ideal has 40 components. 30 of these contain one of the paramaters $c_0, \dots, c_4$ (each parameter in 6 components); the other 10 do not contain any linear combination of the parameters. This means that if exactly one of the parameters is zero, the intersection $H(f) \cap X_2$ consists of 6 lines, whereas if all the c_i are nonzero, it consists of 10 points. After identifying $H(f)$ with $\mathbb{P}^3$ using the Hessian matrix (as in Sections 1.2 and 1.3), the 10 points in $H(f)$ are $[1:0:0:0], [0:1:0:0], [0:0:1:0], [0:0:0:1], [1:-1:0:0], [1:0:-1:0], [1:0:0:-1], [0:1:-1:0], [0:1:0:-1], [0:0:1:-1]$.

Remark 2.6. Proposition 2.4 can also be shown using apolarity: for a general cubic surface

$$f := L_1^3 + L_2^3 + L_3^3 + L_4^3 + L_5^3 = 0$$

(with the L_i in general position) there is an apolar scheme $Y = \{L_1, \dots, L_5\}$. This can easily be verified directly, but also follows from the so-called *apolarity lemma* [6, Lemma 1.15], which states that a homogeneous degree d polynomial f can be written as a linear combination of powers $L_1^d, \dots, L_s^d$ of linear forms if and only if $\{L_1, \dots, L_s\}$ is an apolar scheme to f. It now follows from Observation 1.8 that $H(f)$ is contained in the linear span $\langle v_2(Y) \rangle$ of the second Veronese embedding of Y.

Clearly, $\langle v_2(Y)\rangle \cap X_2$ contains the 10 lines through the $\langle L_i^2, L_j^2\rangle$, and (using the fact that L_i are in general position) it is easy to verify that this is in fact an equality. Now, $H(f)\cap X_2$ is the intersection of $\langle v_2(Y)\rangle \cap X_2$ with a hyperplane H. Since $H(f)$ does not contain any of the L_i^2 (indeed: this would imply that there is a g such that $g\circ L_i = 0$ for 4 out 5 of the L_i, contradicting the general position), H intersects every line $\langle L_i^2, L_j^2\rangle$ in one point, and these points are distinct. These are the 10 points of $H(f)\cap X$.

2.2. Rank six cubics

Proposition 2.7 (See also [13, §91])**.** For a cubic f of the form (2.2), the scheme-theoretic intersection $H(f)\cap X_2$ consists of 4 simple and 3 double points. In particular, f lies on the Hessian discriminant.

Proof. The scheme $H(f)\cap X_2$ is supported at the 7 points $x_1^2-\lambda_1 x_0^2$, $x_2^2-\lambda_2 x_0^2$, $x_3^2-\lambda_3 x_0^2$, $\lambda_1 x_2^2-\lambda_2 x_1^2$, $\lambda_1 x_3^2-\lambda_3 x_1^2$, $\lambda_2 x_3^2-\lambda_3 x_2^2$, $x_0(\mu x_0+2\lambda_1 x_1+2\lambda_2 x_2+2\lambda_3 x_3)$, where the first three are double points. This can be verified using our code [1]. □

Remark 2.8. After identifying $H(f)$ with $\mathbb{P}^3$, the 7 points in $H(f)$ are $[0:1:0:0],[0:0:1:0],[0:0:0:1],[0:\lambda_2:-\lambda_1:0],[0:\lambda_3:0:-\lambda_1],[0:0:\lambda_2:-\lambda_3],[1:0:0:0]$, where the first 3 are double points.

Remark 2.9. There is an intuitive explanation why $H(f)\cap X_2$ contains 3 double points. As we will see in Section 3.1, a cubic of the form (2.2) can be obtained as a limit of cubics of the form $\sum_{i=1}^5 L_i^3$, where in the limit the points $L_4, L_5 \in \mathbb{P}^3$ crash together. In Remark 2.6 we saw that for cubics in pentahedral form, the 10 points in $H(f)\cap X_2$ are in bijection with the 10 lines between the 5 points $L_i^2 \in \mathbb{P}^9$. Now if 2 of our points crash together, these 10 lines become 4 single lines and 3 double lines. We will now make this more precise.

For a general rank 6 cubic surface

$$f := L_1^3 + L_2^3 + L_3^3 + L_4^2 M = 0$$

(with L_1, L_2, L_3, L_4, M in general position) let Z be the nonreduced scheme of length 2 supported at L_4 in direction M, i.e. $I(Z) = I(L_4)^2 + I(\langle L_4, M\rangle)$. Then $Y = L_1 \cup L_2 \cup L_3 \cup Z$ is a length 5 apolar scheme of f. Hence $H(f) \subset \langle v_2(Y)\rangle$.

Note that $v_2(Y) = \langle L_1^2, L_2^2, L_3^2, L_4^2, L_4 M\rangle$. From this we can see that $\langle v_2(Y)\rangle \cap X_2$ contains the 4 lines $\langle L_1^2, L_2^2\rangle$, $\langle L_1^2, L_3^2\rangle$, $\langle L_2^2, L_3^2\rangle$ and $\langle L_4^2, L_4 M\rangle$, as well as three double lines defined by $I(\langle L_i^2, L_4^2\rangle)^2 + I(\langle L_i^2, L_4^2, L_4 M\rangle)$. As before, from the fact that L_i and M are general we can see that $\langle v_2(Y)\rangle \cap X_2$ consists precisely of these lines.

Now, $H(f)\cap X_2$ is the intersection of $\langle v_2(Y)\rangle \cap X_2$ with a hyperplane H. Since $H(f)$ does not contain any of the L_i^2, H intersects every of our 7 lines in 1 (possibly fat) point. These are the 7 points of $H(f)\cap X_2$.

Remark 2.10. For cubics of the form (2.3), we can use our code to show that $H(f)\cap X_2$ consists of 3 triple points and one single point. After identifying $H(f)$ with $\mathbb{P}^3$, the 4 points in $H(f)$ are $[0:1:0:0], [0:0:1:0], [0:0:0:1], [0:1:-\mu_1:0]$, where the first 3 are triple points. In particular, cubics of the form (2.3) also lie on the Hessian discriminant, and we recover Observation 1.2.

2.3. Generic singular cubics

Proposition 2.11. A general singular cubic does not lie on the Hessian discriminant.

Proof. It suffices to find one singular cubic that does not lie on the Hessian discriminant. One way of doing this is by generating a random one of the form (2.5) and using our code. Here we will instead exhibit a very specific example: the Cayley cubic, given by

$$f := x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0,$$

with 4 singular points $[1:0:0:0]$, $[0:1:0:0]$, $[0:0:1:0]$, $[0:0:0:1]$.

Then $H(f)$ is the linear span of the quadratic forms

$$x_1x_2 + x_1x_3 + x_2x_3, x_0x_2 + x_0x_3 + x_2x_3, x_0x_1 + x_0x_3 + x_1x_3, x_0x_1 + x_0x_2 + x_1x_2,$$

and $H(f)\cap X_2$ consists of the following 10 distinct points:

$$\begin{aligned} x_0(x_1+x_2+x_3), x_1(x_0+x_2+x_3), x_2(x_0+x_1+x_3), x_3(x_0+x_1+x_2),\\ (x_0-x_1)(x_2+x_3), (x_0-x_2)(x_1+x_3), (x_0-x_3)(x_1+x_2),\\ (x_1-x_2)(x_0+x_3), (x_1-x_3)(x_0+x_2), (x_2-x_3)(x_0+x_1). \end{aligned}$$

This shows that the f does not lie on the Hessian discriminant. □

Remark 2.12. After identifying $H(f)$ with $\mathbb{P}^3$, the 10 points in $H(f)$ are $[1:1:1:-1], [1:1:-1:1], [1:-1:1:1], [-1:1:1:1], [1:-1:0:0], [1:0:-1:0], [1:0:0:-1], [0:1:-1:0], [0:1:0:-1], [0:0:1:-1]$. It can easily be verified that these 10 points, together with the 4 singular points of f, are precisely the 14 singular points of the Hessian surface of f.

2.4. Proof of Proposition 1.3

We now have the required background to give a proof of Proposition 1.3.

Proof of Proposition 1.3. We can verify the proposition separately for the four cases in Theorem 2.1. For the first three cases, the singularities of the Hessian surface were computed in [3, §§1.5, 5.2, 5.3]; they agree with the points in $H(f) \cap X_2$ that we obtained in Remarks 2.5,2.8 and 2.10. For the final case, the Hessian matrix is equal to

$$3\begin{pmatrix} 2x_0 & 0 & 0 & 0 \\ 0 & 2x_1 & -\lambda x_3 & -\lambda x_2 \\ 0 & -\lambda x_3 & 2x_2 & -\lambda x_1 \\ 0 & -\lambda x_2 & -\lambda x_1 & 2x_3 \end{pmatrix},$$

Hence the Hessian surface $\mathrm{Hess}(f)$ is the union of the plane L defined by $x_0 = 0$ and cone C defined by $(4-\lambda^3)x_1x_2x_3 - \lambda^2(x_1^3 + x_2^3 + x_3^3) = 0$. If $\lambda \neq 0$, one checks (using the assumption $\lambda^3 + 8 \neq 0$) that $\mathrm{Sing}(\mathrm{Hess}(f)) = (L \cap C) \cup \{[1 : 0 : 0 : 0]\}$; if $\lambda = 0$ then $\mathrm{Sing}(\mathrm{Hess}(f))$ is the union of the six lines $x_i = x_j = 0$. In both cases we see that every singular point in the Hessian surface of f gives a rank ≤ 2 matrix, proving our result. □

3. Fundamental invariants

The natural action of $SL(4)$ on V induces an action on the space $S^3(V^*)$ of quaternary cubics, which in turn induces an action on the polynomial ring $R = \mathbb{C}[c_{000}, \ldots, c_{333}]$ in the 20 coefficients of a quaternary cubic. Then the invariant ring $R^{SL(4)}$ is the ring of all polynomials in the coefficients of a cubic surface that are invariant under a (determinant 1) linear change of coordinates. It was shown by Salmon [10] that $R^{SL(4)}$ is generated by polynomials $I_8, I_{16}, I_{24}, I_{32}, I_{40}, I_{100}$, where I_d has degree d. The first 5 are algebraically independent and I_{100}^2 can be written as a polynomial in $I_8, I_{16}, I_{24}, I_{32}, I_{40}$. Using the connectedness of $SL(4)$, one can show that the fundamental invariants I_d are irreducible. The expressions for I_d in terms of c_{ijk} are hard to obtain and too long to write down here. However, it is easy to write them down for cubics in Sylvester normal form.

For a cubic of the form (2.1), we write

$$\begin{aligned}
\sigma_1 = &\quad c_0 + c_1 + c_2 + c_3 + c_4 \\
\sigma_2 = &\quad c_0c_1 + c_0c_2 + c_0c_3 + c_0c_4 + c_1c_2 + c_1c_3 + c_1c_4 + c_2c_3 + c_2c_4 + c_3c_4 \\
\sigma_3 = &\quad c_0c_1c_2 + c_0c_1c_3 + c_0c_1c_4 + c_0c_2c_3 + c_0c_2c_4 \\
&\quad + c_0c_3c_4 + c_1c_2c_3 + c_1c_2c_4 + c_1c_3c_4 + c_2c_3c_4 \\
\sigma_4 = &\quad c_0c_1c_2c_3 + c_0c_1c_2c_4 + c_0c_1c_3c_4 + c_0c_2c_3c_4 + c_1c_2c_3c_4 \\
\sigma_5 = &\quad c_0c_1c_2c_3c_4.
\end{aligned}$$

Then we can write the fundamental invariants as follows:

$$\begin{aligned}
I_8 = &\quad \sigma_4^2 - 4\sigma_3\sigma_5 \\
I_{16} = &\quad \sigma_5^3\sigma_1 \\
I_{24} = &\quad \sigma_5^4\sigma_4 \\
I_{32} = &\quad \sigma_5^6\sigma_2 \\
I_{40} = &\quad \sigma_5^8.
\end{aligned}$$

Remark 3.1. The tuple $[I_8, I_{16}, I_{24}, I_{32}, I_{40}]$ gives a point in weighted projective space $\mathbb{P}(1,2,3,4,5)$. We will denote this point by $\mathrm{Inv}(\mathcal{C})$.

3.1. Computing invariants for cubics of higher rank

A general cubic of the form (2.2) has Waring rank 6 [14], i.e. cannot be written as a sum of 5 cubes. However, since a generic quaternary cubic can be brought in Sylvester normal form, *any* quaternary cubic $\mathcal{C}$ can be arbitrarily closely approximated by cubics in Sylvester normal form.

We do this for cubics of the form (2.2). Fix a cubic $\mathcal{C}$ with equation

$$f(x_0,x_1,x_2,x_3) := x_1^3 + x_2^3 + x_3^3 - x_0^2(\mu x_0 + 3\lambda_1 x_1 + 3\lambda_2 x_2 + 3\lambda_3 x_3) = 0.$$

For every $\varepsilon \in \mathbb{C}^*$, we define a cubic $\mathcal{C}_\varepsilon$ with equation

$$\begin{aligned}
f_\varepsilon(x_0,x_1,x_2,x_3) := &\frac{1}{\lambda_1^3\varepsilon^3}(\varepsilon\lambda_1x_1)^3 + \frac{1}{\lambda_2^3\varepsilon^3}(\varepsilon\lambda_2x_2)^3 + \frac{1}{\lambda_3^3\varepsilon^3}(\varepsilon\lambda_3x_3)^3 + \\
&(\frac{1}{\varepsilon} - \mu)x_0^3 + \frac{1}{\varepsilon}(-x_0 - \varepsilon\lambda_1x_1 - \varepsilon\lambda_2x_2 - \varepsilon\lambda_3x_3)^3 = 0.
\end{aligned}$$

Note that $\lim_{\varepsilon\to 0} f_\varepsilon = f$.

For every fixed ε, we can compute $\mathrm{Inv}(\mathcal{C}_\varepsilon) \in \mathbb{P}(1,2,3,4,5)$ (see our code at [1]). Taking the limit $\varepsilon \to 0$ gives us

$$\begin{aligned}
\mathrm{Inv}(\mathcal{C}) = \quad &[\mu^2 - 4(\lambda_0^3 + \lambda_1^3 + \lambda_2^3) : \lambda_0^3\lambda_1^3 + \lambda_0^3\lambda_2^3 + \lambda_1^3\lambda_2^3 : 2\lambda_0^3\lambda_1^3\lambda_2^3 : \\
&\lambda_0^3\lambda_1^3\lambda_2^3(\lambda_0^3 + \lambda_1^3 + \lambda_2^3) : 0],
\end{aligned}$$

as was already computed in [3, Theorem 6.6]. In particular, we can deduce the following result (see also [4, Chapter 9.4.5]):

Proposition 3.2. For a general smooth cubic of rank 6, it holds that $I_{40} = 0$.

4. Proof of the main theorem

Theorem 4.1. Let HD be the degree 120 polynomial in $c_{000},\dots,c_{333}$ obtained by evaluating the Hurwitz form Hu_X of the variety of rank 2 matrices in the Plücker coordinates of $H(f)$, where f defines a general cubic surface. Then $HD = I_{40}^3$, where I_{40} is the degree 40 Salmon invariant.

Proof. We will show that $V(HD) = V(I_{40})$. Then the result follows from the fact that $\deg(HD) = 3\deg(I_{40})$.

The set of cubics that can be brought in the form (2.2) is of codimension one (see Theorem 2.1), lies on the Hessian discriminant by Proposition 2.7, and satisfies $I_{40} = 0$ by Proposition 3.2. This, together with irreducibility of I_{40}, implies that $V(HD) \supseteq V(I_{40})$.

Now if this were a strict inclusion, this would mean that $HD = I_{40}\cdot g$, and so $V(HD) = V(I_{40}) \cup V(g)$, with $V(g) \neq V(I_{40})$. Then $V(g)$ is a codimension one set of cubics lying on the Hessian discriminant. But Corollary 2.3, and Propositions 2.4 and 2.11 show that the set of cubics on $V(HD)$ that cannot be brought in the form (2.2) is of codimension greater than one, so we reach a contradiction. ☐

As pointed out in [7], there is an intuitive reason why HD must be a cube: it follows from the fact that as soon as f lies on the Hessian discriminant, the number of points in $H(f)\cap X_2$ drops from 10 to 7, and not to 9 as expected. This should somehow mean that "f is a triple zero of HD". We can make this intuition precise using the following general lemma about the Hurwitz form:

Lemma 4.2. Let $X \subseteq \mathbb{P}^n$ be a variety of codimension d and degree p, and let Hu_X be its Hurwitz form in Stiefel coordinates. Denote $\mathcal{H} = V(\mathrm{Hu}_X) \subseteq \mathbb{P}^{d(n+1)-1}$, and assume that for a general point L in $\mathcal{H}$, it holds that $L\cap X$ has exactly 1 double point and $p-2$ simple points. Let furthermore $\mathbb{P}^\ell$ be a linear subspace of $\mathbb{P}^{d(n+1)-1}$ such that for a general point L in $\mathcal{H}\cap\mathbb{P}^\ell$, it holds that $L\cap X$ has exactly k double points and $p-2k$ simple points. Then $\deg(\mathcal{H}) = k\deg(\mathcal{H}\cap\mathbb{P}^\ell)$, where $\mathcal{H}\cap\mathbb{P}^\ell$ is the set-theoretic intersection. Hence $\mathrm{Hu}_X|_{\mathbb{P}^\ell}$ is a k-th power.

Proof. We consider the incidence correspondence

$$\Phi = \{(p,L) | p \in L\cap X \text{ is a double point}\} \subseteq \mathbb{P}^n \times \mathbb{P}^{d(n+1)-1},$$

and

$$\Phi' = \Phi \cap (\mathbb{P}^n \times \mathbb{P}^\ell).$$

For a subvariety Y of $\mathbb{P}^a \times \mathbb{P}^b$, we write $\deg_2(Y)$ for the number of points in the intersection of Y with a general linear space $\mathbb{P}^a \times M$ of the correct codimension.

Then

$$\deg(\mathcal{H}) = \deg_2(\Phi) = \deg_2(\Phi') = k\deg(\mathcal{H} \cap \mathbb{P}^\ell).$$

□

Corollary 2.3 and Propositions 2.4, 2.7 and 2.11 imply that if we choose $X = X_2$ and $\mathbb{P}^\ell = \{H(f) | f \in S^3(V^*)\}$, the conditions of Lemma 4.2 are satisfied, hence HD must be a cube.

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