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A homotopy theorem for Oka theory

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Abstract

We prove a homotopy theorem for sheaves. Its application shortens and simplifies the proof of many Oka principles such as Gromov's Oka principle for elliptic submersions.

1 Introduction

1.1 Motivation

Oka theory is the art of reducing proofs in complex geometry to purely topological statements. Its applications reach beyond complex geometry; for example to the study of minimal surfaces [1]. The power of the theory lies in the fact that there are many problems—some of them almost a century old—for which Oka theory provides the only known approaches. Examples include the following theorems.

Theorem (Grauert [10]) *Two complex analytic vector bundles over a Stein base which are isomorphic as complex topological vector bundles are complex analytically isomorphic.*

Theorem (Forster, Ramspott [5]) *The ideal sheaf of a smooth complex analytic curve in a Stein manifold X of dimension $n \geq 3$ is generated by $n - 1$ holomorphic functions $X \to \mathbb{C}$.*

Theorem (Gromov [12]) *Every continuous map from a Stein manifold X to $\mathbb{C}^n \setminus Y$ is homotopic to a holomorphic map given that $Y \subset \mathbb{C}^n$ is an algebraic subvariety of codimension at least 2.*

Theorem (Leiterer [16]) *For holomorphic maps $a, b \colon X \to \mathrm{Mat}(n \times n, \mathbb{C})$ defined on a Stein manifold X the equation $f(x)a(x)f(x)^{-1} = b(x)$, $x \in X$ has a holomorphic solution $f \colon X \to \mathrm{GL}_n(\mathbb{C})$ if there is a smooth solution of the same equation.*

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Theorem (Kutzschebauch, Lárusson, Schwarz [15]) *A holomorphic action of a complex reductive Lie group G on $\mathbb{C}^n$ is linearizable if there is a smooth G-diffeomorphism from $\mathbb{C}^n$ to a G-module which induces a biholomorphism on the corresponding categorical quotients, and on every reduced fiber of the quotient map.*

All known proofs of these results depend on a specific Oka principle. That is, roughly speaking, on a theorem which states that there are only topological obstructions to a complex analytic solution of an associated problem. In concrete terms, Grauert's result is proved using the Oka principle for principal G-bundles [3,10]. All others depend on extensions of Grauert's work, namely on the Oka principle for admissible pairs of sheaves [4] in the case of Forster and Ramspott's and Leiterer's results, on the Oka principle for elliptic submersions [9,12] in the case of Gromov's result, and on the Oka principle for equivariant isomorphisms [14] in the case of the result due to Kutzschebauch, Lárusson and Schwarz. The first three theorems can be proved alternatively with Forstnerič's Oka principles for stratified fiber bundles [7], a generalization of Gromov's work to stratified settings including more general fibers and possibly non-smooth base spaces. Complete proofs of these powerful tools fill a book. However, a careful study of the literature reveals that all proofs of the cited work can be divided into a rather analytic first part and a purely topological second part; and that the topological part can be formulated very generally, thus providing a reduction of the proofs to the analytic key difficulties. This general topological statement is Theorem 1 from the present text: its assumptions state which key properties one has to show in the first part of the proof of an Oka principle, its conclusion is an Oka principle. Theorem 1 extends Gromov's homomorphism theorem from [11] so that it applies in complex analytic settings and carries out ideas sketched in [12]. Its proof builds on ideas of Gromov [11,12] and on the work of Forstnerič and Prezelj [9], who have carried out many steps of the proof of Theorem 1 in the special case of elliptic submersions. A detailed exposition can be found in Chapter 6 of [8]. References to the sections in [3,4,9,12,14] providing the analytic key ingredients from the assumptions in Theorem 1 are given in the appendix.

1.2 Results

To formulate Theorem 1 let us recall some basic notions of complex geometry. Let X be a reduced complex space. A compact $C \subset X$ is called a Stein compact if it admits a basis of Stein neighborhoods in X. If $C \subset B$ for some $B \subset X$, then the compact C is called $\mathcal{O}(B)$-convex if $C = \{p \in B \colon |f(p)| \leq \max_{x \in C} |f(x)|, \ f \in \mathcal{O}(B)\}$. Here, $\mathcal{O}(B)$ denotes the set of all holomorphic functions defined on unspecified neighborhoods of B.

Definition 1.1 Let X be a complex space and let $A, B \subset X$. The ordered pair (A, B) is a $\mathcal{C}$-*pair* if

(1) A, B, $A \cap B$ and $A \cup B$ are Stein compacts,
(2) $A \cap B$ is $\mathcal{O}(B)$-convex, and
(3) $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$.

A convenient notion to formulate any question arising naturally in Oka theory is the notion of a sheaf of topological spaces.

Definition 1.2 Let X be a topological space. A sheaf of topological spaces Φ on X is a sheaf $U \mapsto \Phi(U)$, $U \subset X$ open, whose sets of local sections are topological spaces, whose restrictions are continuous, and which is well-behaved in the sense that for the closed unit ball $\mathbb{D} \subset \mathbb{R}^n$ of any real dimension $n \geq 1$ the presheaf $U \mapsto \Phi^{\mathbb{D}}(U)$ is in fact a sheaf. Here $\Phi^{\mathbb{D}}(U)$ denotes the set of continuous maps $\mathbb{D} \to \Phi(U)$.

It is more common to ask for $U \to \Phi^Y(U)$ being a sheaf for any topological space Y. However, we stick to the given definition since it reflects what we need in this text. A sheaf of topological spaces is said to be metric if every set of local sections is a metric space, and a metric sheaf is said to be complete if every set of local sections is a complete metric space. We say that $\Phi \hookrightarrow \Psi$ is an inclusion of sheaves of topological spaces if Φ is a subsheaf of the sheaf of topological spaces Ψ whose sets of local sections are endowed with the subspace topology of the sets of local sections of Ψ.

We now define those properties for sheaves of topological spaces which the first part of a proof of an Oka principle is usually concerned with (Definitions 1.3 and 1.5). In the following s, t denote numbers in the unit interval and by using them as subscripts we always indicate a homotopy in the common sense. We use the convention that a singleton (with empty boundary) is the closed unit ball of dimension $n = 0$. For a subset A of X we denote by A° its interior. A restriction $\Psi(U) \to \Psi(V)$ of a sheaf Ψ will be denoted by r_V if there is no ambiguity regarding the domain.

Definition 1.3 An inclusion of sheaves of topological spaces $\Phi \hookrightarrow \Psi$ over a topological space X is a local weak homotopy equivalence if for every point $p \in X$, every open neighborhood U of p and every continuous map f from the closed unit ball $\mathbb{D} \subset \mathbb{R}^n$ of any real dimension $n \geq 0$ to $\Psi(U)$ with $f(\partial \mathbb{D}) \subset \Phi(U)$, there is a neighborhood $p \in V \subset U$ and a homotopy $f_t \colon \mathbb{D} \to \Psi(V)$ such that $f_0 = r_V \circ f$, $f_t|\partial\mathbb{D}$ is independent of t and f_1 has values in $\Phi(V)$.

Remark 1.4 Local weak homotopy equivalences were introduced in [11]. Heuristically a local weak homotopy equivalence of sheaves $\Phi \hookrightarrow \Psi$ is a weak homotopy equivalence at the level of stalks, meaning that $\Phi_p \hookrightarrow \Psi_p$ is a weak homotopy equivalence for every $p \in X$. However there is no suitable topology on the stalks to make this heuristic a precise statement.

Definition 1.5 Let Φ be a sheaf of topological spaces on X and let $A, B \subset X$ be compact. The ordered pair (A, B) is called weakly flexible for Φ if the following holds. Given open neighborhoods U, V resp. W of A, B resp. $A \cap B$ and a triple of maps $a \colon \mathbb{D} \to \Phi(U)$, $b \colon \mathbb{D} \to \Phi(V)$ and $c_s \colon \mathbb{D} \to \Phi(W)$ such that $c_0 = r_W \circ a$, $c_1 = r_W \circ b$ and $c_s|\partial\mathbb{D}$ is independent of s, there are smaller neighborhoods $A \subset U' \subset U$, $B \subset V' \subset V$ resp. $A \cap B \subset W' \subset W$ and homotopies $a_t \colon \mathbb{D} \to \Phi(U')$, $b_t \colon \mathbb{D} \to \Phi(V')$ and $c_{s,t} \colon \mathbb{D} \to \Phi(W')$ with $a_0 = r_{U'} \circ a$, $b_0 = r_{V'} \circ b$ and $c_{s,0} = r_{W'} \circ c_s$ such that

(1) $c_{0,t} = r_{W'} \circ a_t$ and $c_{1,t} = r_{W'} \circ b_t$,
(2) the restrictions $a_t|\partial\mathbb{D}$, $b_t|\partial\mathbb{D}$ and $c_{s,t}|\partial\mathbb{D}$ are independent of t,

(3) $c_{s,1}$ is independent of s, and
(4) $r_{A^\circ} \circ a_t$ is in a prescribed neighborhood of $r_{A^\circ} \circ a_0 \colon \mathbb{D} \to \Phi(A^\circ)$ with respect to the compact open topology for all t.

If (A, B) is a weakly flexible pair for Φ such that the homotopy a_t from the conclusion of the definition can be chosen to satisfy $r_{A^\circ} \circ a_t = r_{A^\circ} \circ a_0$ for all t, then (A, B) is called an ordered flexible pair.

Remark 1.6 Showing that $\mathcal{C}$-pairs (A, B) are weakly flexible for a given sheaf Φ is often the main work in the proof of an Oka principle.

Theorem 1 *Let X be a second countable reduced Stein space and let $\Phi \hookrightarrow \Psi$ be a local weak homotopy equivalence of sheaves of topological spaces on X. Assume that one of the two following statements holds:*

(1) *Φ is complete metric and every point $p \in X$ has a neighborhood U such that every $\mathcal{C}$-pair (A, B) with $B \subset U$ is weakly flexible for Φ.*
(2) *Every point $p \in X$ has a neighborhood U such that every $\mathcal{C}$-pair (A, B) with $B \subset U$ is ordered flexible for Φ.*

Then $\Phi(X) \neq \emptyset$ if and only if $\Psi(X) \neq \emptyset$. Moreover, if Ψ is likewise in either of the two classes of sheaves, then $\Phi(X) \hookrightarrow \Psi(X)$ is a weak homotopy equivalence.

Remark 1.7 Since analytic continuation is unique, assumption (2) from Theorem 1 is too strong to be satisfied by a sheaf Φ of analytic maps, even in the most basic case $\Phi = \mathcal{O}_{\mathbb{C}}$—the sheaf of holomorphic functions in one complex variable. In particular assumption (1) is the property that one tries to show if Φ is a complex analytic sheaf.

Remark 1.8 Many sheaves of interest in Oka theory are complete metric. However, there are exceptions. Such exceptions—discussed briefly in the subsection ordered flexibility of the appendix—motivated to include assumption (2) as an alternative to assumption (1) in Theorem 1.

As we will see Theorem 1 follows from a more abstract homotopy theorem, namely Theorem 2. To state it we need

Definition 1.9 A weakly flexible string for Φ of length $n \geq 2$ is recursively defined as a finite sequence $(A_1, A_2, A_3, \ldots, A_n)$ of compacts of X such that

(1) $(A_1 \cup \cdots \cup A_{n-1}, A_n)$ is a weakly flexible pair for Φ, and, if $n \geq 3$, then
(2) $(A_1, \ldots, A_{n-1})$ and $(A_1 \cap A_n, \ldots, A_{n-1} \cap A_n)$ are weakly flexible strings for Φ.

A weakly flexible cover for Φ is a locally finite cover $(A_1, A_2, A_3, \ldots)$ of X such that for every $n \in \mathbb{N}$ $(A_1, A_2, \ldots, A_n)$ is a weakly flexible string for Φ. We say that Φ is weakly flexible if every open cover of X can be refined by a weakly flexible cover for Φ. Similarly we define ordered flexible sheaves. That is, ordered flexible strings, covers and sheaves are defined by substituting each occurrence of weakly by ordered in the definition of weakly flexible strings, covers and sheaves.

In the following the dimension of a topological space always means the covering dimension.

Theorem 2 *Let X be a paracompact Hausdorff space that has an exhaustion by finite dimensional compact subsets and let $\Phi \hookrightarrow \Psi$ be a local weak homotopy equivalence of sheaves of topological spaces on X. Assume that Φ is either*

(1) *complete metric and weakly flexible, or*
(2) *ordered flexible.*

Then $\Phi(X) \neq \emptyset$ if and only if $\Psi(X) \neq \emptyset$. Moreover, if Ψ is likewise in either of the two classes of sheaves, then $\Phi(X) \hookrightarrow \Psi(X)$ is a weak homotopy equivalence.

Remark 1.10 For a sheaf of topological spaces Φ one can show that flexibility in the sense of Gromov (for a definition see [11]) implies ordered flexibility, that ordered flexibility implies weak flexibility, and that both converse implications are wrong.

Remark 1.11 Gromov's *homomorphism theorem* [11], p. 77 says that if $\Phi \hookrightarrow \Psi$ is a local weak homotopy equivalence and both Φ and Ψ are flexible, then $\Phi(X) \hookrightarrow \Psi(X)$ is a weak homotopy equivalence. Therefore Theorem 2 extends Gromov's result and it follows from Remarks 1.7 and 1.10 that this extension is necessary when working in analytic settings.

2 Proofs

In the first two subsections we define relevant notions and establish preliminaries. In the third subsection the so-called initial complex is constructed (Proposition 2.9), and in the fourth subsection we glue families of local sections to a global one (Proposition 2.15). This part follows at some points closely Prezelj [18], where the corresponding work is done in the special case of sheaves of sections of elliptic submersions, see also [8,9]. In the last subsection Theorems 1 and 2 are proved.

2.1 Parametric sheaves

For topological spaces Y and Z, Z^Y denotes the space of continuous maps $Y \to Z$ equipped with the compact open topology.

Lemma 2.1 *Let P, Y, Z be topological spaces where Y is locally compact Hausdorff. Then a map $f : P \times Y \to Z$ is continuous if and only if the map $\tilde{f} : P \to Z^Y$ defined by $\tilde{f}(p) := f(p, \cdot)$ is continuous.*

Proof See e.g. Theorem 46.11 in [17]. □

Let Φ be a sheaf of topological spaces on a topological space X and let $\mathbb{D} \subset \mathbb{R}^n$ be the closed unit ball of any real dimension $n \geq 0$. Then $U \mapsto \Phi^{\mathbb{D}}(U)$ is a sheaf by definition. We equip the sets of local sections with the compact open topology and note that the restrictions are continuous.

Lemma 2.2 *The sheaf $\Phi^{\mathbb{D}}$ is a sheaf of topological spaces.*

Proof We have to show that $U \mapsto (\Phi(U)^{\mathbb{D}})^{\mathbb{D}'}$ is a sheaf for the closed unit ball $\mathbb{D}' \subset \mathbb{R}^m$ of any real dimension $m \geq 0$. By applying Lemma 2.1 to $P = \mathbb{D}'$, $Y = \mathbb{D}$ and $Z = \Phi(U)$ we get a natural correspondence of $(\Phi^{\mathbb{D}})^{\mathbb{D}'}(U)$ and $\Phi^{\mathbb{D}' \times \mathbb{D}}(U)$. This correspondence yields an isomorphism of presheaves

$$(\Phi^{\mathbb{D}})^{\mathbb{D}'} \cong \Phi^{\mathbb{D}' \times \mathbb{D}}.$$

Since Φ is a sheaf of topological spaces and $\mathbb{D}' \times \mathbb{D}$ is homeomorphic to the closed unit ball of dimension $n + m$, $\Phi^{\mathbb{D}' \times \mathbb{D}}$ is a sheaf, hence so is $(\Phi^{\mathbb{D}})^{\mathbb{D}'}$.   □

For a continuous map $\alpha \colon \partial \mathbb{D} \to \Phi(X)$ defined on the boundary of the closed unit ball $\mathbb{D} \subset \mathbb{R}^n$ set $\Phi_\alpha(U) = \{f \in \Phi^{\mathbb{D}}(U) \colon f|\partial\mathbb{D} = r_U \circ \alpha\}$ for an open $U \subset X$. Note that these sets of local sections define a subsheaf Φ_α of $\Phi^{\mathbb{D}}$ and inherit the structure of a sheaf of topological spaces by taking the subspace topology on $\Phi_\alpha(U) \subset \Phi^{\mathbb{D}}(U)$ on each set of local sections.

We give an alternative definition of a weakly flexible pair (Lemma 2.3). Let $\Phi(U, V)$ denote the path product of $\Phi(U)$ and $\Phi(V)$, that is

$$\{(a, b, c) \in \Phi(U) \times \Phi(V) \times \Phi^{[0,1]}(U \cap V) \colon r_{U \cap V}(a) = c(0),\ r_{U \cap V}(b) = c(1)\},$$

and let us equip $\Phi(U, V)$ with the subspace topology induced by the product topology on $\Phi(U) \times \Phi(V) \times \Phi^{[0,1]}(U \cap V)$. Moreover we identify the image of the inclusion

$$\Phi(U \cup V) \hookrightarrow \Phi(U, V), \quad f \mapsto (r_U(f), r_V(f), r_{U \cap V}(f))$$

with $\Phi(U \cup V)$. For $U' \subset U$ and $V' \subset V$ the map $r_{U',V'} \colon \Phi(U, V) \to \Phi(U', V')$ is defined as $r_{U',V'}(f_1, f_2, f_3) = (r_{U'}(f_1), r_{V'}(f_2), r_{U' \cap V'}(f_3))$. Moreover let $p \colon \Phi(U, V) \to \Phi(U)$ be the projection onto the first component. The following Lemma is straightforward. We leave the proof to the reader.

Lemma 2.3 *Let Φ be a sheaf of topological spaces on X and let $A, B \subset X$ be compact. The pair (A, B) is weakly flexible for Φ if and only if the following holds. Given open neighborhoods U and V of A and B and a map $f \colon \mathbb{D} \to \Phi(U, V)$ with $f(\partial\mathbb{D}) \subset \Phi(U \cup V)$, there are smaller neighborhoods U' and V' of A and B and a homotopy $g_t \colon I \times \mathbb{D} \to \Phi(U', V')$ such that $g_0 = r_{U',V'} \circ f$, g_1 has values in $\Phi(U' \cup V')$, $g_t|\partial\mathbb{D}$ is independent of t and $r_A \circ p \circ g_t$ stays in a prescribed neighborhood of $r_A \circ p \circ g_0$ for all $t \in [0, 1]$.*

Lemma 2.4 *Let Φ be a sheaf of topological spaces on a topological space X and let $\alpha \colon \partial\mathbb{D} \to \Phi(X)$ be continuous, where $\mathbb{D} \subset \mathbb{R}^n$ is the closed unit ball of any real dimension $n \geq 0$. Then, if any of the following properties holds for $\Theta = \Phi$, the same property holds likewise for $\Theta = \Phi_\alpha$:*

(1) *Θ is a sheaf of topological spaces.*
(2) *Θ has the structure of a complete metric sheaf.*
(3) *A family of compacts $(A_1, A_2, \ldots, A_n)$ is weakly flexible for Θ.*
(4) *A family of compacts $(A_1, A_2, \ldots, A_n)$ is ordered flexible for Θ.*

(5) Θ *is weakly flexible.*
(6) Θ *is ordered flexible.*

Moreover, if $\Phi \hookrightarrow \Psi$ *is a local weak homotopy equivalence of sheaves, then so is* $\Phi_\alpha \hookrightarrow \Psi_\alpha$.

Proof It has been discussed already that Φ_α is a sheaf of topological spaces if Φ is. If $\Phi(U)$ is a metric space with metric d, then

$$d_\alpha(f, g) := \sup\{d(f(p), g(p)) \colon p \in \mathbb{D}\}$$

defines a metric on $\Phi_\alpha(U)$ which agrees with the compact open topology. Moreover if $\Phi(U)$ is complete, then so is $\Phi_\alpha(U)$ since limits of continuous maps from $\mathbb{D}$ to complete spaces are continuous. For (3) and (5) it suffices to show that any weakly flexible pair (A, B) for Φ is weakly flexible for Φ_α. We use the alternative definition of a weakly flexible pair given in Lemma 2.3. Let $f \colon \mathbb{D}' \mapsto \Phi_\alpha(U, V)$ be a map as in this assumption (see Lemma 2.3), where $\mathbb{D}' \subset \mathbb{R}^m$ denotes the closed unit ball of dimension $m \geq 0$. By Lemma 2.1 f can be canonically identified with a continuous map $g \colon \mathbb{D}' \times \mathbb{D} \to \Phi(U, V)$ with

$$g(\partial(\mathbb{D}' \times \mathbb{D})) = g(\partial\mathbb{D}' \times \mathbb{D} \cup \mathbb{D}' \times \partial\mathbb{D}) \subset \Phi(U \cup V).$$

Since (A, B) is a weakly flexible pair for Φ and $\mathbb{D}' \times \mathbb{D}$ is a disc of dimension $m + n$, there are neighborhoods U' and V' of A resp. B and a homotopy g_t connecting $r_{U', V'} \circ g$ to some g_1 with values in $\Phi(U' \cup V')$, g_t is independent of t when restricted to the boundary, and $r_{A\circ} \circ p \circ g_t$ stays in a prescribed neighborhood of $r_{A\circ} \circ p \circ g_0$. Since g_t is independent of t when restricted to $\mathbb{D}' \times \partial\mathbb{D}$, g_t is then again canonically identified with a homotopy

$$f_t \colon \mathbb{D}' \to \Phi_\alpha(U', V')$$

satisfying $f_0 = r_{U', V'} \circ f$. Since g_t is also independent of t when restricted to $\partial\mathbb{D}' \times \mathbb{D}$, $f_t|\partial\mathbb{D}'$ is independent of t as well. It follows from g_1 having values in $\Phi(U' \cup V')$ that f_1 has values in $\Phi_\alpha(U' \cup V')$. The fact that $r_{A\circ} \circ p \circ f_t$ approximates $r_{A\circ} \circ p \circ f_0$ as well as we desire follows immediately from the corresponding approximation property of g_t. This shows that (A, B) is weakly flexible for Φ_α and concludes the proof of (3) and (5). The proof that an ordered flexible pair (A, B) for Φ is also ordered flexible for Φ_α is similar (but a little simpler). Statement (4) and (6) follow immediately. Also the proof that $\Phi_\alpha \hookrightarrow \Psi_\alpha$ is a weak homotopy equivalence if $\Phi \hookrightarrow \Psi$ is can be obtained in the same manner as the proof that weakly flexible pairs for Φ are weakly flexible for Φ_α. We leave the details to the reader. $\qquad\square$

2.2 Complexes

In this subsection we define the notion of a complex. The definition is a natural generalization of the one used by Fortnerič and Prezelj, see [9] and Chapter 6 of [8].

Let $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}}$ be a point-finite family of subsets of some topological space X (that is, every element of X is contained in at most finitely many elements of $\mathcal{A}$). The *nerve* $N(\mathcal{A})$ of $\mathcal{A}$ is defined as the set of (finite) subsets I of the index set $\mathcal{I}$ for which the intersection $A_I := \cap_{i \in I} A_i$ is non-empty. Let E be the real vector space spanned by linearly independent vectors $\{e_i\}_{i \in \mathcal{I}}$ and let us topologize E by the final topology induced by injective linear maps $\mathbb{R}^n \to E$, $n \in \mathbb{N}$. Clearly any finite dimensional subspace of E carries the natural topology. For $I \in N(\mathcal{A})$, denote by $|I|$ the simplex defined by the convex hull of $\{e_i : i \in I\} \subset E$. Note that the union $N|\mathcal{A}|$ of all $|I|$ with $I \in N(\mathcal{A})$ is the geometric realization of the abstract simplicial complex $N(\mathcal{A})$. We also set $N_k(\mathcal{A}) := \{I \in N(\mathcal{A}) : \dim |I| \le k\}$ and $N_k|\mathcal{A}|$ the corresponding geometric realization for $k \ge 0$. A family $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ is a *neighborhood* of $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}}$ if U_i is a neighborhood of A_i for every $i \in \mathcal{I}$. $\mathcal{U}$ is called *faithful* if its nerve is equal to the one of $\mathcal{A}$.

Lemma 2.5 *Let X be normal and $\mathcal{A}$ a countable locally finite closed cover of X. Then $\mathcal{A}$ has a closed faithful neighborhood.*

Proof Let $A \in \mathcal{A} = \{A_i\}_{i \in \mathbb{N}}$ be fixed. Recall that the union of the elements of a locally finite closed family is closed. Since $\mathcal{A}$ is locally finite, $\{A_I : I \in N(\mathcal{A})\}$ is locally finite too. In particular

$$U := X \backslash \bigcup_{\substack{I \in N(\mathcal{A}), \\ A \cap A_I = \emptyset}} A_I,$$

is an open neighborhood of A. Note that $\{U\} \cup \mathcal{A} \backslash \{A\}$ has the same nerve as $\mathcal{A}$. If $B \subset U$ is a closed neighborhood of A, then $\{B\} \cup \mathcal{A} \backslash \{A\}$ is a locally finite closed cover with the same nerve as $\mathcal{A}$. Replacing inductively A_n, $n \in \mathbb{N}$ in $\{B_1, B_2, \ldots, B_{n-1}, A_n, A_{n+1}, \ldots\}$ by such a neighborhood B_n yields for $n \to \infty$ a closed faithful neighborhood $\mathcal{B}$ of $\mathcal{A}$. $\qquad\square$

For a sheaf Ψ we say a map has values in Ψ when we actually mean a map with values in the disjoint union of all sets of local sections of Ψ. For an open set $V \subset X$, Ψ_V denotes the disjoint union of those $\Psi(U)$ with $V \subset U$. Then $r_V : \Psi_V \to \Psi(V)$ denotes the map that is given by the restriction morphisms $\Psi(U) \to \Psi(V)$ of Ψ for $V \subset U$ open. We call r_V a *restriction* and note that the new use of the symbol r_V generalizes our earlier use consistently.

Definition 2.6 Let $\mathcal{A}$ be a point-finite family of X. A *homotopy of complexes* over $\mathcal{A}$ (with respect to the sheaf Ψ) is a map

$$f : [0, 1] \times N|\mathcal{A}| \to \Psi$$

with the property that there is an open neighborhood $\mathcal{U}$ of $\mathcal{A}$ such that $f([0, 1] \times |I|) \subset \Psi_{U_I}$ and $r_{U_I} \circ f|[0, 1] \times |I|$ is continuous for all $I \in N(\mathcal{A})$. A *complex* over $\mathcal{A}$ is a map $N|\mathcal{A}| \to \Psi$ with the properties obtained by fixing $t \in [0, 1]$ in the definition of a homotopy of complexes.

If we replace the domain $N|\mathcal{A}|$ by $N_k|\mathcal{A}|$ in the definition of a complex we call the resulting map a *k-skeleton* instead. If the family $\mathcal{A}$ or the sheaf Ψ corresponding to a complex $f\colon N|\mathcal{A}| \to \Psi$ are clear from the context we will sometimes omit to mention these.

Remark 2.7 If a point-finite cover $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}}$ refines a point-finite cover $\mathcal{U} = \{U_j\}_{j \in \mathcal{J}}$, then a complex $f\colon N|\mathcal{U}| \to \Psi$ induces a complex $g\colon N|\mathcal{A}| \to \Psi$. Explicitly this can be seen by setting $g = f \circ \lambda$ where $\lambda\colon N|\mathcal{A}| \to N|\mathcal{U}|$ is the restriction of a linear map given by $e_i \mapsto e_{j(i)}$ for some $j(i)$ with $A_i \subset U_{j(i)}$.

Definition 2.8 Two complexes $f, g\colon N|\mathcal{A}| \to \Psi$ are called *equivalent* if there is a neighborhood $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ of $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}}$ such that $r_{U_I} \circ f\|I\| = r_{U_I} \circ g\|I\|$ for $I \in N(\mathcal{A})$.

It is straightforward to see that equivalence of complexes over $\mathcal{A}$ is an equivalence relation. Since in this text we care about complexes only up to equivalence, we write $f = g$ if f and g are equivalent. Note that we can compose homotopies of complexes f_t and g_t (over $\mathcal{A}$) if f_1 is equivalent to g_0 by setting $h_{t/2} = f_t$ for $t \in [0, 1]$ and $h_{t/2+1/2} = g_t$ for $t \in (0, 1]$.

2.3 The initial complex

In this subsection we prove

Proposition 2.9 *Let X be a paracompact Hausdorff space that has an exhaustion by finite dimensional closed subsets. Let $\Phi \hookrightarrow \Psi$ be a local weak homotopy equivalence of topological sheaves on X and let $f \in \Psi(X)$. Then there is a homotopy of complexes h with values in Ψ connecting the complex h_0 induced by f to a complex h_1 with values in Φ.*

The proof of Proposition 2.9 depends on some topological properties of X. For a topological space X and an open cover $\{V_j\}_{j \in \mathcal{J}}$, a *shrinking* of $\{V_j\}_{j \in \mathcal{J}}$ is an open cover $\{S_j\}_{j \in \mathcal{J}}$ (indexed by the same set $\mathcal{J}$) such that $\overline{S}_j \subset V_j$ for all $j \in \mathcal{J}$.

Remark 2.10 It is well-known that in a paracompact Hausdorff space any open cover has a shrinking.

In the following we set for a family of subset $\mathcal{U}$ of X and $A \subset X$, $\mathcal{U}_A := \{U \in \mathcal{U}\colon A \cap U \neq \emptyset\}$. Moreover $\operatorname{ord} \mathcal{U} = \sup_{x \in X} |\mathcal{U}_{\{x\}}| \in \mathbb{N} \cup \{\infty\}$ denotes the order of $\mathcal{U}$. We use the conventions $\operatorname{ord} \emptyset = 0$.

Lemma 2.11 *Every open cover $\mathcal{U}$ of a finite dimensional paracompact Hausdorff space B has an locally finite open refinement of order at most $\dim B + 1$.*

Proof Pick a locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$ and then an open refinement $\mathcal{W}$ of $\mathcal{V}$ of order at most $\dim B + 1$. Fix for every element $W \in \mathcal{W}$ an element $V(W) \in \mathcal{V}$ with $W \subset V(W)$, set

$$V' := \bigcup_{V(W)=V} W, \quad V \in \mathcal{V}$$

and $\mathcal{V}' := \{V' : V \in \mathcal{V}\}$. The open family $\mathcal{V}'$ covers B since $\mathcal{W}$ covers B. It is locally finite since $\mathcal{V}$ is locally finite, and it is easy to see that ord $\mathcal{V}' \leq$ ord $\mathcal{W} \leq \dim B + 1$. This finishes the proof. $\qquad\square$

Lemma 2.12 *Let X be a paracompact Hausdorff space, $A, B \subset X$ closed such that $A \subset B°$ and $\dim B < \infty$ and let $W \subset A$ open. Then any open family $\mathcal{U}$ of X which covers $X \backslash W$ can be refined by a locally finite open family $\mathcal{A}$ which covers $X \backslash W$ such that $\mathcal{A}_A$ is of order at most $\dim B + 1$.*

Proof Let $\mathcal{U}$ be an open family which covers $X \backslash W$. Since the union of two locally finite families remains locally finite, it suffice to find (1) a locally finite open family which refines $\mathcal{U}$, covers $X \backslash B°$ and whose elements do not intersect A and (2) a locally finite open family of order at most $\dim B + 1$ which refines $\mathcal{U}$ and covers $B° \backslash W$. For (1) we can refine $\mathcal{U} \cup \{W\}$ by a locally finite open cover of X and intersect the elements of the resulting family with the open set $X \backslash A$. For (2) apply Lemma 2.11 to the open cover $\{W\} \cup \{U \cap B : U \in \mathcal{U}\}$ of B, remove from the obtained family the elements which are contained in W and intersect the remaining elements with $B°$. The resulting family does the job. $\qquad\square$

Lemma 2.13 *Let X be a topological space, $\mathcal{V} = \{V_i\}_{i \in \mathcal{I}}$ a locally finite open cover and $\mathcal{S} = \{S_i\}_{i \in \mathcal{I}}$ a shrinking of $\{V_i\}_{i \in \mathcal{I}}$. Then for every point $x \in X$ there is an open neighborhood W of x with the property $W \subset V_I$ whenever $W \cap S_I \neq \emptyset$.*

Proof Since $\mathcal{V}$ is locally finite and $\mathcal{S}$ is a shrinking of $\mathcal{V}$ there is a neighborhood U of x that meets only finitely many $\overline{S}_i, i \in \mathcal{I}$. Therefore

$$W := U \cap \bigcap_{x \in \overline{S}_i} V_i \cap \bigcap_{\substack{x \notin \overline{S}_i, \\ U \cap \overline{S}_i \neq \emptyset}} X \backslash \overline{S}_i$$

is a finite intersection of open sets and hence an open neighborhood of x. One can check that W does the job. $\qquad\square$

The main work of the proof of Proposition 2.9 is done in Lemma 2.14, in whose proof a suitable homotopy of $k + 1$-skeletons is constructed from a given homotopy of k-skeletons.

Lemma 2.14 *Let $A_0 = A_1 = \emptyset$, let $A_2 \subset A_3 \subset A_4 \subset \cdots$ be an exhaustion of finite dimensional closed subsets of X and let $\Phi \hookrightarrow \Psi$ be a local weak homotopy equivalence of sheaves on X. Let $k \geq 0$, $\mathcal{V}$ a locally finite open cover of X such that each element of $\mathcal{V}$ is contained in $A_{i+1} \backslash A_{i-1}$ for suitable $i \in \mathbb{N}$, and let $n \in \mathbb{N}$ be some fixed number such that $\mathrm{ord}\, \mathcal{V}_{A_n} \leq k$. Moreover let*

$$g : [0, 1] \times N_k |\mathcal{V}| \rightarrow \Psi$$

be a homotopy of k-skeletons which connects the sectionally constant complex given by f to a k-skeleton with values in Φ. Then there is a locally finite open cover $\mathcal{U}$ of X refining $\mathcal{V}$ and a $k + 1$-skeleton

$$h \colon [0, 1] \times N_{k+1}|\mathcal{U}| \to \Psi$$

connecting the sectionally constant complex given by f to a $k+1$-skeleton with values in Φ satisfying the following properties: The elements of $\mathcal{U}$ are contained in $A_{i+1}\backslash A_{i-1}$ for suitable $i \in \mathbb{N}$, ord $\mathcal{U}_{A_n} \leq$ ord $\mathcal{V}_{A_n}$, ord $\mathcal{U}_{A_{n+1}} \leq$ ord $\mathcal{V}_{A_n} + \dim A_{n+2} + 1$, $\mathcal{U}_{A_{n-1}} = \mathcal{V}_{A_{n-1}}$ and

$$g|[0, 1] \times N|\mathcal{V}_{A_{n-1}}| = h|[0, 1] \times N|\mathcal{U}_{A_{n-1}}|.$$

Proof Take a shrinking $\mathcal{S}$ of $\mathcal{V} = \{V_j\}_{j\in\mathcal{J}}$ and set

$$W := \bigcup_{S\in\mathcal{S}_{A_n}} S.$$

We have $A_n \subset W$ since $\mathcal{S}$ covers X, and $W \subset A_{n+1}$ since $\mathcal{S}$ is a shrinking of $\mathcal{V}$ and the elements of $\mathcal{V}_{A_n}$ are contained in A_{n+1} by our assumption on $\mathcal{V}$. Clearly W is open. Let $x \in X\backslash W$ be fixed for the moment. We are going to choose a suitable (small) neighborhood U_x of x. By Lemma 2.13 we may pick a neighborhood $W_x \subset X\backslash A_n$ of x such that $W_x \subset V_J$ whenever $W_x \cap S_J$ is non-empty. In addition, after possibly shrinking W_x a bit, we may assume that W_x is contained in some element of $\mathcal{S}$. If for all $|J|$ of dimension $k+1$ the sets W_x and S_J are disjoint, W_x is a suitable neighborhood of x for our purpose and we set $U_x := W_x$. Otherwise consider for every $|J|$ of dimension $k + 1$ with $W_x \cap S_J \neq \emptyset$ the restriction of g to

$$Q_J = \{0\} \times |J| \cup [0, 1] \times \partial|J| \subset [0, 1] \times N|\mathcal{V}|.$$

From $W_x \cap S_J \neq \emptyset$ we get $W_x \subset V_J$ by the choice of W_x. Therefore $r_{W_x} \circ g|Q_J$ is well defined and continuous. Since the pair $\Phi \hookrightarrow \Psi$ is a local weak homotopy equivalence there is a neighborhood $U_{J,x} \subset W_x$ of x such that the map $r_{U_{J,x}} \circ g|Q_J$ extends to $[0, 1] \times |J|$ with values in $\Psi(U_{J,x})$ and the restriction to $\{1\} \times |J|$ is mapped to $\Phi(U_{J,x})$. Since x is contained in at most finitely many $S_j \in \mathcal{S}$, x is also contained in at most finitely many S_J with dim $|J| = k + 1$. Therefore we can replace every $U_{J,x}$ by the finite intersection and hence open neighborhood

$$U_x = \bigcap_{x\in S_J} U_{J,x} \subset W_x$$

of x. We summarize that if the dimension of $|J|$ is $k + 1$ and $U_x \cap S_J \neq \emptyset$, then $U_x \subset V_J$ and $r_{U_x} \circ g|Q_J$ extends to a map on $[0, 1] \times |J|$ with values in $\Psi(U_x)$ such that $\{1\} \times |J|$ is mapped to $\Phi(U_x)$. Moreover, simply by shrinking U_x more, we may assume that each U_x is contained in some $A_{i+1}\backslash A_{i-1}$ for suitable $i \geq n + 1$. By Lemma 2.12 there is a locally finite open family $\mathcal{A}$ which refines $\{U_x : x \in X\backslash W\}$, covers $X\backslash W$ and such that the order of $\mathcal{A}_{A_{n+1}}$ is at most $\dim A_{n+2} + 1$. Define

$$\mathcal{U} := \mathcal{V}_{A_{n-1}} \cup (\mathcal{S}_{A_n}\backslash\mathcal{S}_{A_{n-1}}) \cup \mathcal{A}$$

and let us check that $\mathcal{U}$ satisfies all desired properties: $\mathcal{U}$ covers X because $\mathcal{A}$ covers $X \setminus W$, and $\mathcal{S}_{A_n}$ covers W, hence $\mathcal{V}_{A_{n-1}} \cup (\mathcal{S}_{A_n} \setminus \mathcal{S}_{A_{n-1}})$ covers W too. By construction we have $\mathcal{U}_{A_{n-1}} = \mathcal{V}_{A_{n-1}}$. Every element of $\mathcal{U}$ is contained in some $A_{i+1} \setminus A_{i-1}$ for suitable $i \in \mathbb{N}$ since this is the case for the elements of $\mathcal{A}$, $\mathcal{V}$ and $\mathcal{S}$. Moreover, since the elements of $\mathcal{A}$ do not meet A_n, we have $\operatorname{ord} \mathcal{U}_{A_n} \leq \operatorname{ord} \mathcal{V}_{A_n}$ and

$$\operatorname{ord}(\mathcal{U}_{A_{n+1}}) \leq \operatorname{ord}(\mathcal{V}_{A_n}) + \operatorname{ord}(\mathcal{A}_{A_{n+1}}) \leq \operatorname{ord}(\mathcal{V}_{A_n}) + \dim A_{n+2} + 1.$$

That $\mathcal{U}$ refines $\mathcal{V}$ is clear since the elements of $\mathcal{A}$ refine $\mathcal{S}$. Let us define the $k + 1$-skeleton h. We have the indexed cover $\mathcal{V} = \{V_j\}_{j \in \mathcal{J}}$ and let us index $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ such that $\mathcal{I} \cap \mathcal{J}$ are the indices corresponding to $\mathcal{V}_{A_n}$, and such that for $i \in \mathcal{I} \cap \mathcal{J}$ we have $U_i = S_i$ if $U_i \in \mathcal{S}_{A_n} \setminus \mathcal{S}_{A_{n-1}}$ and $U_i = V_i$ if $U_i \in \mathcal{V}_{A_{n-1}}$. Let $\{e_i : i \in \mathcal{I} \cup \mathcal{J}\}$ denote linearly independent vectors, let E be the real vector space spanned by $\{e_i : i \in \mathcal{I}\}$ and E' the one spanned by $\{e_j : j \in \mathcal{J}\}$. We have $N|\mathcal{U}| \subset E$ and $N|\mathcal{V}| \subset E'$. Pick for each $i \in \mathcal{I} \setminus \mathcal{J}$ an index $j(i) \in \mathcal{J}$ such that $U_i \subset S_{j(i)} \subset V_{j(i)}$, and set $j(i) = i$ for $i \in \mathcal{I} \cap \mathcal{J}$. Moreover let $\lambda : N|\mathcal{U}| \to N|\mathcal{V}|$ be the restriction of the linear map $E \to E'$ given by $e_i \mapsto e_{j(i)}$. From $U_i \subset V_{j(i)}$ it follows that the homotopy of k-skeletons

$$h : [0, 1] \times N_k|\mathcal{U}| \to \Psi, \quad h(t, x) := g(t, \lambda(x))$$

is well-defined. Moreover h_0 is still the sectionally constant complex given by f and h_1 has values in Φ. The definition of λ yields $h|[0, 1] \times N|\mathcal{U}_{A_{n-1}}| = g|[0, 1] \times N|\mathcal{V}_{A_{n-1}}|$. We are left to extend h to a homotopy of $k + 1$-skeletons such that the time-1 map has values in Φ. To do this we are well prepared. Let $I \in N|\mathcal{U}|$ be of length $k + 1$. By assumption $\operatorname{ord} \mathcal{V}_{A_n}$ is smaller or equal to k, hence I contains an index i_0 with $U_{i_0} \in \mathcal{A}$ by the definition of $\mathcal{U}$. Set $|J(I)| := \lambda(|I|)$ and let $J(I)$ be the corresponding element of the nerve $N(\mathcal{V})$. In the case where $\dim |J(I)| \leq k$, this extension exists since $U_I \subset V_{J(I)}$ and g is defined on $N_k(\mathcal{V})$ by assumption. Otherwise choose U_x with $U_{i_0} \subset U_x$. Since $U_i \subset S_{j(i)}$ for all $i \in I$, we have $U_I \subset S_{J(I)}$. This implies

$$U_x \cap S_{J(I)} \supset U_I \cap S_{J(I)} = U_I \neq \emptyset,$$

hence $U_x \subset V_{J(I)}$ and $r_{U_x} \circ g|Q_{J(I)}$ extends to a continuous map $h_I : [0, 1] \times |J(I)| \to \Psi(U_x)$ with the desired properties by construction. Since $U_I \subset U_x$, we get an extension of h to $[0, 1] \times |I|$ by composing the restriction

$$\lambda : [0, 1] \times |I| \to [0, 1] \times |J(I)|, \quad (t, x) \mapsto (t, \lambda(x))$$

with h_I. This finishes the proof. $\qquad \square$

Proof of Proposition 2.9 First construct a suitable 0-skeleton so that we can apply Lemma 2.14 (with $n = 1$) to that 0-skeleton. As in the proof of Lemma 2.14, this depends on $\Phi \hookrightarrow \Psi$ being a local weak homotopy equivalence. In the following $\mathcal{V} = \mathcal{V}_n$ denotes the cover of X corresponding to a homotopy of skeletons obtained in the n-th step. We repress the subscript n to avoid ugly notations. In particular $\mathcal{V}$ may change with every step. Apply Lemma 2.14 (with $n = 1$) in a

first step inductively dim $A_3 + 1$ times to get a homotopy of dim $A_3 + 1$-skeletons whose restriction to $[0, 1] \times N|\mathcal{V}_{A_2}|$ is already a homotopy of complexes. Then apply Lemma 2.14 (with $n = 2$) in a second step for dim $A_4 + 1$ more times to obtain a homotopy of dim $A_3 + \dim A_4 + 2$-skeletons which is already a homotopy of complexes when restricted to $[0, 1] \times N|\mathcal{V}_{A_3}|$. In the third such step we get a homotopy of dim $A_3 + \dim A_4 + \dim A_5 + 3$-skeletons which is a homotopy of complexes on $[0, 1] \times N|\mathcal{V}_{A_4}|$ and the restriction to $[0, 1] \times N|\mathcal{V}_{A_2}|$ (including the cover $\mathcal{V}_{A_2}$) has not been changed in this step. In the n-th such step we get a homotopy of skeletons which is a homotopy of complexes if restricted to $[0, 1] \times N|\mathcal{V}_{A_{n+1}}|$ and the restriction to $[0, 1] \times N|\mathcal{V}_{A_{n-1}}|$ (and $\mathcal{V}_{A_{n-1}}$) has not been changed since the last step. Moreover we get that the elements of $\mathcal{V}$ are contained in $A_{i+1} \backslash A_{i-1}$ for suitable $i \in \mathbb{N}$. Set $\mathcal{U}_{n-1} := \mathcal{V}_{A_{n-1}}$ and let h_{n-1} be the restriction to $[0, 1] \times N|\mathcal{V}_{A_{n-1}}|$ of the homotopy obtained in the n-th step for $n \geq 3$. We get that $\mathcal{U}_n$ covers A_n, that $\mathcal{U}_n \subset \mathcal{U}_{n+1}$, and that $h_{n+1}|[0, 1] \times N|\mathcal{U}_n| = h_n$ for $n \geq 2$. Moreover we get that the elements of $\mathcal{U}_n$ are contained in $A_{i+1} \backslash A_{i-1}$ for suitable $i \in \mathbb{N}$, which guarantees that the union $\mathcal{U}$ of all $\mathcal{U}_n$, $n \geq 2$ is locally finite and that $N|\mathcal{U}|$ is equal to the union of all $N|\mathcal{U}_n|$, $n \geq 2$. Now

$$h \colon [0, 1] \times N|\mathcal{U}| \to \Psi, \quad h(t, x) := h_n(t, x) \text{ for some } n \text{ with } x \in N|\mathcal{U}_n|$$

is the desired homotopy of complexes. This finishes the proof. $\qquad\square$

2.4 Gluing sections

In this subsection we prove

Proposition 2.15 *Let $\mathcal{A} = (A_n)_{n \in \mathbb{N}}$ be an ordered cover of X and let Φ be a sheaf of topological spaces on X. Assume that*

(1) Φ *is complete metric and $\mathcal{A}$ is weakly flexible for Φ, or*

(2) $\mathcal{A}$ *is ordered flexible for Φ.*

Then any complex $f \colon N|\mathcal{A}| \to \Phi$ is homotopic through a homotopy of complexes $f_t \colon N|\mathcal{A}| \to \Phi$ to a sectionally constant complex.

The proof is based on work published in [9], where the task is done for the sheaf of holomorphic sections of elliptic submersions. This was developed in the thesis of Prezelj [18], see also [8]. In the following the symbol Φ denotes always a sheaf of topological spaces on a given space X.

Lemma 2.16 *Let $f \colon N|\mathcal{A}| \to \Phi$ a complex and $g_t \colon N|\mathcal{B}| \to \Phi$ a homotopy of complexes with $g_0 = f|N|\mathcal{B}|$ for some $\mathcal{B} \subset \mathcal{A}$. Then there is a homotopy of complexes $f_t \colon N|\mathcal{A}| \to \Phi$ with $f_0 = f$ and $f_t|N|\mathcal{B}| = g_t$.*

Proof Note that we can write an arbitrary element of $N|\mathcal{A}|$ uniquely as $sx + (1-s)y$, where $x \in N|\mathcal{B}|$, $y \in N|\mathcal{A} \backslash \mathcal{B}|$ and $s \in [0, 1]$. Define the extension f_t of g_t for $sx + (1-s)y \in N|\mathcal{A}|$ by

$$f_t(sx + (1-s)y) = \begin{cases} f((1+t)sx + (1 - (1+t)s)y) & \text{if } (1+t)s \leq 1 \\ g_{(1+t)s-1}(x) & \text{if } (1+t)s > 1. \end{cases}$$

We get $f_0 = f$, $f_t(x) = g_t(x)$ and $f_t(y) = f(y)$, hence $f_t|N|\mathcal{B}| = g_t$ and $f_t|N|\mathcal{A}\backslash\mathcal{B}| = f|N|\mathcal{A}\backslash\mathcal{B}|$. For $I \cup J \in N(\mathcal{A})$ with $I \in N(\mathcal{B})$ and $J \in N(\mathcal{A}\backslash\mathcal{B})$ we have $|I \cup J| = \{sx + (1 - s)y : s \in [0, 1], x \in |I|, y \in |J|\}$. On the set $S \subset [0, 1] \times |I \cup J|$ given by $(1 + t)s \leq 1$, f_t is given by the composition of a continuous map $S \to |I \cup J|$ and f, hence inherits the requested properties from f, whereas on the set $S' \subset [0, 1] \times |I \cup J|$ given by $(1 + t)s > 1$, f_t is given by the composition of a continuous map $S' \to [0, 1] \times |I|$ and g_t, hence the requested properties follow from those of g_t and the fact that $I \subset I \cup J$. This implies together with $f_t(sx + (1 - s)y) = f(x) = g_0(x)$ for $(1 + t)s = 1$ that f_t is indeed a homotopy of complexes. $\qquad\square$

Lemma 2.17 *Let $f : N|\mathcal{A}| \to \Phi$ be a complex over a weakly or ordered flexible string $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ for Φ in X, $n \geq 2$, and set $\mathcal{B} = (A_1, \ldots, A_{n-1})$ and $A = A_1 \cup \cdots \cup A_{n-1}$. Then*

(1) *There is a homotopy of complexes f_t over $\mathcal{A}$ connecting $f_0 = f$ to a sectionally constant complex f_1.*
(2) *If $f|N|\mathcal{B}|$ is sectionally constant, then f_t can be chosen such that $f_t|N|\mathcal{B}|$ is sectionally constant too; and such that $r_{A\circ} \circ g_t$ stays in a prescribed neighborhood of $r_{A\circ} \circ g_0$, where $g_t \in \Phi(U)$, $t \in [0, 1]$ for some neighborhood U of A denotes the homotopy induced by $f_t|N|\mathcal{B}|$.*
(3) *If $\mathcal{A}$ is ordered flexible and $f|N|\mathcal{B}|$ is sectionally constant we can strengthen (2) to $r_{A\circ} \circ g_t$ being independent of t.*

Proof We proceed by induction on n. For $n = 2$ the statement is trivial since (A_1, A_2) is a weakly (resp. ordered) flexible pair by assumption. Suppose statement (1) is true for some n with $n - 1 \geq 2$ and let $f : N|\mathcal{A}| \to \Phi$ be a complex over $\mathcal{A}$. Since $\mathcal{B} = (A_1, \ldots, A_{n-1})$ is a weakly flexible string of length $n - 1$ for Φ, we find by the inductive assumption a homotopy of complexes $\tilde{f}_t : N|\mathcal{B}| \to \Phi$ connecting $\tilde{f}_0 = f|N|\mathcal{B}|$ to a sectionally constant complex $\tilde{f}_1$. By Lemma 2.16 there is a homotopy of complexes f_t that extends $\tilde{f}_t$ to a homotopy of complexes over $\mathcal{A}$ such that $f_0 = f$, and $f_1|N|\mathcal{B}| = \tilde{f}_1$ is sectionally constant. Set $\mathcal{C} = (A_1 \cap A_n, \ldots, A_{n-1} \cap A_n)$, define $\lambda : [0, 1] \times N|\mathcal{C}| \to N|\mathcal{A}|$ by $\lambda(s, e) = (1 - s)e + se_n$ and note that λ maps $[0, 1] \times |I|$ for $I \in N(\mathcal{C})$ to $|I \cup \{n\}|$. Therefore $f \circ \lambda$ is a homotopy of complexes over $\mathcal{C}$. Moreover $f \circ \lambda$ is sectionally constant when restricted to $s = \{0, 1\}$. In particular we may view $f \circ \lambda$ as a complex over $\mathcal{C}$ for the sheaf Φ_α, where $\alpha : \{0, 1\} \to \Phi(V)$ is the map induced by $f \circ \lambda$ restricted to the set given by $s \in \{0, 1\}$ for a sufficiently small neighborhood V of the union of the elements of $\mathcal{C}$. Since $\mathcal{C}$ is weakly (resp. ordered) flexible for Φ, $\mathcal{C}$ is likewise weakly (resp. ordered) flexible for Φ_α (see Lemma 2.4). This implies by our inductive assumption that there is a homotopy h_t connecting the complex given by $f \circ \lambda : N|\mathcal{C}| \to \Phi_\alpha$ to a sectionally constant one. Let us consider the homotopy $h_t : N|\mathcal{C}| \to \Phi_\alpha$, $t \in [0, 1]$ as a homotopy $h_t : [0, 1] \times N|\mathcal{C}| \to \Phi$, $(t, s, e') \mapsto h_t(s, e')$, which is sectionally constant and independent of t for $s \in \{0, 1\}$ and set

$$f_t(e) = \begin{cases} h_t \circ \lambda^{-1}(e), & \text{if } e \in \lambda((0, 1) \times N|\mathcal{C}|) = N|\mathcal{A}|\backslash(N|\mathcal{B}| \cup \{e_n\}), \\ f(e), & \text{otherwise.} \end{cases}$$

Note that $f_0 = f$. Moreover f_t defines a homotopy of complexes over $\mathcal{A}$ since h_t is independent of $t \in [0, 1]$ when restricted to $\{0, 1\} \times N|\mathcal{C}|$. We have $f_t|N|\mathcal{B}| = f|N|\mathcal{B}|$ by definition, and since h_1 is sectionally constant, f_1 yields a complex f' over the pair $(A_1 \cup \cdots \cup A_{n-1}, A_n)$. We are now in the situation of the inductive start and get a homotopy $f'_t \colon [0, 1] \to \Phi$, $t \in [0, 1]$ which connects f' to a sectionally constant complex f'_1 and satisfies the approximation property (2) in the weakly flexible case and the interpolation property (3) in the ordered flexible case. The homotopy f'_t yields the desired homotopy of complexes f_t, which can be seen explicitly by setting $f_t((1 - s)e + se_n) := f'_t(s)$ for $(1 - s)e + se_n \in N|\mathcal{A}|$. This finishes the proof of (1). For the proof of (2) and (3) note that $h_t|N|\mathcal{B}|$ was independent of t and that the homotopy of complexes f_t from the last step satisfies the approximation resp. interpolation property in question since f'_t does. This finishes the proof. $\qquad\square$

Lemma 2.18 *Let $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ be a countable ordered locally finite cover of X by compacts. Then there is a subsequence n_i, $i \in \mathbb{N}$ of $\mathbb{N}$ such that $K_i := A_1 \cup \cdots \cup A_{n_i}$ defines an exhaustion $K_1 \subset K_2 \subset K_3 \subset \cdots$ of X.*

Proof It suffices to show that for $n \in \mathbb{N}$ the union $K := A_1 \cup \cdots \cup A_n$ admits an open neighborhood $U \supset K$ which is contained in a finite union of elements of $\mathcal{A}$. Pick for each $p \in K$ a neighborhood U_p of p which meets at most finitely many elements of $\mathcal{A}$ and then a finite subfamily $\{U_1, \ldots, U_l\} \subset \{U_p \colon p \in K\}$ which covers K. The union $U := U_1 \cup \cdots \cup U_l$ is an open neighborhood of K which meets at most finitely many elements of $\mathcal{A}$. Since $\mathcal{A}$ covers X the finite subfamily $\mathcal{A}_U$ of elements which meet U covers U. This finishes the proof. $\qquad\square$

Proof of Proposition 2.15 We first consider assumption (1), that is the case where $\mathcal{A}$ is weakly flexible. Let $\epsilon > 0$, set $\mathcal{A}_n = (A_1, A_2, \ldots, A_n)$ and $U_n = (A_1 \cup A_2 \cup \cdots \cup A_n)^\circ$ and denote the metric on $\Phi(U_n)$ by d_{U_n}. Applying Lemma 2.17 (2) and Lemma 2.16 inductively ensures the existence of homotopies of complexes f^n_t, $n \in \mathbb{N}$ satisfying (i) $f^1_0 = f$, $f^{n+1}_0 = f^n_1$, (ii) $f^n_t|N|\mathcal{A}_n|$ is sectionally constant, and (iii) the homotopy of maps g^n_t induced by $f^n_t|N|\mathcal{A}_n|$ defined on a neighborhood of $A_1 \cup \cdots \cup A_n$ satisfies $d_{U_m}(r_{U_m} \circ g^n_s, r_{U_m} \circ g^n_t) < \epsilon/2^n$ for all $s, t \in [0, 1]$, $m \le n$ ((iii) can be achieved since the restrictions of Φ are continuous and since there are only finitely many $m \le n$ if $n \in \mathbb{N}$ is fixed). Moreover, by Lemma 2.18 there is for every $n \in \mathbb{N}$ some $m > n$ with $A_n \subset U_m$. Since $f^k||N|\mathcal{A}_m|$ is sectionally constant for $k \ge m$ we may assume that the open neighborhood $\mathcal{U}^k = (U^k_n)_{n\in\mathbb{N}}$ of $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ corresponding to f^k_t is such that $U^k_n \supset U_m$ for $k \ge m$. Write the half open unit interval as

$$\bigcup_{n\in\mathbb{N}}[t_n, t_{n+1}] = [0, 1), \quad \text{where } t_1 := 0 \text{ and } t_{n+1} := \sum_{i=1}^n 1/2^i \text{ for } n > 0$$

and rescale the homotopy parameter of f^n_t to $[t_n, t_{n+1}]$. Let us denote the composition of the homotopies $f^1_t, f^2_t, f^3_t, \ldots$ with these rescaled parameters by f_t. Since $A_n \subset U_m \subset U^k_n$ for sufficiently large $m > n$ and all $k \ge m$, the intersection $\cap_{k\in\mathbb{N}}U^k_n$ contains an open neighborhood of A_n. This guarantees that the infinite composition f_t of homotopies of complexes is again a homotopy of complexes with parameter

$t \in [0, 1)$. We extend f_t to $t = 1$ by passing to the limit. For a fixed $I \in N(\mathcal{A})$ let m be such that $A_I \subset U_m$, hence $r_{U_m} \circ f_t||I|$ is well defined and continuous for $t \geq t_m$; and set $f_1(e) = \lim_{t \to 1} r_{U_m} \circ f_t(e)$ for $e \in |I|$. This limit exists by (iii) and the fact that $\Phi(U_m)$ is a complete metric space by assumption. Moreover, by (iii) we have uniform convergence of $r_{U_m} \circ f_t||I|$ to $f_1||I|$, and hence $f_1||I|$ is a continuous extension of $r_{U_m} \circ f_t||I|$, $t \geq t_m$ to $[t_m, 1] \times |I|$. f_1 is sectionally constant since $f_t|N|\mathcal{A}_n|$ is sectionally constant for $t \geq t_n$ and all $n \in \mathbb{N}$. In particular f_t is a homotopy of complexes connecting $f = f_0$ to a sectionally constant complex f_1. The case where $\mathcal{A}$ is ordered flexible is much easier since instead of (iii) we can achieve the strengthening (iii'): The homotopy of maps g_t^n induced by $f_t^n|N|\mathcal{A}_n|$ defined on a neighborhood of $A_1 \cup \cdots \cup A_n$ satisfies $r_{U_m} \circ g_s^n = r_{U_m} \circ g_t^n$ for all $s, t \in [0, 1]$, $m \leq n$. As an effect of this strengthening, passing to the limit does not require the sets of local sections to be complete metric since the homotopy is independent of t when restricted to a fixed compact and sufficiently big $t < 1$. This finishes the proof. $\square$

2.5 The proofs of Theorems 1 and 2

Proof of Theorem 2 Let $\Phi \hookrightarrow \Psi$ be the inclusion of sheaves on X corresponding to the assumptions of Theorem 2. First we prove a non-parametric version. For given $f \in \Psi(X)$ Proposition 2.9 yields a homotopy of complexes g_t, $t \in [0, 1/2]$ over an open cover $\mathcal{U}$ with values in Ψ such that g_0 is the sectionally constant complex given by f and $g_{1/2}$ has values in Φ. By Remark 2.7, since Φ is either weakly flexible or ordered flexible, we may exchange the cover $\mathcal{U}$ corresponding to g_t by a weakly flexible resp. ordered flexible cover for Φ. Now Proposition 2.15 yields a homotopy of complexes with values in Φ connecting $g_{1/2}$ to a sectionally constant complex g_1. This shows in particular that $\Phi(X) \neq \emptyset$ if $\Psi(X) \neq \emptyset$ under the weaker assumptions stated in Theorem 2. Including the assumption that Ψ is likewise complete metric weakly flexible (or ordered flexible), note that the Ψ-valued homotopy of complexes g_t is a sectionally constant complex when restricted to $t = 0$ and $t = 1$, hence defines a map

$$\beta \colon \{0, 1\} \to \Psi(X)$$

with $\beta(0) = f$ given by g_0 and $\beta(1) \in \Phi(X)$ given by g_1. In particular g_t yields a complex g with values in Ψ_β. Since Ψ_β is complete metric weakly flexible (resp. ordered flexible) if Ψ is (see Lemma 2.4), we may assume that the cover corresponding to g is weakly flexible (resp. ordered flexible) for Ψ_β by Lemma 2.5 and Remark 2.7. Now Proposition 2.15 yields a global section $h \in \Psi_\beta(X)$, which is a path with values in $\Psi(X)$ connecting f to an element of $\Phi(X)$. This finishes the proof of the non-parametric version. To pass to the parametric version, note that $\Phi(X) \hookrightarrow \Psi(X)$ is a weak homotopy equivalence if and only if for every $\alpha \colon \partial \mathbb{D} \to \Phi(X)$ and every $f \in \Psi_\alpha(X)$ there is a path in $\Psi_\alpha(X)$ connecting f to an element of $\Phi_\alpha(X)$. Therefore Lemma 2.4 reduces the proof to the proved non-parametric version. This finishes the proof. $\square$

To prove Theorem 1 let us define $\mathcal{C}$-strings and $\mathcal{C}$-covers as in [9]. These notions appeared first in [12]. They are defined in terms of $\mathcal{C}$-pairs analogous to how we defined weakly flexible strings and weakly flexible covers in terms of weakly flexible pairs. That is

Definition 2.19 Let X be a complex space. A $\mathcal{C}$-*string* of length $n \geq 2$ is recursively defined as a finite sequence $(A_1, A_2, A_3, \ldots, A_n)$ of subsets of X such that

(1) $(A_1 \cup \cdots \cup A_{n-1}, A_n)$ is a $\mathcal{C}$-pair, and if $n \geq 3$, then
(2) $(A_1, \ldots, A_{n-1})$ and $(A_1 \cap A_n, \ldots, A_{n-1} \cap A_n)$ are $\mathcal{C}$-strings.

A $\mathcal{C}$-*cover* is a locally finite cover $(A_n)_{n \in \mathbb{N}}$ of X such that for every $n \in \mathbb{N}$ $(A_1, A_2, \ldots, A_n)$ is a $\mathcal{C}$-string.

The crucial fact is that every Stein space admits arbitrarily fine $\mathcal{C}$-covers, a result which relies on *Grauert's bump method*. In the proof of an Oka principle this technique was initially applied by Henkin and Leiterer in [13]. A good and modern formulation of the required tool, which includes the case where X is singular, is formulated in [8], p. 294, a reference for

Proposition 2.20 *Let X be a second countable reduced Stein space and $\mathcal{U}$ an open cover of X. Then there is a $\mathcal{C}$-cover $(A_n)_{n \in \mathbb{N}}$ which refines $\mathcal{U}$.*

For Stein spaces Proposition 2.20 implies the following sufficient assumption for weak flexibility.

Lemma 2.21 *Let Φ be a sheaf of topological spaces on a second countable reduced Stein space X. Assume every point $p \in X$ has a neighborhood U such that every $\mathcal{C}$-pair (A, B) with $B \subset U$ is weakly flexible for Φ. Then Φ is weakly flexible. The analogous statement holds if weak flexibility is replaced by ordered flexibility in the assumption and the conclusion.*

Proof Let $\mathcal{U}$ be an open cover of X. Pick for every point $p \in X$ a neighborhood U_p such that each $\mathcal{C}$-pair (A, B) with $B \subset U_p$ is weakly flexible. By Proposition 2.20 there is a $\mathcal{C}$-cover $\mathcal{A} = (A_n)_{n \in \mathbb{N}}$ which refines $\{U_p \cap U : p \in X, U \in \mathcal{U}\}$. Clearly $\mathcal{A}$ refines $\mathcal{U}$, hence we are left to show that $\mathcal{A}$ is a weakly flexible cover for Φ. Note that every $\mathcal{C}$-pair (A, B) emerging from $\mathcal{A}$ by applying the recursion in Definition 2.19 satisfies $B \subset A_n \subset U_p$ for suitable $n \in \mathbb{N}$ and $p \in X$ and is therefore a weakly flexible pair by assumption. Moreover the recursions in the definitions of $\mathcal{C}$-covers and weakly flexible covers are the same up to replacing every occurence of $\mathcal{C}$ by *weakly flexible*. From these two facts the result follows immediately. □

Proof of Theorem 1 Sheaves satisfying assumption (1) resp. (2) in Theorem 1 are weakly resp. ordered flexible by Lemma 2.21. Theorem 1 is therefore a special case of Theorem 2. □

Appendix A. Applying Theorem 1 in Oka theory

In this appendix we give references for the proofs of the assumptions of Theorem 1 in the settings of the Oka principles cited in the introduction. The intention is to give the reader a hint where the analytic challenges providing the assumptions of Theorem 1 are tackled in the original work. In some cases the cited work needs some adjustments, which will be pointed out. These adjustments were part of the author's thesis [19], but cost too many lines to be included here.

A.1. Weak flexibility

The proof of the weak flexibility of a $\mathcal{C}$-pair (A, B) with respect to a given complex analytic sheaf is usually proved in two steps:

(1) a parametric Runge approximation property, and
(2) a gluing property.

The Oka principle for elliptic submersions: To show the weak flexibility assumptions of Theorem 1 in this setting one has to show that if $h\colon Z \to X$ is a holomorphic submersion onto a reduced Stein space X and $U \subset X$ is an open set such that the restriction $h\colon h^{-1}(U) \to U$ admits a dominating spray, then every $\mathcal{C}$-pair (A, B) with $B \subset U$ is weakly flexible for the sheaf of holomorphic sections of h. This was discovered by Gromov [12]. Detailed proofs of Gromov's insight have been given by Forstnerič and Prezelj (see e.g. [9]). A convenient source is [8]: The weak flexibility of the pair (A, B) follows the Runge approximation property stated in Theorem 6.2.2, p. 284 and the gluing property stated in Proposition 6.7.2, p. 288.

The Oka principle for principal G-bundles: This Oka principle is a special case of the Oka principle for elliptic submersions. However, since the two remaining Oka principles build strongly on Cartan's exposition of Grauert's work [3], it makes sense to give references. The required Runge approximation property and gluing property which yield the weak flexibility of $\mathcal{C}$-pairs follow in this case from a Runge approximation property and a splitting lemma in an associated sheaf of groups. These two key results are in Cartan's exposition of Grauert's work Proposition 1 and 2 (see [3], p. 109).

The Oka principle for admissible pairs of sheaves: This Oka principle builds on Cartan's exposition of Grauert's work. The necessary extensions of Cartan's Proposition 1 and 2 from [3] are Lemma 2 and 3 from Forster and Ramspott's work (see [4], p. 271 and p. 273).

The Oka principle for equivariant isomorphisms: This Oka principle builds likewise on Cartan's text [3]. The necessary extensions of Cartan's Proposition 1 and 2 in the work of Kutzschebauch, Lárusson and Schwarz are Proposition 10.2 and 10.3 (see [14], p. 7293).

The key results from the last three Oka principles, i.e. from those Oka principles which build on Cartan's exposition of Grauert's work, need some adjustments to yield complete proofs of the weak flexibility of $\mathcal{C}$-pairs. These adjustments can be found in [19], Chapter 5 and 6.

A.2. Local weak homotopy equivalences

The difficulty of the proof that a given inclusion of sheaves $\Phi \hookrightarrow \Psi$ is a local weak homotopy equivalence depends strongly on the setting.

The Oka principle for elliptic submersions: In this setting it suffices to show that if $h \colon Z \to X$ is a holomorphic submersion onto a reduced complex space and $\Phi \hookrightarrow \Psi$ is the inclusion of the sheaf of holomorphic sections to the sheaf of continuous sections of h, then $\Phi \hookrightarrow \Psi$ is a local weak homotopy equivalence. Gromov seems to have taken this result for granted in [12]. In the more detailed work [9] local weak homotopy equivalences are not introduced. Instead, an analogue of our Proposition 2.9 is stated in the special case of holomorphic submersions, namely Proposition 4.7. The validity of Proposition 4.7 in [9] has been carefully checked in the thesis of Jasna Prezelj [18], which yields implicitly a proof of the fact that $\Phi \hookrightarrow \Psi$ is a local weak homotopy equivalence in the mentioned case.

The Oka principle for principal G-bundles: This is a special case of the above.

The Oka principle for admissible pairs of sheaves: In the work of Forster and Ramspott [4] there is a slight weakening of $\Phi \hookrightarrow \Psi$ being a local weak homotopy equivalence in the assumption, namely the homotopy property (PH) from Satz 1, p. 267. Using (PH), the fact that an inclusion of admissible pairs of sheaves $\Phi \hookrightarrow \Psi$ in the sense of Forster and Ramspott is a local weak homotopy equivalence is a corollary to Lemma 1, p. 269 in [4].

The Oka principle for equivariant isomorphisms: In this setting it is hard to show that the given inclusion $\Phi \hookrightarrow \Psi$ is a local weak homotopy equivalence. Theorem 1.3, p. 7253 in [14] reduces the proof to the case where we have $X = Y$ for given Stein G-manifolds X and Y, where G is a complex reductive Lie group. Having this, one needs to extend some Lemmata from [14] in Section 3 and 5 to analogous parametric versions. These necessary adaptations are simple once Section 3 and 5 in [14] are understood.

For more details that the inclusions $\Phi \hookrightarrow \Psi$ from the mentioned Oka principles are local weak homotopy equivalences see [19], Chapter 4.

A.3. Sheaves of topological spaces

Equipping all sets of local sections from the sheaves corresponding to the Oka principles from the introduction with the compact open topology turns these into sheaves of topological spaces. This is used in all the quoted work. To see that there is no pathological behavior when dealing with parametric sheaves, see Lemma 2.1, a basic fact which is usually taken for granted in Oka theory. Recall that (complete) metric sheaves are defined as those sheaves of topological spaces whose sets of local sections are equipped with a (complete) metric which induces the topology. That the complex analytic sheaves from the mentioned Oka principles are complete metric depends on the following two facts.

Fact 1 Let X be a space which admits an exhaustion $K_1 \subset K_2 \subset K_3 \subset \cdots$ by compacts and (Y, d) a complete metric space. Then

$$d_{C(X,Y)}(f, g) = \sum_{n \geq 1} \frac{1}{2^n} \frac{d_n(f, g)}{1 + d_n(f, g)}, \quad d_n(f, g) = \max_{x \in K_n} d(f(x), g(x))$$

for continuous maps $f, g: X \to Y$ defines a complete metric on the space of continuous maps $X \to Y$ which induces the compact open topology.

Fact 2 Let X, Y be two locally connected locally compact second countable metric spaces. Then $d_H(f, g) = d_{C(X,Y)}(f, g) + d_{C(Y,X)}(f^{-1}, g^{-1})$, where $d_{C(X,Y)}$ and $d_{C(Y,X)}$ are as in Fact 1, defines a complete metric on the set of homeomorphisms $X \to Y$ which induces the compact open topology.

Fact 1 is well known, and it is easy to show that d_H from Fact 2 turns the set of homeomorphisms into a complete metric space. It is not obvious (and for locally disconnected topological spaces generally false) that the topology induced by d_H is not finer than the compact open topology. The proof depends on the following

Theorem (Arens [2]) *The homeomorphism group of a locally compact locally connected Hausdorff space equipped with the compact open topology is a topological group.*

To show that d_H induces the compact open topology it suffices to show that $d_H(\cdot, g)$ is continuous with respect to the compact open topology for a fixed homeomorphism $g: X \to Y$; and this is the case if $I: \text{Homeo}(Y, Z) \to \text{Homeo}(Z, Y)$, $I(f) = f^{-1}$ is continuous. The latter follows from the fact that I is composition of the three continuous maps given by $f \mapsto g^{-1} \circ f$, $f \mapsto f^{-1}$ and $f \mapsto f \circ g^{-1}$, where the continuity of the second factor is due to Arens result.

Fact 1 and 2 imply that the considered complex analytic sheaves from the mentioned Oka principles are complete metric in the following way: It is known that the sets of local sections of the complex analytic sheaves from the mentioned Oka principles are closed subspaces either of a space of continuous maps $X \to Y$ for suitable X and Y or from the space of homeomorphisms $X \to Y$ for suitable X and Y. In any case, Fact 1 and 2 imply that the sets of local sections are closed subsets of a complete metric space and hence complete if equipped with the suitable metric from Fact 1 resp. Fact 2.

A.4. Ordered flexibility

Ordered flexibility is—in the context of Oka theory—most of times rather easy to show. Proofs shall be given elsewhere. Instead, we would like to discuss the examples addressed in Remark 1.8. In most known Oka principles one looks at inclusions of sheaves $\Phi \hookrightarrow \Psi$, where Φ lives in the complex analytic category and Ψ lives in the category of topological spaces. However, in recent advances (see e.g. [14,16]) one is forced to place Ψ in the smooth category instead. In the smooth category—opposed to the complex analytic and the topological category—completeness is more

delicate. One can turn e.g. the space of smooth functions $\mathbb{C} \to \mathbb{C}$ into a complete metric space by including all higher derivatives in the definition of the pseudometrics from Fact 1. However, the resulting topology is finer than the compact open topology. This is a disadvantage for proofs in Oka theory since Lemma 2.1 does not apply anymore. A similar example emerges from [6], where one is forced to look at a sheaf Ψ of continuous sections which are holomorphic in a neighborhood of some fixed subvariety. There seems to be no (natural) way to turn this sheaf into a complete metric sheaf without refining the compact open topology. In these examples one benefits from assumption (2) in Theorem 1 as an alternative to assumption (1), since in (2) no completeness is asked.

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