1-loop matching of a thermal Lorentz force

M. Laine

AEC, Institute for Theoretical Physics, University of Bern,
Sidlerstrasse 5, Bern CH-3012, Switzerland

E-mail: laine@itp.unibe.ch

ABSTRACT: Studying the diffusion and kinetic equilibration of heavy quarks within a hot QCD medium profits from the knowledge of a coloured Lorentz force that acts on them. Starting from the spatial components of the vector current, and carrying out two matching computations, one for the heavy quark mass scale (M) and another for thermal scales ($\sqrt{MT}, T$), we determine 1-loop matching coefficients for the electric and magnetic parts of a Lorentz force. The magnetic part has a non-zero anomalous dimension, which agrees with that extracted from two other considerations, one thermal and the other in vacuum. The matching coefficient could enable a lattice study of a colour-magnetic 2-point correlator.

Contents

1   Introduction

The motion of heavy probe particles is a classic tool for extracting information about the microscopic properties of an interacting statistical system. In heavy ion collision experiments, one manifestation of this philosophy is to inspect how efficiently heavy flavours (charm and bottom quarks) participate in hydrodynamic flow (cf., e.g., ref. [1]). In cosmology, assuming that dark matter is made of weakly interacting massive particles, it would be important to know for how long they stay in kinetic equilibrium with the other particles, as this may affect, amongst others, structure formation (cf., e.g., ref. [2]).

To be concrete, consider a particle whose mass M is much larger than the temperature T. Given that the average (equilibrium) velocity is below unity, $v^2 \sim 3T/M \ll 1$, and the (equilibrium) density is exponentially suppressed, $n \sim \left(\frac{MT}{2\pi}\right)^{3/2} e^{-M/T}$, we find ourselves in a non-relativistic dilute regime. Thinking of a single such particle, and assuming that it carries the gauge charge g, the classical Lorentz force acting on it reads

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}t} = g F^{\mu\nu} v_\nu \,, \tag{1.1}$$

where p^μ is the four-momentum and $v^\mu \equiv (1, \mathbf{v})$ is the velocity. The Lorentz force contains an electric part ($\sim g\mathbf{E}$) and a magnetic one ($\sim g\mathbf{v} \times \mathbf{B}$). It has thus been argued that at zeroth order in $\mathbf{v}$, heavy quarks are affected by colour-electric forces [3, 4], whereas at first order in $\mathbf{v}$, corrections originate from colour-magnetic ones [5]. For dark matter, we could similarly consider the forces originating from the weak gauge group.

Being a classical description, eq. (1.1) is guaranteed to hold only at large time scales where phase decoherence has taken place, $t \gg 1/(\alpha^2 T)$, where $\alpha = g^2/(4\pi)$. Due to their large inertia, the time scale associated with the kinetic equilibration of heavy particles is

$\sim M/(\alpha^2 T^2)$ [6]. For $M \gg T$, there should thus be a broad range of time scales for which eq. (1.1) is valid. At the same time, thermal effects break Lorentz invariance and distinguish between electric and magnetic fields, modifying the respective couplings (cf. eq. (2.5)). In fact, we recover an unmodified eq. (1.1) only in vacuum,[1] where the decoherence argument does not apply, but $M \gg \Lambda_{\overline{\text{MS}}}$ still provides for a hierarchy of time scales (cf. eq. (3.22)).

Given that colour interactions are strong in QCD, their effects should be investigated up to the non-perturbative level. For colour-electric forces, large-scale lattice simulations have indeed been carried out in recent years [8–13], whereas for the colour-magnetic corrections, the challenge lies ahead of us. In preparation for this task, the goal of the current study is to clarify the renormalization of the colour-magnetic part of eq. (1.1). Specifically, we show how a divergence found in ref. [5], cf. eq. (7.6), gets cancelled after the inclusion of the proper matching coefficient.

2 Outline of a procedure

Let us consider the vector current, $J_\mu^{\text{QCD}} = \bar{\psi}\gamma_\mu\psi$, associated with one heavy flavour in QCD.[2] The spatial integral over the zeroth component, $\int_{\mathbf{x}} J_0^{\text{QCD}}$, measures the net number of this species (particles minus antiparticles), and is conserved in the absence of weak interactions. In contrast, the spatial components, $\int_{\mathbf{x}} J_i^{\text{QCD}}$, are not conserved. They measure velocities, and velocities can be changed by elastic reactions.

Following eq. (1.1), our focus here is on time derivatives of velocities, i.e. accelerations. The QCD operator that we are interested in can formally be expressed as $\partial_0 \int_{\mathbf{x}} J_i^{\text{QCD}}$. In a vacuum setting, we could take matrix elements of this operator in the presence of a background gauge field $\bar{A}(Q)$ [14], where $Q = (q_0, \mathbf{q})$ is a four-momentum. As we are aiming at an infrared (IR) description, Q is considered small compared with other energy scales. Schematically, then, we could consider matrix elements like

$$\left\langle \mathbf{p}_1 \left| \left[\partial_0 \int_{\mathbf{x}} J_i^{\text{QCD}} \right]_{\bar{A}(Q)} \right| \mathbf{p}_2 \right\rangle \simeq \delta^{(3)}(\mathbf{p}_2 + \mathbf{q} - \mathbf{p}_1)\, \mathcal{A}_i^{\text{QCD}}[\bar{A}(Q)] + \mathcal{O}(q_0^2, \mathbf{q}^2, \mathbf{v}^2)\,, \quad (2.1)$$

$$\left\langle \mathbf{p}_1 \left| \int_{\mathbf{x}} J_0^{\text{QCD}} \right| \mathbf{p}_2 \right\rangle \simeq \delta^{(3)}(\mathbf{p}_2 - \mathbf{p}_1)\, \mathcal{N}_0^{\text{QCD}} + \mathcal{O}(\mathbf{v}^2)\,, \quad (2.2)$$

where the precise way to extract the external states will be discussed presently, and $\mathbf{v}$ is the heavy-quark velocity in the medium rest frame.

The matrix elements in eqs. (2.1) and (2.2) are subject to wave function renormalization, which drops out in the ratio

$$a_i^{\text{QCD}} \equiv \frac{\mathcal{A}_i^{\text{QCD}}}{\mathcal{N}_0^{\text{QCD}}}\,. \quad (2.3)$$

---

[1]There is a famous history of quantum-mechanical derivations of the Lorentz force, cf. e.g. ref. [7].

[2]We do not elaborate on the overall factors $\pm i, \pm 1$ of the various operators, on one hand because these play no role in the end, on the other because we work in Euclidean spacetime, with Euclidean Dirac matrices, and then additional factors may originate from the time coordinate, temporal gauge field components, spatial Dirac matrices, and raising/lowering of indices. It would be a distraction to discuss all of them.

It is for the cause of such an acceleration, multiplied by a (thermally corrected) pole mass M, that we would like to find an operator reminiscent of the Lorentz force.

Before proceeding, we note that for the thermal effects that we are mostly concerned with, the notion of matrix elements such as eqs. (2.1) and (2.2) is ambiguous. Therefore, we generalize the definitions to certain "partition functions", defined in configuration space. Let the Euclidean time coordinate be τ and a generic spatially averaged operator $O(\tau)$. The time direction is compact and is chosen to lie in the interval $\tau \in \left(-\frac{\beta}{2}, \frac{\beta}{2}\right)$, where $\beta \equiv \frac{1}{T}$ is the inverse temperature. In this language, we may consider the 3-point correlator

$$\left\langle \mathrm{Tr} \left\{ \int_{\mathbf{y}} \psi\left(\frac{\beta}{2}, \mathbf{y}\right) e^{-i\mathbf{p}_1 \cdot \mathbf{y}} \left[O(0)\right]_{\bar{A}(Q)} \int_{\mathbf{x}} \bar{\psi}\left(-\frac{\beta}{2}, \mathbf{x}\right) e^{i\mathbf{p}_2 \cdot \mathbf{x}} \right\} \right\rangle_{T,\mathrm{c}}, \qquad (2.4)$$

where $\langle \ldots \rangle_{T,\mathrm{c}}$ is a thermal average, and c stands for connected contractions. We take a trace in Dirac space, given that the operator we are interested in, cf. eq. (2.5), is spin-independent. The part of this correlator proportional to $e^{-\beta M}$ originates from the single heavy quark sector of the Hilbert space, and gives the effects that we are interested in. In a vacuum setting, we may replace $\beta/2 \to +\infty$ and $-\beta/2 \to -\infty$. The leading asymptotics picks up the desired states in this case, and matrix elements analogous to eqs. (2.1), (2.2) are obtained as coefficients of the exponential fall-off, up to overall factors that drop out in eq. (2.3).

Let now θ represent a non-relativistic $2N_\mathrm{c}$-component spinor, defined in the sense of Heavy Quark Effective Theory (HQET) (cf., e.g., refs. [15–17] and references therein). This brings in *two* new ways to define the acceleration. The first is that we consider components of the Noether current, which now read $J_0^{\mathrm{HQET}} = \theta^\dagger \theta$, $J_i^{\mathrm{HQET}} = -\theta^\dagger (i\overleftrightarrow{D}_i)\theta/(2M) + \mathcal{O}(1/M^2)$, and then compute matrix elements of $\partial_0 \int_{\mathbf{x}} J_i^{\mathrm{HQET}}$ and $\int_{\mathbf{x}} J_0^{\mathrm{HQET}}$, just like in eqs. (2.1) and (2.2).

However, one can envisage a more radical reduction, to which we refer as an infrared (IR) description. This involves an operator reminiscent of the Lorentz force in eq. (1.1),

$$F_i^{\mathrm{IR}} \equiv -ig_{\mathrm{B}} \theta^\dagger \left\{ Z_E F_{i0} V_0 + Z_B F_{ij} V_j \right\} \theta, \qquad (2.5)$$

where g_{B} denotes the bare gauge coupling, $-ig_{\mathrm{B}} F_{\mu\nu} \equiv [D_\mu, D_\nu]$ is a field strength, and $V = (i, \mathbf{v})$ is the (Euclidean) heavy-quark velocity. It is important to stress that in the static picture of eq. (2.5), the velocity $\mathbf{v}$ appears as an "external" parameter, whose thermal distribution is fixed later on from separate considerations (cf. section 7).

Defining matrix elements on the IR side as

$$\left\langle \mathbf{p}_1 \left| \left[\int_{\mathbf{x}} F_i^{\mathrm{IR}}\right]_{\bar{A}(Q)} \right| \mathbf{p}_2 \right\rangle \simeq \delta^{(3)}(\mathbf{p}_2 + \mathbf{q} - \mathbf{p}_1) \mathcal{F}_i^{\mathrm{IR}}[\bar{A}(Q)] + \mathcal{O}(q_0^2, \mathbf{q}^2, \mathbf{v}^2), \qquad (2.6)$$

$$\left\langle \mathbf{p}_1 \left| \int_{\mathbf{x}} J_0^{\mathrm{HQET}} \right| \mathbf{p}_2 \right\rangle \simeq \delta^{(3)}(\mathbf{p}_2 - \mathbf{p}_1) \mathcal{N}_0^{\mathrm{IR}} + \mathcal{O}(\mathbf{v}^2), \qquad (2.7)$$

the goal would be to find matching coefficients $Z_{E,B}$ such that (up to possible signature issues)

$$\frac{M \mathcal{A}_i^{\mathrm{QCD}}}{\mathcal{N}_0^{\mathrm{QCD}}} = \frac{\mathcal{F}_i^{\mathrm{IR}}}{\mathcal{N}_0^{\mathrm{IR}}}. \qquad (2.8)$$

This establishes the principal viability of a dynamics like that in eq. (1.1).[3] Such dynamics has already been employed for deriving purely gluonic 2-point imaginary-time correlators, permitting to study features of heavy quark diffusion and kinetic equilibration [3–5].

3 QCD vacuum contribution

The purpose of the present section is to see how the objects of eqs. (2.1)–(2.3) look like at 1-loop level in vacuum QCD. Physically speaking, this amounts to accounting for the heavy quark mass scale, M. Even if the result will be quite simple (cf. eq. (3.22)), we hope that a detailed exposition can set the technical stage for the subsequent sections. The inverse of a heavy quark propagator is denoted by

$$\Delta_P \equiv P^2 + M^2\,, \tag{3.1}$$

and P normally denotes an on-shell four-momentum, i.e. $P^2 = -M^2$.

To get going, we evaluate the 3-point correlator of eq. (2.4) at leading order (LO), with the sink and source placed at $\beta/2 \to y_0$ and $-\beta/2 \to x_0$, respectively. For the denominator, the operator reads $O(0) = \int_{\mathbf{x}} \bar\psi\gamma_0\psi$. The Wick contractions yield

$$\delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2) \int_{\omega_1,\omega_2} e^{i(\omega_1 y_0 - \omega_2 x_0)} \frac{\text{Tr}\,[(-i\not{P}_1 + M)\gamma_0(-i\not{P}_2 + M)]}{(\omega_1^2 + \epsilon_{p_1}^2)(\omega_2^2 + \epsilon_{p_2}^2)}\,, \tag{3.2}$$

where $\epsilon_p \equiv \sqrt{p^2 + M^2}$ and $\int_{\omega_i} \equiv \int_{-\infty}^{\infty} \mathrm{d}\omega_i/(2\pi)$. Sending $y_0 \to +\infty$, $x_0 \to -\infty$, the integrals over $\omega_{1,2}$ pick up the poles at

$$\omega_1 = i\epsilon_{p_1}\,, \quad \omega_2 = i\epsilon_{p_2}\,, \tag{3.3}$$

respectively. As momentum conservation sets the two momenta equal, we denote $\mathbf{p} \equiv \mathbf{p}_1 = \mathbf{p}_2$. The asymptotic wave functions $e^{-\epsilon_p y_0} \times e^{\epsilon_p x_0}$ are factored out, and this defines what we mean by the remaining matrix element. Taking the trace and expanding to leading order in $\mathbf{v} \equiv \mathbf{p}/\epsilon_p$, in accordance with eq. (2.2), we then obtain

$$\mathcal{N}_0^{\text{QCD,vac}} = 2 + \mathcal{O}(g_{\text{B}}^2)\,, \tag{3.4}$$

multiplied by a unit matrix in colour space that is suppressed from the notation.

Proceeding to the numerator, the operator can be expressed in momentum space as

$$\partial_0 \int_{\mathbf{x}} J_i^{\text{QCD}} = \int_{P_3,P_4} i(\omega_4 - \omega_3)\,\bar\psi(P_3)\gamma_i\psi(P_4)\,\delta^{(3)}(\mathbf{p}_4 - \mathbf{p}_3)\,, \quad P_i = (\omega_i, \mathbf{p}_i)\,. \tag{3.5}$$

The diagrams to be computed are shown on the first row of figure 1. The key feature is that, after contracting the momenta to the external ones, i.e. P_1 and P_2, the prefactor $\omega_4 - \omega_3 = q_0 + \omega_2 - \omega_1$ in eq. (3.5) is of $\mathcal{O}(Q)$, but there is an internal propagator (between

---

[3]As alluded to above and demonstrated explicitly in the later sections, the cancellation of singular propagators from the numerator and denominator separately requires the inclusion of thermal corrections in the pole mass M, however these effects are power-suppressed by $g^3 T/M$ or $g^2 T^2/M^2$, and in fact irrelevant for the definition of $Z_{E,B}$, which comprise of corrections only suppressed by g^2.
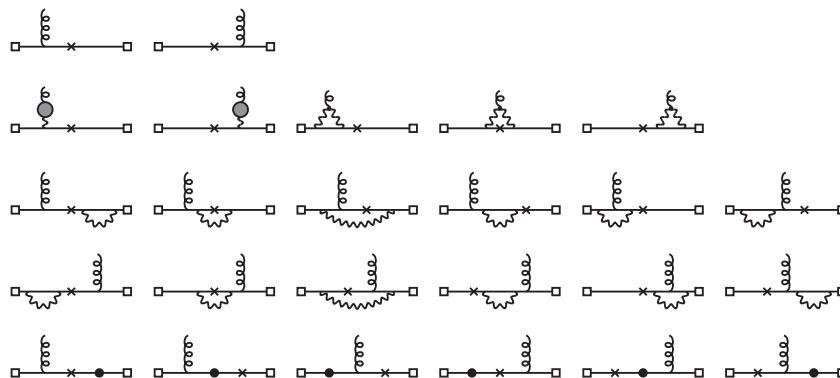
Figure 1. The LO and NLO graphs contributing to the 3-point correlator in full QCD. A solid line denotes a heavy quark, a curly line an external gauge field, a wavy line a dynamical gauge field, a grey blob a 1-loop gauge field self-energy, a solid circle a mass counterterm, open squares a source and a sink, and a cross the operator related to the conserved current or its time derivative.

the external gauge field and the operator) which is of $\mathcal{O}(1/Q)$. These leading singularities cancel, leaving over terms of $\mathcal{O}(1)$:

$$\frac{q_0 + \omega_2 - \omega_1}{\Delta_{P_1 - Q}} = \frac{1}{q_0 - i\epsilon_{p_1} - i\epsilon_{p_2}}, \quad \frac{q_0 + \omega_2 - \omega_1}{\Delta_{P_2 + Q}} = \frac{1}{q_0 + i\epsilon_{p_1} + i\epsilon_{p_2}}. \tag{3.6}$$

Here we made use of the overall momentum constraint $\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{q}$ and put the states on-shell according to eq. (3.3). Subsequently we can insert

$$\mathbf{p}_1 = \mathbf{p} + \frac{\mathbf{q}}{2}, \quad \mathbf{p}_2 = \mathbf{p} - \frac{\mathbf{q}}{2}, \quad \epsilon_{p_1} \approx \epsilon_p + \frac{\mathbf{v} \cdot \mathbf{q}}{2}, \quad \epsilon_{p_2} \approx \epsilon_p - \frac{\mathbf{v} \cdot \mathbf{q}}{2}, \tag{3.7}$$

and Taylor-expand to first order in q_0, $\mathbf{q}$ and $\mathbf{v}$. For future convenience, we split electric fields into two parts, introducing (while being again unconventional about factors of i)[4]

$$E_i^{(A)} \equiv iq_i\bar{A}_0, \quad E_i^{(B)} \equiv iq_0\bar{A}_i, \quad \mathbf{v} \times \mathbf{B}_i \equiv q_i\mathbf{v} \cdot \bar{\mathbf{A}} - \bar{A}_i\mathbf{v} \cdot \mathbf{q}. \tag{3.8}$$

Factoring out the same wave functions as above eq. (3.4), this leads to

$$M\,\mathcal{A}_i^{\text{QCD,vac}} = -2ig_{\text{B}}\left[E_i^{(A)} - E_i^{(B)} + \mathbf{v} \times \mathbf{B}_i\right] + \mathcal{O}(g_{\text{B}}^3). \tag{3.9}$$

Up to overall signature, the ratio of eqs. (3.9) and (3.4) yields a Lorentz force like in eq. (1.1).

The task then is to proceed to next-to-leading order (NLO). For $\mathcal{N}_0$, the computation is relatively straightforward. To remain consistently within the perturbative expansion, the 1-loop correction is evaluated at the location of the tree-level poles, i.e. terms proportional to $P_1^2 + M^2$ or $P_2^2 + M^2$ are omitted. Here we simply state the result,

$$\frac{\mathcal{N}_0^{\text{QCD,vac}}}{2} = 1 - g_{\text{B}}^2 C_{\text{F}} \int_R \left[\frac{4M^2}{R^2\Delta_{P-R}^2} + \left(\frac{2}{\Delta_{P-R}} - \frac{D-2}{2M^2}\right)\left(\frac{1}{\Delta_{P-R}} - \frac{1}{R^2}\right)\right] + \mathcal{O}(g_{\text{B}}^4), \tag{3.10}$$

---

[4]As depicted in figures 1–3, we compute to linear order in the external gauge field $\bar{A}_\mu \equiv \bar{A}_\mu^a T^a$, whereby only the Abelian part appears in the external field strength. Here T^a are Hermitean generators of SU(N_{c}).

where $C_{\mathrm{F}} \equiv (N_{\mathrm{c}}^2 - 1)/(2N_{\mathrm{c}})$, $D = 4 - 2\epsilon$ is the dimension of spacetime, R is a gluon four-momentum, and P is an on-shell heavy quark four-momentum, with $P^2 = -M^2$. Noting that scaleless integrals vanish in dimensional regularization, and inserting non-vanishing master integrals from eq. (3.20), the explicit expression reads

$$\frac{\mathcal{N}_0^{\mathrm{QCD,vac}}}{2} = 1 + \frac{g_{\mathrm{B}}^2 C_{\mathrm{F}} \mu^{-2\epsilon}}{(4\pi)^2} \left(\frac{1}{\epsilon} + \ln \frac{\bar{\mu}^2}{M^2} + 4 \right) + \mathcal{O}(\epsilon g_{\mathrm{B}}^2, g_{\mathrm{B}}^4) \,, \tag{3.11}$$

where $\bar{\mu}^2 \equiv 4\pi \mu^2 e^{-\gamma_{\mathrm{E}}}$ is the scale parameter of the $\overline{\mathrm{MS}}$ scheme.

For the NLO computation of the numerator, let us give some more details. The diagrams are shown in figure 1. Actually, the gluon self-energy diagram is not needed, as it contains scaleless integrals after the Taylor expansion in Q, and therefore vanishes in dimensional regularization. The only exception is the loop containing the heavy quark itself. The effect from here amounts to the contribution that the heavy quark gives to the running of the gauge coupling. As the low-energy side of our matching is a theory without the heavy quark, and we normally refer to the gauge coupling of that theory, this effect is trivially included.

Carrying out colour contractions in the other diagrams, there are two parts, one proportional to C_{F} and the other to $C_{\mathrm{A}} \equiv N_{\mathrm{c}}$. The part proportional to C_{F} is quite IR sensitive: whereas at leading order there is one propagator of $\mathcal{O}(1/Q)$, now there are two such propagators. These poles cancel only by working in the pole mass scheme, whereby the mass counterterm is chosen as $(M_{\mathrm{B}}^2 = M^2 + \delta M^2)$

$$\delta M^2 = -g_{\mathrm{B}}^2 C_{\mathrm{F}} \int_R \left[\frac{4M^2}{R^2 \Delta_{P-R}} + (D-2) \left(\frac{1}{R^2} - \frac{1}{\Delta_{P-R}} \right) \right] + \mathcal{O}(g_{\mathrm{B}}^4) \,. \tag{3.12}$$

We denote again the on-shell momenta of the external legs by P_1 and P_2, with $P_i = (\omega_i, \mathbf{p}_i)$; the dynamical gluon momentum by R; and the external gluon momentum by Q. After taking the Dirac trace, the first step is to eliminate scalar products like $R \cdot P_i$ or $Q \cdot R$, by completing squares and cancelling against denominators. The key issue is to verify that, after including the mass counterterm from eq. (3.12), all singular propagators, $1/\Delta_{P_i}$ and $1/\Delta_{P_i \pm Q}$, drop out. To achieve this it is important to make use of the fact the certain differences are of $\mathcal{O}(Q)$ and cancel against would-be poles, notably

$$\frac{1}{\Delta_{P_1-Q}} \left[\frac{1}{\Delta_{P_1-Q-R}} - \frac{1}{\Delta_{P_2-R}} \right] = -\frac{\omega_1 + \omega_2 - q_0 - 2r_0}{(\omega_1 + \omega_2 - q_0)\,\Delta_{P_1-Q-R}\,\Delta_{P_2-R}} \,, \tag{3.13}$$

$$\frac{1}{\Delta_{P_2+Q}} \left[\frac{1}{\Delta_{P_2+Q-R}} - \frac{1}{\Delta_{P_1-R}} \right] = -\frac{\omega_1 + \omega_2 + q_0 - 2r_0}{(\omega_1 + \omega_2 + q_0)\,\Delta_{P_2+Q-R}\,\Delta_{P_1-R}} \,, \tag{3.14}$$

where the right-hand sides are non-singular. After the elimination of the singular propagators, the non-singular ones $(1/\Delta_{P_i-R}$, $1/\Delta_{P_i \pm Q-R})$ can be Taylor-expanded in Q, with the leading terms given by $1/\Delta_{P-R}$. The gluonic propagator $1/(Q-R)^2$ can likewise be expanded. Left over are tensor integrals of the type

$$\int_R \frac{R_\mu R_\nu}{(R^2)^{i_1} \Delta_{P-R}^{i_2}} = A_{i_1 i_2} \delta_{\mu\nu} + B_{i_1 i_2} P_\mu P_\nu \,, \quad \int_R \frac{R_\mu}{(R^2)^{i_1} \Delta_{P-R}^{i_2}} = C_{i_1 i_2} P_\mu \,, \tag{3.15}$$

contracted with four-vectors like $\bar{A}(Q)$, V or Q, or with $\delta_{\mu i}$, where the index i originates from the operator. The tensor integrals can be reduced to scalar ones with the usual Passarino-Veltman reduction, e.g.

$$A_{i_1 i_2} = \frac{c_{i_1-1,i_2}}{D-1} + \frac{c_{i_1-2,i_2} - 2c_{i_1-1,i_2-1} + c_{i_1,i_2-2}}{4(D-1)M^2}\,, \tag{3.16}$$

$$B_{i_1 i_2} = \frac{c_{i_1-1,i_2}}{(D-1)M^2} + \frac{D(c_{i_1-2,i_2} - 2c_{i_1-1,i_2-1} + c_{i_1,i_2-2})}{4(D-1)M^4}\,, \tag{3.17}$$

$$C_{i_1} = \frac{c_{i_1,i_2-1} - c_{i_1-1,i_2}}{2M^2}\,, \tag{3.18}$$

where

$$c_{i_1,i_2} \equiv \int_R \frac{1}{(R^2)^{i_1}\Delta_{P-R}^{i_2}}\,. \tag{3.19}$$

Negative powers of i_1 can be dealt with by completing squares, e.g. $c_{-1,i_2} = -2M^2 c_{0,i_2} + c_{0,i_2-1}$. After this reduction, we are faced with the integrals $c_{0,1}$, $c_{0,2}$, $c_{0,3}$, $c_{1,1}$ and $c_{1,2}$. In dimensional regularization, these are related by

$$c_{0,1} = \frac{M^2 c_{0,2}}{1-\frac{D}{2}}\,, \quad c_{0,3} = \frac{\left(1-\frac{D}{4}\right)c_{0,2}}{M^2}\,, \quad c_{1,1} = \frac{c_{0,2}}{D-3}\,, \quad c_{1,2} = -\frac{c_{0,2}}{2M^2}\,, \tag{3.20}$$

where $c_{0,2} = \Gamma\left(2-\frac{D}{2}\right)/(4\pi)^{\frac{D}{2}}/(M^2)^{2-\frac{D}{2}}$.

After inserting the relations between the masters integrals, we find that all gauge dependence cancels (i.e. terms proportional to $1/\xi$, ξ, ξ^2). Moreover all terms proportional to C_A cancel in D dimensions. Terms proportional to C_F do not cancel, but they come in the same combination of electric and magnetic fields as the LO result in eq. (3.9). Furthermore the relative correction,

$$M\mathcal{A}_i^{\text{QCD,vac}} = -2ig_B\left[E_i^{(A)} - E_i^{(B)} + \mathbf{v}\times\mathbf{B}_i\right]\left\{1 - g_B^2 C_F \int_R \frac{1}{\Delta_{P-R}^2}\frac{D-5}{D-3}\right\} + \mathcal{O}(g_B^5)\,, \tag{3.21}$$

exactly matches that obtained from eq. (3.10) after inserting the relations between the masters from eq. (3.20). Therefore the ratio defined in eq. (2.3) receives no correction at NLO,

$$\frac{M\mathcal{A}_i^{\text{QCD,vac}}}{\mathcal{N}_0^{\text{QCD,vac}}} = -ig_B\left[E_i^{(A)} - E_i^{(B)} + \mathbf{v}\times\mathbf{B}_i\right] + \mathcal{O}(g_B^5)\,. \tag{3.22}$$

4 QCD thermal contribution

The next step is to repeat the computation of section 3 at finite temperature. Much remains unchanged, notably the diagrams, the Dirac contractions, and algebraic steps such as completions of squares. What changes is that the gluon four-momentum is now thermal, $R = (r_n, \mathbf{r})$, where r_n is a bosonic Matsubara frequency. Integration over R gets replaced by a Matsubara sum-integral, denoted by $\sumint_R$. As a consequence of the different measure, "scaleless" sum-integrals no longer vanish, as the temperature sets a new scale. In addition,

the symmetry group that permits to eliminate numerators from sum-integrals is smaller. Most of this section concerns how to evaluate these new master sum-integrals.

The first important issue, however, is to note that care is needed when Taylor expanding with respect to the external gluon four-momentum, which at finite temperature takes the form $Q = (q_n, \mathbf{q})$, where q_n is a bosonic Matsubara frequency. It is well-known, for instance from the context of Hard Thermal Loop effective theories [18–21], that after carrying out the Matsubara sum over r_n, gluon loops $\sim 1/[R^2(R-Q)^2]$ turn into structures like $\sim n_{\mathrm{B}}(r)/\{r[iq_n r \pm \mathbf{q} \cdot \mathbf{r} + \mathcal{O}(Q^2)]\}$. We could carry out an analytic continuation to Minkowskian frequencies, $iq_n \to q_0$. It is then clear that the result is non-analytic, e.g. with a branch cut in the domain $q > |q_0|$, leading physically to the phenomenon of Landau damping. Even though the same non-analyticities arise on the IR side of matching, it is extremely tedious to track them in an already complicated computation. These problems are absent from the Matsubara zero mode sector, $q_n = 0$. In the language of the Euclidean formulation, non-zero Matsubara modes $\sim 2\pi nT$ carry large energies, and therefore cannot be expanded in; the low-energy mode $q_n = 0$ suffers from no such problem. All in all, we therefore restrict to the Matsubara zero mode of the external gauge field in the thermal computations, viz.

$$q_n = 0 \,. \tag{4.1}$$

We note from eq. (3.8) that, consequently, the electric field denoted by $E_i^{(B)}$ is not available, but this represents no problem, because the counterpart $E_i^{(A)}$ remains present. To avoid confusion, let us stress again that the four-momentum of the *dynamical* (non-external) gauge field, denoted by R, does carry all its Matsubara frequencies.

After this elaboration, let us turn to the sum-integrals present, obtained after carrying out the Taylor expansion in $\mathbf{q}$ and $\mathbf{v}$. There are three classes of them: those sensitive only to the gluon four-momentum (R); those sensitive only to the heavy quark four-momentum $(P - R)$; and those containing both types of propagators. We discuss these in turn.

The structures only containing the gluon propagator, $\sim 1/R^2$, vanish in vacuum as scaleless integrals, but are non-zero at finite temperature. Any spatial momenta appearing in the numerator can be eliminated by Passarino-Veltman type reduction but applied in $d = 3 - 2\epsilon$ dimensions. Dimensional regularization permits also to relate a number of integrals, such as $\int_{\mathbf{r}} \frac{\mathbf{r}^2}{(R^2)^{i_1}} = \frac{d}{2(i_1-1)} \int_{\mathbf{r}} \frac{1}{(R^2)^{i_1-1}}$. The remaining 1-loop sum-integrals can be solved in terms of the Riemann ζ-function, and expansions in ϵ yield familiar expressions, e.g.

$$\sum_R \!\!\!\!\!\! \int \frac{1}{R^2} = \frac{2T\,\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{d/2}} \frac{\zeta(2-d)}{(2\pi T)^{2-d}} = \frac{T^2}{12} + \mathcal{O}(\epsilon) \,, \tag{4.2}$$

$$\sum_R \!\!\!\!\!\! \int \frac{1}{R^4} = \frac{2T\,\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{d/2}} \frac{\zeta(4-d)}{(2\pi T)^{4-d}} = \frac{\mu^{-2\epsilon}}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2\ln\left(\frac{\bar{\mu}e^{\gamma_{\mathrm{E}}}}{4\pi T}\right) + \mathcal{O}(\epsilon)\right] \,, \tag{4.3}$$

$$\sum_R \!\!\!\!\!\! \int \frac{1}{R^2 r^2} = \int_{\mathbf{r}} \frac{\frac{1}{2} + n_{\mathrm{B}}(r)}{r^3}$$

$$= -\frac{2T\,\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{d/2}} \frac{\zeta(4-d)}{(2\pi T)^{4-d}} = \frac{2\mu^{-2\epsilon}}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2\ln\left(\frac{\bar{\mu}e^{\gamma_{\mathrm{E}}}}{4\pi T}\right) + 2 + \mathcal{O}(\epsilon)\right] \,. \tag{4.4}$$

The sum-integral in eq. (4.4) originates in connection with mixed structures (see below), and we have shown a representation in terms of the Bose distribution n_{B} for later convenience.

The second class contains only heavy quark propagators,

$$\sumint_R \frac{R_\mu R_\nu \cdots}{\Delta_{P-R}^{i_1}}\,. \tag{4.5}$$

After substituting $R \to P - R$, we are faced with a fermionic Matsubara sum. Carrying it out, the thermal part of the result comes with the Fermi distribution n_{F}, which is exponentially suppressed by $\sim e^{-M/T}$. Therefore, eq. (4.5) can be replaced by its vacuum part, $\int_R R_\mu R_\nu \ldots / \Delta_{P-R}^{i_1}$, and it then evaluates to the same value as in section 3.

The third class contains mixed structures. To see what happens with them, we note that

$$\sumint_R \frac{\phi(\mathbf{r})}{(R^2 + \lambda^2)\Delta_{P-R}} = \int_R \frac{\phi(\mathbf{r})}{(R^2 + \lambda^2)\Delta_{P-R}}$$

$$+ \int_{\mathbf{r}} \frac{n_{\mathrm{B}}(\epsilon_r)}{2\epsilon_r} \left[\frac{\phi(\mathbf{r})}{\epsilon_{pr}^2 - (i\omega_n + \epsilon_r)^2} + \frac{\phi(\mathbf{r})}{\epsilon_{pr}^2 - (i\omega_n - \epsilon_r)^2} \right] \tag{4.6}$$

$$- \int_{\mathbf{r}} \frac{n_{\mathrm{F}}(\epsilon_{pr})}{2\epsilon_{pr}} \left[\frac{\phi(\mathbf{r})}{\epsilon_r^2 - (i\omega_n + \epsilon_{pr})^2} + \frac{\phi(\mathbf{r})}{\epsilon_r^2 - (i\omega_n - \epsilon_{pr})^2} \right],$$

where we denoted $P = (\omega_n, \mathbf{p})$, $\epsilon_r \equiv \sqrt{r^2 + \lambda^2}$ and $\epsilon_{pr} \equiv \sqrt{(\mathbf{p} - \mathbf{r})^2 + M^2}$. Taking derivatives with respect to λ^2 and M^2 permits to generate powers of propagators. The first term on the right-hand side of eq. (4.6) is a vacuum integral, and reproduces the effects found in section 3. The last term is exponentially suppressed like the thermal effects originating from eq. (4.5), and can be omitted. Relevant contributions originate from the middle term of eq. (4.6). The same exercise can be repeated for the case that r_n appears in the numerator, and then the middle term reads

$$\sumint_R \frac{\phi(\mathbf{r})\, r_n}{(R^2 + \lambda^2)\Delta_{P-R}} \supset \int_{\mathbf{r}} \frac{n_{\mathrm{B}}(\epsilon_r)}{2\epsilon_r} \left[\frac{\phi(\mathbf{r})\, i\epsilon_r}{\epsilon_{pr}^2 - (i\omega_n + \epsilon_r)^2} - \frac{\phi(\mathbf{r})\, i\epsilon_r}{\epsilon_{pr}^2 - (i\omega_n - \epsilon_r)^2} \right]. \tag{4.7}$$

Subsequently, we set the heavy quarks on-shell, $\omega_n \to i\epsilon_p$ like in eq. (3.3),[5] and expand the result in $\mathbf{v} = \mathbf{p}/\epsilon_p$ and T/M, where the temperature originates from the fact that $\epsilon_r \sim T$, as dictated by the Bose distribution. In this way we find that, effectively,

$$\frac{c_0 + c_1\, r_n}{\Delta_{P-R}} \xrightarrow{\text{mixed term}} \frac{c_0\, \mathbf{r} \cdot \mathbf{v} + c_1\, i\epsilon_r^2}{M\epsilon_r^2} + \ldots, \tag{4.8}$$

$$\frac{c_0 + c_1\, r_n}{\Delta_{P-R}^2} \xrightarrow{\text{mixed term}} \frac{c_0 + c_1\, 2i\mathbf{r} \cdot \mathbf{v}}{2M^2\epsilon_r^2} + \ldots, \tag{4.9}$$

$$\frac{c_0 + c_1\, r_n}{\Delta_{P-R}^3} \xrightarrow{\text{mixed term}} \frac{c_0\, 3\mathbf{r} \cdot \mathbf{v} + c_1\, i\epsilon_r^2}{4M^3\epsilon_r^4} + \ldots, \tag{4.10}$$

appearing together with $\int_{\mathbf{r}} n_{\mathrm{B}}(\epsilon_r)/(2\epsilon_r)$ that was factored out in eqs. (4.6) and (4.7).

---

[5]The precise justification for this in the thermal context is provided in section 5.

A few further remarks are in order. First, we note that if r_n^2 appears in the numerator, it can be written as $r_n^2 = R^2 + \lambda^2 - \epsilon_r^2$, and thus represented as a linear combination of the structures that were already considered. A case to watch out for is if the function $\phi(\mathbf{r})$, perhaps in combination with the right-hand sides of eqs. (4.8)–(4.10), leads to a spatial momentum squared, e.g. $r_i r_j \to \delta_{ij} r^2/d$. We may now write $r^2 = \epsilon_r^2 - \lambda^2$. If this appears in a structure with a quadratic gluon propagator, $1/R^4 = -\lim_{\lambda \to 0} \mathrm{d}/\mathrm{d}\lambda^2 \{1/(R^2 + \lambda^2)\}$, then the derivative can act on the numerator as well, implying that $r^2/[R^4 (\epsilon_r^2)^{i_1}] \to 1/[R^2 (\epsilon_r^2)^{i_1}] + 1/[R^4 (\epsilon_r^2)^{i_1 - 1}]$.

To summarize, when we send $\lambda \to 0$, thermal parts of mixed sum-integrals can be represented in terms of eqs. (4.2)–(4.4). After inserting all this to the diagrams of figure 1, we obtain results for the contribution from thermal scales. We postpone their discussion till the end of section 5, where the main result, given in eq. (5.21), is obtained in a different way.

5 Non-relativistic determination of the thermal contribution

The purpose of this section is to re-derive the result of section 4 in a different way. For practical applications, there is thus nothing new; however, on the formal side, we hope that an independent derivation can serve as a crosscheck and an illustration of the general methodology. Moreover this approach brings us in several ways rather close to section 6.

The idea is to use a non-relativistic effective theory for the computation. Whereas full QCD has two scales that we treated separately, M in section 3 and T in section 4, the scale M has essentially been eliminated from the effective theory. This permits to simplify some aspects of the computation (for instance, spin plays a trivial role and Dirac matrices do not appear), even if there is also an overhead, namely an increased number of elementary vertices.

The Euclidean action of the non-relativistic theory reads

$$S_{\mathrm{E}} = \int_X \theta^\dagger \left(D_0 + M - \frac{\mathbf{D}^2 + c_B\, g_{\mathrm{B}} \sigma \cdot \mathbf{B}}{2M} + \dots \right) \theta \,, \tag{5.1}$$

where $\int_X \equiv \int \mathrm{d}\tau \int_{\mathbf{x}}$, and $c_B = 1 + \mathcal{O}(g_{\mathrm{B}}^2)$ is a matching coefficient. Spin-dependent effects are mass-suppressed and do not contribute to our actual computation, however we have shown the term multiplied by c_B because it is needed in section 7. Even if we mentioned above that the scale M has essentially been eliminated, it is important for thermal computations to keep the rest mass explicit in eq. (5.1), as otherwise Boltzmann factors $e^{-M/T}$ go amiss.

The reason for an increased number of vertices is that eq. (5.1) contains not only a linear appearance of gauge fields, as is the case in the heavy-quark part of the QCD action, but higher powers as well. Likewise, the spatial Noether current,

$$J_i^{\mathrm{HQET}} = -\frac{\theta^\dagger (i \overleftrightarrow{D_i}) \theta}{2M} + \mathcal{O}\left(\frac{1}{M^2}\right), \tag{5.2}$$

involves terms with and without gauge fields. We note that all terms of $\mathcal{O}(1/M)$ and $\mathcal{O}(\mathbf{v}/M)$ need to be included, as the acceleration is multiplied by M in eq. (2.8).

In the non-relativistic theory, free propagators take the form

$$\langle \theta(P_1)\, \theta^\dagger(P_2) \rangle = \frac{\bar{\delta}(P_1 - P_2)}{\Omega_{P_1}}\,, \quad \Omega_{P_1} \equiv i\omega_{1n} + \epsilon_{p_1}\,, \tag{5.3}$$

where $P_1 = (\omega_{1n}, \mathbf{p}_1)$, ω_{1n} denotes a fermionic Matsubara frequency, $\epsilon_{p_1} = M + \mathbf{p}_1^2/(2M) + \ldots$, and $\oint_{P_1} \bar{\delta}(P_1) = 1$. We assume all dependence on $1/M$ to be Taylor-expanded to a given order. In the end, propagators therefore appear in a static form, i.e. as inverses of

$$\Lambda_{P_1} \equiv i\omega_{1n} + M\,. \tag{5.4}$$

Let us start with LO computations. For the denominator, where the operator reads $\int_{\mathbf{x}} \theta^\dagger(0, \mathbf{x})\theta(0, \mathbf{x}) = \oint_{P_3, P_4} \theta^\dagger(P_3)\theta(P_4)\delta^{(3)}(\mathbf{p}_3 - \mathbf{p}_4)$, eq. (2.4) leads to

$$T\sum_{\omega_{1n}} e^{\frac{i\beta\omega_{1n}}{2}}\, T\sum_{\omega_{2n}} e^{\frac{i\beta\omega_{2n}}{2}}\, \frac{2\,\delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2)}{(i\omega_{1n} + M)(i\omega_{2n} + M)} + \mathcal{O}\left(\frac{1}{M}\right)\,. \tag{5.5}$$

The Matsubara sums yield $e^{-M/T}$. Factoring out this exponential, as well as $\delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2)$, the "amplitude" corresponding to eq. (2.2) is now extracted as

$$\mathcal{N}_0^{\mathrm{HQET}} = 2 + \mathcal{O}(g_{\mathrm{B}}^2)\,. \tag{5.6}$$

For the numerator, the momentum space operator becomes

$$\partial_0 \int_{\mathbf{x}} J_i^{\mathrm{HQET}} = \oint_{P_3, P_4} i(\omega_{4n} - \omega_{3n})\, \theta^\dagger(P_3) \frac{p_{3i} + p_{4i}}{2M}\, \theta(P_4)\, \delta^{(3)}(\mathbf{p}_4 - \mathbf{p}_3) \tag{5.7}$$

$$- \oint_{Q, P_3, P_4} i(q_n + \omega_{4n} - \omega_{3n})\, \theta^\dagger(P_3) \frac{gA_i(Q)}{M}\, \theta(P_4)\, \delta^{(3)}(\mathbf{q} + \mathbf{p}_4 - \mathbf{p}_3) + \ldots\,,$$

where corrections start at $\mathcal{O}(1/M^2)$. We get a contribution from three diagrams at leading order, illustrated on the first rows of figures 1 and 2. There is an issue with singularities, similar to that discussed around eq. (3.6), but with non-relativistic propagators the cancellation is simpler,[6]

$$\frac{q_n + \omega_{2n} - \omega_{1n}}{\Omega_{P_1 - Q}} = i\,, \quad \frac{q_n + \omega_{2n} - \omega_{1n}}{\Omega_{P_2 + Q}} = -i\,. \tag{5.8}$$

After inserting the small-momentum approximations from eq. (3.7), setting $q_n = 0$ for the external gauge field as explained around eq. (4.1), Taylor-expanding, factoring out external states like around eq. (5.6), and making use of the notation in eq. (3.8), we find

$$M\,\mathcal{A}_i^{\mathrm{HQET}} = 2g_{\mathrm{B}}\left[E_i^{(A)} + \mathbf{v} \times \mathbf{B}_i\right] + \mathcal{O}(g_{\mathrm{B}}^3)\,. \tag{5.9}$$

The ratio of eqs. (5.9) and (5.6) yields a structure similar to the Lorentz force in eq. (1.1).

---

[6]To justify the use of on-shell conditions here, i.e. $\omega_{\mathrm{in}} = i\epsilon_{p_i}$, we note that by adding and subtracting a term, e.g. $\omega_{1n}/(i\omega_{1n} + \epsilon_{p_1}) = -i + i\epsilon_{p_1}/(i\omega_{1n} + \epsilon_{p_1})$ in the term containing $1/\Omega_{P_2 + Q}$, we are left with a Matsubara sum like in eq. (5.5), but with one of the terms (here, $-i$) being independent of one of the summation variables (here, ω_{1n}). These terms vanish in connection with the exponentials.
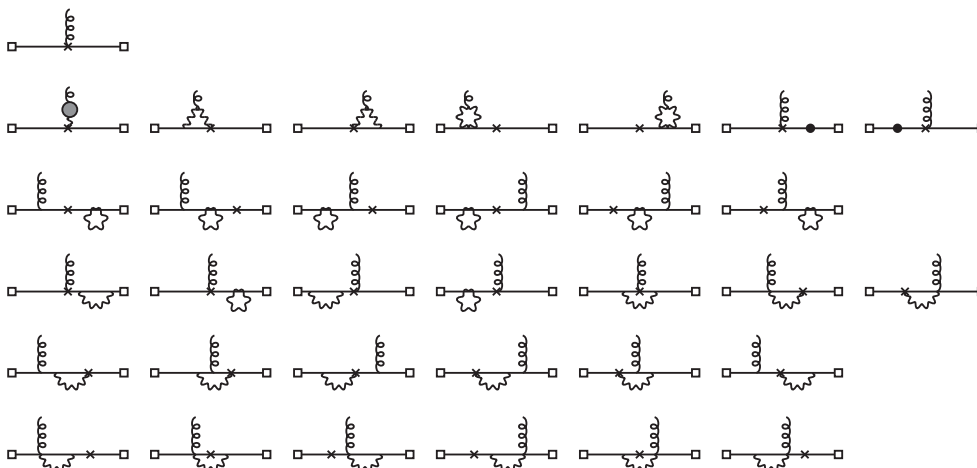
Figure 2. The *additional* LO and NLO graphs contributing to the 3-point correlator in the non-relativistic description of section 5. The notation is the same as in figure 1. We note that in a thermal medium, gluon tadpoles give a finite contribution, proportional to T^2, and must thus be included.

Proceeding to NLO, we start with the denominator, deferring the discussion of technical details to the numerator. Evaluating the NLO correction at the tree-level on-shell point, the final result reads

$$\frac{\mathcal{N}_0^{\mathrm{HQET}}}{2} = 1 - g_{\mathrm{B}}^2 C_{\mathrm{F}} \sum_R \!\!\!\!\!\!\!\! \int \left[\frac{1}{R^2 \Lambda_{P-R}^2} + \frac{1-\xi}{R^4} \right] + \mathcal{O}(g_{\mathrm{B}}^4), \tag{5.10}$$

where Λ is the inverse static propagator from eq. (5.4), and ξ is a gauge parameter. After the insertion of master sum-integrals from eqs. (4.3), (4.4) and (5.18), we obtain

$$\frac{\mathcal{N}_0^{\mathrm{HQET}}}{2} = 1 - \frac{g_{\mathrm{B}}^2 C_{\mathrm{F}} \mu^{-2\epsilon}}{(4\pi)^2} \left\{ (3-\xi) \left[\frac{1}{\epsilon} + 2\ln\left(\frac{\bar{\mu} e^{\gamma_{\mathrm{E}}}}{4\pi T} \right) \right] + 4 \right\} + \mathcal{O}(\epsilon g_{\mathrm{B}}^2, g_{\mathrm{B}}^4). \tag{5.11}$$

The gauge parameter appears because eq. (2.4) is not explicitly gauge invariant, and its ultimate cancellation serves as an important crosscheck of the computation.

Turning to the numerator, let us first discuss the mass counterterm. In order to cancel all singular propagators ($1/\Omega_{P_i}$, $1/\Omega_{P_i \pm Q}$), the mass counterterm needs to be chosen such that we are in an on-shell scheme. In the non-relativistic theory, the counterterm is analogous to that in eq. (3.12) but now with a thermal sum-integral ($M_{\mathrm{B}} = M + \delta M$),

$$\delta M = -g_{\mathrm{B}}^2 C_{\mathrm{F}} \sum_R \!\!\!\!\!\!\!\! \int \left[\frac{1}{R^2 \Omega_{P-R}} + \frac{D-1}{2MR^2} \right] + \mathcal{O}(g_{\mathrm{B}}^4). \tag{5.12}$$

In the main computation it is convenient to use this in unexpanded form, in order to guarantee that the cancellation outlined in eqs. (5.15), (5.16) takes place at an early stage, but we note in passing that if we wanted an explicit value, we could expand the propagator as

$$\sum_R \!\!\!\!\!\!\!\! \int \frac{1}{R^2 \Omega_{P-R}} = \sum_R \!\!\!\!\!\!\!\! \int \frac{1}{R^2} \left(\frac{1}{\Lambda_{P-R}} - \frac{\mathbf{r}^2}{2M} \frac{1}{\Lambda_{P-R}^2} \right) + \mathcal{O}\left(\mathbf{v}^2, \frac{T^3}{M^2} \right). \tag{5.13}$$

The sums can be performed (cf. eqs. (5.17), (5.18)) and subsequently related to that in eq. (4.2). The upshot is that the vacuum pole mass is shifted by a well-known thermal correction [22],

$$M|_T = M|_{T=0} + \frac{g^2 T^2 C_{\mathrm{F}}}{12M} \,. \tag{5.14}$$

With the mass counterterm from eq. (5.12), the cancellation of singular propagators requires the use of identities analogous to eqs. (3.13) and (3.14). In the non-relativistic theory, their form is simplified to

$$\frac{1}{\Omega_{P_1-Q}} \left[\frac{1}{\Omega_{P_1-Q-R}} - \frac{1}{\Omega_{P_2-R}} \right] = -\frac{1}{\Omega_{P_1-Q-R}\Omega_{P_2-R}} \,, \tag{5.15}$$

$$\frac{1}{\Omega_{P_2+Q}} \left[\frac{1}{\Omega_{P_2+Q-R}} - \frac{1}{\Omega_{P_1-R}} \right] = -\frac{1}{\Omega_{P_2+Q-R}\Omega_{P_1-R}} \,. \tag{5.16}$$

After the cancellation of singularities, we can Taylor-expand the non-singular propagators, obtaining powers of $1/\Lambda_{P-R}$. As explained around eq. (4.1), the Taylor expansion of the gluon propagator $1/(Q-R)^2$ is sensible only with respect to spatial momentum $\mathbf{q}$, so we restrict to the Matsubara zero mode $q_n = 0$. In order to handle the large number of diagrams, shown in figures 1 and 2, and the many terms generated by their Taylor expansions, we have made extensive use of FORM [23].

As far as the Matsubara sums go, we need to replace eq. (4.6) with its non-relativistic counterpart. The sum now reads

$$T\sum_{r_n} \frac{1}{(R^2+\lambda^2)\Lambda_{P-R}} = T\sum_{r_n} \frac{1}{(r_n^2+\epsilon_r^2)[i(\omega_n-r_n)+M]}$$
$$= \frac{1}{(i\omega_n+M)^2-\epsilon_r^2} \left\{ \frac{i\omega_n+M}{\epsilon_r} \left[\frac{1}{2}+n_{\mathrm{B}}(\epsilon_r) \right] - \left[\frac{1}{2}-n_{\mathrm{F}}(M) \right] \right\} \,. \tag{5.17}$$

Taking a derivative with respect to M and going subsequently on-shell, $\omega_n \to iM$, leads to

$$T\sum_{r_n} \frac{1}{(R^2+\lambda^2)\Lambda_{P-R}^2} \bigg|_{\omega_n=iM} = \frac{1}{\epsilon_r^2} \left[\frac{\frac{1}{2}+n_{\mathrm{B}}(\epsilon_r)}{\epsilon_r} + n_{\mathrm{F}}'(M) \right] \,. \tag{5.18}$$

For $M \gg T$, the term proportional to n_{F}' is exponentially suppressed, so after $\lambda \to 0$ we are left over with the purely bosonic term in eq. (4.4). For non-trivial numerators, e.g.

$$\sideset{}{\mkern-6mu/}\sum_R \frac{r_i r_j}{(R^2)^{i_1}\Lambda_{P-R}^{i_2}} \,, \tag{5.19}$$

the discussion in the paragraph below eq. (4.10) applies.

As a final technical remark, we note that gluon self-energy contributions, shown on the second rows of figures 1 and 2, do not need to be included. The reason is that they yield precisely the same contribution as in the IR description, whose graphs are shown in figure 3. Therefore the self-energy contribution drops out in the matching step, discussed in section 7.
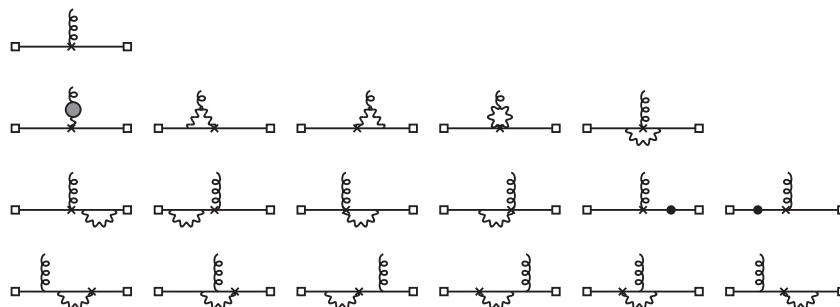
Figure 3. The LO and NLO graphs contributing to the 3-point correlator in the IR description of section 6. The notation is the same as in figure 1.

All in all the thermal NLO result for the numerator can be expressed as

$$
\frac{M\mathcal{A}_i^{\mathrm{HQET}}}{2} = g_{\mathrm{B}} \left[E_i^{(A)} + \mathbf{v} \times \mathbf{B}_i \right] \left\{ 1 - g_{\mathrm{B}}^2 C_{\mathrm{F}} \sum\!\!\!\!\!\!\int_R \left[\frac{1}{R^2 \Lambda_{P-R}^2} + \frac{1-\xi}{R^4} \right] \right\}
$$
$$
+ \frac{g_{\mathrm{B}}^3 C_{\mathrm{A}}}{2} \sum\!\!\!\!\!\!\int_R \left\{ \left[\frac{1}{R^2 \Lambda_{P-R}^2} + \frac{(d-3)(\xi-1)-2}{R^4} \right] E_i^{(A)} + \left[\frac{1-\frac{2}{d}}{R^2 \Lambda_{P-R}^2} - \frac{\frac{2}{d}}{R^4} \right] \mathbf{v} \times \mathbf{B}_i \right\}
$$
$$
+ (\text{gluon self-energy}) + \mathcal{O}(g_{\mathrm{B}}^5). \tag{5.20}
$$

The term proportional to C_{F} agrees with eq. (5.10), and thus drops out in the ratio considered in eq. (2.8). The coefficient of $\mathbf{v} \times \mathbf{B}_i$ on the second row of eq. (5.20) cancels exactly, given that the sum-integrals in eqs. (4.3) and (4.4) differ by a factor $d/2 - 1$. When the same relation is inserted into the coefficient of $E_i^{(A)}$, the result does not cancel but is proportional to $d - 3$. Because the coefficient function has a pole $\sim 1/\epsilon$, this leaves over a finite contribution,

$$
\frac{M\mathcal{A}_i^{\mathrm{HQET}}}{\mathcal{N}_0^{\mathrm{HQET}}} = g_{\mathrm{B}} \left\{ E_i^{(A)} \left[1 + \frac{g_{\mathrm{B}}^2 C_{\mathrm{A}}(3-\xi)}{(4\pi)^2} \right] + \mathbf{v} \times \mathbf{B}_i \right\}
$$
$$
+ (\text{gluon self-energy}) + \mathcal{O}(\epsilon g_{\mathrm{B}}^3, g_{\mathrm{B}}^5). \tag{5.21}
$$

Amusingly, a finite term proportional to $3 - \xi$ is familiar from rescalings discussed in the context of the effective potential for $\bar{A}_0$, cf. eqs. (3.17)–(3.18) of ref. [24].

6   Infrared side of the matching

In the preceding sections, we have computed the contributions of the vacuum ($\sim M$) and thermal ($\sim \sqrt{MT}, T$) scales to the left-hand side of eq. (2.8). The last ingredient needed for matching is to determine the right-hand side of eq. (2.8), by making use of the IR description. This is defined by restricting to a strictly static HQET action,

$$
S_{\mathrm{E}} \equiv \int_X \theta^\dagger \left(D_0 + M \right) \theta. \tag{6.1}
$$

Consequently heavy quark propagators are straight Wilson lines in the time direction. In momentum space, the inverse propagator takes the form of eq. (5.4).

We note that the IR physics of the thermal gluon sector is non-trivial, leading to non-analyticities as discussed around eq. (4.1) and requiring resummations in order to generate a consistent weak-coupling series. However, since we are matching two different computations, these IR issues drop out, as long as they have been treated in the same way on both sides of the matching. We implement this by restricting to $q_n = 0$ and by carrying out unresummed computations throughout. Even after these simplifications, nice crosschecks do remain, in particular that the gauge-dependent electric field normalization visible on the second row of eq. (5.20) is reproduced in d dimensions (cf. eq. (6.5)).

The operator for the denominator reads $\int_{\mathbf{x}} \theta^\dagger \theta$, and that for the numerator $\int_{\mathbf{x}} F_i^{\mathrm{IR}}$, where F_i^{IR} is given in eq. (2.5). As discussed below eq. (2.5), in this description $\mathbf{v}$ appears as an external parameter, whose value is fixed later on from a separate consideration (cf. section 7).

For calibration, we may once again start with LO results. For the denominator, eqs. (5.5) and (5.6) continue to hold, i.e. $\mathcal{N}_0^{\mathrm{IR}} = 2 + \mathcal{O}(g_{\mathrm{B}}^2)$. For the numerator, extracting external states like between eqs. (5.5) and (5.6), the amplitude from eq. (2.6) evaluates to

$$\mathcal{F}_i^{\mathrm{IR}} = 2g_{\mathrm{B}} \left[E_i^{(A)} + \mathbf{v} \times \mathbf{B}_i \right] + \mathcal{O}(g_{\mathrm{B}}^3). \tag{6.2}$$

The ratio of eq. (6.2) and $\mathcal{N}_0^{\mathrm{IR}}$ yields a Lorentz force like in eq. (1.1).

Proceeding to NLO, let us start by elaborating on the issue of the mass counterterm, which previously played an important role in cancelling singular propagators $\sim \mathcal{O}(1/Q)$. The counterterm takes a form obtained from the $M \to \infty$ limit of eq. (5.12), viz.

$$\delta M = -g_{\mathrm{B}}^2 C_{\mathrm{F}} \sum_R \!\!\!\!\! \int \frac{1}{R^2 \Lambda_{P-R}} + \mathcal{O}(g_{\mathrm{B}}^4). \tag{6.3}$$

The Matsubara sum can be extracted from eq. (5.17). At the on-shell point, $\omega_n \to iM$, and omitting exponentially small terms $\sim e^{-M/T}$, this yields

$$\delta M = -g_{\mathrm{B}}^2 C_{\mathrm{F}} \int_{\mathbf{r}} \frac{1}{2\epsilon_r^2} + \mathcal{O}(g_{\mathrm{B}}^4). \tag{6.4}$$

Recalling that after resummation the temporal gauge field components, which are responsible for eq. (6.3), carry a thermal mass m_{D}, eq. (6.3) corresponds to a well-known correction to a heavy-quark mass, $M|_T \supset -g^2 C_{\mathrm{F}} m_{\mathrm{D}}/(8\pi)$ [18]. However, as explained above, we do not need to carry out resummation in our actual computation. Therefore the mass counterterm gives no contribution in dimensional regularization.

With this framework, the denominator remains at the value of eq. (5.10), viz. $\mathcal{N}_0^{\mathrm{IR}} = \mathcal{N}_0^{\mathrm{HQET}}$. The numerator is determined by the graphs in figure 3. Many terms proportional to C_{F} vanish, for the same reason that the mass counterterm does not contribute. The gluon self-energy can be set aside, as it agrees with that on the high-energy side and therefore drops out in the matching. The other diagrams on the second row of figure 3 produce a non-vanishing contribution proportional to C_{A}.

The sum-integrals obtained after a Taylor expansion are in the same class as those discussed in section 5. Writing $Z_{E,B} = 1 + \delta Z_{E,B}$ where $\delta Z_{E,B} \sim \mathcal{O}(g_B^2)$, we are left with

$$
\frac{\mathcal{F}_i^{\text{IR}}}{2} = g_B \left[E_i^{(A)} + \mathbf{v} \times \mathbf{B}_i \right] \left\{ 1 - g_B^2 C_F \sum\!\!\!\!\!\!\!\int_R \left[\frac{1}{R^2 \Lambda_{P-R}^2} + \frac{1-\xi}{R^4} \right] \right\}
$$
$$
+ \frac{g_B^3 C_A}{2} \sum\!\!\!\!\!\!\!\int_R \left\{ \left[\frac{1}{R^2 \Lambda_{P-R}^2} + \frac{(d-3)(\xi-1)-2}{R^4} \right] E_i^{(A)} + \left[\frac{1}{R^2 \Lambda_{P-R}^2} - \frac{4}{R^4} \right] \mathbf{v} \times \mathbf{B}_i \right\}
$$
$$
+ g_B E_i^{(A)} \delta Z_E + g_B \mathbf{v} \times \mathbf{B}_i \, \delta Z_B + \text{(gluon self-energy)} + \mathcal{O}(g_B^5). \tag{6.5}
$$

The correction proportional to C_F agrees with that in eq. (5.10), and therefore drops out in the ratio of eq. (2.8). Inserting the values of the master sum-integrals from eqs. (4.3), (4.4) and (5.18), finally yields

$$
\frac{\mathcal{F}_i^{\text{IR}}}{\mathcal{N}_0^{\text{IR}}} = g_B E_i^{(A)} \left\{ 1 + \delta Z_E + \frac{g_B^2 C_A (3-\xi)}{(4\pi)^2} \right\}
$$
$$
+ g_B \mathbf{v} \times \mathbf{B}_i \left\{ 1 + \delta Z_B - \frac{g_B^2 C_A \, \mu^{-2\epsilon}}{(4\pi)^2} \left[\frac{1}{\epsilon} + 2\ln\left(\frac{\bar{\mu} e^{\gamma_E}}{4\pi T} \right) - 2 \right] \right\}
$$
$$
+ \text{(gluon self-energy)} + \mathcal{O}(\epsilon g_B^3, g_B^5) \tag{6.6}
$$

7 Result and discussion

In the preceding sections we have computed the objects appearing in eq. (2.8) in three different ways: in vacuum, whereby would-be loop effects originate from the scale $\sim M$ but all cancel in the end (cf. section 3); at finite temperature, by making use of a Noether current and its time derivative, thereby incorporating effects from the thermal scales $\sim \sqrt{MT}, T$ (cf. sections 4 and 5); and in an IR description, which makes use of a Lorentz force operator rather than a Noether current (cf. section 6). By requiring that the results agree, we can determine the renormalization constants of the Lorentz force operator, defined through eq. (2.5). Concretely, a comparison of eqs. (5.21) and (6.6) yields

$$
Z_E = 1 + \delta Z_E = 1 + \mathcal{O}(g^4), \tag{7.1}
$$
$$
Z_B = 1 + \delta Z_B = 1 + \frac{g^2 C_A}{(4\pi)^2} \left[\frac{1}{\epsilon} + 2\ln\left(\frac{\bar{\mu} e^{\gamma_E}}{4\pi T} \right) - 2 \right] + \mathcal{O}(g^4), \tag{7.2}
$$

where we have replaced the bare coupling g_B^2 by its renormalized value, viz.

$$
g_B^2 \, \mu^{-2\epsilon} = g^2 + \frac{g^4}{(4\pi)^2} \frac{2N_f - 11 N_c}{3\epsilon} + \mathcal{O}(g^6). \tag{7.3}
$$

It is a little bit subtle to see which scales have been integrated out through the matching steps that we have presented. Indeed, even though T appears inside the logarithm in eq. (7.2), it has not been fully eliminated, but still affects the low-energy observables that could be measured with eq. (2.5), such as eq. (7.4). Rather, what has been eliminated are

the heavy quark spatial momenta. These appear explicitly in the HQET Noether current, J_i^{HQET}, which contains derivatives acting on $\theta^\dagger$ and θ, but are absent from eq. (2.5), where $\mathbf{v}$ appears as an external parameter. In a 2-point correlator, the velocity appears in the form $\langle \mathbf{v}^2 \rangle$. It has been pointed out in ref. [4] that a field-theoretic interpretation for this average is given by the τ-independent part of the 2-point imaginary-time correlator of the vector current,[7] $\int_\mathbf{x} J_i^{\mathrm{QCD}}$, normalized to the susceptibility. This quantity has been computed up to NLO in eqs. (3.4), (3.5), (4.1) and (4.5) of ref. [25]. It accounts for the dynamics at the momentum scale $p \sim \sqrt{MT}$, and is finite after mass renormalization, indicating that this physics does not mix with the renormalization of the magnetic field at this order. The present computation has thus accounted for thermal gauge modes kicking the heavy quarks in spatial directions, and left over are thermal gauge modes not involved in such momentum transfer.

Given the subtle interpretation, it is comforting that the $1/\epsilon$-parts of eqs. (7.1) and (7.2) can be compared with literature. In the thermal context one considers 2-point correlators of the Lorentz force, normalized to the 2-point correlator of the Noether charge (i.e. susceptibility). For the magnetic field this leads to [5]

$$G_B(\tau) \;\equiv\; \frac{g_{\mathrm{B}}^2 \sum_i \mathrm{Re\,Tr} \left\langle U(\beta; \tau)\, B_i(\tau)\, U(\tau; 0)\, B_i(0)\, \right\rangle}{3\,\mathrm{Re\,Tr}\,\langle U(\beta; 0)\rangle}\,, \tag{7.4}$$

where U is a timelike Wilson line and the trace is now in colour space. The imaginary-time correlator is conveniently viewed in a spectral representation,

$$G_B(\tau) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi}\, \rho_B(\omega)\, \frac{\cosh\left[\omega\left(\frac{\beta}{2} - \tau\right)\right]}{\sinh\left[\frac{\omega\beta}{2}\right]}\,. \tag{7.5}$$

For the electric counterpart, a general argument [4] as well as a 1-loop computation [26] show that the spectral function ρ_E is rendered finite through gauge coupling renormalization, and this is consistent with eq. (7.1).[8] In contrast, for G_B, a 1-loop computation [5] shows that after gauge coupling renormalization, the spectral function is *not finite*, but rather reads

$$\rho_B(\omega) = \frac{g^2 C_{\mathrm{F}} \omega^3}{6\pi} \left[1 - \frac{g^2 C_{\mathrm{A}}}{(4\pi)^2} \frac{2}{\epsilon} + (\text{finite})\right] + \mathcal{O}(g^6)\,. \tag{7.6}$$

We now see from eq. (7.2) that multiplying the magnetic fields by Z_B, i.e. considering the correlator $Z_B^2\, G_B(\tau)$, the divergence in eq. (7.6) duly cancels.

A completely different crosscheck originates from vacuum computations, concerning the operator multiplied by c_B in eq. (5.1), known as the chromomagnetic moment. In our notation, the 1-loop result for c_B [28] can be expressed as

$$c_B = 1 + \frac{g^2}{(4\pi)^2} \left\{ C_{\mathrm{A}} \left[\frac{1}{\epsilon} + \ln\frac{\bar{\mu}^2}{M^2} + 2\right] + 2C_{\mathrm{F}} \right\} + \mathcal{O}(g^4)\,. \tag{7.7}$$

---

[7]Or, in real frequency space, by the area under the transport peak in the corresponding spectral function.
[8]In lattice regularization, a finite renormalization factor of $\mathcal{O}(g^2)$ is however needed [27].

Even if the chromomagnetic moment concerns spin-dependent effects, the magnetic field appears in the same form in eqs. (5.1) and (2.5), $\sim g_{\rm B}\mathbf{B}$. Indeed the anomalous dimension visible in eq. (7.7) agrees with that in eq. (7.2).

Going to higher orders, we could possibly profit from the fact that the anomalous dimension of the chromomagnetic moment has been determined up to 2-loop [29, 30] and 3-loop level [31]. Furthermore, non-perturbative renormalization in terms of a renormalization group invariant (RGI) operator has been worked out [32], corresponding to $\bar{\mu} \to \infty$. After such a non-perturbative renormalization, results should be run down to the $\overline{\rm MS}$ scale $\bar{\mu} \simeq 4\pi T e^{1-\gamma_{\rm E}} \approx 19.179 T$ according to eq. (7.2). Given that no pole mass ambiguity appears, unlike in eq. (7.7), and that there is a large numerical prefactor, a reasonable precision could be hoped for.

To summarize, all ingredients needed for estimating the influence of magnetic interactions on heavy quark diffusion should now be available, at least in an approximate form.

Acknowledgments

References

[1] R. Rapp et al., *Extraction of Heavy-Flavor Transport Coefficients in QCD Matter*, *Nucl. Phys. A* **979** (2018) 21 [arXiv:1803.03824] [inSPIRE].

[2] T. Bringmann, *Particle Models and the Small-Scale Structure of Dark Matter*, *New J. Phys.* **11** (2009) 105027 [arXiv:0903.0189] [inSPIRE].

[3] J. Casalderrey-Solana and D. Teaney, *Heavy quark diffusion in strongly coupled $\mathcal{N} = 4$ Yang-Mills*, *Phys. Rev. D* **74** (2006) 085012 [hep-ph/0605199] [inSPIRE].

[4] S. Caron-Huot, M. Laine and G.D. Moore, *A Way to estimate the heavy quark thermalization rate from the lattice*, *JHEP* **04** (2009) 053 [arXiv:0901.1195] [inSPIRE].

[5] A. Bouttefeux and M. Laine, *Mass-suppressed effects in heavy quark diffusion*, *JHEP* **12** (2020) 150 [arXiv:2010.07316] [inSPIRE].

[6] G.D. Moore and D. Teaney, *How much do heavy quarks thermalize in a heavy ion collision?*, *Phys. Rev. C* **71** (2005) 064904 [hep-ph/0412346] [inSPIRE].

[7] F.J. Dyson, *Feynman's proof of the Maxwell equations*, *Am. J. Phys.* **58** (1990) 209 [inSPIRE].

[8] H.B. Meyer, *The errant life of a heavy quark in the quark-gluon plasma*, *New J. Phys.* **13** (2011) 035008 [arXiv:1012.0234] [inSPIRE].

[9] A. Francis, O. Kaczmarek, M. Laine and J. Langelage, *Towards a non-perturbative measurement of the heavy quark momentum diffusion coefficient*, *PoS* **LATTICE 2011** (2011) 202 [arXiv:1109.3941] [inSPIRE].

[10] D. Banerjee, S. Datta, R. Gavai and P. Majumdar, *Heavy Quark Momentum Diffusion Coefficient from Lattice QCD*, *Phys. Rev. D* **85** (2012) 014510 [arXiv:1109.5738] [INSPIRE].

[11] A. Francis, O. Kaczmarek, M. Laine, T. Neuhaus and H. Ohno, *Nonperturbative estimate of the heavy quark momentum diffusion coefficient*, *Phys. Rev. D* **92** (2015) 116003 [arXiv:1508.04543] [INSPIRE].

[12] N. Brambilla, V. Leino, P. Petreczky and A. Vairo, *Lattice QCD constraints on the heavy quark diffusion coefficient*, *Phys. Rev. D* **102** (2020) 074503 [arXiv:2007.10078] [INSPIRE].

[13] L. Altenkort, A.M. Eller, O. Kaczmarek, L. Mazur, G.D. Moore and H.-T. Shu, *Heavy quark momentum diffusion from the lattice using gradient flow*, *Phys. Rev. D* **103** (2021) 014511 [arXiv:2009.13553] [INSPIRE].

[14] L.F. Abbott, *The Background Field Method Beyond One Loop*, *Nucl. Phys. B* **185** (1981) 189 [INSPIRE].

[15] A.V. Manohar and M.B. Wise, *Heavy quark physics*, in *Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology* **10**, Cambridge University Press, Cambridge U.K. (2000).

[16] A.G. Grozin, *Heavy quark effective theory*, in *Springer Tracts in Modern Physics* **201**, Springer (2004).

[17] R. Sommer, *Heavy Quark Effective Theory: A Predictive EFT on the Lattice*, in *Lecture Notes of the Les Houches Summer School* **108**, Oxford University Press, Oxford U.K. (2020).

[18] R.D. Pisarski, *Scattering Amplitudes in Hot Gauge Theories*, *Phys. Rev. Lett.* **63** (1989) 1129 [INSPIRE].

[19] J. Frenkel and J.C. Taylor, *High Temperature Limit of Thermal QCD*, *Nucl. Phys. B* **334** (1990) 199 [INSPIRE].

[20] E. Braaten and R.D. Pisarski, *Soft Amplitudes in Hot Gauge Theories: A General Analysis*, *Nucl. Phys. B* **337** (1990) 569 [INSPIRE].

[21] J.C. Taylor and S.M.H. Wong, *The Effective Action of Hard Thermal Loops in QCD*, *Nucl. Phys. B* **346** (1990) 115 [INSPIRE].

[22] J.F. Donoghue, B.R. Holstein and R.W. Robinett, *Quantum Electrodynamics at Finite Temperature*, *Annals Phys.* **164** (1985) 233 [*Erratum ibid.* **172** (1986) 483] [INSPIRE].

[23] J. Kuipers, T. Ueda, J.A.M. Vermaseren and J. Vollinga, *FORM version 4.0*, *Comput. Phys. Commun.* **184** (2013) 1453 [arXiv:1203.6543] [INSPIRE].

[24] C.P. Korthals Altes, *Constrained effective potential in hot QCD*, *Nucl. Phys. B* **420** (1994) 637 [hep-th/9310195] [INSPIRE].

[25] Y. Burnier and M. Laine, *Massive vector current correlator in thermal QCD*, *JHEP* **11** (2012) 086 [arXiv:1210.1064] [INSPIRE].

[26] Y. Burnier, M. Laine, J. Langelage and L. Mether, *Colour-electric spectral function at next-to-leading order*, *JHEP* **08** (2010) 094 [arXiv:1006.0867] [INSPIRE].

[27] C. Christensen and M. Laine, *Perturbative renormalization of the electric field correlator*, *Phys. Lett. B* **755** (2016) 316 [arXiv:1601.01573] [INSPIRE].

[28] E. Eichten and B.R. Hill, *Static effective field theory: 1/m corrections*, *Phys. Lett. B* **243** (1990) 427 [INSPIRE].

[29] G. Amorós, M. Beneke and M. Neubert, *Two loop anomalous dimension of the chromomagnetic moment of a heavy quark*, *Phys. Lett. B* **401** (1997) 81 [hep-ph/9701375] [INSPIRE].

[30] A. Czarnecki and A.G. Grozin, *HQET chromomagnetic interaction at two loops*, *Phys. Lett. B* **405** (1997) 142 [*Erratum ibid.* **650** (2007) 447] [hep-ph/9701415] [INSPIRE].

[31] A.G. Grozin, P. Marquard, J.H. Piclum and M. Steinhauser, *Three-loop chromomagnetic interaction in HQET*, *Nucl. Phys. B* **789** (2008) 277 [arXiv:0707.1388] [INSPIRE].

[32] D. Guazzini, H.B. Meyer and R. Sommer, *Non-perturbative renormalization of the chromo-magnetic operator in Heavy Quark Effective Theory and the $B^* - B$ mass splitting*, *JHEP* **10** (2007) 081 [arXiv:0705.1809] [INSPIRE].